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Boundary regularity of minimizers of $p(x)$-energy functionals

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Abstract

The paper is devoted to the study of the regularity on the boundary $\partial\Omega$ of a bounded open set $\Omega \subset \mathbb{R}^m$ for minimizers u for $p(x)$-energy functionals of the following type

$$\mathcal{E}(u;\Omega) := \int_{\Omega} \left(g^{\alpha\beta}(x) G_{ij}(u) D_\alpha u^i(x) D_\beta u^j(x)\right)^{p(x)/2} dx$$

where $(g^{\alpha\beta}(x))$ and $(G_{ij}(u))$ are symmetric positive definite matrices whose entries are continuous functions and $p(x) \geq 2$ is a continuous function. The authors prove that such minimizers u have no singular points on the boundary.

Résumé

Dans cet article, les auteurs étudient la régularité sur la frontière $\partial\Omega$ d'un ouvert borné $\Omega \subset \mathbb{R}^m$ des minimiseurs u des fonctionnelles d'énergie $p(x)$ du type suivant :

$$\mathcal{E}(u;\Omega) := \int_{\Omega} \left(g^{\alpha\beta}(x) G_{ij}(u) D_\alpha u^i(x) D_\beta u^j(x)\right)^{p(x)/2} dx,$$

où $(g^{\alpha\beta}(x))$ et $(G_{ij}(u))$ sont des matrices symétriques définies positives dont les éléments sont des fonctions continues et $p(x) \geq 2$ est une fonction continue. Les auteurs prouvent que ces minimiseurs u n'ont pas de point singulier sur la frontière $\partial\Omega$.

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1. Introduction

Let $\Omega \subset \mathbb{R}^m$ $(m \geq 2)$ be a bounded open set. For maps $u : \Omega \to \mathbb{R}^n$ we consider the $p(x)$-*energy functional* defined as

$$\mathcal{E}(u; \Omega) := \int_\Omega \left(g^{\alpha\beta}(x) G_{ij}(u) D_\alpha u^i(x) D_\beta u^j(x)\right)^{p(x)/2} dx, \tag{1.1}$$

where $(g^{\alpha\beta}(x))$ and $(G_{ij}(u))$ are symmetric positive definite matrices whose entries are continuous functions defined on Ω and $\mathbb{R}^n$ respectively, and $p(x)$ in a continuous function on Ω with $p(x) \geq 2$. Greek indices $\alpha, \beta, \ldots$ are to be summed from 1 to m, and Latin indices $i, j, \ldots$ from 1 to n. The Einstein summation convention is used. In the following we write, for the integrand of (1.1),

$$e(u)(x) := g^{\alpha\beta}(x) G_{ij}(u) D_\alpha u^i(x) D_\beta u^j(x). \tag{1.2}$$

The aim of this paper is to study the boundary regularity of the minimizers of the $p(x)$-energy functionals.

The functional $\mathcal{E}$ is a particular case of the functionals of the type

$$\mathcal{F}(u; \Omega) = \int_\Omega f(x, u, Du) dx, \tag{1.3}$$

where $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}$ is a Carathéodory function satisfying the following so-called (p, q)-*growth* condition: there exist constants $\Lambda \geq \lambda > 0$, $q \geq p \geq 1$ such that

$$\lambda |\xi|^p \leq f(x, u, \xi) \leq \Lambda\left(1 + |\xi|^q\right) \tag{1.4}$$

for all $(x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$. We call $\mathcal{F}$ a functional with *standard growth* if $p = q$, and with *non-standard growth* if $q > p$. If the integrand $f = f_{p(x)}$ satisfies

$$\lambda |\xi|^{p(x)} \leq f_{p(x)}(x, u, \xi) \leq \Lambda\left(1 + |\xi|^{p(x)}\right), \tag{1.5}$$

for all $(x, u, \xi) \in \Omega \times R^n \times \mathbb{R}^{mn}$, then

$$\mathcal{F}_{p(x)}(u; \Omega) := \int_\Omega f_{p(x)}(x, u, Du) dx \tag{1.6}$$

is called a functional with $p(x)$-*growth*. The $p(x)$-energy functional $\mathcal{E}$ is a $p(x)$-growth functional with a special structure.

Non-standard growth problems are attracting great interest, since Marcellini treated them in [13]. Especially, in the last two decades, about the regularity of minimizers for $p(x)$-growth functionals, considerable progress has been made. In 1995, Zhikov [17] studied Lavrentiev phenomenon for the functional

$$\mathcal{D}_{p(x)}(u) := \int_\Omega |Du|^{p(x)} dx. \tag{1.7}$$

He also obtained higher integrability results for the minimizers of $\mathcal{D}_{p(x)}$ in [18]. On the regularity of minimizers of $\mathcal{D}_{p(x)}$, a fundamental result was established by Coscia and Mingione [4] in 1999. They proved that a minimizer u of $\mathcal{D}_{p(x)}$ is in the class $C^{1,\alpha}(\Omega)$ under the condition that $p(x)$ is Hölder continuous.

For general $p(x)$-growth functionals, interior partial regularity results are obtained in [1–3,7–9].

For $p(x)$-energy $\mathcal{E}$, Ragusa, Tachikawa and Takabayashi [15] obtained interior partial regularity of minimizers; they showed that the singular set S_u of a minimizer u can have Hausdorff dimension $\dim^{\mathcal{H}}(S_u)$ at most m-$\inf p(x)$. In [14] the interior everywhere regularity was shown under the so-called *one-sided condition*. In [16], assuming the boundedness of a minimizer u, the second author improved the estimate on the Hausdorff dimension of S_u as $\dim_{\mathcal{H}}(S_u) \leq m$-$[\inf p(x)] - 1$, where [ ] stands for the Gauss symbol.

In this paper, we treat boundary regularity of minimizers for $p(x)$-energy $\mathcal{E}$. For standard growth case, Jost and Meier [12] proved that a minimizers for certain quadratic functionals cannot have singular points on the boundary.

Duzaar, Grotowski and Kronz [6] generalized this result to general p-energy functionals

$$\int_\Omega \big(g^{\alpha\beta}(x)G_{ij}(x,u)D_\alpha u^i(x)D_\beta u^j(x)\big)^{p/2}dx,$$

for $p>1$. The aim of this paper is to obtain such kind of boundary regularity results for $p(x)$-energy.

Now, let us introduce some conditions and definitions in order to state the main result. We consider the following conditions on $g^{\alpha\beta}(x)$, $G_{ij}(u)$ and $p(x)$.

(C1) There exist positive constants λ_g, Λ_g, λ_G, Λ_G such that

$$\lambda_g|\zeta|^2 \le g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \le \Lambda_g|\zeta|^2, \qquad \lambda_G|\eta|^2 \le G_{ij}(u)\eta^i\eta^j \le \Lambda_G|\eta|^2 \tag{1.8}$$

for all $x\in\Omega$, $\zeta\in\mathbb{R}^m$ and $u,\eta\in\mathbb{R}^n$.

(C2) The exponent $p(x)$ and the coefficients $g^{\alpha\beta}(x)$, $G_{ij}(u)$ are Hölder continuous; there exist positive constants $\tau,\tau',\sigma<1$, L_p, L_g, and L_h such that

$$\big|p(x)-p(y)\big| \le L_p|x-y|^\sigma/2 =: \omega_p\big(|x-y|/2\big) \quad \text{for all } x,y\in\Omega, \tag{1.9}$$

$$\big|g^{\alpha\beta}(x)-g^{\alpha\beta}(y)\big| \le L_g|x-y|^\tau =: \omega_g\big(|x-y|\big) \quad \text{for all } x,y\in\Omega, \tag{1.10}$$

$$\big|G_{ij}(u)-G_{ij}(v)\big| \le L_h|u-v|^{\tau'} =: \omega_G\big(|u-v|^2\big) \quad \text{for all } u,v\in\mathbb{R}^n. \tag{1.11}$$

(C3) The exponent $p(x)$ satisfies

$$2\le\gamma_1 := \inf_{x\in\Omega} p(x) \le \sup_{x\in\Omega} p(x) =: \gamma_2 < +\infty. \tag{1.12}$$

Let $\Omega\subset\mathbb{R}^m$ be a bounded domain with smooth boundary $\partial\Omega$. In the following, for a function $w:\Omega\to\mathbb{R}^k$ and a measurable set $D\subset\Omega$, we write

$$⨍_D w(x)dx := \frac{1}{|D|}\int_D w(x)dx,$$

where $|D|$ denotes the Lebesgue measure of D. For a ball $B(y,r) := \{x\in\mathbb{R}^m;\ |x-y|<0\}$, we write

$$w_{y,r} := ⨍_{B(y,r)\cap\Omega} w(x)dx.$$

When there is no doubt of confusion, we omit the center y and set $w_r := w_{y,r}$.

Let us define some function spaces. For a bounded open set $\Omega\subset\mathbb{R}^m$ and a function $p:\Omega\to[1,+\infty)$, we define $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ as follows:

$$L^{p(x)} := \left\{u\in L^1(\Omega);\ \int_\Omega |u|^{p(x)}dx < +\infty\right\}.$$

$$W^{1,p(x)} := \big\{u\in L^{p(x)}\cap W^{1,1}(\Omega);\ Du\in L^{p(x)}(\Omega)\big\}.$$

We also define $L^{p(x)}_{\mathrm{loc}}(\Omega)$ and $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ similarly.

As mentioned in [5], if $p(x)$ is uniformly continuous and $\partial\Omega$ satisfies uniform cone property, then

$$W^{1,p(x)}(\Omega) = \big\{u\in W^{1,1}(\Omega);\ Du\in L^{p(x)}(\Omega)\big\}.$$

In any case, if $p(x)$ is continuous in Ω, we have

$$W^{1,p(x)}_{\mathrm{loc}}(\Omega) = \big\{u\in W^{1,1}_{\mathrm{loc}}(\Omega);\ |Du|^{p(x)}\in L^1_{\mathrm{loc}}(\Omega)\big\}.$$

We also define

$$W^{1,p(x)}_0(\Omega) := \left\{u\in W^{1,1}_0(\Omega);\ \int_\Omega |Du|^{p(x)}dx < \infty\right\},$$

and for a given map φ

$$\varphi + W_0^{1,p(x)}(\Omega) := \{u \in W^{1,p(x)}(\Omega);\ u - \varphi \in W_0^{1,p(x)}(\Omega)\}.$$

A map $u \in W_{\text{loc}}^{1,p(x)}(\Omega)$ is called to be a *local minimizer* of $\mathcal{F}_{p(x)}$ if it satisfies

$$\mathcal{F}_{p(x)}(u; \operatorname{supp}\varphi) \le \mathcal{F}_{p(x)}(u + \varphi; \operatorname{supp}\varphi),$$

for any $\varphi \in W_0^{1,p(x)}(\Omega)$ with compact support in Ω.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Assume that $g^{\alpha\beta}(x)$, $G_{ij}(u)$ and $p(x)$ satisfy the conditions (C1)–(C3) on Ω. Let $u \in W^{1,p(x)}(\Omega)$ be a bounded minimizer of the functional $\mathcal{E}(v;\Omega)$ defined by*

$$\mathcal{E}(v;\Omega) := \int_\Omega g^{\alpha\beta}(x) G_{ij}(u) D_\alpha u^i(x) D_\beta u^j(x)\,dx,$$

in the class

$$h + W_0^{1,p(x)}(\Omega) := \{v \in W^{1,p(x)}(\Omega);\ v - h \in W_0^{1,p(x)}(\Omega)\}, \tag{1.13}$$

for a given boundary data $h \in W^{1,s}(\Omega)$ for some $s > m$. Then u is Hölder continuous near the boundary $\partial\Omega$.

2. Notation and preliminary results

Throughout this paper we use the following notation: for $x_0 = (x_0^1, \dots, x_0^{m-1}, 0)$ and $r > 0$, we put

$$B^+(x_0, r) := \{x \in \mathbb{R}^m;\ |x - x_0| < r,\ x^m > 0\},$$
$$\Gamma(x_0, r) := \{x \in \mathbb{R}^m;\ |x - x_0| < r,\ x^m = 0\},$$
$$\partial^+ B^+(x_0, r) := \partial B^+(x_0, r) \setminus \Gamma(x_0, r).$$

When $x_0 = 0$, we omit the center $x_0 = 0$ and write simply

$$B^+(r) := B^+(0, r), \qquad \Gamma(r) := \Gamma(0, r), \qquad \partial^+ B^+(r) := \partial^+ B^+(0, r).$$

For $x \in B^+(x_0, R)$ and $r < \operatorname{dist}(x, \partial B(x_0, R)) = R - |x - x_0|$, we put

$$\Omega(x, r) := B(x, r) \cap B^+(x_0, R),$$
$$p_1(x, r) := \inf_{\Omega(x,r)} p(y), \qquad p_2(x, r) := \sup_{\Omega(x,r)} p(y). \tag{2.1}$$

For p_1 and p_2, when the center $x = x_0$ is clearly understood, we abbreviate as

$$p_1(r) := p_1(x, r), \qquad p_2(r) := p_2(x, r).$$

When we consider the behavior of the solution near the boundary point $x_0 \in \partial\Omega$, we flatten the $\partial\Omega$ so that $x_0 = (0, \dots, 0)$, $B(x_0, R_1) \cap \Omega = B^+(0, R_1)$ for some $R_1 > 0$ and $\partial\Omega \cap B(x_0, R_1) = \Gamma(0, R_1)$.

We use c without subscript as generic constants, which may change from line to line, but does not depend on the crucial quantities.

Let $\omega_1 : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function with $\omega_1(0) = 0$ which represents the modulus of continuity, namely ω_1 satisfies

$$|p(x) - p(y)| \le \omega_1(|x - y|). \tag{2.2}$$

Let us consider the following condition on ω_1.

$$\lim_{r \to 0} \omega_1(r) \log\left(\frac{1}{r}\right) = \mu_0 < +\infty. \tag{2.3}$$

The above condition implies

$$(1/t)^{\omega_1(t)} = \exp\bigl(-\log t\,\omega_1(t)\bigr) \to e^{\mu_0} \quad \text{as } t \to 0^+. \tag{2.4}$$

When ω_1 satisfies (2.3), we say that $p(x)$ is *logarithmic continuous*. We mention also that if $p(x)$ is Hölder continuous, then the condition (2.3) is fulfilled.

For a continuous function $p(x) > 1$ on Ω satisfying (2.2) with (2.3), let $f_{p(x)}(x, u, \xi)$ be a Carathéodory function on $\Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$ which satisfies the growth condition (1.5). We define $\mathcal{F}_{p(x)}(w, \Omega)$ by (1.6).

Let us begin with remembering the following higher integrability results on local minimizers that is originally shown by Zhikov [18] and is generalized by Acerbi and Mingione [1].

Proposition 2.1. *(See [7, Theorem 3.1].) Let $\mathcal{F}_{p(x)}$ be a functional as above. Assume that the exponent $p(x) > 1$ has modulus of continuity ω_1 which satisfies (2.3). Let $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a local minimizer of $\mathcal{F}_{p(x)}$. Then, there exists a constant $\delta > 0$ such that $|Du|^{(1+\delta)p(x)} \in L^1_{\mathrm{loc}}(\Omega)$. Moreover, the estimate*

$$\fint_{B(y,R)} |Du|^{(1+\delta)p(x)} dx \le c_0 \biggl(\fint_{B(y,2R)} \bigl(1 + |Du|^2\bigr)^{p(x)/2} dx \biggr)^{(1+\delta)} \tag{2.5}$$

holds for any $B(y, 2R) \Subset \Omega$.

When we consider the functional $\mathcal{F}_{p(x)}$ on $B^+(T)$, as in [12] for the case $p(x) = 2$, let us call a map $v \in \bigcap_{T'<T} W^{1,p(x)}(B^+(T'), \mathbb{R}^n)$ a *local minimizer* of the functional

$$\mathcal{F}_{p(x)}\bigl(w, B^+(T)\bigr) = \int_{B^+(T)} f_{p(x)}(x, w, Dw)dx$$

in $B^+(T) \cup \Gamma(T)$, if for every $T' < T$ and any $\varphi \in W^{1,1}_0(B^+(T'), \mathbb{R}^n)$ the following inequality holds:

$$\mathcal{F}_{p(x)}\bigl(v, B^+\bigl(T'\bigr)\bigr) \le \mathcal{F}_{p(x)}\bigl(v + \varphi, B^+\bigl(T'\bigr)\bigr).$$

Then we have the following lemma on the higher integrability of such local minimizers.

Proposition 2.2. *(See [15, Lemma 3.2].) Assume that the exponent $p(x) > 1$ has modulus of continuity ω_1 which satisfies (2.3). Let $p_1 := \inf_{B^+(T)} p(x)$ and $p_2 := \sup_{B^+(T)} p(x)$, and suppose that*

$$(p_2)_* = \frac{mp_2}{m + p_2} < p_1 \quad \biggl(\text{or equivalently } p_1^* = \frac{mp_1}{m - p_1} > p_2\biggr). \tag{2.6}$$

For some $\varepsilon > 0$, let h be a given map in the class $W^{1,(1+\varepsilon)p(x)}(B^+(T))$. Let v be a local minimizer of $\mathcal{F}_{p(x)}$ in the class

$$\bigl\{w \in W^{1,p(x)}\bigl(B^+(T), \mathbb{R}^n\bigr);\ w = h \text{ on } \Gamma(T)\bigr\}.$$

Then, there exists a positive constant $\hat{\delta} < \varepsilon$ such that for any $\delta \in (0, \hat{\delta})$ the local minimizer v satisfies $v \in W^{1,(1+\delta)p(x)}(B^+(T'))$ for any $T' < T$. Moreover, if $x_0 \in B^+(T') \cup \Gamma(T')$ and $r < T - T'$, we have

$$\biggl(\fint_{\Omega(x_0,r/2)} \bigl(1 + |Dv|^2\bigr)^{(1+\delta)p(x)/2} dx \biggr)^{1/(1+\delta)}$$
$$\le c_1 \fint_{\Omega(x_0,r)} \bigl(1 + |Dv|^2\bigr)^{p(x)/2} dx + c_2 \biggl(\fint_{\Omega(x_0,r)} \bigl(1 + |Dh|^2\bigr)^{(1+\delta)p(x)/2} dx \biggr)^{1/(1+\delta)}, \tag{2.7}$$

where we put $\Omega(y, \rho) := B(y, \rho) \cap B^+(T)$.

By virtue of Propositions 2.1 and 2.2, we have the following estimate of the minimizer.

Corollary 2.3. *(See [15, Corollary 3.3].) Let $D \subset \mathbb{R}^m$ be a bounded domain with smooth boundary ∂D. Let $S > 0$ be a positive number which satisfies the following conditions.*

(S1) *$p_1 = p_1(x, 4S)$ and $p_2 = p_2(x, 4S)$ satisfy (2.6).*
(S2) *There is a diffeomorphism $\psi : B(y, 4S) \to B(T)$ which satisfies*

$$\psi\big(B(y,4S)\cap D\big) \subset B^+(T) \quad \text{and} \quad \psi\big(B(y,4S)\cap \partial D\big) = \Gamma(T).$$

Assume that $p(x)$, $h(x)$ satisfy assumptions in Proposition 2.2, v be a minimizer of $\mathcal{F}_{p(x)}(\cdot, D)$ in the class

$$\big\{w \in W^{1,p(x)}\big(D, \mathbb{R}^n\big);\ w - h \in W_0^{1,1}\big(D; \mathbb{R}^n\big)\big\}.$$

Then there exists a constant $\hat{\delta} \in (0, \varepsilon)$ such that for any $\delta \in (0, \hat{\delta}]$, we have that $v \in W^{p(x)(1+\delta)}(D, \mathbb{R}^n)$ and that

$$\int_D \big(1 + |Dv|^2\big)^{(1+\delta)p(x)/2} dx \le c_3\big(1 + |D|^\delta S^{-m\delta}\big) \int_D \big(1 + |Dh|^2\big)^{(1+\delta)p(x)/2} dx, \tag{2.8}$$

where c_3 depends only on m, λ, Λ, $p(x)$, $\mathcal{F}_{p(x)}(h, D)$.

We also mention that we have Caccioppoli-type inequality with boundary value by [15, (3.14)].

Lemma 2.4. *Let v be a minimizer of $\mathcal{F}_{p(x)}(\cdot, \Omega)$ in the class*

$$\big\{w \in W^{1,p(x)}\big(\Omega, \mathbb{R}^n\big);\ w = h \text{ on } \partial\Omega\big\}.$$

Then we have

$$\int_{\Omega_{r/2}} |Dv|^{p(x)} dx \le c_4\bigg(\int_{\Omega_r} \bigg(\frac{|v-h|}{r}\bigg)^{p(x)} dx + \int_{\Omega_r} |Dh|^{p(x)} dx\bigg), \tag{2.9}$$

where c_4 depends only on λ, Λ and $p(x)$.

Using the above lemma and Corollary 2.3 with $D = \Omega$, it comes out the following estimates for the derivatives of bounded minimizers.

Corollary 2.5. *Let v be as in the previous lemma. Assume that the boundary condition h satisfies*

$$⨍_{\Omega(y,r)} |Dh|^{(1+\varepsilon)p(x)} dx \le c_h r^{-\gamma_1} \tag{2.10}$$

for some constants $\varepsilon \in (0, 1)$ and $c_h > 0$, and that for some positive constant M

$$\operatorname*{esssup}_{x\in\Omega}\big|v(x)\big|, \qquad \operatorname*{esssup}_{x\in\Omega}\big|h(x)\big| \le M$$

hold for some positive constant M. Then, we have the following estimates for some constants c_5 and c_6 depending only on λ, Λ, $p(x)$, and c_h.

$$⨍_{\Omega(y,r)} |Dv|^{p(x)} dx \le c_5 r^{-p_2(y,2r)}, \tag{2.11}$$

$$⨍_{\Omega(y,r)} |Dv|^{(1+\delta)p(x)} dx \le c_6 r^{-p_2(y,4r)(1+\delta)}, \tag{2.12}$$

where δ is arbitrary constant with $\delta \in (0, \hat{\delta})$ for $\hat{\delta}$ in Corollary 2.3.

Proof. Without loss in generality we can assume that c_h, $M \ge 1$ and that $r \in (0, 1)$. Since $p_2(2r) = p_2(y, 2r) \ge p(x) \ge \gamma_1$ in $\Omega(y, 2r)$, from (2.9) and the assumptions on v and h, we have

$$\begin{aligned}
\fint_{\Omega(y,r)} |Dv|^{p(x)}dx &\le c\Bigl(\fint_{\Omega(y,2r)} \left|\frac{v(x)-h(x)}{2r}\right|^{p(x)} dx + c_h(2r)^{-\gamma_1}\Bigr)\\
&\le c\bigl(M^{p_2(2r)}r^{-p_2(2r)} + c_h(2r)^{-p_2(2r)}\bigr)\\
&\le c_5 r^{-p_2(2r)},
\end{aligned}$$

for some positive constant c_5 depending only on M and c_h. This is nothing but (2.11).

From (2.7), (2.10) and (2.11), we have

$$\begin{aligned}
\fint_{\Omega(y,2r)} |Dv|^{(1+\delta)p(x)}dx &\le c_1\Bigl(\fint_{\Omega(y,2r)} \bigl(1+|Dv|^2\bigr)^{p(x)/2}dx\Bigr)^{(1+\delta)}\\
&\quad + c_2\Bigl(\fint_{\Omega(y,r)} \bigl(1+|Dh|^2\bigr)^{(1+\delta)p(x)/2}dx\Bigr)^{1/(1+\delta)}\\
&\le c_6 r^{-p_2(4r)(1+\delta)}.
\end{aligned}$$

Thus we get (2.12) also. □

In what follows, we are fixing a constant $\delta \in (0,1)$ so that the above lemmas and propositions hold and that

$$m\delta < \sigma \tag{2.13}$$

We prepare the boundary version of the regularity result by Coscia and Mingione [4] for minimizers of the functional

$$\mathcal{D}_{p(x)}(w, D) := \int_D |Dw|^{p(x)}dx. \tag{2.14}$$

Theorem 2.6. *Assume that $p(x)$ satisfies (1.9). Let $R > 0$ be sufficiently small so that*

$$\Bigl(1+\frac{\delta}{2}\Bigr)p_2(2R) \le (1+\delta)p_1(2R). \tag{2.15}$$

Let $v \in W^{1,p(x)}(B^+(R), \mathbb{R}^n)$ a local minimizer of $\mathcal{D}_{p(x)}$ in the class

$$\bigl\{w \in W^{1,p(x)};\ w = h \text{ on } \Gamma(R)\bigr\},$$

where h is a given boundary data in the class $W^{1,s}(B^+(R), \mathbb{R}^n)$ $s > (1+\delta)p_2$. Assume that $\mathcal{D}_{p(x)}(v) \le K$ for some positive constant K. Then, for any $\varepsilon \in (0, mp_2(2R)/s)$, we have

$$\begin{aligned}
\int_{B^+(\rho)} |Dv|^{p_2(2R)}dx &\le c_7\biggl[\Bigl(\frac{\rho}{R}\Bigr)^{m-\varepsilon} \int_{B^+(R)} |Dv|^{p_2(2R)}dx\\
&\quad + \rho^{m-mp_2(2R)/s}\Bigl(\int_{B^+(2R)} \bigl(1+|Dh|^2\bigr)^{s/2}dx\Bigr)^{p_2(2R)/s}\biggr].
\end{aligned} \tag{2.16}$$

Proof. In this proof, we abbreviate $p_2(2R)$ to p_2. Let us define a frozen functional as

$$\mathcal{D}_0(w) := \int_{B^+(R)} |Dw|^{p_2}dx, \tag{2.17}$$

and let $w \in W^{1,p_2}(B^+(R))$ be a minimizer of $\mathcal{D}_0$ with $w = v$ on $\partial B^+(R)$.

Since we are supposing (2.15), by virtue of Proposition 2.2, we see that $v \in W^{1,(1+\delta)p(x)}(B^+(R)) \subset W^{1,(1+\delta/2)p_2}(B^+(R))$. So, using Corollary 2.3 with $D = B^+(R)$ and $S = R/k$ for a suitable $k > 0$, we have

$$\int_{B^+(R)} |Dw|^{(1+\delta/2)p_2}dx \le c \int_{B^+(R)} \bigl(1+|Du|^2\bigr)^{(1+\delta/2)p_2/2}dx. \tag{2.18}$$

On the other hand, by boundary regularity results for minimizers of functionals of standard growth (see for example [6, p. 446, l.-5]), we have for any $k \in (0, 1)$

$$\int_{B_\rho^+} |Dw|^{p_2} dx \le c\left[\left(\left(\frac{\rho}{R}\right)^m + k\right) \int_{B^+(R)} |Dw|^{p_2} dx + k^{1-p_2} R^{m(1-p_2/s)} \left(\int_{B^+(2R)} |Dh|^s dx\right)^{p_2/s}\right]. \tag{2.19}$$

As in [4, (9)], the minimality of v implies that

$$\mathcal{D}_0(v) - \mathcal{D}_0(w) \ge c \int_{B^+(R)} \big(|Dv| + |Dw|\big)^{p_2-2} |Dv - Dw|^2 dx.$$

(Although in [4] the integrand is $(|Du| - |Dv|)^{p_2-2} \dots$, the minus sign in the parentheses is clearly a typo.) So, we have

$$\int_{B^+(R)} |Dv - Dw|^{p_2} dx \le \mathcal{D}_0(v) - \mathcal{D}_0(w). \tag{2.20}$$

Since v minimize $\mathcal{D}_{p(x)}$,

$$\mathcal{D}_0(v) - \mathcal{D}_0(w) \le \mathcal{D}_0(v) - \mathcal{D}_{p(x)}(v) + \mathcal{D}_{p(x)}(w) - \mathcal{D}_0(w). \tag{2.21}$$

In order to estimate the right-hand side of the above inequality, we mention that as (7) in [4] for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\big|t^r - t^s\big| \le C(\varepsilon)(s - r)\big(1 + t^{(1+\varepsilon)s}\big), \quad \text{for all } t \ge 0 \text{ and } s \ge r \ge 1. \tag{2.22}$$

By virtue of the above inequality, Hölder continuity of $p(x)$, and the assumption (2.15), using Proposition 2.2, we can estimate as follows:

$$\begin{aligned}
&\big|\mathcal{D}_0(v) - \mathcal{D}_{p(x)}(v)\big| \\
&\quad \le cR^\sigma \int_{B^+(R)} \big(1 + |Dv|^2\big)^{(1+\delta/2)p_2/2} dx \\
&\quad \le c_7 R^\sigma \int_{B^+(R)} \big(1 + |Dv|^2\big)^{(1+\delta)p(x)/2} dx \\
&\quad \le cR^{\sigma - m\delta}\left(\int_{B^+(2R)} \big(1 + |Dv|^2\big)^{p(x)/2} dx\right)^{1+\delta} + cR^\sigma \int_{B^+(2R)} \big(1 + |Dh|^2\big)^{(1+\delta)p(x)/2} dx \\
&\quad \le cR^{\sigma - m\delta} \int_{B^+(2R)} \big(1 + |Dv|^2\big)^{p(x)/2} dx + cR^{\sigma + m(1-(1+\delta)p_2/s)} \left(\int_{B^+(2R)} \big(1 + |Dh|^2\big)^{s/2} dx\right)^{(1+\delta)p_2/s}.
\end{aligned} \tag{2.23}$$

Here, we used the assumption $\int |Dv|^{p(x)} dx \le K$ for the last inequality.

Using Corollary 2.3 with $p(x) = p_2$, $v = w$ and $h = v$ and the latter half of the above estimates, we can estimate $|\mathcal{D}_{p(x)}(w) - \mathcal{D}_0(w)|$ similarly.

$$\begin{aligned}
&\big|\mathcal{D}_{p(x)}(w) - \mathcal{D}_0(w)\big| \\
&\quad \le cR^\sigma \int_{B^+(R)} \big(1 + |Dw|^2\big)^{(1+\delta/2)p_2/2} dx \\
&\quad \le cR^\sigma \int_{B^+(R)} \big(1 + |Dv|^2\big)^{(1+\delta/2)p_2/2} dx \\
&\quad \le cR^{\sigma - m\delta} \int_{B^+(2R)} \big(1 + |Dv|^2\big)^{p(x)/2} dx + cR^{\sigma + m(1-(1+\delta)p_2/s)} \left(\int_{B^+(2R)} \big(1 + |Dh|^2\big)^{s/2} dx\right)^{(1+\delta)p_2/s}
\end{aligned} \tag{2.24}$$

Combining (2.20), (2.21), (2.23) and (2.24), we get

$$\int_{B^+(R)} |Dv - Dw|^{p_2} dx$$
$$\leq cR^{\sigma - m\delta} \int_{B^+(2R)} \left(1 + |Dv|^2\right)^{p(x)/2} dx + cR^{\sigma + m(\sigma - (1+\delta)p_2/s)} \left(\int_{B^+(2R)} \left(1 + |Dh|^2\right)^{s/2} dx \right)^{(1+\delta)p_2/s}. \tag{2.25}$$

Combining (2.19) with the above estimate, we obtain

$$\int_{B^+(\rho)} |Dv|^{p_2} dx$$
$$\leq c\left[\left(\frac{\rho}{R}\right)^m + k\right] \int_{B^+(2R)} |Dw|^{p_2} dx + ck^{1-p_2} R^{m(1-p_2/s)} \left(\int_{B^+(R)} |Dh|^s dx \right)^{p_2/s}$$
$$+ cR^{\sigma - m\delta} \int_{B^+(2R)} \left(1 + |Dv|^2\right)^{p(x)/2} dx + cR^{\sigma + m(1-(1+\delta)p_2/s)} \left(\int_{B^+(2R)} \left(1 + |Dh|^2\right)^{s/2} dx \right)^{(1+\delta)p_2/s}.$$

So, taking $R \leq 1$ sufficiently small so that $\int_{B^+(2R)} (1 + |Dh|^2)^{s/2} dx \leq 1$, and remarking that $\sigma - m(1+\delta)p_2/s \geq -mp_2/s$, and using the minimality of w, we see that

$$\int_{B^+(\rho)} |Dv|^{p_2} dx$$
$$\leq c\left[\left(\frac{\rho}{R}\right)^m + k + R^{\sigma - m\delta}\right] \int_{B^+(2R)} |Dv|^{p_2} dx + c\left(k^{1-p_2} + 1\right) R^{m - p_2 m/s} \left(\int_{B^+(2R)} \left(1 + |Dh|^2\right)^{s/2} dx \right)^{p_2/s}$$

holds. Now, by virtue of a well-known lemma (see, for example [10, Lemma 5.13] taking k and R sufficiently small, we get the decay estimate (2.16). □

3. Partial regularity up to the boundary

In this section we consider the boundary analogue of the result of [15].

For a map $u : \Omega \to \mathbb{R}^n$ under consideration, we introduce the following quantities:

$$\Phi(x, r, p) := r\left(r^{-m} \int_{\Omega(x,r)} \left(1 + \left|Du(y)\right|^2\right)^{p/2} dy \right)^{1/p}, \tag{3.1}$$

$$\Psi(x, r) := \Phi\left(x, r, p_2(x, r)\right). \tag{3.2}$$

For these quantities we prepare the following simple estimates.

Lemma 3.1. *For $\gamma_1 \leq p < q \leq \gamma_2$, we have*

$$\Phi(x, r, p) \leq \omega_m^{(1/\gamma_1)-(1/\gamma_2)} \Phi(x, r, q), \tag{3.3}$$

where ω_m stands for the volume of the m-dimensional unit ball.

For some $x, x' \in \Omega$, $r > 0$ and $k > 1$, suppose that $B(x, r) \subset B(x', kr)$, then we see that

$$\Psi(x, r) \leq \omega_m^{(1/\gamma_1)-(1/\gamma_2)} k^{(m/\gamma_1)-1} \Psi\left(x', kr\right). \tag{3.4}$$

Proof. Using Hölder's inequality, we see that

$$\begin{aligned}
\Psi(x,r,p) &= r\left(r^{-m}\int_{\Omega(x,r)}\left(1+|Du|^2\right)^{p/2}dy\right)^{1/p}\\
&\le r\left[r^{-m}\left(\int_{B(x,r)}1dy\right)^{1-p/q}\left(\int_{\Omega(x,r)}\left(1+|Du|^2\right)^{q/2}dx\right)^{p/q}\right]^{1/p}\\
&\le r\left[r^{-m}\left(\omega_m r^m\right)^{1-p/q}\left(\int_{\Omega(x,r)}\left(1+|Du|^2\right)^{q/2}dy\right)^{p/q}\right]^{1/p}\\
&= \omega_m^{(1/p)-(1/q)}r\left(r^{-m}\int_{\Omega(x,r)}\left(1+|Du|^2\right)^{q/2}dy\right)^{1/q}\\
&= \omega_m^{(1/\gamma_1)-(1/\gamma_2)}\Phi(x,r,q),
\end{aligned}$$

where we also used the fact that $\omega_m > 1$. Thus we get (3.3).

Since the inclusion $B(x,r)\subset B(x',kr)$ implies that $p_2(x,r)\le p_2(x',kr)$, using (3.3), we see that

$$\begin{aligned}
\Phi(x,r) &\le \left(k^{m-p_2(x,r)}(kr)^{p_2(x,r)-m}\int_{\Omega(x',kr)}\left(1+|Du|^2\right)^{p_2(x,r)}dx\right)^{1/p_2(x,r)}\\
&= k^{(m/p_2(x,r))-1}\Phi\left(x',kr,p_2(x,r)\right)\\
&\le k^{(m/p_2(x,r))-1}\omega_m^{(1/\gamma_1)-(1/\gamma_2)}\Psi\left(x',kr\right)\\
&\le k^{(m/\gamma_1)-1}\omega_m^{(1/\gamma_1)-(1/\gamma_2)}\Psi\left(x',kr\right).
\end{aligned}$$

Thus we get (3.4). □

In the following we abbreviate

$$C_* := \omega_m^{(1/\gamma_1)-(1/\gamma_2)}, \qquad \gamma_3 := \frac{m}{\gamma_1}-1. \tag{3.5}$$

Theorem 3.2. *Let $R_1 > R_2$ be positive constants. Assume that $g^{\alpha\beta}$, $G_{ij}(u)$ and $p(x)$ satisfy the conditions (C1)–(C3) in $B^+(R_1)=B^+(0,R_1)$, and that*

$$\omega_p(R_1)\le\delta, \qquad \left(1+\frac{\delta}{2}\right)p_2(R_1)\le(1+\delta)p_1(R_1). \tag{3.6}$$

(For the constant δ see the comments before (2.13).) Let $u\in W^{1,p(x)}(B^+(R_1))$ be a local minimizer of the functional $\mathcal{E}(v;B^+(R_1))$ in the class

$$\left\{v\in W^{1,p(x)};\ v=h \text{ on } \Gamma(R_1)\right\},$$

for a given boundary data $h\in W^{1,s}(B^+(R_1))$ with $s>(1+\delta)\max\{m,\gamma_2\}$.

Then, there exist positive constants r_1 and ε_0 with the following property: if for some $x\in B^+(R_2)$ and $r_0\in(0,r_1)$ we have $\Psi(x,r_0)\le\varepsilon_0$, then u satisfies for some $\alpha\in(0,1)$

$$\Psi(x,\rho)\le c\rho^\alpha, \quad \text{for any } \rho\in(0,r_0). \tag{3.7}$$

Proof. Take a point $x_1\in\Gamma(R_2)$ arbitrarily. For $r<R_1-R_2$ let us put

$$B_r^+ := B^+(x_1,r), \tag{3.8}$$

$$p_1(r) := p_1(x_1,r)=\inf_{B_r^+}p(x), \qquad p_2(r) := p_2(x_1,r)=\sup_{B_r^+}p(x). \tag{3.9}$$

Let R be a positive constant with $R \le (R_1 - R_2)/2$. As in [15], we define two types of *frozen functionals*.

$$\mathcal{F}_1(v) := \int_{B_R^+} \left(g_R^{\alpha\beta} G_{ij}(u_R) D_\alpha v^i D_\beta v^j\right)^{p(x)/2} dx, \tag{3.10}$$

$$\mathcal{F}_2(v) := \int_{B_R^+} \left(g_R^{\alpha\beta} G_{ij}(u_R) D_\alpha v^i D_\beta v^j\right)^{p_2(2R)/2} dx, \tag{3.11}$$

where we are writing

$$g_R^{\alpha\beta} = g_{x_1,R}^{\alpha\beta} := \fint_{B_R^+} g^{\alpha\beta}(x)dx \quad \text{and} \quad u_R = u_{x_1,R} := \fint_{B_R^+} u(x)dx.$$

Let v be a minimizer of $\mathcal{F}_1$ in the class

$$u + W_0^{1,p(x)}(B_R^+) := \{w \in W^{1,p(x)}(B_R^+);\ w - u \in W_0^{1,p(x)}(B_R^+)\}.$$

Then, using Corollary 2.3 with $D = B_R^+$ and $h = u$, we have for any $\varepsilon \in (0, \delta]$

$$\int_{B_R^+} \left(1 + |Dv|^2\right)^{(1+\varepsilon)p(x)/2} dx \le c \int_{B_R^+} \left(1 + |Du|^2\right)^{(1+\varepsilon)p(x)/2} dx. \tag{3.12}$$

On the other hand, for any $\beta \in (0, mp_2(2R)/s)$, from Theorem 2.6, we can see that the following estimate holds for any $0 < \rho < R/2$.

$$\int_{B_\rho^+} |Dv|^{p_2(2R)} dx \le c\left[\left(\frac{\rho}{R}\right)^{m-\beta} \int_{B_R^+} |Dv|^{p_2(2R)} dx + \rho^{m-mp_2(2R)/s}\left(\int_{B_{2R}^+} \left(1 + |Dh|^2\right)^{s/2} dx\right)^{p_2(2R)/s}\right]. \tag{3.13}$$

Using (3.12) with $\varepsilon = \omega_p(2R)(< \omega_p(2R_1) \le \delta)$ and Proposition 2.2, we can estimate the integral of the first term of the right hand side of (3.13) as

$$\begin{aligned}
&\int_{B_R^+} |Dv|^{p_2(2R)} dx \\
&\quad \le c \int_{B_R^+} \left(1 + |Dv|^2\right)^{(1+\omega_1(2R))p(x)/2} dx \\
&\quad \le c \int_{B_R^+} \left(1 + |Du|^2\right)^{(1+\omega_1(2R))p(x)/2} dx \\
&\quad \le cR^{-m\omega_1(2R)} \left(\int_{B_{2R}^+} \left(1 + |Du|^2\right)^{p(x)/2} dx\right)^{1+\omega_1(2R)} + c\int_{B_{2R}^+} \left(1 + |Dh|^2\right)^{(1+\omega_1(2R))p(x)/2} dx \\
&\quad \le cR^{-m\omega_1(2R)} \int_{B_{2R}^+} \left(1 + |Du|^2\right)^{p(x)/2} dx + c\int_{B_{2R}^+} \left(1 + |Dh|^2\right)^{(1+\delta)p_2(2R)/2} dx \\
&\quad \le cR^{-m\omega_1(2R)} \int_{B_{2R}^+} \left(1 + |Du|^2\right)^{p_2(2R)/2} dx + cR^{m(1-(1+\delta)p_2(2R)/s)}\left(\int_{B_{2R}^+} \left(1 + |Dh|^2\right)^{s/2} dx\right)^{(1+\delta)p_2(2R)/s}.
\end{aligned}$$

Here, we used (3.12) for the second inequality, Proposition 2.2 for the third and boundedness of $\int_\Omega (1 + |Du|^2)^{p(x)/2} dx$ for the fourth. In what follows, we abbreviate as $p_2 = p_2(2R)$ and $\overline{p}_2 = (1 + \delta)p_2(2R)$. Since we see that $R^{-m\omega_1(2R)}$ is bounded by virtue of (2.4), from (3.13) and the above estimate, we obtain for $\beta \in (0, mp_2/s)$

$$\int_{B_\rho^+} |Dv|^{p_2} dx \le c\left(\frac{\rho}{R}\right)^{m-\beta} \int_{B_{2R}^+} \left(1+|Du|^2\right)^{p_2/2} dx$$
$$+ c\rho^{m-mp_2/s}\left(\int_{B_{2R}^+} \left(1+|Dh|^2\right)^{s/2} dx\right)^{\bar{p}_2/s} + c\rho^{m-mp_2/s}\left(\int_{B_{2R}^+} \left(1+|Dh|^2\right)^{s/2}\right)^{p_2/s}. \tag{3.14}$$

Let us write

$$K(h) := \left(\int_{B^+(R_1)} \left(1+|Dh|^2\right)^{s/2} dx\right)^{1/s}, \qquad \hat{K}(h) := \max\left\{K(h)^{p_2},\ K(h)^{\bar{p}_2}\right\}. \tag{3.15}$$

Then, from (3.14), we have for some positive constants K_1 and K_2 that

$$\int_{B_\rho^+} |Dv|^{p_2} dx \le K_1\left(\frac{\rho}{R}\right)^{m-\beta} \int_{B_{2R}^+} \left(1+|Du|^2\right)^{p_2/2} dx + K_2 \rho^{m-m\bar{p}_2/s} \hat{K}(h). \tag{3.16}$$

On the other hand, proceeding as in [15, pp. 3343–3344], we can estimate $\int_{B_R^+} |Du-Dv|^{p_2(2R)} dx$ as follows:

$$\begin{aligned}
&\int_{B_R^+} |Du-Dv|^{p_2(2R)} dx \\
&\quad \le c_6\big(\mathcal{F}_2(u)-\mathcal{F}_2(v)\big) + cR^\sigma \int_{B_R^+} \left(1+|Dv|^2\right)^{(1+\varepsilon)p_2/2} dx \\
&\quad \le c_5\big(\mathcal{F}_2(u)-\mathcal{F}_1(u)+\mathcal{F}_1(u)-\mathcal{F}(u)+\mathcal{F}(u)-\mathcal{F}(v) \\
&\qquad + \mathcal{F}(v)-\mathcal{F}_1(v)+\mathcal{F}_1(v)-\mathcal{F}_2(v)\big) + cR^\sigma \int_{B_R^+} \left(1+|Dv|^2\right)^{(1+\varepsilon)p_2/2} dx \\
&\quad \le c_4\big\{\big(\mathcal{F}_2(u)-\mathcal{F}_1(u)\big)+\big(\mathcal{F}_1(u)-\mathcal{F}(u)\big)+\big(\mathcal{F}(v)-\mathcal{F}_1(v)\big) \\
&\qquad + \big(\mathcal{F}_1(v)-\mathcal{F}_2(v)\big)\big\} + cR^\sigma \int_{B_R^+} \left(1+|Dv|^2\right)^{(1+\varepsilon)p_2/2} dx \\
&\quad =: I + II + III + IV + V.
\end{aligned} \tag{3.17}$$

In order to estimate $|I|$ and $|IV|$, we use (2.22) with $s = p_2(2R)/2$ and $r = p(x)/2$ and get

$$|I| \le cR^\sigma \int_{B_R^+} \left(1+|Du|^2\right)^{(1+\varepsilon)p_2(2R)/2} dx,$$

$$|IV| \le cR^\sigma \int_{B_R^+} \left(1+|Dv|^2\right)^{(1+\varepsilon)p_2(2R)/2} dx.$$

Let us take $\varepsilon < \delta/2$, then by the assumption (3.6), we have

$$(1+\varepsilon)p_2(2R) \le \left(1+\frac{\delta}{2}\right)p_2(2R) < (1+\delta)p_1(2R) \le (1+\delta)p(x). \tag{3.18}$$

So, we can estimate I as

$$|I| \le cR^\sigma \int_{B_R^+} \left(1+|Du|^2\right)^{(1+\delta)p(x)/2} dx. \tag{3.19}$$

Using Corollary 2.3, we have

$$|IV|, |V| \le cR^\sigma \int_{B_R^+} \left(1+|Dv|^2\right)^{(1+\delta)p(x)/2} dx \le cR^\sigma \int_{B_R^+} \left(1+|Du|^2\right)^{(1+\delta)p(x)/2} dx. \tag{3.20}$$

Using Proposition 2.2 we can see that

$$\begin{aligned} R^\sigma &\int_{B_R^+} \left(1+|Du|^2\right)^{(1+\delta)p(x)/2} dx \\ &\le cR^{\sigma-m\delta}\left(\int_{B_{2R}^+} \left(1+|Du|^2\right)^{p(x)/2} dx\right)^{1+\delta} + cR^\sigma \int_{B_{2R}^+} \left(1+|Dh|^2\right)^{p(x)/2} dx \\ &\le cR^{\sigma-m\delta} \int_{B_{2R}^+} \left(1+|Du|^2\right)^{p(x)/2} dx + cR^{\sigma+m-mp_2/s} K(h)^{p_2}, \end{aligned} \tag{3.21}$$

where we used the boundedness of $\int |Du|^{p(x)} dx$. Combining (3.19) and (3.20) with (3.21), we obtain

$$|I|, |IV|, |V| \le cR^{\sigma-m\delta}\left[\int_{B_{2R}^+} \left(1+|Du|^2\right)^{p(x)/2} dx + cR^{m-mp_2/s} K(h)^{p_2}\right], \tag{3.22}$$

where we used the fact that $R^\sigma < R^{\sigma-m\delta}$.

Let us estimate $|II+III|$. Writing $q = 1+\delta$ and $q' = q/(q-1) = (1+\delta)/\delta$, using Hölder's inequality, and remembering the condition (C2), we have

$$\begin{aligned} |II+III| \le\; & c\left[\left(\int_{B_R^+} \omega_G^{q'}\left(|u-u_R|^2\right) dx\right)^{1/q'} + \left(\int_{B_R^+} |g(x)-g_R|^{q'} dx\right)^{1/q'}\right]\left(\int_{B_R^+} |Du|^{qp(x)} dx\right)^{1/q} \\ & + c\left[\left(\int_{B_R^+} \omega_G^{q'}\left(|v-u_R|^2\right) dx\right)^{1/q'} + \left(\int_{B_R^+} |g(x)-g_R|^{q'} dx\right)^{1/q'}\right]\left(\int_{B_R^+} |Dv|^{qp(x)} dx\right)^{1/q} \\ =:\; & c\left(II' + III'\right). \end{aligned} \tag{3.23}$$

Here and in the sequel, we write

$$g(x) = \left(g^{\alpha\beta}(x)\right), \quad \text{and} \quad \left|g(x)-g_R\right| = \left\{\sum_{\alpha\beta}\left(g^{\alpha\beta}(x) - g_R^{\alpha\beta}\right)^2\right\}^{1/2}.$$

Since ω_G and g are bounded, using Proposition 2.2, Jensen's inequality, Hölder's inequality and the Sobolev–Poincaré inequality, we can estimate II' as follows:

$$\begin{aligned} II' \le\; & c\left[\omega_G^{1/q'}\left(cR^{2-m}\int_{B_R^+} |Du|^2 dx\right) + \left(⨍_{B_R^+} |g(x)-g_R| dx\right)^{1/q'}\right] \\ & \times\left[\int_{B_{2R}^+} \left(1+|Du|^2\right)^{p(x)/2} dx + \left(R^{\delta m}\int_{B_{2R}^+} \left(1+|Dh|^2\right)^{qp(x)/2} dx\right)^{1/q}\right] \end{aligned}$$

$$
\begin{aligned}
&\le c\Big[\omega_G^{1/q'}\Big(c\Big\{R^{p_2(2R)-m}\int_{B_R^+}|Du|^{p_2}dx\Big\}^{2/p_2}\Big)+c\omega_g(2R)^{1/q'}\Big]\\
&\quad\times\Big[\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p(x)/2}dx+\Big(R^{\delta m}\int_{B_{2R}^+}\big(1+|Dh|^2\big)^{\bar p_2/2}dx\Big)^{1/q}\Big]\\
&\le c\Big[\omega_G^{1/q'}\Big(c\Big\{R^{p_2(2R)-m}\int_{B_R^+}|Du|^{p_2}dx\Big\}^{2/p_2}\Big)+c\omega_g^{1/q'}\Big]\\
&\quad\times\Big[\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p(x)/2}dx+R^{m(1-p_2/s)}K(h)^{p_2}\Big].
\end{aligned}
\tag{3.24}
$$

Using (3.12) and proceeding as above, we estimate III' as

$$
\begin{aligned}
III'&\le c\Big[\Big(\int_{B_R^+}\omega_G\big(|u-u_R|^2+|u-v|^2\big)dx\Big)^{1/q'}+\Big(\int_{B_R^+}\big|g(x)-g_R\big|dx\Big)^{1/q'}\Big]\Big(\int_{B_R^+}\big(1+|Du|^2\big)^{qp(x)/2}dx\Big)^{1/q}\\
&\le c\Big[\omega_G^{1/q'}\Big(\fint_{B_R^+}|u-u_R|^2dx+\fint_{B_R^+}|u-v|^2dx\Big)\\
&\quad+\Big(\fint_{B_R^+}\big|g(x)-g_R\big|dx\Big)^{1/q'}\Big]\Big[\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p(x)/2}dx+\Big(R^{\delta m}\int_{B_{2R}^+}\big(1+|Dh|^2\big)^{qp(x)/2}dx\Big)^{1/q}\Big]\\
&\le c\Big[\omega_G^{1/q'}\Big(cR^{2-m}\int_{B_R^+}|Du|^2dx+cR^{2-m}\int_{B_R^+}|Dv|^2dx\Big)\\
&\quad+\Big(\fint_{B_R^+}\big|g(x)-g_R\big|dx\Big)^{1/q'}\Big]\Big[\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p(x)/2}dx+R^{m(1-p_2/s)}K(h)^{p_2}\Big].
\end{aligned}
\tag{3.25}
$$

Again with (3.12) and Proposition 2.2, we can estimate the second term in ω_G as follows:

$$
\begin{aligned}
R^{2-m}\int_{B_R^+}|Dv|^2dx&\le c\Big(R^{p_2(2R)-m}\int_{B_R^+}|Dv|^{p_2(2R)}dx\Big)^{2/p_2}\\
&\le c\Big(R^{p_2-m}\int_{B_R^+}\big(1+|Dv|^2\big)^{(1+\omega_p(2R))p(x)/2}dx\Big)^{2/p_2}\\
&\le c\Big(R^{p_2-m}\int_{B_R^+}\big(1+|Du|^2\big)^{(1+\omega_p(2R))p(x)/2}dx\Big)^{2/p_2}\\
&\le c\Big[R^{p_2-m}\Big\{R^{-\omega_p(2R)m}\Big(\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p(x)/2}dx\Big)^{1+\omega_p(2R)}\\
&\quad+\int_{B_{2R}^+}\big(1+|Dh|^2\big)^{(1+\omega_p(2R))p(x)dx}\Big\}\Big]^{2/p_2},
\end{aligned}
$$

where we used Proposition 2.2 with $\delta=\omega_p(2R)$ for the fourth inequality. Since $R^{-\omega_1(2R)}$ and $\int|Du|^{p(x)}dx$ are bounded, from the above estimate, we obtain

$$
\begin{aligned}
&R^{2-m}\int_{B_R^+}|Dv|^2dx\\
&\le c\Big(R^{p_2-m}\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p(x)/2}dx\Big)^{2/p_2}+c\Big(R^{p_2-m}\int_{B_{2R}^+}\big(1+|Dh|^2\big)^{\bar p_2}dx\Big)^{2/p_2}\\
&\le c\Big(R^{p_2-m}\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p_2/2}dx\Big)^{2/p_2}+c\Big[R^{p_2-m}R^{m-mp_2/s}\Big(\int_{B_{2R}^+}\big(1+|Dh|^2\big)^{s}dx\Big)^{\bar p_2/s}\Big]^{2/p_2}\\
&\le c\Big(R^{p_2-m}\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p_2/2}dx\Big)^{2/p_2}+cR^{2(1-qm/s)}K(h)^{2q}.
\end{aligned}\tag{3.26}
$$

Here, mention that by the assumption that $s>(1+\delta)m$ we have $1-mq/s=1-m(1+\delta)/s>0$.

From (3.25) and (3.26) we obtain

$$
\begin{aligned}
III'\le{}& c\big[\omega_G^{1/q'}\big(c_8\Psi(2R)^2+c_9R^{2(1-qm/s)}K(h)^{2q}\big)+\omega_g^{1/q'}(2R)\big]\\
&\times\Big[\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p(x)/2}dx+R^{m(1-p_2/s)}K(h)^{p_2}\Big].
\end{aligned}\tag{3.27}
$$

Now, combining (3.17), (3.22), (3.23), (3.24) and (3.27), we obtain

$$
\begin{aligned}
&\int_{B_R^+}|Du-Dv|^{p_2}dx\\
&\le cR^{\sigma-m\delta}\Big[\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p_2/2}+R^{m(1+\delta-p_2/s)}K(h)^{p_2}\Big]\\
&\quad+c\big[\omega_G^{1/q'}\big(c_8\Psi(2R)^2+c_9R^{2(1-qm/s)}K(h)^{2q}\big)+\omega_g^{1/q'}(2R)\big]\\
&\quad\times\Big[\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p(x)/2}dx+R^{m(1-p_2/s)}K(h)^{p_2}\Big]\\
&\le c\big[R^{\sigma-m\delta}+\omega_G^{1/q'}\big(c_8\Psi(2R)^2+c_9R^{2(1-qm/s)}K(h)^{2q}\big)+\omega_g^{1/q'}(2R)\big]\\
&\quad\times\Big[\int_{B_{2R}^+}\big(1+|Du|^2\big)^{p(x)/2}dx+R^{m(1-p_2/s)}K(h)^{p_2}\Big].
\end{aligned}\tag{3.28}
$$

Here, for the last inequality, we used the fact that $R^\delta<1$.

Now, putting $r=2R$, $\hat\omega_G(t)=\omega_G^{1/q'}(\max\{c_8,c_9\}\cdot t)$ and $\hat\omega_g=\omega_g^{1/q'}$, from (3.16) and (3.28) we obtain

$$
\begin{aligned}
&\int_{B_\rho^+}\big(1+|Du|^2\big)^{p_2/2}dx\\
&\le K_1\Big(\frac{\rho}{r}\Big)^{m-\beta}\int_{B_r^+}\big(1+|Du|^2\big)^{p_2/2}dx+K_2\rho^{m(1-\bar p_2/s)}\hat K(h)\\
&\quad+c\big[r^{\sigma-m\delta}+\hat\omega_G+\hat\omega_g\big]\times\Big[\int_{B_r^+}\big(1+|Du|^2\big)^{p_2/2}dx+r^{m(1-p_2/s)}K(h)^{p_2}\Big]
\end{aligned}
$$

$$\leq K_3\left[\left(\frac{\rho}{r}\right)^{m-\beta} + r^{\sigma-m\delta} + \hat{\omega}_G + \hat{\omega}_g\right] \int_{B_r^+} \left(1+|Du|^2\right)^{p_2/2} dx$$
$$+ K_4\left[1 + r^{\sigma-m\delta} + \hat{\omega}_G + \hat{\omega}_g\right] r^{m(1-\overline{p}_2/s)} \hat{K}(h), \tag{3.29}$$

for some constants K_3 and K_4. Here, for the second inequality we used the fact that $r < 1$ and $\overline{p}_2 > p_2$.

For $\tau \in (0,1)$ which will be specified later, put $\rho = \tau r$ in the above estimate and multiply both sides by $(\tau r)^{p_2-m}$, then we have

$$(\tau r)^{p_2-m} \int_{B_{\tau r}^+} \left(1+|Du|^2\right)^{p_2/2} dx$$
$$\leq K_3\left[\tau^{p_2-\beta} + \tau^{p_2-m} r^{\sigma-m\delta} + \tau^{p_2-m}\hat{\omega}_G + \tau^{p_2-m}\hat{\omega}_g\right] r^{p_2-m} \int_{B_r^+} \left(1+|Du|^2\right)^{p_2/2} dx$$
$$+ K_4\left[\tau^{p_2-m} + \tau^{p_2-m} r^{\sigma-m\delta} + \tau^{p_2-m}\hat{\omega}_G + \tau^{p_2-m}\hat{\omega}_g\right] r^{p_2-m\overline{p}_2/s} \hat{K}(h). \tag{3.30}$$

Remembering the definitions of Φ and Ψ, and mentioning (3.3) and (3.5), from the above estimate we get

$$\begin{aligned}
\left(\Psi(x_1,\tau r)\right)^{p_2(r)} &= \Phi\left(x_1, \tau r, p_2(\tau r)\right)^{p_2(r)} \\
&\leq C_*^{p_2(r)} \Phi\left(x_1, \tau r, p_2(r)\right)^{p_2(r)} \\
&\leq K_3^{p_2(r)} C_*^{p_2(r)} \left[\tau^{p_2-\beta} + \tau^{p_2-m}\left\{r^{\sigma-m\delta}\right.\right. \\
&\quad \left.\left. + \hat{\omega}_G\left(\Psi(x_1,r)^2 + r^{2(1-q/s)}K(h)^{2q}\right) + \hat{\omega}_g(r)\right\}\right] \times \Phi(x_1,r)^{p_2} \\
&\quad + \tau^{p_2-m} r^{p_2-m\overline{p}_2/s} C(g,G,p,h) \\
&= K_3^{p_2(r)} C_*^{p_2(r)} \tau^{p_2-\beta}\left[1 + \tau^{\beta-m}\left\{r^{\sigma-m\delta}\right.\right. \\
&\quad \left.\left. + \hat{\omega}_G\left(\Psi(x_1,r)^2 + r^{2(1-q/s)}K(h)^{2q}\right) + \hat{\omega}_g(r)\right\}\right] \times \Phi(x_1,r)^{p_2} \\
&\quad + \tau^{p_2-m} r^{p_2-m\overline{p}_2/s} C(g,G,p,h),
\end{aligned} \tag{3.31}$$

where $C(g,G,p,h)$ is a positive constant depending only on $g^{\alpha\beta}(x)$, $G_{ij}(u)$, $p(x)$ and $h(x)$. So, we obtain

$$\begin{aligned}
\Psi(x_1,\tau r) &= K_5 \tau^{1-\beta/p_2}\left[1 + \tau^{(\beta-m)/p_2}\left\{r^{(\sigma-m\delta)/p_2}\right.\right. \\
&\quad \left.\left. + \hat{\omega}_G^{1/p_2}\left(\Psi(x_1,r)^2 + r^{2(1-q/s)}K(h)^{2q}\right) + \hat{\omega}_g^{1/p_2}(r)\right\}\right] \times \Phi(x_1,r) \\
&\quad + \tau^{1-m/p_2} r^{1-mq/s} C_0(g,G,p,h),
\end{aligned} \tag{3.32}$$

where $K_5 = K_3 C_*$ and $C_0(g,G,p,h) = C(g,G,p,h)^{1/p_2}$.

Since $0 < \beta < 1$, $m > 2$, $\gamma_1 \leq p_2 = p_2(r) \leq \gamma_2$, and $\tau < 1$, we have

$$\tau^{(\beta-m)/p_2(r)} \leq \tau^{(\beta-m)/\gamma_1}. \tag{3.33}$$

Without loss of generality we can assume that $0 < r < 1$, so we see that

$$r^{(\sigma-m\delta)/p_2(r)} \leq r^{(\sigma-m\delta)/\gamma_2}, \qquad (\tau r)^{1-(\beta/p_2(r))} \leq (\tau r)^{1-(\beta/\gamma_1)}. \tag{3.34}$$

In the following, since we consider the case that ω_G and ω_g are sufficiently small, we can assume that $\omega_G, \omega_g < 1$. So, we have

$$\hat{\omega}_G^{1/p_2(r)} \leq \hat{\omega}_G^{1/\gamma_2}, \qquad \hat{\omega}_g^{1/p_2(r)} \leq \hat{\omega}_g^{\,1/\gamma_2}. \tag{3.35}$$

For the sake of simplicity, let us put

$$\mu_1 := 1 - \frac{\beta}{\gamma_1}, \qquad \mu_2 := 1 - \frac{mq}{s}, \qquad \tilde{\omega}_G := \hat{\omega}_G^{1/\gamma_2}, \qquad \tilde{\omega}_g := \hat{\omega}_g^{1/\gamma_2}.$$

Then, from (3.32), assuming $\Psi(r) < 1$, we get

$$\Psi(\tau r) \le K_5\tau^{\mu_1}\big[1+\tau^{(\beta-m)/\gamma_1}\big\{r^{(\sigma-m\delta)/\gamma_2}+\tilde{\omega}_G\big(\Psi(r)+r^{2\mu_2}K(h)^{2q}\big)+\tilde{\omega}_g\big\}\big]\Psi(r) \\ +\tau^{1-m/p_2}r^{\mu_2}C_0(g,G,p,h) \tag{3.36}$$

Now, let us take $\beta < m\gamma_1/s$, then we have $\mu_1 = 1-\beta/\gamma_1 > 1-qm/s = \mu_2$. Fix $\nu \in (\mu_2,\mu_1)$ and choose $\tau \in (0,1)$ so that $K_5\tau^{\mu_1} \le \tau^{\nu}/5$. Take $\varepsilon_1 > 0$ such that

$$\tilde{\omega}_G(2\varepsilon_1) < \tau^{(m-\beta)/\gamma_1}. \tag{3.37}$$

Finally, let $r_0 > 0$ be a sufficiently small constant for which the following inequalities hold:

$$\tau^{(\beta-m)\gamma_1}r_0^{(\sigma-m\delta)/\gamma_2}, \qquad \tau^{(\beta-m)/\gamma_1}\tilde{\omega}_g(r_0) \le 1 \\ \tau^{1-m/\gamma_1}r_0^{\mu_2}C_0(g,G,p,h) \le \varepsilon_1/5, \qquad r_0^{2\mu_2}K(h)^{2q} \le \varepsilon_1. \tag{3.38}$$

Now, assume that $\Psi(x_1,r) \le \varepsilon_1$ for some $r \in (0,r_0)$, we obtain from (3.36)

$$\Psi(x_1,\tau r) \le \frac{\tau^{\nu}}{5}[1+1+1+1]\Psi(x_1,r)+\frac{\varepsilon_1}{5} \\ = \frac{4}{5}\tau^{\nu}\Psi(x_1,r)+\frac{\varepsilon_1}{5} \\ \le \varepsilon_0. \tag{3.39}$$

The above estimate enables us to use an iteration argument to get

$$\Psi\big(x_1,\tau^{k+1}r\big) = \tau^{(k+1)\nu}\Psi(x_1,r)+C_1r^{\mu_2}\tau^{k\mu_2}\sum_{j=0}^{k}\tau^{(\nu-\mu_2)j} \\ \le \tau^{(k+1)\nu}\Psi(x_1,r)+C_2\big(\tau^k r\big)^{\mu_2}, \tag{3.40}$$

where $C_1 = \tau^{1-m/\gamma_1}C_0$ and $C_2 = C_1/(1-\tau^{\nu-\mu_2})$.

For any $t \in (0,r)$, there exists a nonnegative integer k such that $\tau^{k+1}r < t \le \tau^k r$, and we have

$$\Psi(x_1,t) \le t\bigg(t^{-m}\big(\tau^k r\big)^m\big(\tau^k r\big)^{-m}\int_{B^+_{\tau^k r}}\big(1+|Du|^2\big)^{p_2(t)}dx\bigg)^{1/p_2(t)} \\ \le \bigg(\frac{\tau^k r}{t}\bigg)^{(m/p_2(t))-1}\Phi\big(x_1,\tau^k r,p_2(t)\big) \\ \le C_*\bigg(\frac{\tau^k r}{t}\bigg)^{(m/p_2(t))-1}\Psi\big(x_,\tau^k r\big) \\ \le C_*\tau^{1-m/\gamma_1}\big(\tau^{-\nu}\tau^{(k+1)\nu}\Psi(x_1,r)+\tau^{-\mu_2}C_2\big(\tau^{k+1}r\big)^{\mu_2}\big) \\ \le C_*\tau^{1-(m/\gamma_1)-\nu}\bigg(\bigg(\frac{t}{r}\bigg)^{\nu}\Psi(x_1,r)+C_2\tau^{\mu_2}\bigg) \tag{3.41}$$

For an interior point $x_1 \in B^+(R_2)$ and for $0 < t < r \le \min\{R_1-R_2, x_1^m\}$, proceeding as above without the boundary condition h or as in [15], we can get an estimate similar to (3.41). Consequently, we see that there are positive constants $r_0 \in (0,(R_1-R_2)/2)$, $\varepsilon_1 > 0$, $\alpha \in (0,1)$, C_A and C_B such that if

(a) $x_1 \in \Gamma(R_2)$ and $\Psi(x_1,r) \le \varepsilon_1$ for some $r \in (0,r_0)$,

or

(b) $x_1 \in B^+(0,R_2)$, $B(x_1,r) \in B^+(0,R_1)$ and $\Psi(x_1,r) \le \varepsilon_1$ for some $r \in (0,r_0)$, then $\Psi(x_1, \rho)$ satisfies the following decay estimate:

$$\Psi(x_1, t) \le C_A \left(\frac{t}{r}\right)^{\alpha} \Psi(x, r) + C_B t^{\alpha}. \tag{3.42}$$

Now, by a standard argument (see, for example, [11, pp. 317–319]), we can see that (3.42) holds for any $x_1 \in B^+(0, R_2) \cup \Gamma(R_2)$ and $r \in (0, r_0)$. Thus, by the Morrey's theorem on the Dirichlet growth, we can deduce the assertion. □

4. Convergence lemma and boundary regularity

Lemma 4.1 *(Convergence lemma with boundary value). Let $B^+ := B^+(1)$ and $\Gamma := \Gamma(1)$. Let $A^{(\nu)}(x,u) = A^{(\nu)\alpha\beta}_{ij}(x,u)$ be a sequence of continuous functions defined on $B^+ \times \mathbb{R}^n$ converging uniformly to $A(x,u) = A^{\alpha\beta}_{ij}(x,u)$ and satisfying the following inequalities for positive constants K, λ_A and a bounded continuous concave function ω_A with $\omega_A(0) = 0$.*

(A-1) $|A^{(\nu)}(x,u)| \le K$,
(A-2) $A^{(\nu)}\xi\xi := A^{(\nu)\alpha\beta}_{ij}\xi^i_\alpha \xi^j_\beta \ge \lambda_A |\xi|^2$ *for all* $(x,u,\xi) \in B^+ \times \mathbb{R}^n \times \mathbb{R}^{mn}$,
(A-3) $|A^{(\nu)}(x,u) - A^{(\nu)}(y,v)| \le \omega_A(|x-y|^2 + |u-v|^2)$.

Let $p_\nu(x)$ be a sequence of continuous functions on B^+ converging uniformly to a constant $p_0 \ge 2$ which satisfies the following conditions.

(P-1) $p_\nu(x) \ge 2$,
(P-2) $|p_\nu(x) - p_\nu(y)| \le \omega_1(|x-y|/2) = c_p|x-y|^\sigma$ *for constants* $c_p > 0$ *and* $\sigma \in (0,1)$.

For some fixed $s > m$, let $\{h^{(\nu)}\}$ be a sequence in $W^{1,s}(B^+)$ converging to h in $W^{1,s}(B^+)$ weakly. For each $\nu \in \mathbb{N}$, let $u^{(\nu)} \in W^{1,p_\nu(x)}$ be a local minimizer of

$$\mathcal{F}^{(\nu)}\big(v; B^+\big) := \int_{B^+} \big(A^{(\nu)}(x,v)DvDv\big)^{p_\nu(x)/2} dx$$

in the class

$$\big\{w \in W^{1,p_\nu(x)}\big(B^+\big);\ w = h^{(\nu)} \text{ on } \Gamma\big\}.$$

Suppose that $u^{(\nu)} \rightharpoonup v$ in $L^2(B^+)$ and that $\|u^{(\nu)}\|_\infty \le M$ for some positive constant M. Then, u^ν, or a subsequence that we also denote by the same symbol, is such that $u^{(\nu)} \rightharpoonup v$ in $W^{1,(1+\varepsilon)p_0}(B^+(R))$ for some $\varepsilon > 0$ and any $R \in (0,1)$, and v minimizes the functional

$$\mathcal{F}_0\big(w; B^+(R)\big) := \int_{B^+(R)} \big(A(x,w)DwDw\big)^{p_0/2} dx$$

in the class

$$\big\{w \in W^{1,p_0}\big(B^+(R)\big);\ w = h \text{ on } \Gamma(R)\big\}.$$

Moreover, if x_ν is a singular point of $u^{(\nu)}$ and $x_\nu \to \bar{x}$, then $\bar{x}$ is a singular point of v.

Proof. We divide the proof into 3 parts.

Part 1 *(Preliminary estimates and the convergence of $u(\nu)$).* Since all assumptions are independent on the number ν, all results in Section 2 are valid with common constants for all $u^{(\nu)}$. So, by Proposition 2.2 there exists a constant

$\delta_0 > 0$ such that $|Du^{(\nu)}|^{(1+\delta_0)p_\nu(x)} \in L^1_{\mathrm{loc}}(B^+)$, and by Corollary 2.5 we also have

$$\int_{B^+(R)} \left|Du^{(\nu)}\right|^{(1+\delta_0)p_\nu(x)} dx \le C_3(R) \tag{4.1}$$

for some constant $C_3(R)$ which depends on R, but does not on ν. Fixing such a constant $\delta_0 > 0$, let us choose $\delta_2 \in (0, \delta_0)$ for which Corollary 2.3 holds. In what follows, let δ be a positive constant with $\delta < \delta_2$.

Since we are assuming that $p_\nu(x)$ converge uniformly to p_0 on B^+, we can assume without loss of generality that

(P-3) $p_\nu(x)$ satisfies that $2 \le q_1 \le p_\nu(x) \le q_2$ on B^+ for some constants q_1 and q_2 with

$$q_1(1+\delta) \ge q_2\left(1+\frac{\delta}{2}\right), \qquad p_0\left(1+\frac{\delta}{2}\right) \ge q_2. \tag{4.2}$$

By virtue of (4.1), (4.2) and the choice of δ, we have

$$\int_{B^+(R)} \left|Du^{(\nu)}\right|^{(1+\delta/2)q_2} dx \le C_4(R). \tag{4.3}$$

Since we are assuming that $\|u^{(\nu)}\|_\infty \le M$, the estimate (4.3) implies that $u^{(\nu)} \rightharpoonup \tilde{v}$ in $W^{1,(1+\delta/2)q_2}(B^+(R))$ for some $\tilde{v} \in W^{1,(1+\delta/2)q_2}(B^+(R))$ taking subsequence if necessary. On the other hand we are assuming that $u^{(\nu)} \rightharpoonup v$ in L^2, so we see that $v = \tilde{v}$ and that

$$u^{(\nu)} \to v \quad \text{in } L^{(1+\delta/2)q_2}\big(B^+(R)\big), \tag{4.4}$$

$$Du^{(\nu)} \rightharpoonup Dv \quad \text{in } L^{(1+\delta/2)q_2}\big(B^+(R)\big). \tag{4.5}$$

Thus, we get the assertion on the convergence of $u^{(\nu)}$.

Moreover, by virtue of the lower semicontinuity of the norm with respect to weak convergence, we have

$$\int_{B^+(R)} |Dv|^{(1+\delta/2)q_2} dx \le C_4(R). \tag{4.6}$$

Part 2 *(Minimality of v)*. Now, let us prove that v minimizes $\mathcal{F}_0$ relative to the boundary value h on $\Gamma(R)$. For this purpose, as the first step we are going to show that

$$\mathcal{F}_0\big(v; B^+(R)\big) \le \liminf_{\nu\to\infty} \mathcal{F}^{(\nu)}\big(u^{(\nu)}; B^+(R)\big). \tag{4.7}$$

Observing that

$$\mathcal{F}^{(\nu)}\big(u^{(\nu)}; B^+(R)\big) = \mathcal{F}_0\big(u^{(\nu)}; B^+(R)\big) + \mathcal{F}^{(\nu)}\big(u^{(\nu)}; B^+(R)\big) - \mathcal{F}_0\big(u^{(\nu)}; B^+(R)\big), \tag{4.8}$$

and mentioning the lower semicontinuity of $\mathcal{F}_0$ with respect to the weak convergence in $W^{1,p_0}(B^+(R))$, we see that it is enough to show that

$$\left|\mathcal{F}^{(\nu)}\big(u^{(\nu)}; B^+(R)\big) - \mathcal{F}_0\big(u^{(\nu)}; B^+(R)\big)\right| \to 0 \quad \text{as } \nu \to \infty. \tag{4.9}$$

Let us put

$$e_\nu := A^{(\nu)}\big(x, u^{(\nu)}\big) Du^{(\nu)} Du^{(\nu)}, \tag{4.10}$$

$$e_1 := A\big(x, u^{(\nu)}\big) Du^{(\nu)} Du^{(\nu)}. \tag{4.11}$$

Then we have

$$\begin{aligned} &\left|\mathcal{F}^{(\nu)}\big(u^{(\nu)}; B^+(R)\big) - \mathcal{F}_0\big(u^{(\nu)}; B^+(R)\big)\right| \\ &\quad \le \int_{B^+(R)} \left|e_\nu^{p_\nu(x)/2} - e_\nu^{p_0/2}\right| dx + \int_{B^+(R)} \left|e_\nu^{p_0/2} - e_1^{p_0/2}\right| dx \\ &\quad =: I + II. \end{aligned} \tag{4.12}$$

Put

$$\tilde{p}_\nu(x) := \max\{p_\nu(x), p_0\} \quad (\leq q_2). \tag{4.13}$$

Then, by virtue of (2.22) and (4.3), taking $\varepsilon \leq \delta/2$, we can see that

$$\begin{aligned} I &\leq c_e(\varepsilon) \int_{B^+(R)} |p_\nu(x) - p_0|(1+e_\nu)^{\tilde{p}_\nu(1+\varepsilon)/2} dx \\ &\leq c_e(\varepsilon) \sup_{B^+(R)} |p_\nu(x) - p_0| \int_{B^+(R)} (1+e_\nu)^{q_2(1+\delta/2)/2} dx \\ &\leq c(\varepsilon, R) \sup_{B^+(R)} |p_\nu(x) - p_0| \to 0 \quad \text{as } \nu \to \infty. \end{aligned} \tag{4.14}$$

In order to estimate II, we mention that for $q \geq 1$

$$|s^q - t^q| \leq q|s-t|(s^{q-1} + t^{q-1}) \tag{4.15}$$

holds for any $s, t \geq 0$. Then, using (4.3) also, and mentioning that $q_2 \geq p_0$, we can estimate II as

$$\begin{aligned} II &\leq c \int_{B^+(R)} |A^{(\nu)}(x, u^{(\nu)}) - A(x, u^{(\nu)})| \cdot (1 + |Du^{(\nu)}|)^{q_2} dx \\ &\leq c\left(\int_{B^+(R)} |A^{(\nu)}(x, u^{(\nu)}) - A(x, u^{(\nu)})|^{(2+\delta)/\delta} dx\right)^{\delta/(2+\delta)} \left(\int_{B^+(R)} (1 + |Du^{(\nu)}|)^{(1+\delta/2)q_2} dx\right)^{2/(2+\delta)} \\ &\leq c(R)\left(\int_{B^+(R)} |A^{(\nu)}(x, u^{(\nu)}) - A(x, u^{(\nu)})|^{(2+\delta)/\delta} dx\right)^{\delta/(2+\delta)}. \end{aligned}$$

Since (4.4) implies $u^{(\nu)}(x) \to v(x)$ almost every x, taking subsequence if necessary, from the assumption that $A^{(\nu)}(x, u)$ converges uniformly to $A(x, u)$, by virtue of Lebesgue's dominated convergent theorem, we have that

$$\int_{B^+(R)} |A^{(\nu)}(x, u^{(\nu)}) - A(x, u^{(\nu)})|^{(2+\delta)/\delta} dx \to 0.$$

Thus we see that

$$II \to 0 \quad \text{as } \nu \to \infty. \tag{4.16}$$

From (4.12), (4.14) and (4.16) we get (4.9), so we see that (4.7) holds.

Now, let us prove that v is a local minimizer of $\mathcal{F}_0$. Let $w \in W^{1,p_0}(B^+(R))$ be a minimizer of $\mathcal{F}_0$ on $B^+(R)$ with $w = v$ on $\partial B^+(R)$. We mention that the w satisfies the same boundary condition that v satisfies on $\Gamma(R)$, namely $w = h$ on $\Gamma(R)$.

In the following part of the proof, taking ν sufficiently large, we suppose always that

$$\left(1 + \frac{\delta}{2}\right) p_0 \geq \left(1 + \frac{\delta}{4}\right) \sup_{B^+} p_\nu(x). \tag{4.17}$$

On the other hand, by (4.4) and (4.5), we have that

$$v \in W^{1,(1+\delta/2)q_2}(B^+(R)) \subset W^{1,(1+\delta/2)p_0}(B^+(R)). \tag{4.18}$$

Then, using Corollary 2.3 with $p(x) = p_0$, we see that

$$w \in W^{1,(1+\delta/2)p_0}(B^+(R)) \subset W^{1,(1+\delta/4)p_\nu(x)}(B^+(R)) \cap W^{1,q_2}(B^+(R)). \tag{4.19}$$

Here, we used (4.2) and (4.17) for the last inclusion. Moreover, using (4.17), Corollary 2.3 and (4.6), we see that w satisfies

$$\int_{B^+(R)} |Dw|^{(1+\delta/4)p_\nu(x)}dx \le c \int_{B^+(R)} \big(1+|Dw|\big)^{(1+\delta/2)p_0}dx \le c \int_{B^+(R)} \big(1+|Dv|\big)^{(1+\delta/2)p_0}dx \le c(R). \tag{4.20}$$

Fixing $R \in (0,1)$, for $\rho \in (R/2, R)$ put

$$T_\rho := \big\{x \in B^+(\rho);\ x^m > R-\rho\big\}, \tag{4.21}$$

and let $\eta \in C_0^1(B^+(R))$ be a cut-off function satisfying that

$$0 \le \eta \le 1 \quad \text{on } B^+(R), \qquad \eta \equiv 1 \quad \text{on } T_\rho, \qquad |D\eta| \le \frac{2}{R-\rho} \quad \text{on } B^+(R).$$

If necessary, we extend η outside $B^+(R)$ by 0. Let us put

$$\psi := (1-\eta)\big(u^{(\nu)}-v\big), \qquad v^{(\nu)} := w+\psi. \tag{4.22}$$

From the assumption that $w = v$ on $\partial B^+(R)$, we have

$$v^{(\nu)} = w + \big(u^{(\nu)}-v\big) = u^{(\nu)} \quad \text{on } \partial B^+(R).$$

So, the minimality of $u^{(\nu)}$ for $\mathcal{F}^{(\nu)}$, we see that

$$\mathcal{F}^{(\nu)}\big(u^{(\nu)}; B^+(R)\big) \le \mathcal{F}^{(\nu)}\big(v^{(\nu)}; B^+(R)\big). \tag{4.23}$$

Now, as in [6, pp. 458–460], by estimating $|\mathcal{F}^{(\nu)}(v^{(\nu)}; B^+(R)) - \mathcal{F}_0(v^{(\nu)}; B^+(R))|$ and $|\mathcal{F}_0(v^{(\nu)}; B^+(R)) - \mathcal{F}_0(w; B^+(R))|$, we show that $\mathcal{F}^{(\nu)}(v^{(\nu)}; B^+(R)) \to \mathcal{F}_0(w; B^+(R))$.

First, let us estimate $|\mathcal{F}^{(\nu)}(v^{(\nu)}; B^+(R)) - \mathcal{F}_0(v^{(\nu)}; B^+(R))|$.

$$\begin{aligned}
&\big|\mathcal{F}^{(\nu)}\big(v^{(\nu)}; B^+(R)\big) - \mathcal{F}_0\big(v^{(\nu)}; B^+(R)\big)\big| \\
&\le \int_{B^+(R)} \Big|\big(A^{(\nu)}\big(x, v^{(\nu)}\big)Dv^{(\nu)}Dv^{(\nu)}\big)^{p_\nu(x)/2} - \big(A\big(x, v^{(\nu)}\big)Dv^{(\nu)}Dv^{(\nu)}\big)^{p_\nu(x)/2}\Big|dx \\
&\quad + \int_{B^+(R)} \Big|\big(A\big(x, v^{(\nu)}\big)Dv^{(\nu)}Dv^{(\nu)}\big)^{p_\nu(x)/2} - \big(A\big(x, v^{(\nu)}\big)Dv^{(\nu)}Dv^{(\nu)}\big)^{p_0/2}\Big|dx \\
&\le \int_{B^+(R)} \big|A^{(\nu)}\big(x, v^{(\nu)}\big) - A\big(x, v(\nu)\big)\big| \cdot \big|Dv^{(\nu)}\big|^{p_\nu(x)}dx \\
&\quad + C(\varepsilon) \sup_{B^+(R)} \big|p_\nu(x) - p_0\big| \int_{B^+(R)} \big(1+\big|Dv^{(\nu)}\big|^2\big)^{(1+\delta/2)p_0/2}dx,
\end{aligned} \tag{4.24}$$

where we used (4.15) and (2.22). By the definition of $v^{(\nu)}$, we see that

$$\big|Dv^{(\nu)}\big| = \big|Dw + D(1-\eta)\big(u^{(\nu)}-v\big)\big| \le |Dw| + (1-\eta)\big|D\big(u^{(\nu)}-v\big)\big| + \frac{2}{R-\rho}\big|u^{(\nu)}-v\big|. \tag{4.25}$$

So,we have that

$$\begin{aligned}
&\int_{B^+(R)} \big|Dv^{(\nu)}\big|^{(1+\delta/2)p_0/2}dx \\
&\le c(p_0)\Bigg[\int_{B^+(R)} |Dw|^{(1+\delta/2)p_0/2} + \int_{B^+(R)} \big|Du^{(\nu)}\big|^{(1+\delta/2)p_0/2}dx + \int_{B^+(R)} |Dv|^{(1+\delta/2)p_0/2}dx \\
&\quad + \left(\frac{2}{R-\rho}\right)^{(1+\delta/2)p_0/2} \int_{B^+(R)} \big|u^{(\nu)}-v\big|^{(1+\delta/2)p_0/2}dx\Bigg].
\end{aligned} \tag{4.26}$$

By virtue of (4.3), (4.4), (4.6) and (4.20), all terms of the right hand side can be estimated by some constant $C(R)$ depending on R. Thus we get

$$\int_{B^+(R)} \left(1+\left|Dv^{(\nu)}\right|^2\right)^{(1+\delta/2)p_0/2} dx \le C(R). \tag{4.27}$$

By Hölder's inequality, (4.17) and (4.27) we can estimate the first term of the right-hand side of (4.24) as

$$\begin{aligned}
&\int_{B^+(R)} \left|A^{(\nu)}\left(x, v^{(\nu)}\right) - A\left(x, v^{(\nu)}\right)\right| \cdot \left|Dv^{(\nu)}\right|^{p_\nu(x)} dx \\
&\le \left(\int_{B^+(R)} \left|A^{(\nu)}\left(x, v^{(\nu)}\right) - A\left(x, v^{(\nu)}\right)\right|^{(4+\delta)/\delta}\right)^{\delta/(4+\delta)} \left(\int_{B^+(R)} \left|Dv^{(\nu)}\right|^{(1+\delta/4)p_\nu} dx\right)^{4/(4+\delta)} \\
&\le \left(\int_{B^+(R)} \left|A^{(\nu)}\left(x, v^{(\nu)}\right) - A\left(x, v^{(\nu)}\right)\right|^{(4+\delta)/\delta}\right)^{\delta/(4+\delta)} \left(\int_{B^+(R)} \left(1+\left|Dv^{(\nu)}\right|^2\right)^{(1+\delta/2)p_0/2} dx\right)^{4/(4+\delta)} \\
&\le C(R)\left(\int_{B^+(R)} \left|A^{(\nu)}\left(x, v^{(\nu)}\right) - A\left(x, v^{(\nu)}\right)\right|^{(4+\delta)/\delta} dx\right)^{\delta/(4+\delta)}.
\end{aligned} \tag{4.28}$$

By (4.4) and the assumption that $A^{(\nu)}$ converges uniformly to A, we see that the right-hand side of (4.28) tends to 0 as $\nu \to \infty$. From (4.27), we also see that the second term of (4.24) tends to 0 as $\nu \to \infty$ easily. Thus we have

$$\lim_{\nu\to\infty} \left|\mathcal{F}^{(\nu)}\left(v^{(\nu)}; B^+(R)\right) - \mathcal{F}_0\left(v^{(\nu)}; B^+(R)\right)\right| = 0. \tag{4.29}$$

Next, let us estimate $|\mathcal{F}_0(v^{(\nu)}); B^+(R) - \mathcal{F}_0(w; B^+(R))|$. Remarking that w differs from $v^{(\nu)}$ only on $B^+(R) \setminus T_\rho$, and using (4.25), we see that

$$\begin{aligned}
&\left|\mathcal{F}_0\left(v^{(\nu)}; B^+(R)\right) - \mathcal{F}_0\left(w; B^+(R)\right)\right| \\
&= \left|\int_{B^+(R)} \left(A\left(x, v^{(\nu)}\right) Dv^{(\nu)} Dv^{(\nu)}\right)^{p_0/2} dx - \int_{B^+(R)} \left(A(x, w) Dw Dw\right)^{p_0/2} dx\right| \\
&\le K \int_{B^+(R)\setminus T_\rho} \left|Dv^{(\nu)}\right|^{p_0} dx + K \int_{B^+(R)\setminus T_\rho} |Dw|^{p_0} dx \\
&\le 2KC(p_0)\left[\int_{B^+(R)\setminus T_\rho} |Dw|^{p_0} dx + \int_{B^+(R)\setminus T_\rho} \left|Du^{(\nu)}\right|^{p_0} dx \right. \\
&\qquad \left. + \int_{B^+(R)\setminus T_\rho} |Dv|^{p_0} dx + \left(\frac{2}{R-\rho}\right)^{p_0} \int_{B^+(R)\setminus T_\rho} \left|u^{(\nu)} - v\right|^{p_0} dx\right] \\
&=: III + IV + V + VI
\end{aligned} \tag{4.30}$$

where K is a constant which appeared in condition (A-1) and $c(p_0)$ a constant depending only on p_0. Since the weak convergence (4.5) implies uniform boundedness of $L^{(1+\delta/2)q_2}$ norm, we see that there exists a constant M_0 such that

$$\left(\int_{B^+(R)} \left|Du^{(\nu)}\right|^{(1+\delta/2)p_0} dx\right)^{2/(2+\delta)} \le M_0. \tag{4.31}$$

Here, mention that $p_0 \le q_2$. So, using Hölder's inequality, we can estimate IV as

$$IV \le \Big(\int_{B^+(R)\setminus T_\rho} 1 dx \Big)^{\delta/(2+\delta)} \Big(\int_{B^+(R)\setminus T_\rho} \big|Du^{(\nu)}\big|^{(1+\delta/2)p_0} dx \Big)^{2/(2+\delta)}$$
$$\le c\big(R^m - \rho^m\big)^{\delta/(2+\delta)} M_0. \tag{4.32}$$

Similarly, by virtue of (4.18) and (4.19), using Hölder's inequality, we can estimate III and V as follows

$$III \le C\big(R^m - \rho^m\big)^{\delta/(2+\delta)} \|Dw\|^{p_0}_{L^{(1+\delta/2)p_0}(B^+(R))}, \tag{4.33}$$
$$V \le C\big(R^m - \rho^m\big)^{\delta/(2+\delta)} \|Dv\|^{p_0}_{L^{(1+\delta/2)p_0}(B^+(R))}. \tag{4.34}$$

For fixed R and ρ, the strong convergence (4.4) implies that

$$VI \to 0 \quad \text{as } \nu \to \infty. \tag{4.35}$$

Combing (4.30) with (4.32)–(4.35), we see that

$$\limsup_{\nu\to\infty} \big|\mathcal{F}_0\big(v^{(\nu)}; B^+(R)\big) - \mathcal{F}_0(w)\big| \le M_1(M_0, v, w)\big(R^m - \rho^m\big)^{\delta/(2+\delta)}. \tag{4.36}$$

Now, by virtue of (4.7), (4.23), (4.29) and (4.36), we obtain

$$\begin{aligned}\mathcal{F}_0\big(v; B^+(R)\big) &\le \liminf_{\nu\to\infty} \mathcal{F}^{(\nu)}\big(u^{(\nu)}, B^+(R)\big)\\ &\le \liminf_{\nu\to\infty} \mathcal{F}^{(\nu)}\big(v^{(\nu)}, B^+(R)\big)\\ &= \liminf_{\nu\to\infty} \mathcal{F}_0\big(v^{(\nu)}, B^+(R)\big)\\ &\le \mathcal{F}_0\big(w; B^+(R)\big) + CM_1\big(R^m - \rho^m\big)^{\delta/(2+\delta)}\end{aligned} \tag{4.37}$$

Letting $\rho \to R$, we see that $\mathcal{F}_0(v; B^+(R)) \le \mathcal{F}_0(w; B^+(R))$. On the other hand we are assuming that w minimizes $\mathcal{F}_0$ relative to the boundary value $w = v$ on $\partial B^+(R)$. So, we can conclude that v minimizes $\mathcal{F}_0$.

Part 3 *(Proof for the statement on singular points)*. Let $x_\nu \in B^+ \cup \Gamma$ be a singular point of $u^{(\nu)}$ and assume that $x^\nu \to \bar{x}$. We want to show that $\bar{x}$ is a singular point of the limit map v. For the case that $\bar{x} \in B^+$ this assertion is shown in [16]. So let us consider the case $\bar{x} \in \Gamma_R$ for some $R \in (0, 1)$.

Considering sufficiently large ν if necessary, we can assume that $x^\nu \in B^+(R')$ for some $R' \in (R, 1)$.

For $y \in B^+(R'')$ and $r \in (0, 1 - R'')$, let us write

$$p_2^{(\nu)}(y, r) := \sup_{\Omega(y,r)} p_\nu(x). \tag{4.38}$$

By virtue of Theorem 3.2, we can choose $\bar{R} \in (0, 1 - R'')$ so that

$$\Psi_\nu(x_\nu, r) := r^{p_2^{(\nu)}(x_\nu, r) - m} \int_{\Omega(x_\nu, r)} \big(1 + |Du|^2\big)^{p_2^{(\nu)}(x_\nu, r)/2} dx > \varepsilon_0 \tag{4.39}$$

holds for the positive number ε_0 that appears in Theorem 3.2, any $r \in (0, \bar{R})$ and any number $\nu \in \mathbb{N}$. In the following, let us abbreviate

$$p_2^{(\nu)}(r) := p_2^{(\nu)}(x_\nu, r) = \sup_{\Omega(x_\nu, r)} p_\nu(x),$$

and let $\delta' < \delta$ be a positive constant satisfying

$$\delta' q_2 \le \frac{\sigma}{2}. \tag{4.40}$$

Since $p_\nu(x) \to p_0$ uniformly, taking ν sufficiently large, we can assume that

$$\Big(1 + \frac{\delta'}{2}\Big) p_2^{(\nu)}(r) \le \big(1 + \delta'\big) p_\nu(x) \quad \text{for all } x \in \Omega(x_\nu, r) \subset B. \tag{4.41}$$

Using (2.22) and Corollary 2.5, we see that

$$\begin{aligned}
& r^{p_2^{(\nu)}(r)-m}\left|\int_{\Omega(x_\nu,r)} \left(1+\left|Du^{(\nu)}\right|^2\right)^{p_2^{(\nu)}(r)/2}dx - \int_{\Omega(x_\nu,r)} \left(1+\left|Du^{(\nu)}\right|^2\right)^{p_\nu(x)/2}dx\right| \\
&\quad \le r^{p_2^{(\nu)}(r)-m}\omega_1(r)c_e\left(\delta'/2\right)\int_{\Omega(x_\nu,r)} \left(1+\left|Du^{(\nu)}\right|^2\right)^{(1+\delta'/2)p_2^{(\nu)}(r)/2}dx \\
&\quad \le r^{p_2^{(\nu)}(r)}\omega_1(r)c_e\left(\delta'/2\right)\fint_{\Omega(x_\nu,r)} \left(1+\left|Du^{(\nu)}\right|^2\right)^{(1+\delta')p_\nu(x)/2}dx \\
&\quad \le cr^{\sigma}r^{p_2^{(\nu)}(r)}r^{-(1+\delta')p_2^{(\nu)}(4r)} \le cr^{\sigma-(1+\delta')\omega_1(4r)-\delta' q_2} \\
&\quad \le cr^{\sigma/2-(1+\delta')\omega_1(4r)} \to 0 \quad \text{as } r\to 0,
\end{aligned}$$

since $\omega_1(4r)\to 0$ as $r\to 0$. Here, we used (4.41) for the second inequality, (2.12) and assumption (P-2) for the third one and (4.40) for the last one. Thus, (4.39) implies

$$r^{p_2^{(\nu)}(x_\nu,r)-m}\int_{\Omega(x_\nu,r)} \left(1+\left|Du^{(\nu)}\right|^2\right)^{p_\nu(x)/2}dx \ge \varepsilon_0/2, \quad \text{for any } r\in(0,\tilde{R}), \tag{4.42}$$

for sufficiently small $\tilde{R}\in(0,\bar{R})$.

Remarking that

$$(a+b)^q \le 2^{q-1}\left(a^q+b^q\right), \qquad \sqrt{a+b}\le\sqrt{a}+\sqrt{b} \quad \text{for any } a,b\ge 0 \text{ and } q\ge 1,$$

and taking $r>$ so small that

$$2^{q_2-1}r^{p_2^{(\nu)}(r)}\omega_m < \frac{\varepsilon_0}{8},$$

we get

$$r^{p_2^{(\nu)}(x_\nu,r)-m}\int_{\Omega(x_\nu,r)} \left|Du^{(\nu)}\right|^{p_\nu(x)}dx \ge \frac{3}{8\cdot 2^{q_2-1}}\varepsilon_0 = \frac{3}{2^{q_2+2}}\varepsilon_0. \tag{4.43}$$

Here, ω_m stands for the volume of m-dimensional unit ball.

Thus, for singular points x_ν of $u^{(\nu)}$, combining (4.43) with (2.9), we see that

$$c_4 r^{p_2^{(\nu)}(r)-m}\left(\int_{\Omega(x_\nu,2r)} \left|\frac{u^{(\nu)}-h^{(\nu)}}{2r}\right|^{p_\nu(x)}dx + \int_{\Omega(x_\nu,2r)} \left|Dh^{(\nu)}\right|^{p_\nu(x)}\right) \ge \frac{3}{2^{q_2+2}}\varepsilon_0. \tag{4.44}$$

Since we are assuming that $h^{(\nu)}\in W^{1,s}$ for some $s>m$, we have

$$\begin{aligned}
& \int_{\Omega(x_\nu,2r)} \left|Dh^{(\nu)}\right|^{p_\nu(x)} \\
&\quad \le \int_{\Omega(x_\nu,2r)} \left(1+\left|Dh^{(\nu)}\right|\right)^{p_2^{(\nu)}(2r)}dx \\
&\quad \le 2^{p_2^{(\nu)}(2r)}\left[(2r)^m\omega_m + \left[(2r)^m\omega_m\right]^{1-p_2^{(\nu)}(2r)/s}\left(\int_{\Omega(x_\nu,2r)} \left|Dh^{(\nu)}\right|^s dx\right)^{p_2^{(\nu)}(2r)/s}\right].
\end{aligned}$$

On the other hand we are also assuming that $h^{(\nu)}$ converges weakly to h in $W^{1,s}(B^+)$, so $\int_{\Omega(x_\nu,2r)}|Dh^{(\nu)}|^s dx$ are bounded by a constant which does not depend on ν and r. Thus, remarking also that $p_2^{(\nu)}\ge q_1$ and that $r<1$ we have for a constant C_h that

$$r^{p_2^{(\nu)}(2r)-m} \int_{\Omega(x_\nu,2r)} \left|Dh^{(\nu)}\right|^{p_\nu(x)} dx \le C_h r^{q_1(1-m/s)}.$$

Now, choosing $r > 0$ sufficiently small so that

$$c_4 C_h r^{q_1(1-m/s)} \le \frac{1}{2^{q_2+2}} \varepsilon_0,$$

we obtain from (4.44) that

$$r^{p_2^{(\nu)}(r)-m} \int_{\Omega(x_\nu,2r)} \left|\frac{u^{(\nu)} - h^{(\nu)}}{2r}\right|^{p_\nu(x)} dx \ge \frac{1}{2^{q_2+2}c_4} \varepsilon_0. \tag{4.45}$$

On the other hand, since $x_\nu \to \bar{x}$, $h^{(\nu)} \to h$ in L^{q_2}, $p_\nu(x) \rightrightarrows p_0$ and $u^{(\nu)} \to v$ in L^{q_2}, we can see that, as in [16, (3.41)],

$$r^{-m} \int_{\Omega(x_\nu,r)} \left|u^{(\nu)} - h^{(\nu)}\right|^{p_\nu(x)} dx \to r^{-m} \int_{\Omega(\bar{x},R)} |v-h|^{p_0} dx.$$

So, from (4.45) we can deduce that

$$r^{-m} \int_{\Omega(\bar{x},r)} |v-h|^{p_0} dx \ge \frac{1}{2^{q_2+2}c_4} \varepsilon_0 > 0 \tag{4.46}$$

for any $r \in (0, \tilde{R})$ for some $\tilde{R} > 0$. This implies that $\bar{x}$ is a singular point of v, since $v = h$ on the boundary. □

Now, thanks to the above lemma, we can prove full boundary regularity, Theorem 1.1, proceeding as in [6].

Proof of Theorem 1.1. For an arbitrarily fixed point $x_0 \in \partial\Omega$, choose a positive number $R_1 > 0$ sufficiently small so that (3.6) in Theorem 3.2 holds. By considering suitable bi-Lipschitz transformation from $B(x_0, R_1)$ onto $B^+ = B^+(0, 1)$, we can assume, without loss in generality, that $x_0 = 0$, $B^+ = B(x_0, R_1) \cap \Omega$ and that (3.6) holds on B^+. It is enough to show that $x_0 = 0$ is not a singular point of u.

For $\nu \in \mathbb{N}$, let us put

$$u^{(\nu)}(x) := u\left(\nu^{-1}x\right), \qquad h^{(\nu)}(x) := h\left(\nu^{-1}x\right), \qquad p_\nu(x) := p\left(\nu^{-1}x\right)$$
$$A^{(\nu)}(x, v) = A_{ij}^{(\nu)\alpha\beta}(x, v) := \nu^{2-2(p(0)/p_\nu(x))} g^{\alpha\beta}\left(\nu^{-1}x\right) h_{ij}(v).$$

Then, $u^{(\nu)}$ minimizes the functional

$$\mathcal{E}^{(\nu)}\left(v; B^+\right) := \int_{B^+} \left(A_{ij}^{(\nu)\alpha\beta}(x, v) D_\alpha v^i D_\beta v^j\right)^{p_{(\nu)}(x)/2} dx,$$

in the class

$$\left\{v \in W^{1,p_\nu(x)}\left(B^+\right);\ v = h \text{ on } \Gamma\right\}.$$

Since we are assuming that $p(x)$ is Hölder continuous, $\nu^{p_{(\nu)}(x)-p(0)}$ tends to 1 uniformly as $\nu \to \infty$. So, we have that

$$A_{ij}^{(\nu)\alpha\beta}(x, v) \rightrightarrows g^{\alpha\beta}(0) h_{ij}(v).$$

On the other hand, since we are assuming the boundedness of u, $\|u^{(\nu)}\|_\infty$ are uniformly bounded, and therefore, taking subsequence if necessary, $u^{(\nu)} \rightharpoonup u_\infty$ for some u_∞ in $L^2(B^+)$.

About the boundary conditions $h^{(\nu)}$, we can see that $h^{(\nu)} \to h(0)$ strongly in $W^{1,s}(B^+)$ exactly as in [6, p. 465].

Thus, all the assumptions in Lemma 4.1 are satisfied. So, using Lemma 4.1, we see that u_∞ minimizes the functional

$$\mathcal{E}_\infty(v, B^+) := \int_{B^+} \left(g^{\alpha\beta}(0) h_{ij}(v) D_\alpha v^i D_\beta v^j\right)^{p(0)/2} dx,$$

in the class

$$\left\{v \in W^{1,p(0)}(B^+);\ v = h(0) \text{ on } \Gamma\right\},$$

and 0 is a singular point of u_∞. However, [6, Theorem 5.4] says that a minimizer of a standard p-growth functional $(p > 1)$ cannot have singularity on the boundary. This is a contradiction, and we conclude that $x_0 = 0$ cannot be a singular point of u. □

Conflict of interest statement

This article has no conflict of interest.

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