Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 31 (2014) 1101–1107
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Almost reduction and perturbation of matrix cocycles

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Received 31 January 2013; received in revised form 15 August 2013; accepted 23 August 2013

Available online 13 September 2013

Abstract

In this note, we show that if all Lyapunov exponents of a matrix cocycle vanish, then it can be perturbed to become cohomologous to a cocycle taking values in the orthogonal group. This extends a result of Avila, Bochi and Damanik to general base dynamics and arbitrary dimension. We actually prove a fibered version of this result, and apply it to study the existence of dominated splittings into conformal subbundles for general matrix cocycles.

1. From zero Lyapunov exponents to rotation cocycles

1.1. Basic definitions

Let $F:\Omega \to \Omega$ be a homeomorphism of a compact metric space Ω. Let V be a finite-dimensional real vector bundle over Ω, whose fiber over ω is denoted by V_ω. Let $\mathcal{A}$ be a vector-bundle automorphism that fibers over F; this means that the restriction of $\mathcal{A}$ to each fiber V_ω is a linear automorphism $A(\omega)$ onto $V_{F\omega}$. In the case of trivial vector bundles, $\mathcal{A}$ is usually called a *linear cocycle*.

As a convention, automorphisms of V will be denoted by calligraphic letters, and the restrictions to the fibers will be denoted by the corresponding roman letters. Analogously, for any integer n, the restriction of the power $\mathcal{A}^n$ to the fiber V_ω is denoted by $A^n(\omega)$; thus $A^n(\omega) = A(F^{n-1}\omega) \circ \cdots \circ A(\omega)$ for $n > 0$.

A *Riemannian metric* on V is a continuous choice of inner product $\langle \cdot,\cdot \rangle_\omega$ on each fiber V_ω. It induces a *Riemannian norm* $\|v\|_\omega = \sqrt{\langle v, v\rangle_\omega}$. Given a linear map $L: V_\omega \to V_{\omega'}$, its *norm* $\|L\|$ and its *mininorm* $\mathfrak{m}(L)$ are defined respectively as the supremum and the infimum of $\|Lv\|_{\omega'}$ over all unit vectors $v \in V_\omega$.

Let $\mathrm{Aut}(V, F)$ denote the space of all automorphisms of V that fiber over F, endowed with the topology induced by the distance $d(\mathcal{A}, \mathcal{B}) = \sup_\omega \|A(\omega) - B(\omega)\|$, for some choice of a Riemannian norm on V.

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1 Partially supported by CNPq (Brazil) and FAPERJ (Brazil).
2 Partially supported by the Fondecyt Project 1120131 and the “Center of Dynamical Systems and Related Topics” (DySyRF; ACT-Project 1103, Conicyt).

0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2013.08.004

1.2. Uniform subexponential growth and its consequences

Define

$$\lambda^+(\mathcal{A}) = \lim_{n\to+\infty} \frac{1}{n} \sup_{\omega\in\Omega} \log\|A^n(\omega)\| \quad \text{and} \quad \lambda^-(\mathcal{A}) = \lim_{n\to+\infty} \frac{1}{n} \inf_{\omega\in\Omega} \log \mathfrak{m}\big(A^n(\omega)\big),$$

which exist by subadditivity and supraadditivity, respectively.

If μ is an ergodic probability measure for $F: \Omega \to \Omega$, then there are constants $\lambda^+(\mathcal{A}, \mu)$, $\lambda^-(\mathcal{A}, \mu)$, called the *top* and *bottom Lyapunov exponents*, such that, for μ-almost every $\omega \in \Omega$,

$$\frac{1}{n} \log\|A^n(\omega)\| \to \lambda^+(\mathcal{A}, \mu) \quad \text{and} \quad \frac{1}{n} \log \mathfrak{m}\big(A^n(\omega)\big) \to \lambda^-(\mathcal{A}, \mu) \quad \text{as } n \to +\infty.$$

Moreover, the following "variational principle" holds[3]:

$$\lambda^+(\mathcal{A}) = \sup_{\mu} \lambda^+(\mathcal{A}, \mu) \quad \text{and} \quad \lambda^-(\mathcal{A}) = \inf_{\mu} \lambda^-(\mathcal{A}, \mu), \tag{1.1}$$

where μ runs over all invariant ergodic probabilities for F.

Let us say that the automorphism $\mathcal{A}$ has *uniform subexponential growth* if $\lambda^+(\mathcal{A}) = \lambda^-(\mathcal{A}) = 0$. By (1.1), this is equivalent to the vanishing of all Lyapunov exponents with respect to all ergodic probability measures.

Our first result is:

Theorem 1.1. *Assume that $\mathcal{A} \in \mathrm{Aut}(V, F)$ has uniform subexponential growth. Then*:

(a) *For any $\varepsilon > 0$, there exists a Riemannian norm $|||\cdot|||$ on V such that*

$$e^{-\varepsilon} |||v|||_\omega < \big|\big|\big|A(\omega)v\big|\big|\big|_{F\omega} < e^{\varepsilon} |||v|||_\omega, \quad \textit{for all } \omega \in \Omega,\ v \in V_\omega. \tag{1.2}$$

(b) *There exists an arbitrarily small perturbation of $\mathcal{A}$ that preserves some Riemannian norm on V.*

As we will see, part (a) follows from a standard construction in Pesin theory, and part (b) follows form part (a). However, the latter implication is *not* straightforward, because if ε is small then the Riemannian norm constructed in part (a) may be very distorted with respect to a fixed reference Riemannian norm on V.

For a reformulation of the theorem in terms of conjugacy to isometric automorphisms, see Section 1.4.

Despite making stringent assumptions about the automorphism $\mathcal{A}$, Theorem 1.1 can be used to obtain very strong properties for a dense subset D of $\mathrm{Aut}(V, F)$, under the assumption that F is uniquely ergodic (or, in some cases, minimal). More precisely, we show that for every automorphism $\mathcal{A}$ in the subset D there exist a Riemannian metric norm $|||\cdot|||$ on V and a splitting of V as a Whitney sum of $\mathcal{A}$-invariant subbundles where $\mathcal{A}$ acts conformally with respect to the norm $|||\cdot|||$. Moreover, this splitting is either trivial or dominated. See Section 2.2 for details.

In the paper [6], we prove results about cocycles of isometries of spaces of nonpositive curvature that generalize Theorem 1.1. Actually, we first obtained Theorem 1.1 as a corollary of the geometrical results of [6]. Later, we realized that the constructions could be modified or adapted to produce an elementary proof of Theorem 1.1, which we present, together with its applications, in this note.

1.3. Proof of Theorem 1.1

We need a few preliminaries.

Recall that V is a finite-dimensional vector bundle over the compact space Ω. We choose and fix a Riemannian metric $\langle\cdot,\cdot\rangle$ on V. Let $\mathcal{B}$ be an automorphism of V over a homeomorphism $G: \Omega \to \Omega$. The *transpose* of $\mathcal{B}$ is the automorphism $\mathcal{B}^*$ over G^{-1} defined by

$$\big\langle B(\omega)u, v\big\rangle_{G\omega} = \big\langle u, B^*(G\omega)v\big\rangle_\omega, \quad \text{for all } u \in V_\omega,\ v \in V_{G\omega}.$$

[3] This follows from [16, Thm. 1] or [17, Thm. 1.7]. Although these references assume Ω to be compact metrizable, the proofs also work for compact Hausdorff Ω. (See also the proof of Proposition 1 in [1].) A particular case was considered in [10].

If $\mathcal{B}^* = \mathcal{B}$ (and thus G is the identity), then $\mathcal{B}$ is called *symmetric*. An automorphism $\mathcal{P}$ is called *positive* if it is symmetric and $\langle P(\omega)v, v\rangle_\omega > 0$ for all nonzero $v \in V_\omega$. We write $\mathcal{B} < \mathcal{C}$ if $\mathcal{B}$ and $\mathcal{C}$ are symmetric and $\mathcal{C} - \mathcal{B}$ is positive.

The following proposition collects some useful properties:

Proposition 1.2.

(a) *If $\mathcal{A}$ is any automorphism and $\mathcal{B}$ is symmetric, then $\mathcal{A}^*\mathcal{B}\mathcal{A}$ is symmetric; moreover, if $\mathcal{B} < \mathcal{C}$, then $\mathcal{A}^*\mathcal{B}\mathcal{A} < \mathcal{A}^*\mathcal{C}\mathcal{A}$.*
(b) *Each positive automorphism $\mathcal{P}$ has a unique positive square root $\mathcal{P}^{1/2}$; moreover, $\mathcal{P}^{1/2}$ commutes with $\mathcal{P}$, and the map $\mathcal{P} \mapsto \mathcal{P}^{1/2}$ is continuous.*
(c) *The square root map is monotonic: if $\mathcal{P}$, $\mathcal{Q}$ are positive and $\mathcal{P} < \mathcal{Q}$, then $\mathcal{P}^{1/2} < \mathcal{Q}^{1/2}$.*

Properties (a) and (b) above are easy exercises. For a proof of property (c), see [4, p. 9].

Proof of Theorem 1.1. Let $\mathcal{A}$ be an automorphism of V over the homeomorphism F having uniform subexponential growth. Fix a small $\varepsilon > 0$.

To prove part (a), we will use a standard construction in Pesin theory called *Lyapunov norms* (see e.g. [13, p. 667]). Define

$$|||v|||_\omega^2 := \sum_{n\in\mathbb{Z}} e^{-2\varepsilon|n|} \| A^n(\omega)v \|_{F^n\omega}^2. \tag{1.3}$$

Since the cocycle has uniform subexponential growth, the series converges uniformly on compact subsets of V, and hence defines a (continuous) Riemannian norm. Property (1.2) is straightforward to check. This proves part (a).

To prove part (b), let $\langle\langle\cdot,\cdot\rangle\rangle$ be the inner product that induces the norm (1.3). Then there are positive automorphisms $\mathcal{R}$, $\mathcal{Q}$ such that for all $u, v \in V_\omega$,

$$\langle\langle u, v\rangle\rangle_\omega = \langle R(\omega)u, v\rangle_\omega, \tag{1.4}$$

$$\langle\langle A(\omega)u, A(\omega)v\rangle\rangle_{F\omega} = \langle Q(\omega)u, v\rangle_\omega. \tag{1.5}$$

The almost-invariance property (1.2) can now be expressed as:

$$e^{-2\varepsilon}\mathcal{R} < \mathcal{Q} < e^{2\varepsilon}\mathcal{R}. \tag{1.6}$$

We want to find an automorphism $\tilde{\mathcal{A}}$ over F that is close to $\mathcal{A}$ and leaves the inner product $\langle\langle\cdot,\cdot\rangle\rangle$ invariant. As it is straightforward to check, invariance means that the automorphism $\mathcal{P} = \mathcal{A}^{-1}\tilde{\mathcal{A}}$ (over the identity) satisfies:

$$\mathcal{P}^*\mathcal{Q}\mathcal{P} = \mathcal{R}. \tag{1.7}$$

Equivalently,

$$(\mathcal{Q}^{1/2}\mathcal{P}\mathcal{Q}^{1/2})^*(\mathcal{Q}^{1/2}\mathcal{P}\mathcal{Q}^{1/2}) = \mathcal{Q}^{1/2}\mathcal{R}\mathcal{Q}^{1/2}.$$

Let us try to find a *positive* solution $\mathcal{P}$. Then the relation above becomes $(\mathcal{Q}^{1/2}\mathcal{P}\mathcal{Q}^{1/2})^2 = \mathcal{Q}^{1/2}\mathcal{R}\mathcal{Q}^{1/2}$, and using the uniqueness of positive square roots (property (b) in Proposition 1.2), we obtain

$$\mathcal{P} = \mathcal{Q}^{-1/2}(\mathcal{Q}^{1/2}\mathcal{R}\mathcal{Q}^{1/2})^{1/2}\mathcal{Q}^{-1/2}. \tag{1.8}$$

One checks directly that this formula solves the invariance equation (1.7), and thus gives the unique positive solution.

To estimate $\mathcal{P}$, we follow the steps of [14]. By the first inequality in (1.6) and property (a) in Proposition 1.2, we have $\mathcal{Q}^{1/2}\mathcal{R}\mathcal{Q}^{1/2} < e^{2\varepsilon}\mathcal{Q}^2$. So, by property (c) in that proposition, $(\mathcal{Q}^{1/2}\mathcal{R}\mathcal{Q}^{1/2})^{1/2} < e^{\varepsilon}\mathcal{Q}$. Applying property (a) again, we obtain $\mathcal{P} < e^{\varepsilon}\mathcal{I}$, where $\mathcal{I}$ is the identity automorphism. This means that $\|P(\omega)\| < e^{\varepsilon}$ for every ω. An analogous argument starting from the second inequality in (1.6) gives $\mathfrak{m}(P(\omega)) > e^{-\varepsilon}$ for every ω. This shows that $\mathcal{P}$ is close to the identity, and therefore the automorphism $\tilde{\mathcal{A}} := \mathcal{A}\mathcal{P}$ is close to $\mathcal{A}$. As we have seen, $\tilde{\mathcal{A}}$ preserves the new Riemannian metric, thus completing the proof of the theorem. □

Remark 1.3. Eq. (1.7) obviously has infinitely many solutions $\mathcal{P}$, not all of them close to the identity. As we have seen, restricting to positive automorphisms we have a unique solution, which is close to the identity and varies continuously with the data.

In [6], we obtain a generalization of Theorem 1.1 to cocycles of isometries of symmetric spaces of non-positive curvature. If specialized to the present situation, the construction presented in [6] is the same as the one given here for part (b), thus "explaining" the efficiency of positive matrices.

Remark 1.4. Notice that the Riemannian norm and the perturbed automorphism constructed in the proof of Theorem 1.1 depend continuously on the parameter ε and also on the automorphism $\mathcal{A}$ itself. These properties are relevant for the applications obtained in [3].

1.4. Conjugacy

Let us put Theorem 1.1 under a different perspective.

Two automorphisms $\mathcal{A}$, $\mathcal{B} \in \mathrm{Aut}(V, F)$ are said to be *conjugate* if there exists $\mathcal{U} \in \mathrm{Aut}(V, \mathrm{id})$ such that $\mathcal{A} = \mathcal{U}\mathcal{B}\mathcal{U}^{-1}$. (In the case of a trivial vector bundle, we say that the two linear cocycles are *cohomologous*.)

Fixed a Riemannian metric on V, we say that an automorphism $\mathcal{A}$ is *isometric* if it preserves this metric. (In the case of a trivial vector bundle, the cocycle will take values in the orthogonal group, i.e., it will be a *rotation cocycle*.)

Then we have:

Theorem 1.5. *Fix a Riemannian metric on the vector bundle V. Assume that $\mathcal{A} \in \mathrm{Aut}(V, F)$ has uniform subexponential growth. Then:*

(a) *There exists an automorphism conjugate to $\mathcal{A}$ that is close to an isometric automorphism. More precisely, every neighborhood of the set of isometric automorphisms contains a conjugate of $\mathcal{A}$.*
(b) *There exists an automorphism close to $\mathcal{A}$ that is conjugate to an isometric automorphism. More precisely, every neighborhood of $\mathcal{A}$ contains a conjugate of an isometric automorphism.*

For $\mathrm{SL}(2, \mathbb{R})$-cocycles and under extra assumptions on the dynamics F, the result above was shown by Avila, Bochi and Damanik as a step in the proofs of their results about spectra of Schrödinger operators, see [2,3].[4]

In the case of cocycles (i.e., trivial vector bundles), it is natural to look for conditions under which we can improve the conclusion of Theorem 1.5(b) and find a perturbed cocycle cohomologous to a constant rotation, or even to the identity. The case of $\mathrm{SL}(2, \mathbb{R})$-cocycles is studied in [3].

Proof of Theorem 1.5. Let $\varepsilon > 0$ be small. We follow the notation of the proof of Theorem 1.1.

It follows from (1.4) that $|||v|||_\omega = \|R(\omega)^{1/2} v\|_\omega$ for every $v \in V_\omega$. Let $\mathcal{B} := \mathcal{R}^{1/2}\mathcal{A}\mathcal{R}^{-1/2}$. Then, by (1.2),

$$e^{-\varepsilon}\|v\|_\omega < \left\|B(\omega)v\right\|_{F\omega} < e^{\varepsilon}\|v\|_\omega.$$

This implies that $\mathcal{B}$ is close to an isometric isomorphism, thus proving part (a).

To prove part (b), it suffices to notice that $\mathcal{R}^{-1/2}\tilde{\mathcal{A}}\mathcal{R}^{1/2}$ is an isometric isomorphism. □

There is another property which is closely related to what we have seen so far. Let us say that $\mathcal{A} \in \mathrm{Aut}(V, F)$ is *product-bounded* if

$$0 < \inf_{\substack{\omega\in\Omega \\ n\in\mathbb{Z}}} \mathfrak{m}\big(A^n(\omega)\big) \leqslant \sup_{\substack{\omega\in\Omega \\ n\in\mathbb{Z}}} \left\|A^n(\omega)\right\| < \infty.$$

If an automorphism $\mathcal{A}$ is conjugate to an isometric automorphism then $\mathcal{A}$ is product-bounded, as it is easy to check. Although product-bounded cocycles are not always conjugate to isometric automorphisms,[5] this happens whenever F is minimal, according to a result shown by Coronel, Navas and Ponce in [9].[6]

[4] However, they haven't explicitly stated the result: see the proof of Theorem 1 in [2] and Proposition 6.3 in [3].
[5] See e.g. [13, Exercise 2.9.2], [15].
[6] In the non-minimal case, one can still ensure the existence of a bounded and measurable conjugacy.

2. Conformality properties

2.1. Extensions of the previous results for the case of coinciding Lyapunov exponents

The following is an immediate consequence of Theorem 1.1:

Corollary 2.1. *Let $\mathcal{A} \in \mathrm{Aut}(V, F)$ be such that $\lambda^+(\mathcal{A}) = \lambda^-(\mathcal{A}) =: \lambda$. Then there exist an arbitrarily small perturbation $\tilde{\mathcal{A}}$ of $\mathcal{A}$ and a Riemannian norm $|||\cdot|||$ on V such that*

$$\big|\big|\big|\tilde{A}(\omega)v\big|\big|\big|_{F\omega} = e^{\lambda}|||v|||_{\omega}, \quad \textit{for all } \omega \in \Omega,\ v \in V_{\omega}.$$

In other words, if all Lyapunov exponents of an automorphism are equal to some λ, then we can perturb it to become conformal with respect to a new Riemannian metric; moreover it dilates the metric by the constant factor e^{λ}.

Actually, a weaker assumption is sufficient to obtain conformality:

Corollary 2.2. *Let $\mathcal{A} \in \mathrm{Aut}(V, F)$ be such that $\lambda^+(\mathcal{A}, \mu) = \lambda^-(\mathcal{A}, \mu)$ for every ergodic probability measure μ for F. Then there exist an arbitrarily small perturbation $\tilde{\mathcal{A}}$ of $\mathcal{A}$, a Riemannian norm $|||\cdot|||$ on V, and a continuous function $\lambda : \Omega \to \mathbb{R}$ such that*

$$\big|\big|\big|\tilde{A}(\omega)v\big|\big|\big|_{F\omega} = e^{\lambda(\omega)}|||v|||_{\omega}, \quad \textit{for all } \omega \in \Omega,\ v \in V_{\omega}.$$

See [12] for a non-perturbative result with a similar conclusion.

Proof of Corollary 2.2. First of all, notice that for any $\mathcal{A} \in \mathrm{Aut}(V, F)$, we have

$$\mathfrak{m}\big(A(\omega)\big)^d \leqslant \big|\det A(\omega)\big| \leqslant \big\|A(\omega)\big\|^d,$$

where d be the fiber dimension of V. So, by submultiplicativity of norms,

$$\lambda^-(\mathcal{A}, \mu) \leqslant \int \lambda\, d\mu \leqslant \lambda^+(\mathcal{A}, \mu) \quad \text{for every ergodic measure } \mu, \tag{2.1}$$

where

$$\lambda(\omega) := \frac{1}{d}\log\big|\det A_i(\omega)\big|. \tag{2.2}$$

Now assume that equalities hold in (2.1). Let $\mathcal{B} = e^{-\lambda}\mathcal{A}$. Then $\lambda^{\pm}(\mathcal{B}, \mu) = \lambda^{\pm}(\mathcal{A}, \mu) - \int \lambda\, d\mu = 0$ for every ergodic measure μ. By the "variational principle" (1.1), this implies $\lambda^+(\mathcal{B}) = \lambda^-(\mathcal{B}) = 0$. Therefore, by Theorem 1.1, there is a Riemannian norm $|||\cdot|||$ on V that is preserved by a perturbation $\tilde{\mathcal{B}}$ of $\mathcal{B}$.

Let $\tilde{\mathcal{A}} = e^{\lambda}\tilde{\mathcal{B}}$. This is a perturbation of $\mathcal{A}$ with the desired conformality property. □

2.2. Existence of conformal subbundles

Using Corollary 2.1 and a theorem from [7], we will obtain the following result:

Theorem 2.3. *Assume that $F : \Omega \to \Omega$ is a uniquely ergodic homeomorphism with an invariant probability measure of full support. Then for every automorphism $\mathcal{A}$ in a dense subset of $\mathrm{Aut}(V, F)$, there exist:*

- *a Riemannian norm $|||\cdot|||$ on V;*
- *a continuous $\mathcal{A}$-invariant splitting $V = V^1 \oplus \cdots \oplus V^k$ which is orthogonal with respect to the Riemannian norm;*
- *and constants $\lambda_1 > \cdots > \lambda_k$;*

such that

$$\big|\big|\big|A(\omega)v_i\big|\big|\big|_{F\omega} = e^{\lambda_i}|||v_i|||_{\omega}, \quad \textit{for all } \omega \in \Omega,\ i = 1, \ldots, k,\ v_i \in V^i_{\omega}.$$

Weakening the assumption of unique ergodicity to minimality, we have the following result:

Theorem 2.4. *Assume that $F:\Omega\to\Omega$ is a minimal homeomorphism of a compact space of finite dimension.*[7] *Then for every $\mathcal{A}$ in a dense subset of* $\mathrm{Aut}(V,F)$*, there exist*:

- *a Riemannian norm $|||\cdot|||$ on V;*
- *a continuous $\mathcal{A}$-invariant splitting $V=V^1\oplus\cdots\oplus V^k$ which is orthogonal with respect to the Riemannian norm;*
- *and continuous functions $\lambda_1>\cdots>\lambda_k$ on Ω;*

such that

$$\big|\big|\big|A(\omega)v_i\big|\big|\big|_{F\omega}=e^{\lambda_i(\omega)}|||v_i|||_\omega,\quad \text{for all } \omega\in\Omega,\ i=1,\dots,k,\ v_i\in V^i_\omega.$$

As we will see, this has a similar proof as Theorem 2.3, basically replacing Corollary 2.1 by Corollary 2.2 and the result from [7] by the result from [5].

We expect that Theorems 2.3, 2.4 will be useful to answer the following question: *When can a linear cocycle over a uniquely ergodic or minimal base dynamics be approximated by a cocycle with a dominated* (*non-trivial*) *splitting*? Results on the 2-dimensional case were obtained in [2,3].

2.3. Proofs

Proof of Theorem 2.3. Assume that $F:\Omega\to\Omega$ has a unique invariant probability μ, and its support is Ω. Take any $\mathcal{A}\in\mathrm{Aut}(V,F)$; we will explain how to perturb it so that it has the desired properties. First, by [7], one can perturb $\mathcal{A}$ so that along μ-almost every orbit, the Oseledets splitting is trivial or dominated. Let

$$V^1_\omega\oplus\cdots\oplus V^k_\omega=V_\omega,\quad \omega\in\Omega,$$

be the *finest dominated splitting* of the cocycle, that is, the unique everywhere defined global dominated splitting with a maximal number k of bundles (with $k=1$ if there is no dominated splitting).[8]

We claim that for almost every point, there are exactly k different Lyapunov exponents. Indeed, on the one hand, there are at least k different exponents because there is a dominated splitting with k bundles. On the other hand, if there is a positive measure set of points with more than k different Lyapunov exponents, then select an orbit along which the Oseledets splitting is dominated. This orbit is dense on Ω (because the invariant measure has full support). Since dominated splittings extend to the closure (see [8]), one gets a global dominated splitting with more than k bundles; this is a contradiction.

For each $i=1,\dots,k$, let $\mathcal{A}_i$ be the restriction of $\mathcal{A}$ to the bundle V^i; this is a (continuous) vector bundle automorphism. By the claim above,

$$\lambda^+(\mathcal{A}_i)=\lambda^+(\mathcal{A}_i,\mu)=\lambda^-(\mathcal{A}_i,\mu)=\lambda^-(\mathcal{A}_i)=:\lambda_i.$$

Therefore, by Corollary 2.1, for each i there is a perturbation $\tilde{\mathcal{A}}_i$ of $\mathcal{A}_i$ and a Riemannian norm $|||\cdot|||_i$ on V^i such that

$$\big|\big|\big|\tilde{A}_i(\omega)v_i\big|\big|\big|_{i,F\omega}=e^{\lambda}|||v_i|||_{i,\omega},\quad \text{for all } \omega\in\Omega,\ v_i\in V^i_\omega.$$

Let $|||\cdot|||$ be the Riemannian norm that makes the subbundles orthogonal and that coincides with $|||\cdot|||_i$ on V^i. Let $\tilde{\mathcal{A}}$ be the automorphism of V whose restriction to the subbundles V^i are the automorphisms $\tilde{\mathcal{A}}_i$. This automorphism has the desired properties, thus completing the proof. □

For the proof of Theorem 2.4, we need the following result:

Theorem 2.5. *Assume that $F:\Omega\to\Omega$ is a minimal homeomorphism of a compact space of finite dimension. Then every $\mathcal{A}$ in a residual subset of* $\mathrm{Aut}(V,F)$ *has the following property*: *the Oseledets splitting with respect to any invariant probability measure coincides almost everywhere with the finest dominated splitting of $\mathcal{A}$.*

7 We say that Ω has *finite dimension* if it is homeomorphic to a subset of an euclidean space $\mathbb{R}^d$.

8 See [8] for details on finest dominated splittings.

This result is proved in full generality in [5]. (The case of $\mathrm{SL}(2,\mathbb{R})$-cocycles was previously considered in [1].) Notice that, as we have seen in the proof of Theorem 2.3 above, under the additional assumption of unique ergodicity, Theorem 2.5 follows from [7].

Proof of Theorem 2.4. Assume that $F : \Omega \to \Omega$ is minimal. Take any $\mathcal{A} \in \mathrm{Aut}(V, F)$; we will explain how to perturb it so that it has the desired properties. First, perturb $\mathcal{A}$ so that it has the property from Theorem 2.5. This means that if

$$V^1_\omega \oplus \cdots \oplus V^k_\omega = V_\omega \quad (\omega \in \Omega)$$

is the finest dominated splitting of the cocycle and $\mathcal{A}_i$ is the restriction of $\mathcal{A}$ to the bundle V^i, then

$$\lambda^+(\mathcal{A}_i, \mu) = \lambda^-(\mathcal{A}_i, \mu) \quad \text{for every ergodic probability } \mu \text{ for } F. \tag{2.3}$$

By [11], we can choose a Riemannian metric on V that is *adapted* to the dominated splitting, which means that

$$\inf_{\omega \in \Omega} \frac{\mathfrak{m}(A_i(\omega))}{\|A_{i+1}(\omega)\|} > 1, \quad \text{for every } i = 1, 2, \dots, k-1.$$

Let d_i be the fiber dimension of V^i, and let

$$\lambda_i(\omega) := \frac{1}{d_i} \log\left|\det A_i(\omega)\right|; \tag{2.4}$$

here determinants are computed with respect to the adapted metric, and in particular $\lambda_1 > \lambda_2 > \cdots > \lambda_d$ pointwise.[9]

For each i, property (2.3) permits us to apply Corollary 2.2 and find a perturbation $\tilde{\mathcal{A}}_i$ of $\mathcal{A}_i$ that is conformal with respect to some Riemannian norm $|||\cdot|||_i$ on V^i. Recalling formula (2.2) from the proof of Corollary 2.2, we see that $|||A(\omega)v|||_{i,F\omega} = e^{\lambda_i(\omega)} |||v|||_{i,\omega}$ where the function λ_i is given by (2.4).

Let $|||\cdot|||$ be the Riemannian norm that makes the subbundles orthogonal and that coincides with $|||\cdot|||_i$ on V^i. Let $\tilde{\mathcal{A}}$ be the automorphism of V whose restrictions to the subbundles V^i are the automorphisms $\tilde{\mathcal{A}}_i$. This automorphism has the desired properties, thus completing the proof. □

Acknowledgement

We thank the referee for several suggestions and corrections.

References

[1] A. Avila, J. Bochi, A uniform dichotomy for generic cocycles over a minimal base, Bull. Soc. Math. Fr. 135 (3) (2007) 407–417.
[2] A. Avila, J. Bochi, D. Damanik, Cantor spectrum for Schrödinger operators with potentials arising from generalized skew-shifts, Duke Math. J. 146 (2) (2009) 253–280.
[3] A. Avila, J. Bochi, D. Damanik, Opening gaps in the spectrum of strictly ergodic Schrödinger operators, J. Eur. Math. Soc. 14 (1) (2012) 61–106.
[4] R. Bhatia, Positive Definite Matrices, Princeton University Press, Princeton, 2007.
[5] J. Bochi, Generic linear cocycles over a minimal base, Studia Math. (2013), in press, arXiv:1302.5542, preprint.
[6] J. Bochi, A. Navas, A geometric path from zero Lyapunov exponents to rotation cocycles, Ergod. Theory Dyn. Syst. (2013), http://dx.doi.org/10.1017/etds.2013.58, in press, arXiv:1112.0397, preprint.
[7] J. Bochi, M. Viana, The Lyapunov exponents of generic volume preserving and symplectic maps, Ann. Math. 161 (3) (2005) 1423–1485.
[8] C. Bonatti, L.J. Díaz, M. Viana, Dynamics Beyond Uniform Hyperbolicity, Springer, 2005.
[9] D. Coronel, A. Navas, M. Ponce, On bounded cocycles of isometries over a minimal dynamics, J. Mod. Dyn. 7 (1) (2013) 45–77, http://dx.doi.org/10.3934/jmd.2013.7.45, arXiv:1101.3523, preprint.
[10] A. Furman, On the multiplicative ergodic theorem for uniquely ergodic systems, Ann. Inst. Henri Poincaré B, Probab. Stat. 33 (1997) 797–815.
[11] N. Gourmelon, Adapted metrics for dominated splittings, Ergod. Theory Dyn. Syst. 27 (6) (2007) 1839–1849.
[12] B. Kalinin, V. Sadovskaya, Linear cocycles over hyperbolic systems and criteria of conformality, J. Mod. Dyn. 4 (3) (2010) 419–441.
[13] A. Katok, B. Hasselbatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge Univ. Press, 1996.
[14] G.K. Pedersen, M. Takesaki, The operator equation $THT = K$, Proc. Am. Math. Soc. 36 (1) (1972) 311–312.
[15] A.N. Quas, Rigidity of continuous coboundaries, Bull. Lond. Math. Soc. 29 (1997) 595–600.
[16] S.J. Schreiber, On growth rates of subadditive functions for semiflows, J. Differ. Equ. 148 (1998) 334–350.
[17] R. Sturman, J. Stark, Semi-uniform ergodic theorems and applications to forced systems, Nonlinearity 13 (2000) 113–143.

[9] One can avoid using adapted metrics in the proof of Theorem 2.4 by using the following fact: if the functions $\lambda_1, \dots, \lambda_k$ satisfy $\int \lambda_1 \, d\mu > \cdots > \int \lambda_k \, d\mu$ for every ergodic probability μ, then there are functions $\hat{\lambda}_i$ cohomologous to the λ_i's such that $\hat{\lambda}_1 > \cdots > \hat{\lambda}_d$ pointwise.