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Uniform Lipschitz regularity for classes of minimizers in two phase free boundary problems in Orlicz spaces with small density on the negative phase

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Abstract

In this paper we investigate Lipschitz regularity of minimizers for classes of functionals including ones of the type $E_G(u,\Omega)=\int_\Omega[G(|\nabla u|)+f_2\chi_{\{u>0\}}+f_1\chi_{\{u\leqslant 0\}}]\,dx$. We prove that there exists a universal "tolerance" (depending only on the degenerate ellipticity and other intrinsic parameters) for the density of the negative phase along the free boundary under which uniform Lipschitz regularity holds. We also prove density estimates from below for the negative phase on points inside the contact set between the negative and positive free boundaries in the case where Lipschitz regularity fails to be the optimal one.

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1. Introduction

In this paper we investigate the Lipschitz regularity for minimizers of functionals of the type

$$E_G(u,\Omega)=\int_\Omega\big[G\big(|\nabla u|\big)+\lambda(f_1,f_2)(u)\big]\,dx \tag{1.1}$$

where

$$\lambda(f_1,f_2)(u)(x):=f_2(x)\cdot\chi_{\{u>0\}}+f_1(x)\cdot\chi_{\{u<0\}}+\min\big(f_1(x),f_2(x)\big)\cdot\chi_{\{u=0\}},\quad 0\leqslant f_1,f_2\leqslant\mu.$$

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At least formally, the Euler–Lagrange equations for such minimizers are two phase Free Boundary Problems (FBPs) of the type

$$\begin{cases} \mathcal{L}_g u := \operatorname{div}\left(\dfrac{g(|\nabla u|)}{|\nabla u|} \cdot \nabla u\right) = 0 & \text{in } \{u \neq 0\}, \\ H\left(\left|\nabla u^+\right|\right) - H\left(\left|\nabla u^-\right|\right) = f_2(x) - f_1(x) & \text{on } F(u)^\pm := \left(\partial\{u > 0\} \cup \partial\{u < 0\}\right) \cap \Omega, \\ g(t) = G'(t), \quad H(t) = g(t)t - G(t). \end{cases} \tag{1.2}$$

Here, we assume the natural conditions introduced by G.M. Lieberman (see [13]) in the study of the regularity theory of the degenerate/singular elliptic equations of the type $\mathcal{L}_g u = B(x, u, \nabla u)$. The precise conditions are

$$G'(t) = g(t), \quad \text{where } g \in C^0\left([0, +\infty)\right) \cap C^1\left((0, +\infty)\right); \tag{P}$$

and for $0 < \delta \leqslant g_0$ fixed constants,

$$0 < \delta \leqslant Q_g(t) := \frac{tg'(t)}{g(t)} \leqslant g_0, \quad \forall t > 0. \tag{C1}$$

As pointed out in [15], Lieberman's condition says that $\mathcal{L}_g u = 0$ is equivalent to a uniformly elliptic equation (in non-divergence form) with (positive) ellipticity constants depending just on δ, g_0 in the sets where $\nabla u \neq 0$ (Remark 7.1, [15]). They do not imply any kind of homogeneity whatsoever for the function G and they even allow different behaviors for g when $|\nabla u|$ is near zero or infinity. This fact alone makes the study of FBPs like (1.2) more involved as shown in the excellent paper of S. Martinez and N. Wolanski [15]. In this paper, we regard δ, g_0 as the (degenerate) ellipticity constants for the functionals E_G.

Examples of functions satisfying (P) and (C1) are $g(t) = t^p$ with $\delta = g_0 = p$, $g(t) = at^p + bt^q$ with $a, b, p, q > 0$ with $\delta = \min\{p, q\}$ and $g_0 = \max\{p, q\}$ and $g(t) = t^p \log(at + b)$ with $p, a, b > 0$ where in this case $\delta = p$ and $g_0 = p + 1$. Many other examples are discussed in Section 7.

FBPs like (1.2) appear in many applications, in particular, in the study of the flow of two liquids in models of jets and cavities. Lipschitz regularity for minimizers of E_G has been obtained in several circumstances. The general two phase case for $G(t) = t^2$ was proven by H.W. Alt, L.A. Caffarelli and A. Friedman in the seminal papers [2,5]. They also extended the one phase case to quasilinear equations where $g(t) \sim t$ in [3]. Recently, S. Martinez and N. Wolanski studied the general one phase case in the Orlicz spaces setting under (P) and (C1) conditions [15]. These results extended the p-Laplace case for one phase ($G(t) = t^p$) considered by D. Danielli and A. Petrosyan in the nice paper [10].

The situation for the two phase case is much more delicate and less understood in the general setting. The only result essentially known is the one due to H.W. Alt, L.A. Caffarelli and A. Friedman in [5] in the standard case $G(t) = t^2$. The proof relies in a fundamental way on the so-called Alt–Caffarelli–Friedman monotonicity formula. To the best of our knowledge, Lipschitz regularity remains wide-open in two phase scenarios involving more general functionals, except in those cases where a variant of the ACF monotonicity formula exists (see [7], for instance). There is also an interesting result of A. Karakhanyan in [11], for the p-Laplace case, where he showed the existence of a universal constant $C > 0$ depending on p, n and L^∞ norm of u such that if the Lebesgue density of the negative phase along the free boundary is at most C then minimizers are (locally) Lipschitz continuous.

In this paper, our goal is to extend the result of A. Karakhanyan to the context of Orlicz spaces and to present new density estimates for the negative phase at contact free boundary points even in the case where Lipschitz regularity fails to be the optimal one.

We would like to obtain universal estimates for two phase minimizers for (the largest possible class of) functionals that share the same ellipticity, instead of focusing on any particular one. In order to do that we introduce the class,

$$\mathcal{G}(\delta, g_0) := \left\{G : [0, \infty) \to [0, \infty);\ G \text{ is } N\text{-function satisfying (P) and (C1)}\right\}.$$

We show that the graph of one phase minimizers of the functionals E_G with $G \in \mathcal{G}(\delta, g_0)$ hits the free boundary with slope comparable to negative powers of $G(1)$. More precisely, if $G(1)$ is small enough, we have for universal constants $C_1, C_2 > 0$

$$C_1 \cdot \left(\frac{1}{G(1)}\right)^{\frac{1}{1+g_0}} \leqslant |\nabla u| \leqslant C_2 \cdot \left(\frac{1}{G(1)}\right)^{\frac{1}{1+\delta}} \quad \text{along } F(u) := \partial\{u > 0\} \cap \Omega. \tag{1.3}$$

This shows that Lipschitz regularity can only hold for non-degenerate subclasses of $\mathcal{G}(\delta, g_0)$, i.e., subsets where $G(1)$ is bounded from below away from zero, say $G(1) \geqslant \varepsilon_0 > 0$. In this case, ε_0 is regarded as the non-degeneracy constant.

Since a tool like the ACF monotonicity formula is missing in our context, it seems hard to capture any quantitative information to control the gradient of the minimizers close enough to the free boundary. This way, we are led to consider compactness arguments. At this point, unlike the previous cases, the situation is more involved and we have to study compactness in the class of degenerate/singular elliptic operators $\mathcal{L}_g$ represented here by the classes $\mathcal{G}(\delta, g_0)$. We also observe that conditions (P) and (C1) are too weak to provide compactness (as shown in Example 7.1). The main reason for this failure is the absence of a uniform modulus of continuity for the quotient Q_g within $\mathcal{G}(\delta, g_0)$.

The following Morrey type control

$$\int_t^{t+\kappa} \left|Q_g'(s)\right| ds \leqslant \frac{C_0}{t^\beta} \cdot \kappa^\beta \quad \forall t, \kappa > 0, \tag{1.4}$$

supplies what is needed for this compactness to hold in the non-degenerate classes with respect to the $C^{2,\gamma}_{loc}(0,\infty)$, $\gamma < \beta$ and $C^1_{loc}[0,\infty)$ topologies. We believe that this compactness result may be of independent interest.

Our main result follows then by a combination of compactness, invariance of the non-degenerate classes and the Morrey control above by nonlinear normalizing scalings, Martinez–Wolanski stability estimate (Theorem 2.3 in [15]) and $C^{1,\alpha}$ regularity theory for degenerate/singular elliptic equations.

We can roughly summarize the results in the paper as follows. Theorem 2.1 provides a pointwise Lipschitz estimate for any minimizer of any functional E_G with $G \in \mathcal{G}(\delta, g_0)$ satisfying $G(1) \geqslant \varepsilon_0$ and (1.4) at any free boundary point provided the density of the negative phase at that point is smaller than some (small) universal constant $C = C(n, \delta, g_0, \varepsilon_0, \mu, \beta) > 0$. As a consequence of that, if the density stays below this constant in a whole free boundary neighborhood of such a point, the estimate propagates around and a local and uniform Lipschitz estimate is obtained. This is the content of Corollary 2.1. It essentially says that there exists a (small) universal "tolerance" for the size of the negative phase under which the uniform Lipschitz regularity still holds.

Since optimal Lipschitz regularity remains unsettled, we look at the region where it may fail. In the generality treated here, there is no ordering on the phase functions (like $f_2 \geqslant f_1$). Thus, unlike the case studied by Alt–Caffarelli–Friedman in [5], the negative free boundary may, in principle, separate from the positive one, i.e., $\partial\{u < 0\} \setminus \partial\{u > 0\} \cap \Omega \neq \emptyset$.

By using a delicate scaling of Theorem 2.1 (Proposition 8.1 and Remark 9.1), we show that the region where Lipschitz regularity fails is contained in the contact set between the positive and negative free boundaries. Furthermore, at any such point of that region, the negative phase has a universal (upper) density estimate from below. In particular, if Lipschitz regularity fails at a point, then this point belongs to the contact set $\partial\{u < 0\} \cap \partial\{u > 0\}$ and there the negative phase is cusp free. This is proven in Proposition 2.1.

These results complement the ones in [5], where density estimates for the nonnegative phase were proven in the presence of Lipschitz regularity and non-degeneracy properties, absent here. They also corroborate with the idea that free boundaries of minimizers for the cavity flow type functionals are somewhat nicer than the free boundaries of the solutions to the obstacle problem, once cusps cannot develop.

Our approach in this paper differs substantially from the one adopted in [11]. There, minimizers are subsolutions and Cacciopoli/energy estimates yield a p-energy control. Here, due to the fact that G may behave differently at zero and infinity, Cacciopoli/energy estimates render no control whatsoever on the G-energy. Moreover, under no ordering assumptions on the phase functions f_1 and f_2, minimizers may not be neither subsolutions nor supersolutions.

Our paper is organized as follows: In Section 2, we present our results. In Section 3, we quote some background results in the theory of degenerate/singular elliptic equations studied by G.M. Lieberman, S. Martinez and N. Wolanski. In Section 4, we study existence and boundedness for global minimizers. Section 5 is devoted to the uniform C^α estimates for the class of all minimizers of non-degenerate functionals. In Section 6, we motivate the necessity of non-degenerate classes by discussing (1.3) and we also prove the compactness and scaling invariance results. Section 7 is devoted to the presentation of several examples of N-functions in the non-degenerate classes of $\mathcal{G}_\beta(\delta, g_0)$ and $\mathcal{G}_2(\delta, g_0)$ showing they are indeed quite large. Here, we also discuss the lack of the compactness in $\mathcal{G}(\delta, g_0)$ (see Example 7.1). Section 8 is destined to the proof of Theorem 2.1 and its scaled version Proposition 8.1. In Section 9, we provide the proof of Corollary 2.1 which is the local Lipschitz regularity under small density assumption. Finally, in Section 10, we discuss the touching between positive and negative free boundaries and the density estimates for the negative phase at the contact set. There is also an appendix with a short proof of a Morrey's type result in the Orlicz space setting (Lemma 3.1) in Appendix A.

2. Presentation of our results

We begin this section by recalling the definition of N-function. This is a function of the type

$$G(t) = \int_0^t g(s)\,ds,$$

where $g : [0, \infty) \to \mathbb{R}$ is a positive nondecreasing function satisfying the following properties:

a) $g(0) = 0$ and $\lim_{t\to\infty} g(t) = \infty$;
b) g is right continuous, that is, if $t \geqslant 0$ then $\lim_{s\to t+} g(s) = g(t)$.

Definition 2.1. Let $0 < \beta \leqslant 1$. We say that an N-function G belongs to the $\mathcal{G}_\beta(\delta, g_0)$ if

(i) $G \in \mathcal{G}(\delta, g_0)$ with $G' = g \in W^{2,1}_{loc}((0, +\infty))$ and
(ii) for any $t > 0$ and $\kappa > 0$ the following control holds

$$\int_t^{t+\kappa} \left|Q_g'(s)\right| ds \leqslant \frac{C_0}{t^\beta} \cdot \kappa^\beta, \tag{MC-β}$$

where $Q_g(s) := \frac{sg'(s)}{g(s)}$ and here C_0 is some positive constant depending on δ, g_0, and β.

We say that $G \in \mathcal{G}_2(\delta, g_0)$ if (i) is satisfied and (ii) is replaced by

$$0 \leqslant \frac{t^2|g''(t)|}{g(t)} \leqslant \mathcal{C}(\delta, g_0), \quad \text{for a.e. } t > 0. \tag{C2}$$

This way, for every $\beta \in (0, 1]$ it follows that $\mathcal{G}_2(\delta, g_0) \subset \mathcal{G}_\beta(\delta, g_0) \subset \mathcal{G}(\delta, g_0)$ as shown in Section 7.

Definition 2.2 *(Non-degenerate subclasses).* For $\varepsilon_0 > 0$ we define

$$\mathcal{G}(\delta, g_0, \varepsilon_0) = \bigl\{G \in \mathcal{G}(\delta, g_0):\ G(1) \geqslant \varepsilon_0\bigr\},$$
$$\mathcal{G}_\beta(\delta, g_0, \varepsilon_0) = \bigl\{G \in \mathcal{G}_\beta(\delta, g_0):\ G(1) \geqslant \varepsilon_0\bigr\},$$
$$\mathcal{G}_2(\delta, g_0, \varepsilon_0) = \bigl\{G \in \mathcal{G}_2(\delta, g_0):\ G(1) \geqslant \varepsilon_0\bigr\}.$$

With this notation, we can now define the main classes of minimizers for which our results hold.

Definition 2.3. We say a function u belongs to $\mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu)$ if u is a minimizer of a functional of the type

$$E_G(u, \Omega) = \int_\Omega \bigl[G\bigl(|\nabla u|\bigr) + \lambda(f_1, f_2)(u)\bigr]\,dx \tag{2.5}$$

where

$$\lambda(f_1, f_2)(u)(x) := f_2(x) \cdot \chi_{\{u>0\}} + f_1(x) \cdot \chi_{\{u<0\}} + \min\bigl(f_1(x), f_2(x)\bigr) \cdot \chi_{\{u=0\}} \tag{2.6}$$

and

$$G \in \mathcal{G}(\delta, g_0, \varepsilon_0), \quad 0 \leqslant f_1, f_2 \leqslant \mu. \tag{2.7}$$

If condition (2.7) is replaced by

$$G \in \mathcal{G}_\beta(\delta, g_0, \varepsilon_0), \quad 0 \leqslant f_1, f_2 \leqslant \mu, \tag{2.8}$$

we then say $u \in \mathcal{S}^\beta(\Omega, \delta, g_0, \varepsilon_0, \mu)$. Now,

$$\mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu, M) := \Big\{u \in \mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu): \sup_\Omega |u| \leqslant M\Big\},$$

$$\mathcal{S}^\beta(\Omega, \delta, g_0, \varepsilon_0, \mu, M) := \Big\{u \in \mathcal{S}^\beta(\Omega, \delta, g_0, \varepsilon_0, \mu): \sup_\Omega |u| \leqslant M\Big\},$$

$$\mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu)^+, \quad \mathcal{S}^\beta(\Omega, \delta, g_0, \varepsilon_0, \mu)^+, \quad \mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu, M)^+ \quad \text{and} \quad \mathcal{S}^\beta(\Omega, \delta, g_0, \varepsilon_0, \mu, M)^+$$

indicate nonnegative functions in Ω in the respective classes. Additionally, we denote

$$F^+(u) := \partial\{u > 0\} \cap \Omega, \qquad F^-(u) := \partial\{u < 0\} \cap \Omega, \qquad F^\pm(u) := F^+(u) \cup F^-(u)$$

and for $x_0 \in \Omega$

$$\mathcal{S}^\beta(\Omega, \delta, g_0, \varepsilon_0, \mu, M)(x_0) := \big\{u \in \mathcal{S}^\beta(\Omega, \delta, g_0, \varepsilon_0, \mu, M): x_0 \in F^\pm(u)\big\}, \tag{2.9}$$

and for the one phase case

$$\mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu, M)^+(x_0) := \big\{u \in \mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu, M)^+: x_0 \in F^+(u)\big\}. \tag{2.10}$$

For future reference we define the "degenerate" class $\mathcal{S}^*(\Omega, \delta, g_0, \mu, M)^+$ to be

$$\mathcal{S}^*(\Omega, \delta, g_0, \mu, M)^+ = \bigcup_{\varepsilon_0 > 0} \mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu, M)^+.$$

For simplicity of notation, we also use

$$S_r^\beta(\delta, g_0, \varepsilon_0, \mu, M)(x_0) := \mathcal{S}^\beta\big(B_r(x_0), \delta, g_0, \varepsilon_0, \mu, M\big)(x_0),$$

$$S_r^+(\delta, g_0, \varepsilon_0, \mu, M)(x_0) := \mathcal{S}\big(B_r(x_0), \delta, g_0, \varepsilon_0, \mu, M\big)^+(x_0).$$

Consider further Θ_u^- the density function of the negativity set along the free boundaries for u, that is:

$$\Theta_u^-(x_0, r) = \frac{|\{u < 0\} \cap B_r(x_0)|}{|B_r(x_0)|} \quad \text{with } x_0 \in F^\pm(u).$$

In this paper, n will always denote the dimension. The main results of this paper are the following

Theorem 2.1. *There exists a (small) universal constant $C = C(n, \delta, g_0, \varepsilon_0, \mu, \beta) > 0$ such that the estimate*

$$\big|u(x)\big| \leqslant \frac{2 \cdot \max\{M, 1\}}{C} \cdot |x - x_0|, \quad \forall x \in B_1(x_0), \tag{2.11}$$

holds for any $u \in S_1^\beta(\delta, g_0, \varepsilon_0, \mu, M)(x_0)$ provided $\Theta_u^-(x_0, r) \leqslant C$ for all $0 < r < 1$.

Corollary 2.1. *Let $u \in S_1^\beta(\delta, g_0, \varepsilon_0, \mu, M)(0)$ and assume that*

$$\Theta_u^-(z, r) \leqslant C \quad \forall z \in F^\pm(u) \cap B_{3/4} \quad \text{and for all } 0 < r < 1/4.$$

Here $C > 0$ is the constant given by the theorem above. Then u is Lipschitz continuous in $B_{1/2}$ and

$$[u]_{C^{0,1}(B_{1/2})} \leqslant \frac{C_0 \cdot \max\{M, 1\}}{C},$$

for a universal constant $C_0 = C_0(n, \delta, g_0) > 0$.

In the sequel, we use the following notation:

$$[u]_{C^{0,1}(B_\rho(x))}(x) = \sup_{y\in B_\rho(x), y\neq x} \frac{|u(y)-u(x)|}{|y-x|}, \qquad [u]_{C^{0,1}(B_\rho(x))} = \sup_{z,w\in B_\rho(x), z\neq w} \frac{|u(z)-u(w)|}{|z-w|}.$$

Proposition 2.1. *Let $u \in S_1^\beta(\delta, g_0, \varepsilon_0, \mu, M)$ and consider the sets*

$$\mathcal{S}_p := \{x \in \overline{B_{1/2}}\colon [u]_{C^{0,1}(B_r(x))}(x) = \infty \text{ for all } 0 < r < 1/2\},$$
$$\mathcal{S}_l := \{x \in \overline{B_{1/2}}\colon [u]_{C^{0,1}(B_r(x))} = \infty \text{ for all } 0 < r < 1/2\}.$$

Then, $\mathcal{S}_p \subset \mathcal{S}_l \subset F^+(u) \cap F^-(u)$ and

$$\limsup_{F^\pm(u)\times(0,1)\ni(x,r)\to(x_0,0)} \Theta_u^-(x,r) \geqslant D \quad \text{for every } x_0 \in \mathcal{S}_l,$$
$$\limsup_{r\to 0} \Theta_u^-(x_0,r) \geqslant D \quad \text{for every } x_0 \in \mathcal{S}_p,$$

where $D = D(n, \delta, g_0, \beta) > 0$ is a universal constant.

For one phase minimizers, there is actually a better result, in the sense that Lipschitz regularity holds for the class $S_1(\delta, g_0, \varepsilon_0, \mu, M)^+(0)$ with no further assumptions. This is essentially the result of S. Martinez and N. Wolanski, Theorem 4.2 in [15] rewritten in terms of the class $S_1^+(\delta, g_0, \varepsilon_0, \mu, M)(x_0)$ introduced here. We present the result below.

The proof depends essentially on the fact that uniform C^α estimate holds for non-degenerate classes $\mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu, M)$ (Theorem 5.1) and on a modification of the proof of Lemma 4.3 in [15] to put it in the context of classes. The details are left to the reader.

Theorem 2.2 *(One phase case). There exists a universal constant $C = C(n, \delta, g_0, \varepsilon_0, \mu)$ such that*

$$u \in S_1(\delta, g_0, \varepsilon_0, \mu)^+(0) \quad \Rightarrow \quad \|u\|_{C^{0,1}(B_{1/6})} \leqslant C.$$

3. Background results on Orlicz spaces and degenerate/singular elliptic equations

In this section, we present some background results that will be used throughout the paper. They are drawn from the theory of Orlicz–Sobolev spaces and the regularity theory of degenerate/singular elliptic equations of the type $\mathcal{L}_g u = 0$. The proofs can be found in the papers [13] and [15] as indicated below. More details on the theory of Orlicz spaces can be found in [1]. We systematically use definitions, results and basic properties of the functions G as developed in Section 2 of [15]. Throughout this paper $\Omega \subset \mathbb{R}^n$ always denotes a bounded domain with Lipschitz boundary.

We start off by observing that the conditions (P) and (C1) imply the following properties:

(g-1) $\min\{s^\delta, s^{g_0}\}g(t) \leqslant g(st) \leqslant \max\{s^\delta, s^{g_0}\}g(t)$, $\forall s, t > 0$;
(g-2) $\frac{tg(t)}{1+g_0} \leqslant G(t) \leqslant tg(t)$, $\forall t \geqslant 0$;
(G-1) G is convex and C^2;
(G-2) $\frac{1}{1+g_0}\min\{s^{1+\delta}, s^{1+g_0}\}G(t) \leqslant G(st) \leqslant (1+g_0)\max\{s^{1+\delta}, s^{1+g_0}\}G(t)$, $\forall s, t > 0$;
(G-3) $G(a+b) \leqslant 2^{g_0}(1+g_0)(G(a)+G(b))$, $\forall a, b > 0$.

The following result is the version of Morrey's lemma in the Orlicz–Sobolev setting. Part of the proof appears more or less in a paper of G.M. Lieberman [13] inside the proof of Theorem 1.7. We need a sharper version of it, where the relationship between the Hölder semi-norm $[u]_{\alpha,\Omega}$ and the number $G(1)$ is precisely computed. This way, we present the proof in Appendix A. This estimate will be an important point for our compactness argument in Sections 6 and 8.

Lemma 3.1 *(Morrey's type theorem). Let* $u \in W^{1,1}_{loc}(\Omega) \cap L^1(\Omega)$, $G \in \mathcal{G}(\delta, g_0)$ *and* $0 < \alpha < 1$. *Suppose* $\Omega' \Subset \Omega$ *such that*

$$\fint_{B_r(x_0)} G\big(|\nabla u|\big)\,dx \leqslant L r^{\alpha-1} \quad \text{for every } x_0 \in \Omega' \text{ with } 0 < r \leqslant R_0 \leqslant \operatorname{dist}\big(\Omega', \partial\Omega\big).$$

Then, there exist $C_1 = C_1(\alpha, n, g_0) > 0$ *and* $C_2 = C_2(\alpha, n, g_0, R_0) > 0$ *such that*

$$\big|u(x) - u(y)\big| \leqslant \left(C_1 \cdot \max\left\{\frac{L}{G(1)}, 1\right\}\right) \cdot |x-y|^\alpha, \quad \text{for } x, y \in \Omega' \text{ with } |x-y| \leqslant \frac{R_0}{2},$$

$$\|u\|_{L^\infty(\Omega')} \leqslant C_2\big(L + \|u\|_{L^1(\Omega)}\big),$$

$$[u]_{\alpha,\Omega'} \leqslant \max\left(C_1 \cdot \max\left\{\frac{L}{G(1)}, 1\right\}, \frac{2^{\alpha+1} \cdot \|u\|_{L^\infty(\Omega')}}{R_0^\alpha}\right). \tag{3.12}$$

Proof. See Appendix A. □

Definition 3.1. We say that a function $v \in W^{1,G}(\Omega)$ is a weak solution to the equation $\mathcal{L}_g v = 0$ in Ω if for all $\xi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega g\big(|\nabla v|\big)\frac{\nabla v}{|\nabla v|} \cdot \nabla\xi\,dx = 0.$$

The theorem stated below compiles some results on the regularity theory of (weak) solutions to degenerate/singular elliptic equations. These results were proven by G.M. Lieberman, S. Martinez and N. Wolanski in [13] and [15] respectively (see, for instance Theorem 1.7 and Lemma 5.1 in [13] and Lemma 2.7 in [15]). We present a slightly sharper version for our purposes (see Remarks 3.1 and 3.2).

Theorem 3.1. *Let* $G \in \mathcal{G}(\delta, g_0)$ *and* v *be a weak solution to* $\mathcal{L}_g v = 0$ *in* Ω, *where* G *is a primitive of* g. *Then* v *is* $C^{1,\alpha}(\Omega)$ *for some positive constant* $\alpha(n, g_0, \delta) < 1$. *Moreover, for any* $\Omega' \Subset \Omega$, *there exists a constant* $C_0 > 0$, *depending possibly on* $n, \delta, g_0, \operatorname{dist}(\Omega', \partial\Omega)$ *and* $\sup_{\Omega'} |v| > 0$ *such that*

$$\|v\|_{C^{1,\alpha}(\Omega')} \leqslant C_0. \tag{3.13}$$

Also, there exists a constant $C_1 = C_1(n, \delta, g_0) > 0$ *such that for all* $B_r \subset \Omega$,

$$\sup_{B_{r/2}} |\nabla v| \leqslant \frac{C_1}{r} \sup_{B_r} |v|. \tag{3.14}$$

Further, for every $\bar\beta \in (0, n)$, *there exists* $C_2 = C_2(n, \bar\beta, \delta, g_0, \sup_{B_r} |v|) > 0$ *such that*

$$\int_{B_{r/2}} G\big(|\nabla v|\big)\,dx \leqslant C_2 r^{\bar\beta}. \tag{3.15}$$

Remark 3.1. Despite the fact that Theorem 1.7 in [13] states that the constant C_0 above depends on $g(1)$, we observe that for homogeneous equations this dependence can be dropped. In fact, suppose $G \in \mathcal{G}(\delta, g_0)$ and $\mathcal{L}_g u = 0$ in Ω. Then, for any $\alpha > 0$ if we set $G_\alpha(t) := \alpha G(t) \in \mathcal{G}(\delta, g_0)$ then it follows that $\mathcal{L}_{g_\alpha} u = 0$ in Ω. In particular, taking $\alpha_0 = g(1)^{-1} > 0$, we have $g_{\alpha_0}(1) = 1$ and thus, Theorem 1.7 in [13] actually shows that there is no dependence on $g(1)$.

Remark 3.2. If C_2 is the constant in the estimate (3.15) of the previous theorem we have actually

$$C_2\Big(n, \bar\beta, \delta, g_0, \sup_{B_r} |v|\Big) = C_2(n, \bar\beta, \delta, g_0) \cdot \sup_{B_r} |v|.$$

Indeed, first we observe that if v solves $\mathcal{L}_g v = 0$ in B_r, then by setting $w(x) := \alpha v(\beta x)$ for $\alpha, \beta > 0$, we have $\mathcal{L}_{\bar{g}} w = 0$ in $B_{r/\beta}$ where $\bar{g}(t) = g(\frac{t}{\alpha \cdot \beta})$. This way, let us set $w(x) := v(x)/S$ where $S := \|v\|_{L^\infty(B_r)} > 0$. w solves $\mathcal{L}_{\bar{g}} w = 0$ in B_r with $\bar{g}(t) = g(S \cdot t)$ and $\|w\|_{L^\infty(B_r)} = 1$. In this case, it is easy to see that $\overline{G}(t) = \frac{G(S \cdot t)}{S}$. This way

$$r^{\bar{\beta}} \cdot C_2(n, \bar{\beta}, \delta, g_0) \geqslant \int_{B_{r/2}} \overline{G}(|\nabla w|)\, dx = \frac{1}{S} \cdot \int_{B_{r/2}} G(|\nabla v|)\, dx.$$

4. Existence theory and L^∞ estimates for global minimizers of E_G

In this section we establish existence and boundedness of global minimizers for the functionals E_G. As in the one phase case, the existence theory follows the standard procedure but the peculiar properties of the Orlicz spaces settings, like those in Section 2, come into play. Here, we just point out the main differences from the one phase case in [15].

Definition 4.1. We say $u \in W^{1,G}(\Omega)$ is a minimizer of E_G over K_φ if

$$E_G(u, \Omega) = \min_{v \in K_\varphi} E_G(v, \Omega),$$

where $\varphi \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ with $\int_\Omega G(|\nabla \varphi|)\, dx < \infty$ and

$$K_\varphi = \left\{ v \in L^1(\Omega) \colon \int_\Omega G(|\nabla v|)\, dx < \infty, \quad v = \varphi \text{ on } \partial\Omega \right\}.$$

We will also refer to minimizers over K_φ as global minimizers.

Proposition 4.1 *(Existence). Let $G \in \mathcal{G}(\delta, g_0)$ and suppose that $E_G(\varphi, \Omega) < \infty$. Then, the optimization problem*

$$\min_{v \in K_\varphi} E_G(v, \Omega)$$

has at least one minimizer over K_φ.

Proof. Let $\{u_j\}$ be a minimizing sequence for E_G in K_φ. Proceeding as in Theorem 3.1 in [15], we see that $\{u_j\}$ is bounded in $W^{1,G}(\Omega)$ and by reflexivity we may assume

$$u_j \rightharpoonup u \quad \text{weakly in } W^{1,G}(\Omega).$$

From this we conclude that $u = \varphi$ on $\partial\Omega$. Since $W^{1,G}(\Omega) \hookrightarrow L^{1+\delta}(\Omega)$ compactly, once more we can assume $u_j(x) \to u(x)$ for a.e. $x \in \Omega$. We see that,

$$\lambda(f_1, f_2)(u_j)(x) \to \lambda(f_1, f_2)(u)(x) \quad \text{for a.e. } x \in \{u > 0\} \cup \{u < 0\}. \tag{4.16}$$

In the case $u(x) = 0$, we note that

$$\lambda(f_1, f_2)(u)(x) \leqslant \liminf_{j \to \infty} \lambda(f_1, f_2)(u_j)(x). \tag{4.17}$$

By Fatou's lemma together with (4.16) and by (4.17) we have

$$\int_\Omega \lambda(f_1, f_2)(u)\, dx \leqslant \liminf_{j \to \infty} \int_\Omega \lambda(f_1, f_2)(u_j)\, dx.$$

By convexity of G and duality in Orlicz spaces as in Theorem 3.1 in [15] we have

$$\int_\Omega G(|\nabla u|)\, dx \leqslant \liminf_{j \to \infty} \int_\Omega G(|\nabla u_j|)\, dx.$$

This leads us to conclude that $u \in K_\varphi$ and

$$E_G(u,\Omega) \leqslant \inf_{v\in K_\varphi} E_G(v,\Omega),$$

proving that u is a minimum of (2.5). □

Definition 4.2. $u \in W^{1,G}_{loc}(\Omega)$ is said to be a minimizer of the functional $E_G(\cdot,\Omega)$ if for any $\psi \in W^{1,G}_c(\Omega)$, we have

$$E_G(u,\Omega) \leqslant E_G(u+\psi,\Omega).$$

It follows that u is a minimizer for $E_G(\cdot,\Omega)$ if and only if $\forall D \Subset \Omega$ with Lipschitz boundary

$$E_G(u,D) = \min_{v\in K_u} E_G(v,D),$$
$$K_u = \{v \in W^{1,G}(D):\ v = u \text{ on } \partial D\}.$$

It is easy to check that global minimizers are also minimizers.

Remark 4.1 *(Equivalence of functionals).* Let us consider two functionals defined as follows:

$$I^1_G(v) := \int_\Omega \big[G(|\nabla v|) + \lambda_1(f_1,f_2)(v)\big]\,dx \quad \text{and} \quad I^2_G(v) := \int_\Omega \big[G(|\nabla v|) + \lambda_2(f_1,f_2)(v)\big]\,dx,$$

where

$$\lambda_1(f_1,f_2)(v) = (f_1 - f_2)\chi_{\{v<0\}} - (f_1 - f_2)^-\chi_{\{v=0\}} \quad \text{and}$$
$$\lambda_2(f_1,f_2)(v) = (f_2 - f_1)\chi_{\{v>0\}} - (f_2 - f_1)^-\chi_{\{v=0\}}.$$

We observe that,

$$E_G(v) - I^1_G(v) = \int_\Omega f_2(x)\,dx \quad \text{and} \quad E_G(v) - I^2_G(v) = \int_\Omega f_1(x)\,dx.$$

From this, we conclude that u is a minimizer of E_G if and only if it is also a minimizer of the functionals I^1_G and I^2_G. This information will allow us to interchange the functionals conveniently in order to simplify the proofs.

Proposition 4.2. *Let $G \in \mathcal{G}(\delta, g_0)$ and u be a minimizer of E_G over K_φ. Then, $u \in L^\infty(\Omega)$ and we have the following estimate*

$$\inf_\Omega \varphi \leqslant u(x) \leqslant \sup_\Omega \varphi \quad \textit{a.e. } x \in \Omega.$$

Furthermore, if u is a minimizer of E_G then

$$\mathcal{L}_g u = 0 \quad \textit{in } \Omega \setminus \{u=0\}.$$

Proof. We start off by observing that we can assume without loss of generality that $\inf_\Omega \varphi < 0 \leqslant \sup_\Omega \varphi$. Otherwise $u \geqslant 0$ a.e. in Ω and the estimate follows from Lemma 3.2 in [15], since the problem is then reduced to the one phase case. To prove the lower bound, we use $u_\varepsilon(x) = u(x) + \varepsilon(u-m)^-$ with $0 < \varepsilon < 1$ and $m = \inf_\Omega \varphi < 0$. By the minimality of u with respect to I^2_G, we arrive to

$$\int_{\Omega^-_m} \big[G(|\nabla u|) + \lambda_2(f_1,f_2)(u)\big]\,dx \leqslant \int_{\Omega^-_m} \big[(1+g_0)(1-\varepsilon)^{1+\delta} G(|\nabla u|) + \lambda_2(f_1,f_2)(u_\varepsilon)\big]\,dx,$$

where $\Omega^-_m := \{u < m\}$. Once $u < 0$ and $u_\varepsilon < 0$ both in Ω^-_m, we have

$$\int_{\Omega_m^-} G\big(|\nabla u|\big)\,dx \leqslant (1+g_0)(1-\varepsilon)^{1+\delta} \int_{\Omega_m^-} G\big(|\nabla u|\big)\,dx$$

which yields for ε close enough to 1 that $\int_\Omega G(|\nabla(u-m)^-|)\,dx \leqslant 0$, and this implies $u \geqslant m$ a.e. in Ω. The upper bound is proven similarly by considering I_G^1 instead. As it is proven in the next section, u is actually $C^\alpha_{loc}(\Omega)$ and thus the set $\Omega \setminus \{u=0\}$ is open. So, let $\eta \in C_0^\infty(\Omega \setminus \{u=0\})$ and suppose that $K = \operatorname{supp}\eta$. Set $m_0 = \inf_K |u|$ and $M_0 = \max_K |\eta|$. For $0 < |\varepsilon| < m_0/M_0$, we see that

$$\{u>0\} \cap K = \{u+\varepsilon\eta > 0\} \cap K \quad \text{and} \quad \{u<0\} \cap K = \{u+\varepsilon\eta < 0\} \cap K.$$

Thus,

$$E_G(u+\varepsilon\eta) = \int_\Omega \big[G\big(\big|\nabla(u+\varepsilon\eta)\big|\big) + \lambda(f_1,f_2)(u)\big]\,dx.$$

Hence,

$$0 = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\big(E_G(u+\varepsilon\eta) - E_G(u)\big) = \int_{\Omega\setminus\{u=0\}} g\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|}\cdot\nabla\eta\,dx. \qquad \square$$

Remark 4.2. If the phase functions are ordered, i.e., say $f_2 \geqslant f_1$ in Ω, then minimizers are globally subsolutions, i.e., $\mathcal{L}_g u \geqslant 0$ in Ω for the corresponding g. This is proven in Remark 10.1. Similarly, $f_2 \leqslant f_1$ implies that $\mathcal{L}_g u \leqslant 0$ in Ω.

5. The uniform local Hölder regularity for minimizers in $\mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu, M)$

In this section, we prove the uniform (local) Hölder regularity for minimizers in $\mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu, M)$. Our proof follows closely the proof in Theorem 4.1 in [15]. As mentioned by the authors there, this is one of the main proofs in their paper. It is delicate and long. Here, we show it for the two phase case setting and put it in the context of uniform regularity within the class $\mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu, M)$. The non-degeneracy condition (limitation from below on $G(1)$) is crucial for the uniform Hölder estimate.

The key estimates developed in what follows play an important role in the proof of our main result Theorem 2.1 in Section 8. This way, we present essentially the major steps and the key estimates of the proof.

Theorem 5.1. *Let $u \in \mathcal{S}(\Omega, \delta, g_0, \varepsilon_0, \mu, M)$ and $\alpha \in (0,1)$. Then u is uniformly in $C^\alpha_{loc}(\Omega)$. More precisely, for any $\Omega' \Subset \Omega$ there exists a universal constant $\overline{C} = \overline{C}(\delta, g_0, n, \alpha, M, \mu)$ such that*

$$[u]_{C^{0,\alpha}(\Omega')} \leqslant C := \max\left(C_1 \cdot \max\left\{\frac{\overline{C}}{\varepsilon_0}, 1\right\}, \frac{2^{\alpha+1}\cdot M}{\min\{(\frac{1}{2})^{1+1/\varepsilon}, \operatorname{dist}(\Omega',\partial\Omega)\}^\alpha}\right),$$

where

$$\varepsilon := \frac{1}{2}(1-\alpha)/(n+\alpha-1),$$

and $C_1 = C_1(\delta, g_0, n)$. In fact, $C = C(n, \delta, g_0, \varepsilon_0, \mu, M, \alpha, \operatorname{dist}(\Omega', \partial\Omega)) > 0$.

Proof. By definition, there exists a functional of the type

$$E_G(u,\Omega) = \int_\Omega \big[G\big(|\nabla u|\big) + \lambda(f_1,f_2)(u)\big]\,dx$$

such that u is a minimizer of E_G, $\sup_\Omega |u| \leqslant M$ and $0 \leqslant f_1, f_2 \leqslant \mu$. By Lemma 3.1, it is enough to show that for any $0 < \alpha < 1$ and $\Omega' \Subset \Omega$, there exists $0 < \rho_0 \leqslant \operatorname{dist}(\Omega', \partial\Omega)$ such that

$$\int_{B_\rho(y)} G(|\nabla u|)\,dx \leqslant C\rho^{n+\alpha-1} \quad \text{for every } y \in \Omega', \text{ and for all } 0<\rho\leqslant\rho_0\leqslant \operatorname{dist}(\Omega',\partial\Omega).$$

So, let $\alpha \in (0,1)$ be fixed and $\Omega' \Subset \Omega$ with $y \in \Omega'$. Consider $r>0$ such that $B_r(y) \subset \Omega$. To simplify the notation, assume $y=0$. Let w be the solution to the Dirichlet problem

$$\mathcal{L}_g w = 0 \quad \text{in } B_r \quad \text{and} \quad w-u \in W_0^{1,G}(B_r).$$

From Theorem 2.3 of [15] we have

$$\int_{B_r} \big[G(|\nabla u|) - G(|\nabla w|)\big]\,dx \geqslant C(g_0,\delta)\bigg(\int_{A_2} G(|\nabla u - \nabla w|)\,dx + \int_{A_1} F(|\nabla u|)|\nabla u - \nabla w|^2\,dx\bigg), \tag{5.18}$$

where

$$A_1 = \{x \in B_r\colon |\nabla u - \nabla w| \leqslant 2|\nabla u|\}, \qquad A_2 = \{x \in B_r\colon |\nabla u - \nabla w| > 2|\nabla u|\}$$

and $F(t) = g(t)/t$. On the other hand, it follows by the minimality of u that

$$\int_{B_r} \big[G(|\nabla u|) - G(|\nabla w|)\big]\,dx \leqslant \int_{B_r} \big[\lambda(f_1,f_2)(w) - \lambda(f_1,f_2)(u)\big]\,dx \leqslant C(n)\mu r^n. \tag{5.19}$$

A combination of (5.18) and (5.19) reveals that

$$\int_{A_2} G(|\nabla u - \nabla w|)\,dx \leqslant C(n,g_0,\delta)\mu r^n \tag{5.20}$$

and

$$\int_{A_1} F(|\nabla u|)|\nabla u - \nabla w|^2\,dx \leqslant C(n,g_0,\delta)\mu r^n. \tag{5.21}$$

Let $\varepsilon>0$ and suppose that $r^\varepsilon \leqslant 1/2$. Since $\frac{G(t)}{t}$ is increasing, (g-2) and (5.21) yields

$$\begin{aligned}\int_{A_1\cap B_{r^{1+\varepsilon}}} G(|\nabla u - \nabla w|)\,dx &\leqslant C(g_0)\bigg(\int_{A_1} \frac{G(|\nabla u|)}{|\nabla u|^2}|\nabla u - \nabla w|^2\,dx\bigg)^{1/2}\cdot\bigg(\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx\bigg)^{1/2}\\ &\leqslant C\mu^{1/2} r^{n/2}\bigg(\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx\bigg)^{1/2},\end{aligned} \tag{5.22}$$

where $C = C(n,\delta,g_0)$. Combining now (5.20) and (5.22) we arrive again for $C = C(n,\delta,g_0)>0$ at

$$\begin{aligned}\int_{B_{r^{1+\varepsilon}}} G(|\nabla u - \nabla w|)\,dx &= \int_{A_1\cap B_{r^{1+\varepsilon}}} G(|\nabla u - \nabla w|)\,dx + \int_{A_2\cap B_{r^{1+\varepsilon}}} G(|\nabla u - \nabla w|)\,dx\\ &\leqslant C\mu^{1/2}\bigg[\mu^{1/2}r^n + r^{n/2}\bigg(\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx\bigg)^{1/2}\bigg].\end{aligned} \tag{5.23}$$

Note that by the maximum principle (Lemma 2.8 of [15]), we have

$$\|w\|_{L^\infty(B_r)} = \|u\|_{L^\infty(B_r)} \leqslant M. \tag{5.24}$$

Thus for $r \leqslant 1$, it follows by property (3.15), Remark 3.2 and (G-3) that for any $\overline{\beta} \in (0,n)$

$$\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx \leqslant C\bigg\{(1+\mu)r^{\overline{\beta}} + (1+\mu)^{\frac{1}{2}} r^{\overline{\beta}/2}\bigg(\int_{B_{r^{1+\varepsilon}}} G(|\nabla u|)\,dx\bigg)^{1/2}\bigg\}, \tag{5.25}$$

where $C = C(\delta, g_0, n, \overline{\beta}, M)$. Hence as in (4.9) in [15], as long as $r^\varepsilon \leqslant 1/2$, we obtain

$$\int_{B_{r^{1+\varepsilon}}} G\big(|\nabla u|\big)\,dx \leqslant \big[(C+1)^{1/2} + C^{1/2}\big]^2 C(1+\mu) r^{\overline{\beta}}. \tag{5.26}$$

Since $0 < \alpha < 1$ set $\overline{\beta} := (1+\varepsilon)(n-(1-\alpha))$ and $\rho_0 := \min\{(\frac{1}{2})^{1+1/\varepsilon}, \operatorname{dist}(\Omega', \partial\Omega)\}$. Now, we choose ε so that $0 < \varepsilon < (1-\alpha)/(n+\alpha-1)$. This way, $0 < \overline{\beta} < n$. Thus, for $0 < \rho < \rho_0$ with $r = \rho^{1/(1+\varepsilon)}$ we have that $r^\varepsilon \leqslant 1/2$ and therefore (5.26) translates to

$$\int_{B_\rho} G\big(|\nabla u|\big)\,dx \leqslant \big[(C+1)^{1/2} + C^{1/2}\big]^2 C(1+\mu)\rho^{n+\alpha-1} = \overline{C}\rho^{n+\alpha-1},$$

where $\overline{C} = \overline{C}(\delta, g_0, n, \alpha, M, \mu)$. Now by estimate (3.12) in Morrey's type lemma, we have for

$$\varepsilon := \frac{1}{2}(1-\alpha)/(n+\alpha-1) \quad \text{that}$$

$$[u]_{\alpha,\Omega'} \leqslant \max\left(C_1 \cdot \max\left\{\frac{\overline{C}}{\varepsilon_0}, 1\right\}, \frac{2^{\alpha+1}\cdot M}{\min\{(\frac{1}{2})^{1+1/\varepsilon}, \operatorname{dist}(\Omega', \partial\Omega)\}^\alpha}\right). \qquad \square$$

6. Non-degenerate classes: free boundary hitting angle, compactness and scaling

In this section we motivate the definition of the non-degenerate classes presented in the introduction. We proceed by a heuristic argument in the general (Orlicz) setting and show that once full regularity of the free boundary is available (i.e., when classical solutions to FBPs exist) we can make the argument rigorous.

We start off with a heuristic motivation that shows that as far as Lipschitz regularity is concerned, we need in fact to concentrate on some special subsets of $\mathcal{G}(\delta, g_0)$. So, suppose we have a (classical) solution of the following one phase free boundary problem (FBP)

$$\begin{cases} \mathcal{L}_g u = 0 & \text{in } \{u > 0\}, \\ H\big(|\nabla u|\big) = \lambda > 0 & \text{on } F(u) := \partial\{u > 0\} \cap B_1, \\ u \geqslant 0 & \text{in } B_1 \end{cases} \tag{6.27}$$

where

$$H(t) = g(t)t - G(t) \quad \text{for } t \geqslant 0.$$

The gradient of the solution along the free boundary is controlled above and below by negative powers of $G(1)$, provided $G(1)$ is small enough. More precisely, there exist positive constants $C_1 = C_1(\lambda, g_0)$ and $C_2 = C_2(\lambda, \delta, g_0)$ such that

$$C_1 \cdot \left(\frac{1}{G(1)}\right)^{\frac{1}{1+g_0}} \leqslant |\nabla u| \leqslant C_2 \cdot \left(\frac{1}{G(1)}\right)^{\frac{1}{1+\delta}} \text{ along } F(u) \tag{6.28}$$

provided $G(1) < \min\{\lambda(1+g_0)^{-1}, 1\}$ and the solution u and its free boundary $F(u)$ are smooth enough.

Indeed, suppose $x_0 \in F(u)$ and let $\beta = |\nabla u(x_0)| > 0$ (by Hopf's lemma). Then,

$$\beta = \frac{\lambda + G(\beta)}{g(\beta)} \geqslant \frac{\lambda}{g(\beta)} \geqslant \frac{\lambda}{g(1)\cdot \max\{\beta^\delta, \beta^{g_0}\}} \geqslant \frac{\lambda}{1+g_0}\cdot\frac{1}{G(1)}\cdot\frac{1}{\max\{\beta^\delta, \beta^{g_0}\}}.$$

Now since $G(1) < \lambda(1+g_0)^{-1}$ then

$$\beta \geqslant \left(\frac{\lambda}{1+g_0}\right)^{\frac{1}{1+g_0}} \cdot \left(\frac{1}{G(1)}\right)^{\frac{1}{1+g_0}}.$$

To prove the other inequality, we observe that $H'(t) = g'(t)\cdot t \geqslant \delta g(t)$ for $t > 0$ by (C1). This way, from the continuity of g we have $H(t) \geqslant \delta G(t)$ for $t \geqslant 0$. This implies

$$\lambda = H(\beta) \geqslant \delta \cdot G(\beta) \geqslant \frac{\delta \cdot G(1)}{1+g_0} \cdot \min\{\beta^{1+\delta}, \beta^{1+g_0}\}.$$

So, since $G(1) < 1$ then

$$\beta \leqslant \max\left\{\left(\frac{(1+g_0)\cdot\lambda}{\delta}\right)^{\frac{1}{1+g_0}}, \left(\frac{(1+g_0)\cdot\lambda}{\delta}\right)^{\frac{1}{1+\delta}}\right\} \cdot \left(\frac{1}{G(1)}\right)^{\frac{1}{1+\delta}}.$$

This shows heuristically the desired inequality. We observe that the first inequality in (6.28) strongly suggests that solutions to FBPs like (6.27) cannot have a uniform (local) Lipschitz estimate with respect to the class $\mathcal{G}(\delta, g_0)$, since once $G(1)$ is allowed to go to zero within the class then $|\nabla u|$ blows up along $F^+(u)$.

Hence, since FBPs like (6.27) are at least heuristically (see [15]) the "Euler–Lagrange equations" of minimizers of the functionals of the type

$$E_G(u, B_1) = \int_{B_1} \left[G\left(|\nabla u|\right) + \lambda \cdot \chi_{\{u>0\}}\right] dx,$$

we should not expect the same (local) uniform Lipschitz control with respect to $\mathcal{G}(\delta, g_0)$ for such minimizers, unless some control is imposed on $G(1)$ within the class considered. We can actually make this heuristic discussion rigorous, if the full regularity of the free boundary is available. This is done in the next example.

Example 6.1. Let us consider the functionals

$$E_j(u, B_1) = \int_{B_1} \left[G_j\left(|\nabla u|\right) + \chi_{\{u>0\}}\right] dx,$$

where $G_j(t) = \alpha_j^2 \cdot t^2$ and $\alpha_j > 0$. In this case, $\delta = g_0 = \mu = 1$. So, clearly $G_j \in \mathcal{G}(1, 1)$ and $G_j(1) = \alpha_j^2$. We observe that,

$$E_j(u) = \alpha_j^2 \cdot F_j(u) := \alpha_j^2 \left\{ \int_{B_1} \left[|\nabla u|^2 + \frac{1}{\alpha_j^2} \chi_{\{u>0\}} \right] dx \right\}.$$

This way, u is a minimizer of E_j if and only if it is a minimizer of F_j. Let u_j be a minimizer for E_j such that $0 \leqslant \varphi \leqslant M$. This function can be easily constructed by considering the minimization process with some prescribed boundary data $0 \leqslant \varphi \leqslant M$ as done in Section 3, for instance. This way

$$u_j \in \mathcal{S}^*(B_1, 1, 1, 1, M)^+(0) \quad \text{for all } j \geqslant 1.$$

Since u_j also minimizes F_j, it follows by the results of H.W. Alt, L.A. Caffarelli, D. Jerison and C. Kenig in [2] and [8] that in dimension $n = 2$ or $n = 3$ the free boundary is an analytic hypersurface everywhere and by elliptic regularity theory, u_j is a classical solution of the following FBP

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\}, \\ |\nabla u| = \dfrac{1}{\alpha_j} & \text{on } F(u) := \partial\{u > 0\} \cap B_1, \\ u \geqslant 0. \end{cases}$$

So, if the uniform (local) Lipschitz estimate with respect to $\mathcal{S}^*(B_1, 1, 1, 1, M)^+(0)$ holds then we have

$$\frac{1}{\sqrt{G_j(1)}} = \frac{1}{\alpha_j} = \left|\nabla u_j(0)\right| \leqslant [u_j]_{C^{0,1}(B_{1/8})} < \infty.$$

This implies that $G_j(1)$ has to be uniformly bounded away from zero.

This example shows that for the uniform Lipschitz estimate to hold, we must restrict ourselves to deal with some non-degenerate subsets of $\mathcal{G}(\delta, g_0)$, namely, subsets of $\mathcal{G}(\delta, g_0)$ where $G(1)$ is bounded away from zero, i.e., the non-degenerate classes presented in the introduction.

6.1. Compactness of non-degenerate subclasses

We observe that there may be sequences in $\mathcal{G}(\delta, g_0)$ and $\mathcal{G}_\beta(\delta, g_0)$ that converge uniformly to zero in compact subsets of $[0, +\infty)$. In non-degenerate classes this phenomenon does not happen. The next theorem establishes the compactness of non-degenerate subclasses $\mathcal{G}_\beta(\delta, g_0)$ in the appropriate topologies.

Theorem 6.1 *(Compactness in $\mathcal{G}_\beta(\delta, g_0, \varepsilon_0)$). Let $0 < \beta \leqslant 1$ and $\tau, L > 0$. Define*

$$\mathcal{F}_\tau(\varepsilon_0, L) := \{G \in \mathcal{G}_\beta(\delta, g_0, \varepsilon_0):\ G(\tau) \leqslant L\}.$$

If $\{G_j\} \subset \mathcal{F}_\tau(\varepsilon_0, L)$ then there exist a subsequence still denoted by $\{G_j\}$ and $G \in \mathcal{G}(\delta, g_0) \cap C^{2,\beta}(0, \infty)$ such that G_j converge to G in $C^{2,\gamma}$ topology on compact subsets of $(0, +\infty)$ for every $\gamma \in (0, \beta)$ and in the C^1 topology on compact subsets of $[0, +\infty)$.

Proof. Let $\{G_j\}_{j \geqslant 1}$ be a sequence in $\mathcal{F}_\tau(\varepsilon_0, L)$. Since each G_j is increasing we can assume that $0 < \tau \leqslant 1$. By (G-2), $G_j(1) \leqslant (1 + g_0)L\tau^{-(1+g_0)} =: L_0$. Hence, the non-degeneracy condition $G(1) \geqslant \varepsilon_0$ implies

$$\frac{\varepsilon_0}{(g_0+1)} \min\{t^{1+\delta}, t^{1+g_0}\} \leqslant G_j(t) \leqslant L_0(g_0+1) \max\{t^{1+\delta}, t^{1+g_0}\} \quad \forall t \geqslant 0 \tag{6.29}$$

and by (g-1), (g-2) and (C1)

$$\varepsilon_0 \min\{t^\delta, t^{g_0}\} \leqslant g_j(t) \leqslant (1+g_0)L_0 \max\{t^\delta, t^{g_0}\} \quad \forall t \geqslant 0, \tag{6.30}$$

$$0 < g_j'(t) \leqslant g_0(1+g_0)^2 L_0 \max\{t^{\delta-1}, t^{g_0-1}\} \quad \forall t > 0. \tag{6.31}$$

Once G satisfies (MC-β), we observe that g' is locally β-Hölder continuous function. Indeed, we claim that for any fixed closed interval $K \Subset (0, +\infty)$ there exists a constant C_1 depending only on $\delta, g_0, \beta, \tau, L$ and K such that

$$[g_j']_{C^{0,\beta}(K)} \leqslant C_1(\delta, g_0, \beta, \tau, L, K).$$

Indeed, let $Q_j(t) = \frac{t g_j'(t)}{g_j(t)}$. Then since $g_j > 0$ in K by (6.30), $Q_j \in W^{1,1}(K)$ and

$$\forall t, t+\kappa \in I_K = \left[\frac{1}{2}\inf_K t, \sup_K t + \frac{1}{2}\inf_K t\right] \quad \Rightarrow \quad \int_t^{t+\kappa} |Q_j'(s)|\,ds \leqslant \frac{C_0(\delta, g_0, \beta)}{t^\beta} \cdot \kappa^\beta \leqslant \frac{C_0(\delta, g_0, \beta)}{(\frac{1}{2}\inf_K t)^\beta} \cdot \kappa^\beta.$$

Then, Morrey's lemma (see Theorem 1.1 in [14]) provides

$$[Q_j]_{C^{0,\beta}(K)} \leqslant \frac{\widetilde{C_0}(\delta, g_0, \beta)}{(\frac{1}{2}\inf_K t)^\beta}. \tag{6.32}$$

Hence, since $\frac{g_j(t)}{t}$ is Lipschitz continuous in K (once it has bounded derivative there) we conclude that $g_j'(t) = \frac{Q_j(t) \cdot g_j(t)}{t}$ is β-Hölder continuous in K and (6.30), (6.31) and (6.32) imply

$$\begin{aligned} [g_j']_{C^{0,\beta}(K)} &= [Q_j \cdot g_j/t]_{C^{0,\beta}(K)} \\ &\leqslant (\mathrm{diam}(K))^{1-\beta} \|Q_j\|_{L^\infty(K)} \cdot [g_j/t]_{C^{0,1}(K)} + \|g_j/t\|_{L^\infty(K)} \cdot [Q_j]_{C^{0,\beta}(K)} \\ &\leqslant L_0 \cdot \left(\overline{C}(\delta, g_0, \beta, K) + \frac{\widetilde{C}(\delta, g_0, \beta, K)}{(\frac{1}{2}\inf_K t)^\beta}\right) \\ &\leqslant C_1(\delta, g_0, \beta, L, \tau, K). \end{aligned}$$

Thus, all the above estimates imply that for any closed interval $K \Subset (0, \infty)$ we have

$$\|G_j\|_{C^{2,\beta}(K)} \leqslant C_2, \quad C_2 = C_2(\delta, g_0, \beta, L, \tau, K). \tag{6.33}$$

Therefore, there exist a subsequence still denoted by $\{G_j\}$ and a function $G \in C^{2,\beta}((0, +\infty))$ such that

$$G_j \to G \quad \text{in } C^{2,\gamma}_{loc}(0,\infty) \text{ for every } \gamma \in (0,\beta) \text{ where } G \text{ satisfies (C1).}$$

Let us set $g := G'$ in $(0,\infty)$. To finish the proof, we have to extend G to become $C^1[0,\infty)$ and show that $G_j \to G$ in $C^1_{loc}[0,\infty)$. This way, G will belong to $\mathcal{G}(\delta, g_0)$. We do it in the natural way, i.e., $\overline{G}(0) = 0$ and $\overline{G}(t) = G(t)$ if $t > 0$. We observe that $\overline{G} \in C^0[0,\infty)$ since (6.29) implies for $0 < t < 1$,

$$G_j(t) \leqslant L_0(1+g_0)\cdot t^{1+\delta}.$$

Passing to the limit, we have $G(t) \leqslant L_0(1+g_0)\cdot t^{1+\delta}$ for $0 < t < 1$ which yields $\lim_{t\to 0^+} \overline{G}(t) = 0$. Arguing similarly with the estimates

$$0 \leqslant \lim_{t\to 0^+} \frac{G(t)}{t} \leqslant L_0(1+g_0) \lim_{t\to 0^+} t^{\delta} = 0,$$
$$0 < t < 1 \quad \Rightarrow \quad g_j(t) \leqslant (1+g_0)L_0 t^{\delta},$$

we see that $\overline{G} \in C^1[0,\infty)$ and $G_j \to \overline{G}$ in $C^1_{loc}[0,\infty)$. This, finishes the proof. □

6.2. Scaling

We close this section with some observations about scaling. Many estimates for minimizers of functionals of the type E_G involve universal constants, i.e., constants depending on the ellipticity constants δ, g_0. The constants in these estimate sometimes also depend on $G(1)$.

This way, it is very important that our minimizers admit rescaling that normalize the values $G(1)$. This fact is what allows us to run a compactness argument using the theorem above to perform a blow-up analysis in the next section. For $s > 0$ we define the following normalizing rescalings

$$G_s(t) := \frac{G(st)}{sG'(s)} \quad \text{and} \quad G^*_s(t) := G(st). \tag{6.34}$$

Below we record important facts about these scalings in a proposition.

Proposition 6.1 *(Scaling properties).*

$$G \in \mathcal{G}(\delta, g_0) \quad\Rightarrow\quad G_s \in \mathcal{G}\big(\delta, g_0, (1+g_0)^{-1}\big) \quad \textit{and} \quad G_s(1) \leqslant 1, \tag{S-1}$$
$$G \in \mathcal{G}_\beta(\delta, g_0) \quad\Rightarrow\quad G_s \in \mathcal{G}_\beta\big(\delta, g_0, (1+g_0)^{-1}\big) \quad \textit{and} \quad G_s(1) \leqslant 1, \tag{S-2}$$
$$G \in \mathcal{G}(\delta, g_0, \varepsilon_0), \quad s \geqslant 1 \quad\Rightarrow\quad G^*_s \in \mathcal{G}(\delta, g_0, \varepsilon_0), \tag{S-3}$$
$$G \in \mathcal{G}_\beta(\delta, g_0, \varepsilon_0), \quad s \geqslant 1 \quad\Rightarrow\quad G^*_s \in \mathcal{G}(\delta, g_0, \varepsilon_0). \tag{S-4}$$

Proof. Let $G \in \mathcal{G}(\delta, g_0)$. We observe that the rescaling in (6.34) preserves regularity. Let us define

$$Q(t) = \frac{tg'(t)}{g(t)}, \quad g = G', \qquad Q_s(t) = \frac{tg_s'(t)}{g_s(t)}, \quad g_s = G_s' \quad \text{and} \quad Q^*_s(t) = \frac{t\dot{g}^*_s(t)}{g^*_s(t)}, \quad g^*_s = \dot{G}^*_s.$$

(S-1) and (S-3) follow from the fact that G is increasing, (g-2) and

$$Q_s(t) = Q^*_s(t) = Q(st) \quad \forall t \geqslant 0,$$
$$G^*_s(1) = G(s) \geqslant G(1) \geqslant \varepsilon_0.$$

To prove (S-2) and (S-4), we set $H_s(t) := Q_s(t) = Q^*_s(t)$ for $G \in \mathcal{G}_\beta(\delta, g_0)$ and then

$$\int_\tau^{\tau+\kappa} \left|H_s'(t)\right| dt = s \int_\tau^{\tau+\kappa} \left|Q'(st)\right| dt = \int_{s\tau}^{s\tau+s\kappa} \left|Q'(\zeta)\right| d\zeta \leqslant \frac{C_0}{\tau^\beta}\cdot\kappa^\beta.$$

This finishes the proof. □

7. Examples of N-functions in the non-degenerate classes of $\mathcal{G}(\delta, g_0)$

In this section, we present examples of N-functions in the non-degenerate classes $\mathcal{G}_2(\delta, g_0)$ and $\mathcal{G}_\beta(\delta, g_0)$. Moreover, we show that if the control constant $\mathcal{C}(\delta, g_0)$ is chosen appropriately, the class $\mathcal{G}_2$ stays invariant under a variety of elementary operations such as (positive) linear combinations, products, composition and C^2-gluings. This shows that the degenerate classes are quite large indeed.

Let us observe that condition (C1) studied by G.M. Lieberman in the theory of degenerate/singular equations in [13] is not enough to assure compactness in the class $\mathcal{G}(\delta, g_0)$.

Example 7.1 *(Lack of compactness in $\mathcal{G}(\delta, g_0)$).* For each $0 < \varepsilon < 1$ we consider the functions defined by

$$G_\varepsilon(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } 0 \leqslant t \leqslant 1, \\ \frac{1}{6\varepsilon}t^3 + \frac{1}{2}(1 - \frac{1}{\varepsilon})t^2 + \frac{1}{2\varepsilon}t - \frac{1}{6\varepsilon} & \text{if } 1 \leqslant t \leqslant 1 + \varepsilon, \\ t^2 - (1 + \frac{1}{2}\varepsilon)t + \frac{1}{2} + \frac{1}{2}\varepsilon + \frac{1}{6}\varepsilon^2 & \text{if } 1 + \varepsilon \leqslant t. \end{cases}$$

It is easy to see that G_ε is an N-function in $C^2([0, \infty))$. Furthermore, $G_\varepsilon \in \mathcal{G}(1, 2, 1/2)$ for all $0 < \varepsilon < 1$. We can also verify $G_\varepsilon \to G_0$ uniformly in the compact sets of $[0, +\infty)$ as $\varepsilon \to 0$ where

$$G_0(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } 0 \leqslant t \leqslant 1, \\ t^2 - t + \frac{1}{2} & \text{for } t \geqslant 1. \end{cases}$$

Also, $G_0 \in C^{1,1}([0, \infty)) \setminus C^2[0, \infty)$ and hence $G_0 \notin \mathcal{G}(\delta, g_0)$ for whatever $0 < \delta \leqslant g_0$.

We observe also that $\mathcal{G}_2(\delta, g_0) \subsetneq \mathcal{G}_\beta(\delta, g_0)$ for every $\beta \in (0, 1)$. Indeed, let $G \in \mathcal{G}_2(\delta, g_0)$ with $g = G'$. This way,

$$\begin{aligned} \left|Q'(t)\right| &\leqslant \frac{g'(t)}{g(t)} + \frac{t|g''(t)|}{g(t)} + t\left(\frac{g'(t)}{g(t)}\right)^2 \\ &\leqslant \frac{1}{t}\left(g_0 + \mathcal{C}(\delta, g_0) + g_0^2\right) \\ &\leqslant \frac{\widetilde{\mathcal{C}}(\delta, g_0)}{t}. \end{aligned}$$

Hence, for any $t, \kappa > 0$ we have

$$\int_t^{t+\kappa} \left|Q'(s)\right| ds \leqslant \widetilde{\mathcal{C}}(\delta, g_0) \int_t^{t+\kappa} \frac{s^{\beta-1}}{s^\beta} ds \leqslant \frac{\widetilde{\mathcal{C}}(\delta, g_0)}{\beta \cdot t^\beta} \cdot \left[(t+\kappa)^\beta - t^\beta\right] \leqslant \frac{\widetilde{\mathcal{C}}(\delta, g_0, \beta)}{t^\beta} \cdot \kappa^\beta.$$

The fact that the inclusion is strict is shown in the next example.

Example 7.2. Let $\beta \in (0, 1]$ be fixed and define $G_{(\beta)} : [0, \infty) \to [0, \infty)$ by

$$G_{(\beta)}(t) = \begin{cases} -\frac{t^4}{24} + \frac{t^3}{6}, & 0 \leqslant t \leqslant 1, \\ \frac{1}{(\beta+1)(\beta+2)}(t-1)^{\beta+2} + \frac{1}{4}t^2 - \frac{1}{6}t + \frac{1}{24}, & t \geqslant 1. \end{cases}$$

Simple but long computations show that $G_{(\beta)} \in \mathcal{G}_\beta(1, 9/2, 1/8)$ and for any interval K such that $1 \in K \Subset (0, +\infty)$ we have $[G''_{(\beta)}]_{\gamma, K} = \infty$ for every $\beta < \gamma \leqslant 1$. In particular, it follows from the proof of Theorem 6.1 (actually estimate (6.33)) that $G_{(\beta)} \notin \mathcal{G}_\gamma(\delta, g_0)$ for any $\beta < \gamma \leqslant 1$.

We now focus on the classes $\mathcal{G}_2(\delta, g_0)$. In (C2), we take the control constant to be given by $\mathcal{C}(\delta, g_0) = g_0(g_0 - 1)$ with $g_0(g_0 - 1) \geqslant 1/4$. This is not restrictive. We observe actually that this particular choice makes the class invariant under many simple operations (pointed out below) and it is indeed quite natural if we want to include examples such as $G(t) = t^p$ or else $G(t) = t^p + t^q$. Furthermore, since $g_0(g_0 - 1) \to \infty$ as $g_0 \to \infty$, we can eventually enlarge g_0 to contemplate many concrete situations.

In what follows, we mention some concrete examples that belong to $\mathcal{G}_2(\delta, g_0)$.

The first example is $g(t) = t^p$ with for $\delta = p$ and $g_0 = \max\{p, (\sqrt{2}+1)/2\}$. Also, $g(t) = at^p + bt^q$ with $a, b > 0$ and $\delta = \min\{p, q\}$ and $g_0 = \max\{p, q, (\sqrt{2}+1)/2\}$. Two examples involving the logarithm function are: $g(t) = t^a \log_c(bt + d)$, for $a, b > 0$, $c, d > 1$, with $\delta = a$ and $g_0 = \max\{a + r, 7/4 + r\}$, where $r = \max\{1, (\ln d)^{-1}\}$ and $g(t) = t^a / \log_c(bt + d)$, for $b > 0$; $c, d > 1$ and $a > (\ln d)^{-1}$ with constants $\delta = a - (\ln d)^{-1}$ and $g_0 = \max\{a + 2, P((\ln d)^{-1})\}$, where here P is the quadratic polynomial given by $P(t) = 3t^2 + 2t + 1$.

Functions $g \in C^1(0, +\infty)$ of the form: $g(t) = c_1 t^{a_1}$ for $0 \leqslant t \leqslant t_0$ and $g(t) = c_2 t^{a_2} + d$ with $t \geqslant t_0$. For $a_1, a_2, c_1, c_2 > 0$, with constants $\delta = \min\{a_1, a_2\}$ and $g_0 = \max\{a_1, a_2, 2\}$.

Similarly, for any positive constants t_0, a_i, c_j, $i = 1, 2, 3$ and $j = 1, 2$, there exist single constants $c > 0$ and $d \neq 0$ such that the C^1-function given by

$$g(t) = ct^{a_1}, \quad \text{for } 0 \leqslant t \leqslant t_0 \quad \text{and} \quad g(t) = c_1 t^{a_2} + c_2 t^{a_3} + d, \quad \text{for } t \geqslant t_0,$$

satisfies (C2) for $\delta = \min_{i \in \{1,2,3\}}\{a_i\}$ and $g_0 = \max_{i \in \{2,3\}}\{a_1, a_i + (1 + \sqrt{2})/2\} \cdot r$, with $r = 1 - d/g(t_0)$ if $d < 0$ and $\delta = \min\{(1 - \frac{d}{g(t_0)}) \min_{i \in \{2,3\}}\{a_i\}, a_1\}$ and $g_0 = \max_{i \in \{1,2,3\}}\{a_i, (\sqrt{2}+1)/2\}$, if $d > 0$.

Another example that satisfies (C2) is given by $g \in C^1([0, +\infty))$ such that

$$g(t) = c_1 t^{a_1} + c_2 t^{a_2}, \quad \text{for } 0 \leqslant t \leqslant t_0 \quad \text{and} \quad g(t) = c_3 t^{a_3} + d \quad \text{for } t \geqslant t_0,$$

for $\delta = \min_{i \in \{1,2,3\}}\{a_i\}$ and $g_0 = \max\{a_1, a_2, 2a_3, (1 + \sqrt{3})/2\}$ if $d < 0$ and $\delta = \{a_1, a_2, (1 - d/g(t_0))a_3\}$ and $g_0 = \max_{i \in \{1,2,3\}}\{a_i, (\sqrt{2} - 1)/2\}$ if $d > 0$. Here a_i, c_i, t_0 are positive constants.

As mentioned previously, we have the following invariance under elementary operations.

Finite linear combinations with positive coefficients: Let $G_1, \ldots, G_m$ are N-functions such that $G_j \in \mathcal{G}_2(\delta_j, g_{0,j})$ and let $g_j = G'_j$, $1 \leqslant j \leqslant m$, then for $\lambda_j > 0$ define $g = \sum_{j=1}^m \lambda_j g_j$. We have $G(t) = \int_0^t g(s)\,ds \in \mathcal{G}_2(\delta, g_0)$ for $\delta = \min\{\delta_j\}$ and $g_0 = \sum g_{0,j}$;

Product of functions: Let $G_i = \int_0^t g_i(s)\,ds$ for $i = 1, 2$ where $G_j \in \mathcal{G}_2(\delta_j, g_{0,j})$, $j = 1, 2$, then $G = \int_0^t g_1(s) g_2(s)\,ds \in \mathcal{G}_2(\delta, g_0)$ with $\delta = \delta_1 + \delta_2$ and $g_0 = g_{0,1} + g_{0,2}$;

Composition of functions: If $G_j = g'_j$, $j = 1, 2$, with $G_j \in \mathcal{G}_2(\delta_j, g_{0,j})$, then $G(t) = \int_0^t g(s)\,ds$, where $g(t) = g_1(g_2(t))$, belongs to $\mathcal{G}_2(\delta, g_0)$ with $\delta = \delta_1 \cdot \delta_2$ and $g_0 = g_{0,1} \cdot g_{0,2}$; and

Finite C^2 gluing of N-functions: Let $G_1, \ldots, G_m$ be functions such that $G_j \in \mathcal{G}_2(\delta_j, g_{0,j})$, with $G'_j = g_j$ and suppose that there exist $0 < t_1 < t_2 < \cdots < t_{m-1} < \infty$ such that:

$$G_j(t_j) = G_{j+1}(t_j), \qquad g_j(t_j) = g_{j+1}(t_j) \quad \text{and} \quad g'_j(t_j) = g'_{j+1}(t_j).$$

Then $G(t) = \int_0^t g(s)\,ds \in \mathcal{G}_2(\delta, g_0)$ where

$$g(t) = g_i(t), \quad t_{i-1} \leqslant t \leqslant t_i \quad \text{and} \quad g(t) = g_m(t), \quad t_{m-1} \leqslant t,$$

where $i = 0, 1, \ldots, m - 1$, $t_0 = 0$, $\delta = \min\{\delta_j\}$ and $g_0 = \max\{g_{0,j}\}$.

Finally, we observe that for $d_1, d_2, d_3 \geqslant \varepsilon_0$, $p \geqslant 3/2$ and $q \geqslant 4$ it follows that

$$\left\{ d_1 t^\alpha, d_2 t^\beta \ln(t + e), \frac{d_3 t^\gamma}{\ln(t + e)} \right\}_{\alpha, \beta, \gamma \in [p,q]} \subset \mathcal{G}_1(p - 1, q + 2, \varepsilon_0).$$

8. Proof of the main result – Theorem 2.1

This section is devoted to the proof of our main result, Theorem 2.1. In order to proceed, we present a lemma that provides an estimate putting in perspective the Hölder and Lipschitz continuity character of functions. It will be used in the sequel.

Lemma 8.1 *(Hölder/Lipschitz continuity character). Let $w : (0, 1] \to \mathbb{R}$ be a nonnegative and nondecreasing function such that $w(1) \leqslant L$ for some $L > 0$. Suppose $0 < \tau < 1$ and $0 < \alpha \leqslant 1$ are such that*

$$w\left(\tau^{k+1}\right) \leqslant \max_{0\leqslant m\leqslant k}\left\{L\cdot\tau^{\alpha(k+1)}, \tau^{\alpha(m+1)}\cdot w\left(\tau^{k-m}\right)\right\} \quad \textit{for every } k\geqslant 0. \tag{8.35}$$

Then,

$$w(r)\leqslant \tau^{-\alpha}Lr^{\alpha} \quad \textit{for } 0<r\leqslant 1.$$

In particular, if u is a bounded function in B_1 such that $\sup_{B_1}|u|\leqslant L$ and

$$\sup_{B_{\tau^{k+1}}}|u| \leqslant \max_{0\leqslant m\leqslant k}\left\{L\cdot\tau^{\alpha(k+1)}, \tau^{\alpha(m+1)}\cdot \sup_{B_{\tau^{k-m}}}|u|\right\} \quad \textit{for every } k\geqslant 0, \tag{8.36}$$

we have the following estimate,

$$\left|u(x)\right|\leqslant \tau^{-\alpha}L|x|^{\alpha} \quad \textit{for every } x\in B_1.$$

Proof. Indeed, we claim that $w(\tau^k)\leqslant\tau^{\alpha k}L$ for every $k\geqslant 0$. We proceed by induction on k. The claim is true for $k=0$ by assumption. Let us assume that it is also true for all $k\leqslant k_0\in\mathbb{N}\setminus\{0\}$. This way, by the hypothesis of induction, for any $m\in\mathbb{N}$ with $0\leqslant m\leqslant k_0$

$$w\left(\tau^{k_0-m}\right)\tau^{\alpha(m+1)}\leqslant \tau^{\alpha(k_0-m)}L\tau^{\alpha(m+1)} = L\tau^{\alpha(k_0+1)}.$$

This estimate and (8.35) implies

$$w\left(\tau^{k_0+1}\right)\leqslant \max_{0\leqslant m\leqslant k_0}\left\{L\cdot\tau^{\alpha(k_0+1)}, \tau^{\alpha(m+1)}\cdot w\left(\tau^{k_0-m}\right)\right\}\leqslant L\tau^{\alpha(k_0+1)}.$$

This proves the claim. If $0<r\leqslant 1$, we can find $k_0\in\mathbb{N}$ such that $\tau^{k_0+1}\leqslant r<\tau^{k_0}$ and thus by the claim and the monotonicity of w we conclude that

$$w(r)\leqslant w\left(\tau^{k_0}\right)\leqslant\tau^{\alpha k_0}L=\frac{\tau^{\alpha(k_0+1)}L}{\tau^{\alpha}}\leqslant\tau^{-\alpha}Lr^{\alpha}.$$

The second part follows just by observing that $w(r)=\sup_{B_r}|u|$ satisfies the assumptions of the first part of the lemma. □

Remark 8.1. The condition $w(1)\leqslant L$ or $\sup_{B_1}u\leqslant L$ cannot be dropped in the previous lemma. Indeed, consider the function $u(x)=\mu\tau L|x|^{\alpha}$ in B_1 where $\mu\tau^{\alpha+1}>1$, so that $\sup_{B_1}u>L$. Observe that (8.36) is satisfied. Indeed, $\sup_{B_{\tau^{k+1}}}u=\mu L\tau^{\alpha(k+1)+1}$ and once $\mu\tau^{\alpha+1}>1$ we obtain that

$$\max_{0\leqslant m\leqslant k}\left\{L\cdot\tau^{\alpha(k+1)}, \tau^{\alpha(m+1)}\cdot\sup_{B_{\tau^{k-m}}}|u|\right\}=\max\left\{L\left(\tau^{k+1}\right)^{\alpha}, \mu\tau^{\alpha(k+1)+1}L\right\}=\mu\tau^{\alpha(k+1)+1}L.$$

However, $|u(x)|\leqslant\tau^{-\alpha}L|x|^{\alpha}$ does not hold for any $x\in B_1\setminus\{0\}$.

We now have all the ingredients to prove Theorem 2.1.

Proof of Theorem 2.1. Without loss of generality we can assume that $x_0=0$. In fact, suppose that the theorem is true for minimizers in $S_1^{\beta}(\delta, g_0, \varepsilon_0, \mu, M)(0)$ and let $u\in S_1^{\beta}(\delta, g_0, \varepsilon_0, \mu, M)(x_0)$. Then $\bar{u}(x)=u(x+x_0)\in S_1^{\beta}(\delta, g_0, \varepsilon_0, \mu, M)(0)$. Applying the theorem to $\bar{u}$ and getting back to u we obtain

$$\left|u(x)\right|\leqslant\frac{2\cdot\max\{M,1\}}{C}\cdot|x-x_0|, \quad \forall x\in B_1(x_0). \tag{8.37}$$

We divide the proof into two cases:

Case 1. $u\in S_1^{\beta}(\delta, g_0, \varepsilon_0, \mu, 1)(0)$.

Since $0 < C < 1$ (in fact it is a small constant), Lemma 8.1 applied with $L = 1/C$, $\tau = 1/2$ and $\alpha = 1$ shows that in order to prove (8.37) for this case, it is enough to show that there exists a universal constant $C > 0$ such that for any $u \in S_1^\beta(\delta, g_0, \varepsilon_0, \mu, 1)(0)$ satisfying $\Theta_u^-(0, r) \leqslant C$ for all $0 < r < 1$ we have

$$\sup_{B_{2^{-(k+1)}}} |u(x)| \leqslant \max_{0 \leqslant m \leqslant k} \left\{ \frac{1}{C \cdot 2^{k+1}}, \frac{S(k-m)}{2^{m+1}} \right\}, \quad \forall k \geqslant 0 \tag{8.38}$$

where $S(j) := \sup_{B_{2^{-j}}} |u|$. So, let us suppose by contradiction, that (8.38) does not hold. Then, for each $j \in \mathbb{N}$, $j \geqslant 2$, we can find integers $k_j \geqslant 0$ and functions $u_j \in S_1^\beta(\delta, g_0, \varepsilon_0, \mu, 1)(0)$ such that

$$\sup_{B_1} |u_j| \leqslant 1$$

$$\Theta_{u_j}^-\left(0, 2^{-k_j}\right) \leqslant \frac{1}{j} \to 0 \quad \text{as } j \to \infty \tag{8.39}$$

but

$$\sup_{B_{2^{-(k_j+1)}}} |u_j(x)| > \max_{0 \leqslant m \leqslant k_j} \left\{ \frac{j}{2^{k_j+1}}, \frac{S_j(k_j-m)}{2^{m+1}} \right\} \tag{8.40}$$

where

$$S_j(k_j - m) = \sup_{B_{2^{-(k_j-m)}}} |u_j|, \quad 0 \leqslant m \leqslant k_j.$$

Moreover, (8.40) implies that $k_j \geqslant \log_2 j - 1 \to \infty$. Now we consider the following family of auxiliary functions

$$v_j(x) := \frac{u_j(2^{-k_j} x)}{S_j(k_j+1)}, \quad x \in B_{2^{k_j}}. \tag{8.41}$$

Since density is scaling invariant, (8.39) implies that

$$\frac{|\{v_j < 0\} \cap B_1|}{|B_1|} \leqslant \frac{1}{j}. \tag{8.42}$$

For each u_j there exist $G_j \in \mathcal{G}_\beta(\delta, g_0, \varepsilon_0)$ and $0 \leqslant f_{1,j}, f_{2,j} \leqslant \mu$ in B_1 such that u_j is a minimizer of

$$E_{G_j}(u_j) = \int_{B_1} \left[G_j\left(|\nabla w|\right) + \lambda(f_{1,j}, f_{2,j})(u_j) \right] dx.$$

On the other hand, by Remark 4.1, u_j minimizes E_{G_j} if only if u_j minimizes the functional

$$I_{G_j}^2(u_j) := \int_{B_1} \left[G_j\left(|\nabla u_j|\right) + f_j(u_j) \right] dx,$$

where $f_j(u_j) := \lambda_2(f_{1,j}, f_{2,j})(u_j)$. Setting $g_j = G_j'$ we can consider the following family of normalized rescalings

$$\overline{G}_j(t) := G_{\sigma_j}(t) = \frac{G_j(\sigma_j t)}{\sigma_j g_j(\sigma_j)}, \quad \text{where } \sigma_j := 2^{k_j} \cdot S_j(k_j+1). \tag{8.43}$$

Proposition 6.1 implies

$$\overline{G}_j \in \mathcal{G}_\beta\left(\delta, g_0, (1+g_0)^{-1}\right) \quad \text{and} \quad \overline{G}_j(1) \leqslant 1, \tag{8.44}$$

and again by (8.40),

$$\sigma_j \geqslant \frac{j}{2} \to +\infty \quad \text{as } j \to \infty. \tag{8.45}$$

Let us define now for each j the functional

$$\mathcal{F}_j(w) := \int_{B_{2^{k_j}}} \left[\overline{G}_j\big(|\nabla w|\big) + \overline{f}_j(w)\right] dx,$$

where $\overline{f}_j(w)(x) = \lambda_2(\overline{f}_{1,j}, \overline{f}_{2,j})(w)(x)$ and for $i = 1, 2$,

$$\overline{f}_{i,j}(x) = \frac{f_{i,j}(2^{-k_j}x)}{\sigma_j g_j(\sigma_j)}.$$

We claim that v_j is a minimizer for $\mathcal{F}_j$. Indeed, for any $\psi \in W_c^{1,\overline{G}_j}(B_{2^{k_j}})$ we have

$$\widehat{\psi}(x) := S_j(k_j+1) \cdot \widehat{\psi}\big(2^{k_j}x\big) \in W_c^{1,G_j}(B_1).$$

Hence,

$$\mathcal{F}_j(v_j+\psi, B_{2^{k_j}}) = \frac{2^{k_j n}}{\sigma_j g(\sigma_j)} I^2_{G_j}(u_j+\widehat{\psi}, B_1) \geqslant \frac{2^{k_j n}}{\sigma_j g(\sigma_j)} I^2_{G_j}(u_j, B_1) = \mathcal{F}_j(v_j, B_{2^{k_j}}).$$

Once more by (8.40), (8.45), the non-degeneracy condition $G_j(1) \geqslant \varepsilon_0$ and (G-2) imply

$$\|v_j\|_{L^\infty(B_8)} \leqslant \frac{S_j(k_j-3)}{S_j(k_j+1)} \leqslant 16 \tag{8.46}$$

and

$$\big\|\overline{f}_j(w)\big\|_{L^\infty(B_8)} \leqslant \frac{1+g_0}{\varepsilon_0} \cdot 2^{1+\delta} \cdot \frac{\mu}{j^{1+\delta}} \to 0 \quad \text{as } j \to \infty, \tag{8.47}$$

for any function $w : B_{2^{k_j}} \to \mathbb{R}$.

This information together with (8.44) imply that for j large enough $v_j \in \mathcal{S}(B_8, \delta, g_0, (1+g_0)^{-1}, 1, 16)$.

This way, the uniform Hölder estimate – Theorem 5.1 with $\alpha = 1/2$ – implies that

$$\|v_j\|_{C^{1/2}(B_4)} \leqslant 16 + C(n, \delta, g_0).$$

So, we conclude that there exists a $v_\infty \in C^{1/2}(B_4)$ such that

$$v_j \to v_\infty \quad \text{uniformly in } B_4. \tag{8.48}$$

Let us consider now $\{w_j\}_{j\geqslant 1}$ a sequence of solutions in $W^{1,\overline{G}_j}(B_4)$ to the following sequence of Dirichlet problems

$$\mathcal{L}_j w_j = 0 \quad \text{in } B_4 \quad \text{and} \quad w_j = v_j \quad \text{on } \partial B_4,$$

where the operators $\mathcal{L}_j$ are given by

$$\mathcal{L}_j w := \operatorname{div}\left(\bar{g}_j\big(|\nabla w|\big)\frac{\nabla w}{|\nabla w|}\right), \quad \bar{g}_j = \overline{G}'_j.$$

Now, we use a slightly modified version of the stability estimates ((5.23), (5.25), (5.26)) developed in the proof of Theorem 5.1. A careful inspection of these estimates reveals that all of them work with $B_{r^{1+\varepsilon}}$ replaced by B_r and μ replaced by $\|\overline{f}_j(w)\|_{L^\infty(B_8)}$. Also, (5.25) and (5.26) hold with $\overline{\beta}$ replaced by n for $r \geqslant 1$.

This way, combining these estimates together for $r = 4$, we find by (8.47) that

$$\int_{B_4} \overline{G}_j\big(|\nabla v_j - \nabla w_j|\big)\,dx \leqslant C(n,\delta,g_0).\big(\|\overline{f}_j(v_j)\|_{L^\infty(B_8)}\big)^{1/2} \leqslant C(n,\delta,g_0,\varepsilon_0,\mu) \cdot j^{-1/2}. \tag{8.49}$$

Furthermore, for all j, (G-2) and (8.44) yield

$$\frac{1}{(g_0+1)^2}\min\big\{t^{1+\delta}, t^{1+g_0}\big\} \leqslant \overline{G}_j(t) \leqslant (g_0+1)\max\big\{t^{1+\delta}, t^{1+g_0}\big\} \quad \forall t \geqslant 0. \tag{8.50}$$

Hence, by setting

$$B_4^- := B_4 \cap \{|\nabla v_j - \nabla w_j| < 1\} \quad \text{and} \quad B_4^+ := B_4 \cap \{|\nabla v_j - \nabla w_j| \geqslant 1\},$$

by (8.49) and (8.50) we arrive at

$$\frac{C}{j^{1/2}} \geqslant (1+g_0)^{-2}\bigg(\int_{B_4^-} |\nabla v_j - \nabla w_j|^{g_0+1}\,dx + \int_{B_4^+} |\nabla v_j - \nabla w_j|^{\delta+1}\,dx\bigg),$$

and by Hölder's inequality

$$\int_{B_4^-} |\nabla v_j - \nabla w_j|^{1+\delta}\,dx \leqslant |B_4|^{\frac{g_0-\delta}{1+g_0}}\bigg(\int_{B_4^-} |\nabla v_j - \nabla w_j|^{1+g_0}\,dx\bigg)^{(1+\delta)/(1+g_0)}.$$

Thus,

$$\frac{C}{j^{1/2}} \geqslant (1+g_0)^{-2}\bigg[|B_4|^{\frac{\delta-g_0}{1+\delta}}\bigg(\int_{B_4^-} |\nabla v_j - \nabla w_j|^{\delta+1}\,dx\bigg)^{c} + \int_{B_4^+} |\nabla v_j - \nabla w_j|^{\delta+1}\,dx\bigg],$$

where $c = (1+g_0)/(1+\delta) \geqslant 1$. Hence, for $\widetilde{C} = \widetilde{C}(n, \delta, g_0, \varepsilon_0)$

$$\frac{\widetilde{C}}{j^{1/2c}} \geqslant \int_{B_4} |\nabla v_j - \nabla w_j|^{1+\delta}\,dx.$$

Now, it follows by Poincaré's inequality that

$$h_j := v_j - w_j \to 0 \quad \text{strongly in } W_0^{1,\delta+1}(B_4). \tag{8.51}$$

By the maximum principle and (8.46)

$$\|w_j\|_{L^\infty(B_4)} = \|v_j\|_{L^\infty(B_4)} \leqslant 16.$$

Theorem 3.1 guarantees that there exists $C = C(n, \delta, g_0) > 0$ such that $\|w_j\|_{C^{1,\alpha}(B_2)} \leqslant C$. Therefore, we can find $w_\infty \in C^{1,\alpha/2}(B_2)$ such that up to a subsequence

$$w_j \to w_\infty \quad \text{uniformly in } B_2,$$
$$\nabla w_j \to \nabla w_\infty \quad \text{uniformly in } B_2.$$

We conclude this way that $v_\infty = w_\infty$ in B_2 by (8.51). Now, because of (8.44), we can use our compactness result – Theorem 6.1 – to conclude that there exists a $G_\infty \in \mathcal{G}(\delta, g_0) \cap C^{2,\beta}(0,\infty)$ such that, again up to a subsequence,

$$\overline{G}_j \to G_\infty \quad \text{and} \quad \overline{G}_j{}' \to G_\infty' \quad \text{uniformly in compact subsets of } [0,\infty),$$

and

$$\overline{G}_j{}'' \to G_\infty'' \quad \text{uniformly in compact subsets of } (0,\infty).$$

We now claim that v_∞ is a (global) minimizer of $I(v, B_1) := \int_{B_1} G_\infty(|\nabla v|)\,dx$, i.e.,

$$I(v_\infty) \leqslant I(v_\infty + \varphi), \qquad \forall \varphi \in C_0^\infty(B_1).$$

In fact, since v_j is a minimizer of $\mathcal{F}_j$, for any given $\varphi \in C_0^\infty(B_1)$ we have that

$$\int_{B_1} \big[\overline{G}_j\big(|\nabla v_j|\big) + \overline{f}_j(v_j)\big]\,dx \leqslant \int_{B_1} \big[\overline{G}_j\big(|\nabla(v_j+\varphi)|\big) + \overline{f}_j(v_j+\varphi)\big]\,dx. \tag{8.52}$$

Since ∇w_j is uniformly bounded in B_1 by the $C^{1,\alpha}$ estimate, it follows from the uniform convergence that

$$\int_{B_1} \big|\overline{G}_j\big(|\nabla w_j|\big) - G_\infty\big(|\nabla v_\infty|\big)\big|\,dx \to 0. \tag{8.53}$$

Since

$$\overline{G}_j\big(|\nabla v_j|\big) \leqslant 2^{g_0}(1+g_0)\big[\overline{G}_j\big(\big|\nabla(v_j - w_j)\big|\big) + \overline{G}_j\big(|\nabla w_j|\big)\big],$$

we conclude from (8.49) and (8.53) and Theorem 4.9 in [6] that there exists $h \in L^1(B_1)$ such that

$$\overline{G}_j\big(|\nabla v_j|\big) \leqslant h \quad \text{a.e. in } B_1.$$

Once $\nabla v_j \to \nabla v_\infty$ a.e. in B_1, Lebesgue's dominated convergence theorem implies

$$\int_{B_1} \overline{G}_j\big(|\nabla v_j|\big)\,dx \to \int_{B_1} G_\infty\big(|\nabla v_\infty|\big)\,dx.$$

Passing to a subsequence, we find

$$\big|\overline{G}_j\big(\big|\nabla(v_j+\varphi)(x)\big|\big)\big| \leqslant h_\varphi(x) \quad \text{for a.e. } x \in B_1,$$

for

$$h_\varphi := C(g_0)\Big[h + \Big(\Big|1 + \sup_{B_1}|\nabla\varphi|\Big|\Big)^{1+g_0}\Big] \in L^1(B_1).$$

Again by the dominated convergence theorem

$$\int_{B_1} \overline{G}_j\big(\big|\nabla(v_j+\varphi)\big|\big)\,dx \to \int_{B_1} G_\infty\big(\big|\nabla(v_\infty+\varphi)\big|\big)\,dx. \tag{8.54}$$

In addition, by (8.47) we find

$$\int_{B_1} \overline{f}_j(v_j)\,dx \to 0 \quad \text{and} \quad \int_{B_1} \overline{f}_j(v_j+\varphi)\,dx \to 0.$$

Hence passing to the limit in (8.52), we see as claimed that $\forall\varphi \in C_0^\infty(B_1)$

$$\int_{B_1} G_\infty\big(|\nabla v_\infty|\big)\,dx \leqslant \int_{B_1} G_\infty\big(\big|\nabla(v_\infty+\varphi)\big|\big)\,dx.$$

This implies that for $g_\infty := G'_\infty$, v_∞ is a weak solution to

$$\mathcal{L}_\infty v_\infty := \operatorname{div}\bigg(g_\infty\big(|\nabla v_\infty|\big)\frac{\nabla v_\infty}{|\nabla v_\infty|}\bigg) = 0 \quad \text{in } B_1,$$

with

$$G_\infty \in \mathcal{G}(\delta, g_0)$$

and

$$v_\infty(0) = 0 \quad \text{and} \quad 0 \leqslant v_\infty \leqslant 2 \quad \text{and} \quad \sup_{B_{1/2}} v_\infty = 1,$$

where the second fact in the last line follows by the density decay estimate (8.42) and

$$\sup_{B_1}|v_j| = \frac{\sup_{B_1}|u_j(2^{-k_j}x)|}{S_j(k_j+1)} = \frac{S_j(k_j)}{S_j(k_j+1)} < 2,$$

and the last fact

$$\sup_{B_{1/2}} v_\infty = \lim_j \sup_{B_{1/2}}|v_j| = \lim_j \frac{S_j(k_j+1)}{S_j(k_j+1)} = 1.$$

Therefore, by Harnack's inequality (Corollary 1.4 of [13]) we have for $C_0 = C_0(\delta, g_0, n) > 0$

$$1 = \sup_{B_{1/2}} v_\infty \leqslant C_0 \cdot v_\infty(0) = 0,$$

which is a contradiction. This way (8.38) holds and thus so does (8.37).

Case 2. $u \in S_1^\beta(\delta, g_0, \varepsilon_0, \mu, M)(0)$ for $M > 1$.

As before, u is a minimizer of a functional E_G of the type

$$E_G(u, B_1) = \int_{B_1} \big[G\big(|\nabla u|\big) + \lambda(f_1, f_2)(u)\big]\,dx$$

with $G \in \mathcal{G}_\beta(\delta, g_0, \varepsilon_0)$, $\mu \geqslant 0$ and $\sup_{B_1} |u| \leqslant M$. Now, it is easy to see that if we define $v := u/M$, then $\sup_{B_1} |v| \leqslant 1$ and v is a minimizer of $E_{G_M^*}(w) = \int_{B_1}[G_M^*(|\nabla w|) + \lambda(f_1, f_2)(w)]\,dx$. Since $G_M^* \in \mathcal{G}(\delta, g_0, \varepsilon_0)$ by Proposition 6.1, $v \in S_1^\beta(\delta, g_0, \varepsilon_0, \mu, 1)(0)$. Applying Case 1 to v, we are done. □

Now, we present a scaled version of Theorem 2.1 in the proposition below. In particular, we precisely show the dependence between the Lipschitz constant of minimizers and the scale where the density starts to become small enough. This proposition will be used to prove Lipschitz regularity and also density estimates from below of the negative phase in the contact points under the appropriate conditions in the next section.

Proposition 8.1. *There exist a (small) universal constant $D = D(n, \delta, g_0, \beta) > 0$ such that the following estimate*

$$\big|u(x)\big| \leqslant \frac{2 \cdot \max\{M, 1\}}{D \cdot \rho} \cdot |x - x_0|, \quad \forall x \in B_\rho(x_0) \tag{8.55}$$

holds for any $u \in S_\rho^\beta(\delta, g_0, \varepsilon_0, \mu, M)(x_0)$ with $\rho \leqslant \rho_0$ provided $\Theta_u^-(x_0, r) \leqslant D$ for all $0 < r \leqslant \rho$, where $\rho_0 = \rho_0(\delta, g_0, \varepsilon_0, \mu) \leqslant 1$ is also a universal constant.

Proof. We can assume without loss of generality that $x_0 = 0$. Since $u \in S_\rho^\beta(\delta, g_0, \varepsilon_0, \mu, M)(0)$ there exists

$$G \in \mathcal{G}_\beta(\delta, g_0, \varepsilon_0), \quad 0 \leqslant f_1, f_2 \leqslant \mu,$$

such that u is a minimizer of $E_G(u) = \int_{B_\rho}[G(|\nabla u|) + \lambda(f_1, f_2)(u)]\,dx$ where $\lambda(f_1, f_2)(u)$ is given by (2.6). Now we define the rescaled function $v(x) := u(\rho x)$ for $x \in B_1$ and claim that

$$v \in S_1^\beta\big(\delta, g_0, (1 + g_0)^{-1}, 1, M\big)(0) \quad \text{for } \rho \leqslant \rho_0 := \min\left\{1, \left(\frac{\varepsilon_0}{(1 + g_0)\mu}\right)^{(1+\delta)^{-1}}\right\}.$$

In order to show this, it is enough to show that for any ball $B_r(y_0) \subset B_1(0)$ and $w \in W^{1,G}(B_r(y_0))$ with $w = v$ on $\partial B_r(y_0)$ we have

$$\mathcal{J}(v) \leqslant \mathcal{J}(w), \tag{8.56}$$

where

$$\mathcal{J}\big(\eta, B_r(y_0)\big) := \int_{B_r(y_0)} \left[G_{\rho^{-1}}\big(|\nabla \eta|\big) + \frac{\lambda(\widetilde{f}_1, \widetilde{f}_2)(\eta)(x)}{\rho^{-1} g(\rho^{-1})}\right] dx, \quad g = G',$$

$$\widetilde{f}_i(x) = f_i(\rho x), \quad \text{for } x \in B_1 \text{ and } i = 1, 2.$$

It is easy to check that

$$\lambda(\widetilde{f}_1, \widetilde{f}_2)(\eta)(x) = \lambda(f_1, f_2)(\overline{\eta})(\rho x), \quad \text{where } \overline{\eta}(x) := \eta\big(\rho^{-1} x\big),\ x \in B_1.$$

This way, if we define $\overline{w}(x) = w(\rho^{-1} x)$ then $\overline{w} \in W^{1,G}(B_{\rho r}(\rho y_0))$ with $\overline{w} = u$ on $\partial B_{\rho r}(\rho y_0)$. Hence,

$$\begin{aligned}
\mathcal{J}\big(v, B_r(y_0)\big) &= \frac{\rho}{g(\rho^{-1})} \int_{B_r(y_0)} \big[G\big(\rho^{-1}\big|\nabla v(x)\big|\big) + \lambda(\widetilde{f}_1, \widetilde{f}_2)(v)(x)\big]\,dx \\
&= \frac{\rho}{g(\rho^{-1})} \int_{B_r(y_0)} \big[G\big(\big|\nabla u(\rho x)\big|\big) + \lambda(f_1, f_2)(u)(\rho x)\big]\,dx \\
&= \frac{1}{\rho^{n-1} g(\rho^{-1})} \int_{B_{\rho r}(\rho y_0)} \big[G\big(\big|\nabla u(y)\big|\big) + \lambda(f_1, f_2)(u)(y)\big]\,dy \\
&= \frac{1}{\rho^{n-1} g(\rho^{-1})} E_G\big(u, B_{\rho r}(\rho y_0)\big) \\
&\leqslant \frac{1}{\rho^{n-1} g(\rho^{-1})} E_G\big(\overline{w}, B_{\rho r}(\rho y_0)\big) \\
&= \mathcal{J}\big(w, B_r(y_0)\big).
\end{aligned}$$

Observe also that by Proposition 6.1, $G_{\rho^{-1}} \in \mathcal{G}_\beta(\delta, g_0, (1+g_0)^{-1})$.

Furthermore, (g-2) and (G-2) imply for $\rho \leqslant \rho_0$ that

$$\rho^{-1} g\big(\rho^{-1}\big) \geqslant G(\rho) \geqslant \frac{G(1)}{1+g_0} \min\big\{\rho^{-(1+\delta)}, \rho^{-(1+g_0)}\big\} \geqslant \frac{\varepsilon_0}{1+g_0} \rho^{-(1+\delta)} \geqslant \mu,$$

and thus

$$\sup_{x \in B_1} \left\{ \frac{|f_1(\rho x)|}{\rho^{-1} g(\rho^{-1})}, \frac{|f_2(\rho x)|}{\rho^{-1} g(\rho^{-1})} \right\} \leqslant \frac{\mu}{\rho^{-1} g(\rho^{-1})} \leqslant 1.$$

This way, since $\|v\|_{L^\infty(B_1)} \leqslant M$ and $0 \in F^\pm(v)$, the claim is proven. Now, applying Theorem 2.1 to v we obtain

$$\big|v(x)\big| \leqslant \frac{2\max\{M, 1\}}{C} \cdot |x| \quad \forall x \in B_1(0),$$

provided $\Theta_v^-(x_0, r) \leqslant D$ for all $0 < r < 1$, where $D = D(n, \delta, g_0, (1+g_0)^{-1}, 1, \beta) > 0$ is a (small) universal constant. Translating this back in terms of u we finish the proof. □

9. Proof of Corollary 2.1 – Lipschitz regularity under the small density

In this section, we provide the proof of Corollary 2.1.

Proof. Suppose $0 \in F^+(u)$. Consider $x_0 \in B_{1/2}(0)$. We can assume that $u(x_0) > 0$ since the case $u(x_0) < 0$ can be treated analogously. Define $d(x_0) := \operatorname{dist}(x_0, \partial\{u > 0\})$ and let $z_0 \in \partial\{u > 0\}$ such that $d(x_0) = |x_0 - z_0|$. If $|x_0 - z_0| \geqslant 1/4$, then by local gradient estimate, (3.14),

$$\big|\nabla u(x_0)\big| \leqslant \sup_{B_{1/8}(x_0)} \big|\nabla u(x)\big| \leqslant 4C_1 M \leqslant \frac{4C_1 M}{C} \leqslant \frac{4C_1}{C} \max\{M, 1\},$$

since $0 < C < 1$. If $|x_0 - z_0| < 1/4$ we have $|z_0| < 3/4$. Also $u \in \mathcal{S}^\beta_{1/4}(\delta, g_0, \varepsilon_0, \mu, M)(z_0)$ and $\Theta_u^-(z_0, r) \leqslant C$ for $0 \leqslant r \leqslant 1/4$. Thus, by Theorem 2.1, for some universal $C > 0$,

$$u(x_0) \leqslant \frac{2 \cdot \max\{M, 1\}}{C} d(x_0).$$

Once u is a positive solution to $\mathcal{L}_g(u) = 0$ in $B_{d(x_0)}(x_0)$ for some $g = G'$ with $G \in \mathcal{G}_\beta(\delta, g_0, \varepsilon_0)$, Harnack's inequality in [13] implies

$$u \leqslant \frac{2c \cdot \max\{M, 1\}}{C} d(x_0) \quad \text{in } B_{d(x_0)/2}(x_0), \tag{9.57}$$

where $c := c(n, \delta, g_0)$. Again by (3.14) and (9.57), for $r = d(x_0)/4$, we have

$$\left|\nabla u(x_0)\right| \leqslant \sup_{B_{r/2}(x_0)} |\nabla u| \leqslant \frac{C_1}{r} \sup_{B_r(x_0)} u \leqslant \frac{8c \cdot C_1 \cdot \max\{M, 1\}}{C}.$$

The case where $0 \in F^-(u)$ is again treated similarly. □

Combining the last proof and Proposition 8.1, we obtain the following scaled result.

Remark 9.1 *(Scaled version of Corollary 2.1).* Let D and ρ_0 as in Proposition 8.1.

Let $u \in S_\rho^\beta(\delta, g_0, \varepsilon_0, \mu, M)(x_0)$ with $\rho \leqslant \rho_0$ and assume that

$$\Theta_u^-(z, r) \leqslant D \quad \forall z \in F^\pm(u) \cap B_{3\rho/4}(x_0) \text{ and for all } 0 < r < \rho/4.$$

Then,

$$[u]_{C^{0,1}(B_{\rho/2}(x_0))} \leqslant \frac{C_0 \cdot \max\{M, 1\}}{D \cdot \rho} \quad \text{for } C_0 = C_0(n, \delta, g_0) > 0.$$

Remark 9.2 *(Minimizers with sign).* Let D and ρ_0 as in Proposition 8.1. We observe that

$$u \in S_\rho^\beta(\delta, g_0, \varepsilon_0, \mu, M)(x_0) \quad \Rightarrow \quad -u \in S_\rho^\beta(\delta, g_0, \varepsilon_0, \mu, M)(x_0). \tag{9.58}$$

Also, if $u \in S_\rho^\beta(\delta, g_0, \varepsilon_0, \mu, M)(x_0)$ with $\rho \leqslant \rho_0$ and $u \geqslant 0$ or $u \leqslant 0$ in $B_\rho(x_0)$, then

$$[u]_{C^{0,1}(B_{\rho/2}(x_0))} \leqslant \frac{C_0 \cdot \max\{M, 1\}}{D \cdot \rho} \quad \text{for } C_0 = C_0(n, \delta, g_0) > 0.$$

Indeed, by (9.58), we can assume $u \geqslant 0$. Observe that $\Theta_u^-(x, r) = 0 < D$ for all $x \in B_{3\rho/4}(x_0)$ and $0 < r < \rho/4$. The result follows by Remark 9.1. In fact, one can prove Lipschitz regularity for a larger class of minimizers if there is a sign. The estimate can in fact be proven for $u \in S_1(\delta, g_0, \varepsilon_0, \mu)(0)$ once $u \geqslant 0$ or $u \leqslant 0$ by using Theorem 2.2.

10. Regularity and touching behavior of the free boundaries $F^+(u)$ and $F^-(u)$

In the classical paper of Alt, Caffarelli and Friedman [5], they study minimizers of functionals of the type

$$J(u, B_1) = \int_{B_1} \left\{|\nabla u|^2 + \lambda_2 \cdot \chi_{\{u>0\}} + \lambda_1 \cdot \chi_{\{u<0\}} + \min(\lambda_1, \lambda_2) \cdot \chi_{\{u=0\}}\right\} dx,$$

where $\Lambda = \lambda_1 - \lambda_2 \neq 0$, $\lambda_1, \lambda_2 \geqslant 0$. For definiteness, they assume that $\Lambda < 0$ since the other case can be treated similarly. These conditions imply that minimizers are globally subharmonic functions (Theorem 2.3 in [5]) and this fact together with the maximum principle restrict the way the free boundaries $F^+(u)$, $F^-(u)$ may touch. Essentially $F^-(u)$ cannot separate from $F^+(u)$. More precisely, as pointed out in the beginning of Section 6 in [5], the set $F^-(u) \setminus F^+(u)$ is empty. In the general case treated here, if the phase functions are ordered, i.e., say $f_2 \geqslant f_1$ in $B_1(0)$ the same phenomenon happens. This is the content of the remark below.

Remark 10.1. Let u is a minimizer of E_G in B_1 with $f_2 \geqslant f_1$ and $G \in \mathcal{G}(\delta, g_0)$. Then,

$$\mathcal{L}_g(u) \geqslant 0 \quad \text{in } B_1 \quad \text{and} \quad F^-(u) \setminus F^+(u) = \emptyset.$$

Proof. Indeed, given any $0 \leqslant \eta \in C_0^\infty(\Omega)$ and $\varepsilon > 0$, it follows by minimality that

$$I_G^1(u) \leqslant I_G^1(u - \varepsilon\eta),$$

for $\Lambda_0(x) = f_1(x) - f_2(x) \leqslant 0$. Hence,

$$\int_{B_1} \left(G\left(\left|\nabla(u - \varepsilon\eta)\right|\right) - G\left(|\nabla u|\right)\right) dx \geqslant \int_{B_1} \Lambda_0(x)\left(\chi_{\{u\leqslant 0\}} - \chi_{\{u-\varepsilon\eta\leqslant 0\}}\right) dx \geqslant 0,$$

since $\{u \leqslant 0\} \subset \{u - \varepsilon\eta \leqslant 0\}$. Thus, by setting the functional $I(v) = \int_{B_1} G(|\nabla v|)\,dx$, we conclude that

$$0 \leqslant \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\big(I(u - \varepsilon\eta) - I(u)\big) = \frac{d}{d\varepsilon} I(u - \varepsilon\eta)\bigg|_{\varepsilon=0} = -\int_{\Omega} g\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|} \cdot \nabla\eta\,dx. \tag{10.59}$$

This proves that u is an $\mathcal{L}_g$ subsolution. Now, suppose $x_0 \in F^-(u) \setminus F^+(u)$. There exists $\varepsilon > 0$ such that $B_\varepsilon(x_0) \cap F^+(u) = \emptyset$. We observe that $\mathcal{L}_g(u) = 0$ in $\{u < 0\} \cap B_\varepsilon(x_0)$ and $u \leqslant 0$ in $B_\varepsilon(x_0)$. This way, by Lemma 8.1 in [15] we conclude that $\mathcal{L}_g(u) \leqslant 0$ in $B_\varepsilon(x_0)$ and thus $\mathcal{L}_g(u) = 0$ in $B_\varepsilon(x_0)$. Now, the Harnack inequality implies that for a universal constant $C_\varepsilon > 0$ depending on ε we have

$$\sup_{B_{\varepsilon/2}(x_0)} (-u) \leqslant C \cdot \inf_{B_{\varepsilon/2}(x_0)} (-u) \leqslant -C_\varepsilon \cdot u(x_0) = 0.$$

This implies that $u \equiv 0$ in $B_\varepsilon(x_0)$ and so, $x_0 \notin F^-(u)$ a contradiction. Thus, $F^-(u) \setminus F^+(u) = \emptyset$. □

Still in the context of Alt, Caffarelli and Friedman paper [5], by combining the (crucial) Lipschitz regularity with the non-degeneracy property (also $\Lambda < 0$), they are able to prove that the nonnegative phase has positive density from below along the free boundary $F^+(u)$, i.e., there exists a constant $c \in (0, 1)$ such that

$$\frac{|B_r \cap \{u \leqslant 0\}|}{|B_r|} \geqslant c,$$

for any ball $B_r \subset B_1$ with center on the free boundary $F^+(u)$ (see Theorem 7.1 in [5]).

In the case discussed here the situation is more general. In particular, as pointed out before, there is no ordering on the phase functions f_2 and f_1 in the functionals (2.5). So, in principle, minimizers are neither subsolutions nor supersolutions. Thus, in our conditions, the free boundaries $F^+(u)$ and $F^-(u)$ may separate, i.e., $F^-(u) \setminus F^+(u) \neq \emptyset$. A similar situation occurs even in the standard case involving the Laplace operator in an inhomogeneous setting as it appears in flame propagation problems with forcing terms in [12].

It becomes an interesting question to understand the touching of the free boundaries and possibly some geometric information at the touching points. To the best of our knowledge, there are few concrete examples in the literature addressing these geometric issues for minimizers and almost all of them concern the one phase scenario. In that respect, we recently learned about the interesting paper [4] by M. Allen and H.C. Lara where they investigate the touching of the free boundaries (also in the two phase case) in cone vertices in a very much related variational problem on the 2 dimensional sphere.

As a consequence of Theorem 2.1, we can show that if Lipschitz regularity fails to be the optimal regularity for a minimizer $u \in S_1^\beta(\delta, g_0, \varepsilon_0, \mu, M)$ say in a point, then this point must not only be a contact point between the free boundaries $F^+(u)$ and $F^-(u)$ but also the negative phase must have a universal (upper) density from below at that point. In particular, at that contact point the negative phase $\{u < 0\}$ is cusp free. We also show that if u is not locally Lipschitz around x_0, then x_0 is also a contact point between the free boundaries and the negative phase is "asymptotically cusp free" there, in the sense that there is a sequence of points along $F^\pm(u)$ converging to x_0 and scales going to zero, along which the density function Θ_u^- is universally bounded from below.

Proof of Proposition 2.1. Clearly, $\mathcal{S}_p \subset \mathcal{S}_l$ since $[u]_{C^{0,1}(B_r(x_0))}(x_0) \leqslant [u]_{C^{0,1}(B_r(x_0))}$. Now let $x_0 \in \mathcal{S}_l$. We can find two sequences $\{x_n\}_{n\geqslant 1}, \{y_n\}_{n\geqslant 1} \subset B_{1/2}(x_0)$ such that:

$$x_n \to x_0, \qquad y_n \to x_0 \quad \text{as } n \to \infty \quad \text{and} \quad \rho_n := \frac{|u(x_n) - u(y_n)|}{|x_n - y_n|} > n.$$

Since $B_1 = \{u > 0\} \cup \{u < 0\} \cup \{u = 0\}^\circ \cup F^\pm(u)$, we see that $x_0 \notin \{u > 0\} \cup \{u < 0\} \cup \{u = 0\}^\circ$. This follows from interior Lipschitz estimates. Indeed, in that case there would exist $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset \{u > 0\}$ or $B_\varepsilon(x_0) \subset \{u < 0\}$ or else $B_\varepsilon(x_0) \subset \{u = 0\}$. The latter case is impossible since it would imply $0 = \rho_n > n$ for n sufficiently large. In any of the first two cases, since $\mathcal{L}_g(u) = 0$ in $B_\varepsilon(x_0)$ by Proposition 4.2, the gradient estimate in Theorem 3.1 would imply that for a universal constant $C_1 > 0$ and n large enough,

$$n < \frac{|u(x_n) - u(y_n)|}{|x_n - y_n|} \leqslant \sup_{B_{\varepsilon/2}(x_0)} |\nabla u| \leqslant \frac{C_1}{\varepsilon} \sup_{B_\varepsilon(x_0)} |v| = \frac{C_1 \cdot M}{\varepsilon} < \infty,$$

which is a contradiction. This way, $x_0 \in F^{\pm}(u) \cap \overline{B_{1/2}}$. Let us suppose that $x_0 \in F^-(u) \setminus F^+(u)$. In this case, $x_0 \notin F^+(u) \cap \overline{B_{1/2}}$ which is a compact set, and thus, there exists $0 < \varepsilon < \rho_0$ such that $B_\varepsilon(x_0) \cap F^+(u) \cap \overline{B_{1/2}} = \emptyset$. Since $u \leqslant 0$ in $B_\varepsilon(x_0)$ and $u \in S_\varepsilon^\beta(\delta, g_0, \varepsilon_0, \mu, M)(x_0)$, Remark 9.2 gives for n large enough,

$$n < \frac{|u(x_n) - u(x_0)|}{|x_n - x_0|} \leqslant [u]_{C^{0,1}(B_{\varepsilon/2}(x_0))} \leqslant \frac{C_0 \cdot \max\{M, 1\}}{D \cdot \varepsilon} < \infty,$$

which is a contradiction. The case where $x_0 \in F^+(u) \setminus F^-(u)$ is treated similarly. Thus, we conclude that $x_0 \in F^+(u) \cap F^-(u)$. In the case where $x_0 \in \mathcal{S}_l$ then for any $\rho \leqslant \rho_0$ there exist $x_\rho \in B_{\frac{3\rho}{4}}(x_0) \cap F^{\pm}(u)$ and $0 < r_\rho < \rho/4$ such that $\Theta_u^-(x_\rho, r_\rho) > D$ since otherwise Remark 9.1 implies that

$$[u]_{C^{0,1}(B_{\rho/2}(x_0))} \leqslant \frac{C_0 \cdot \max\{M, 1\}}{D \cdot \rho},$$

and thus would imply $x_0 \notin \mathcal{S}_l$. Now if $x_0 \in \mathcal{S}_p$, we also see that for any $0 < \rho \leqslant \rho_0$ there exists $0 < r_\rho \leqslant \rho$ such that $\Theta_u^-(x_0, r_\rho) > D$ since otherwise Proposition 8.1 would imply that

$$[u]_{C^{0,1}(B_\rho(x_0))}(x_0) \leqslant \frac{2 \cdot \max\{M, 1\}}{D \cdot \rho} < \infty,$$

and thus, $x_0 \notin \mathcal{S}_p$. This finishes the proof. □

Remark 10.2. In general, it is not an easy task to prove that some specific function although a (viscosity) solution to some FBP is not a minimizer. There are few examples in the literature the authors are aware of. One of the first important examples was obtained by H.W. Alt and L.A. Caffarelli in Section 2.7 of [2] where some one phase cone type solution to an FBP is shown not to be a minimizer in dimension 3. Many years later, L.A. Caffarelli, D. Jerison and C. Kenig proved that the same solution is not a minimizer up to dimension 6 (Proposition in [8]). After that, D. de Silva and D. Jerison proved a result in [9] showing that this cone type solution is actually a minimizer in dimension 7, providing the first example of a singular free boundary for minimizers in analogy with the Simons cone for the theory of minimal surfaces. All of these developments take place for the case where $G(t) = t^2$. Our result (Proposition 2.1), may be of some use in ruling out (viscosity) solutions of FBPs of the type (1.2) from being minimizers of E_G functionals in the case where (pointwise) Lipschitz regularity fails and cusps in the negative phase develop in the contact points $F^+(u) \cap F^-(u)$.

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Appendix A

In this appendix, we provide short proof for Lemma 3.1.

Proof of Lemma 3.1. By Poincaré's inequality, there exists a dimensional constant that we can suppose $C \geqslant 1$ such that

$$r^{-1} \fint_{B_r(x_0)} \left|u(x) - (u)_{x_0,r}\right| dx \leqslant C \cdot \fint_{B_r(x_0)} \left|\nabla u(x)\right| dx \quad \text{for } x_0 \in \Omega',\ r \leqslant R_0, \tag{A.60}$$

where $(u)_{x_0,r} = \fint_{B_r(x_0)} u(x)\, dx$. Clearly, we can assume $R_0 \leqslant 1$. Let us define for $0 < r \leqslant R_0$ the function

$$\rho(r) = r^{-1} \fint_{B_r(x_0)} \left|u(x) - (u)_{x_0,r}\right| dx.$$

Since G is increasing, convex and it satisfies (G-2), (A.60) implies

$$G\big(\rho(r)\big) \leqslant (1+g_0)\cdot C^{1+g_0}\cdot \fint_{B_r(x_0)} G\big(\big|\nabla u(x)\big|\big)\,dx \leqslant (1+g_0)\cdot C^{1+g_0}\cdot L\cdot r^{\alpha-1}.$$

On one hand, if $\rho(r)\geqslant 1$ we have

$$G\big(\rho(r)\big) \geqslant G(1)\cdot \min\big\{\rho(r)^{1+\delta}, \rho(r)^{1+g_0}\big\} \geqslant G(1)\cdot\rho(r).$$

So, combining these inequalities, we obtain

$$\rho(r) \leqslant \frac{(1+g_0)\cdot C^{1+g_0}\cdot L}{G(1)}\cdot r^{\alpha-1}$$

which lead us to

$$\fint_{B_r(x_0)} \big|u(x)-(u)_{x_0,r}\big|\,dx \leqslant \frac{\widetilde{C}(g_0)}{G(1)}\cdot L\cdot r^{\alpha}\quad \text{for } 0<r\leqslant R_0. \tag{A.61}$$

Now, in the case $\rho(r)\leqslant 1$ since $0<\alpha\leqslant 1$ we have

$$\fint_{B_r(x_0)} \big|u(x)-(u)_{x_0,r}\big|\,dx \leqslant r \leqslant r^{\alpha}\quad \text{for } 0<r\leqslant R_0. \tag{A.62}$$

This way, combining (A.61) and (A.62) we have

$$\fint_{B_r(x_0)} \big|u(x)-(u)_{x_0,r}\big|\,dx \leqslant \max\left\{\frac{\widetilde{C}(g_0)}{G(1)}\cdot L, 1\right\}\cdot r^{\alpha}\quad \text{for } x_0\in\Omega' \text{ with } 0<r\leqslant R_0.$$

The estimates now follow from the Campanato theorem as in Theorem 1.1 in [14]. □

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