



# Regularity of $p(\cdot)$ -superharmonic functions, the Kellogg property and semiregular boundary points

Tomasz Adamowicz<sup>a,b,\*</sup>, Anders Björn<sup>a</sup>, Jana Björn<sup>a</sup>

<sup>a</sup> Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden

<sup>b</sup> Institute of Mathematics, Polish Academy of Sciences, 00-956 Warsaw, Poland

Received 1 February 2013; accepted 31 July 2013

Available online 14 September 2013

## Abstract

We study various boundary and inner regularity questions for  $p(\cdot)$ -(super)harmonic functions in Euclidean domains. In particular, we prove the Kellogg property and introduce a classification of boundary points for  $p(\cdot)$ -harmonic functions into three disjoint classes: regular, semiregular and strongly irregular points. Regular and especially semiregular points are characterized in many ways. The discussion is illustrated by examples.

Along the way, we present a removability result for bounded  $p(\cdot)$ -harmonic functions and give some new characterizations of  $W_0^{1,p(\cdot)}$  spaces. We also show that  $p(\cdot)$ -superharmonic functions are lower semicontinuously regularized, and characterize them in terms of lower semicontinuously regularized supersolutions.

© 2013 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

MSC: primary 35J67; secondary 31C45, 46E35

Keywords: Comparison principle; Kellogg property; lsc-regularized; Nonlinear potential theory; Nonstandard growth equation; Obstacle problem;  $p(\cdot)$ -harmonic; Quasicontinuous; Regular boundary point; Removable singularity; Semiregular point; Sobolev space; Strongly irregular point;  $p(\cdot)$ -superharmonic;  $p(\cdot)$ -supersolution; Trichotomy; Variable exponent

## 1. Introduction

The theory of partial differential equations with nonstandard growth has been a subject of increasing interest in the last decade. Several results known for the model elliptic differential operator of nonlinear analysis, the  $p$ -Laplacian  $\Delta_p := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ , have been established in the variable exponent setting for the so-called  $p(\cdot)$ -Laplace equation and some of its modifications. The  $p(\cdot)$ -Laplace equation

$$\operatorname{div}(p(x)|\nabla u|^{p(x)-2} \nabla u) = 0$$

is the Euler–Lagrange equation for the minimization of the  $p(\cdot)$ -Dirichlet integral

\* Corresponding author.

E-mail addresses: [T.Adamowicz@impan.pl](mailto:T.Adamowicz@impan.pl) (T. Adamowicz), [anders.bjorn@liu.se](mailto:anders.bjorn@liu.se) (A. Björn), [jana.bjorn@liu.se](mailto:jana.bjorn@liu.se) (J. Björn).

$$\int_{\Omega} |\nabla u|^{p(x)} dx$$

among functions with given boundary data. Such minimization problems and equations arise for instance from applications in image processing, see Chen, Levine, and Rao [12], and in the description of electrorheological fluids, see Acerbi and Mingione [1] and Růžička [29].

Variable exponent equations have been studied, among others, in the context of interior regularity of solutions, see e.g. Acerbi and Mingione [2], Fan [14] and Henriques [23], and from the point of view of geometric properties of the solutions, see e.g. Adamowicz and Hästö [3,4]. Also, the nonlinear potential theory associated with variable exponent elliptic equations has recently attracted attention, see e.g. Harjulehto, Kinnunen, and Lukkari [19], Harjulehto, Hästö, Koskenoja, Lukkari, and Marola [16], Latvala, Lukkari, and Toivanen [25] and Lukkari [28]. For a survey of recent results in the field we refer to Harjulehto, Hästö, Lê, and Nuortio [18].

Despite the symbolic similarity to the  $p$ -Laplacian, various unexpected phenomena can occur when the exponent is a function, for instance the minimum of the  $p(\cdot)$ -Dirichlet energy may not exist even in the one-dimensional case for smooth  $p$ , see [18, Section 3], and smooth functions need not be dense in the corresponding variable exponent Sobolev spaces, see the monograph by Diening, Harjulehto, Hästö, and Růžička [13, Chapter 9.2].

In this paper we address several questions regarding boundary regularity of  $p(\cdot)$ -harmonic functions, i.e. the solutions of the  $p(\cdot)$ -Laplace equation. Our focus is on discussing various types of boundary points and on analyzing the structure of sets of such points.

A boundary point  $x_0 \in \partial\Omega$  is *regular* if

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) = f(x_0) \quad \text{for all } f \in C(\partial\Omega),$$

where  $\Omega$  is a nonempty bounded open subset of  $\mathbf{R}^n$  and  $Hf$  is the solution of the  $p(\cdot)$ -Dirichlet problem with boundary values  $f$ . (See later sections for notation and precise definitions.)

**Theorem 1.1** (*The Kellogg property*). *The set of all irregular boundary points has zero  $p(\cdot)$ -capacity.*

The Kellogg property for variable exponents was recently obtained by Latvala, Lukkari, and Toivanen [25] using balayage and the Wiener criterion (the latter being due to Alkhutov and Krasheninnikova [5, Theorem 1.1]). Here we provide a shorter and more elementary proof, which in particular does not depend on the Wiener criterion. It is based on the ideas introduced by Björn, Björn, and Shanmugalingam [11] for their proof of the Kellogg property in metric spaces (with constant  $p$ ). The proof in [11] is based on Newtonian-type Sobolev spaces, but here we have refrained from the Newtonian approach and only use the usual variable exponent Sobolev spaces. Our proof may therefore be of interest also in the constant  $p$  case, for readers who prefer to avoid Newtonian spaces.

That a boundary point is regular can be rephrased in the following way. A point  $x_0 \in \partial\Omega$  is regular if the following two conditions hold:

(a) for all  $f \in C(\partial\Omega)$  the limit

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) \quad \text{exists;} \tag{1.1}$$

(b) for all  $f \in C(\partial\Omega)$  there is a sequence  $\{y_j\}_{j=1}^{\infty}$  such that

$$\Omega \ni y_j \rightarrow x_0 \quad \text{and} \quad Hf(y_j) \rightarrow f(x_0), \quad \text{as } j \rightarrow \infty. \tag{1.2}$$

It turns out that for irregular boundary points *exactly one* of these two properties holds, i.e. it can never happen that both fail. This is the content of the following theorem. We say that  $x_0 \in \partial\Omega$  is *semiregular* if (a) holds but not (b), and *strongly irregular* if (b) holds but not (a).

**Theorem 1.2** (*Trichotomy*). *A boundary point  $x_0 \in \partial\Omega$  is either regular, semiregular or strongly irregular.*

The first example (for  $p = 2$ ) of an irregular boundary point was given by Zaremba [30] in 1911, in which he showed that the centre of a punctured disk is irregular. This is an example of a semiregular point. Shortly afterwards,

Lebesgue [26] presented his famous Lebesgue spine, whose tip is a strongly irregular point (see e.g. Remark 6.6.17 in Armitage and Gardiner [6]).

In the linear case the trichotomy was developed in detail in Lukeš and Malý [27] (in an axiomatic setting), whereas in the nonlinear constant  $p$  case it was first stated by A. Björn [7] who obtained it in metric spaces and also for quasiminimizers. As in [7], there are two main ingredients needed to obtain the trichotomy in the variable exponent case: the Kellogg property above and the following new removability result.

**Theorem 1.3.** *Let  $F \subset \Omega$  be relatively closed and such that  $C_{p(\cdot)}(F) = 0$ . If  $u$  is a bounded  $p(\cdot)$ -harmonic function in  $\Omega \setminus F$ , then it has a unique  $p(\cdot)$ -harmonic extension to  $\Omega$ .*

Here and in Theorem 1.4,  $\Omega$  is allowed to be unbounded.

The paper is organized as follows. In Section 2 we recall some of the basic definitions and theorems from the theory of variable exponent Sobolev spaces, as well as potential theory. We also observe that some of the characterizations of the  $p(\cdot)$ -Sobolev spaces with zero boundary data discussed in [13] can be improved, and these improvements turn out useful for our later results. We also discuss the “squeezing” Lemma 2.6 for variable exponent Sobolev spaces with zero boundary values, which to our best knowledge was not known or formulated in the literature so far.

In Section 3 we discuss  $p(\cdot)$ -supersolutions and the obstacle problem in the variable exponent setting. We discuss the existence and uniqueness of solutions to the obstacle problem and their regularized representatives. In addition, we obtain comparison principles for Dirichlet and obstacle problems, see Lemmas 3.10 and 3.11.

Section 4 is devoted to studying  $p(\cdot)$ -superharmonic functions. Although this notion is well known, also in the  $p(\cdot)$ -setting, we establish the following new characterization of bounded  $p(\cdot)$ -superharmonic functions.

**Theorem 1.4.** *Assume that  $u : \Omega \rightarrow \mathbf{R}$  is locally bounded from above in  $\Omega$ . Then  $u$  is  $p(\cdot)$ -superharmonic if and only if it is an lsc-regularized  $p(\cdot)$ -supersolution.*

For unbounded  $p(\cdot)$ -superharmonic functions we obtain a similar (but necessarily more involved) characterization in Theorem 4.4.

Section 5 is devoted to the Kellogg property, whereas the removability result (Theorem 1.3 above) is obtained in Section 6. In the latter section we also obtain a similar removability result for bounded  $p(\cdot)$ -superharmonic functions. Lukkari [28] studied removability for unbounded  $p(\cdot)$ -harmonic functions, our results are however not included in his treatment.

In Section 7 we obtain the trichotomy (Theorem 1.2) and also provide a number of characterizations of regular points. In the last section we focus on semiregularity and give several characterizations both of semiregular points themselves and of sets of semiregular points, involving capacity and  $p(\cdot)$ -harmonic and  $p(\cdot)$ -superharmonic extensions. In particular, we show that semiregularity is a local property. A similar result for regular points is a direct consequence of the Wiener criterion. It would be interesting to obtain the locality for regular (and thus also for strongly irregular) points more directly, without appealing to the Wiener criterion. Let us again stress the fact that we do not use the Wiener criterion in this paper, except for constructing a few examples in Example 8.6 and Propositions 8.7 and 8.8.

## 2. Preliminaries

A *variable exponent* is a measurable function  $p : \mathbf{R}^n \rightarrow [1, \infty]$ . In this paper we assume that

$$1 < p^- \leq p^+ < \infty, \quad \text{where} \quad p^- = \operatorname{ess\,inf}_{\mathbf{R}^n} p \quad \text{and} \quad p^+ = \operatorname{ess\,sup}_{\mathbf{R}^n} p,$$

and that  $p$  is log-Hölder continuous, i.e. there is a constant  $L > 0$  such that

$$|p(x) - p(y)| \leq \frac{L}{\log(e + 1/|x - y|)} \quad \text{for } x, y \in \mathbf{R}^n. \quad (2.1)$$

In addition, one usually assumes that  $p$  satisfies the log-Hölder decay condition (see Definition 4.1.1 and the discussion in Chapter 4.1 in Diening, Harjulehto, Hästö, and Růžička [13]). However, for the results in this paper no decay

condition is required. We also assume throughout the paper that  $\Omega \subset \mathbf{R}^n$  is a nonempty open set. (In Sections 5–8 as well as in the second half of Section 3 we will further assume that  $\Omega$  is bounded.) For background on variable exponent function spaces we refer to [13].

The *variable exponent Lebesgue space*  $L^{p(\cdot)}(\Omega)$  consists of all measurable functions  $u : \Omega \rightarrow \mathbf{R}$  for which the so-called Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

is finite. Equipped with this norm,  $L^{p(\cdot)}(\Omega)$  becomes a Banach space. The variable exponent Lebesgue space is a special case of a Musielak–Orlicz space. For a constant function  $p$ , it coincides with the standard Lebesgue space.

One of the difficulties when extending results from the constant to variable exponent setting is the lack of functional relationship between the norm and the integral. Nevertheless, we do have the following useful estimates

$$\left( \int_{\Omega} |u(x)|^{p(x)} dx \right)^{1/p^-} \leq \|u\|_{L^{p(\cdot)}(\Omega)} \leq \left( \int_{\Omega} |u(x)|^{p(x)} dx \right)^{1/p^+} \tag{2.2}$$

whenever  $\int_{\Omega} |u(x)|^{p(x)} dx \leq 1$ . For a proof and further discussion we refer to Lemmas 3.2.4 and 3.2.5 in Diening, Harjulehto, Hästö, and Růžička [13]. Note also, that if  $\{u_i\}_{i=1}^{\infty}$  is a sequence of  $L^{p(\cdot)}(\Omega)$ -integrable functions, then from (2.2) we infer that

$$\lim_{i \rightarrow \infty} \int_{\Omega} |u_i(x)|^{p(x)} dx = 0 \iff \lim_{i \rightarrow \infty} \|u_i\|_{L^{p(\cdot)}(\Omega)} = 0. \tag{2.3}$$

If  $p \geq q$  are variable exponents, then  $L^{p(\cdot)}_{\text{loc}}(\Omega)$  embeds into  $L^{q(\cdot)}_{\text{loc}}(\Omega)$ . In particular, every function in  $L^{p(\cdot)}_{\text{loc}}(\Omega)$  also belongs to  $L^{p^-}_{\text{loc}}(\Omega)$  (see Theorem 3.3.1 and the discussion in Section 3.3 in [13]). The Hölder inequality takes the form

$$\int_{\Omega} uv dx \leq 2 \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}, \tag{2.4}$$

where  $p'(\cdot)$  is the pointwise *conjugate exponent*, i.e.  $1/p(x) + 1/p'(x) \equiv 1$ , (see Lemma 3.2.20 in [13]).

The *variable exponent Sobolev space*  $W^{1,p(\cdot)}(\Omega)$  consists of all  $u \in L^{p(\cdot)}(\Omega)$  whose distributional gradient  $\nabla u$  also belongs to  $L^{p(\cdot)}(\Omega)$ . The space  $W^{1,p(\cdot)}(\Omega)$  is a Banach space with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In general, smooth functions are not dense in  $W^{1,p(\cdot)}(\mathbf{R}^n)$  but the log-Hölder condition (2.1) guarantees that they are, see Theorem 9.2.2 in [13] and the discussion following it. We refer to Chapter 9 in [13] for a detailed discussion of this topic.

**Definition 2.1.** The (Sobolev)  $p(\cdot)$ -capacity of a set  $E \subset \mathbf{R}^n$  is defined as

$$C_{p(\cdot)}(E) := \inf_u \int_{\mathbf{R}^n} (|u|^{p(x)} + |\nabla u|^{p(x)}) dx,$$

where the infimum is taken over all  $u \in W^{1,p(\cdot)}(\mathbf{R}^n)$  such that  $u \geq 1$  in a neighbourhood of  $E$ .

The  $p(\cdot)$ -capacity enjoys similar properties as in the constant case, see Theorem 10.1.2 in [13]. We say that a claim holds *quasi*everywhere (*q.e.* for short) if it holds everywhere except for a set with  $p(\cdot)$ -capacity zero.

**Definition 2.2.** A function  $u : \Omega \rightarrow [-\infty, \infty]$  is *quasicontinuous* if for every  $\varepsilon > 0$  there exists an open set  $U \subset \mathbf{R}^n$  with  $C_{p(\cdot)}(U) < \varepsilon$  such that  $u|_{\Omega \setminus U}$  is real-valued and continuous.

Since  $C_{p(\cdot)}$  is an outer capacity (which follows directly from the definition) it is easy to show that if  $u$  is quasicontinuous and  $v = u$  q.e., then  $v$  is also quasicontinuous.

The following lemma sheds more light on quasicontinuous functions. It was obtained by Kilpeläinen for general capacities satisfying two axioms, both of which are easily verified for the  $p(\cdot)$ -capacity.

**Lemma 2.3.** (See Kilpeläinen [24].) *If  $u$  and  $v$  are quasicontinuous in  $\Omega$  and  $u = v$  a.e. in  $\Omega$ , then  $u = v$  q.e. in  $\Omega$ .*

Following Definition 8.1.10 in Diening, Harjulehto, Hästö, and Růžička [13] we define the Sobolev space  $W_0^{1,p(\cdot)}(\Omega)$  with zero boundary values as the closure in  $W^{1,p(\cdot)}(\Omega)$  of  $W^{1,p(\cdot)}(\mathbf{R}^n)$ -functions with compact support in  $\Omega$ . By Proposition 11.2.3 in [13], this is equivalent to taking the closure of  $C_0^\infty(\Omega)$  in  $W^{1,p(\cdot)}(\Omega)$ .

In the rest of this section we give several useful characterizations of  $W_0^{1,p(\cdot)}(\Omega)$  which will be needed later and do not seem to be anywhere else in the literature. The following result improves upon Theorem 11.2.6 in [13], where the same conclusion is obtained if  $u \in W^{1,p(\cdot)}(\mathbf{R}^n)$  and  $u = 0$  q.e. in  $\mathbf{R}^n \setminus \Omega$ .

**Lemma 2.4.** *If  $u \in W^{1,p(\cdot)}(\Omega)$  is quasicontinuous in  $\mathbf{R}^n$  and  $u = 0$  q.e. on  $\partial\Omega$ , then  $u \in W_0^{1,p(\cdot)}(\Omega)$ .*

**Proof.** By definition, we need to show that  $u$  can be approximated in  $W^{1,p(\cdot)}(\Omega)$  by functions from  $W^{1,p(\cdot)}(\Omega)$  with compact support in  $\Omega$ . This can be done in a similar way as the proof of Theorem 11.2.6 in [13]. Let us recall the main points of the argument. By Lemma 9.1.1 in [13],  $u$  can without loss of generality be assumed to be bounded and nonnegative. Multiplying  $u$  by the Lipschitz functions  $\eta_j(x) := \min\{1, (j - |x|)_+\}$  for  $j = 0, 1, \dots$  and noting that  $\|u - u\eta_j\|_{W^{1,p(\cdot)}(\Omega)} \rightarrow 0$  as  $j \rightarrow \infty$ , we can also assume that  $u$  has bounded support.

Let  $\varepsilon > 0$ . By quasicontinuity and the fact that  $u = 0$  q.e. on  $\partial\Omega$ , there exists an open set  $G \subset \mathbf{R}^n$  such that  $C_{p(\cdot)}(G) < \varepsilon$ , the restriction of  $u$  to  $\mathbf{R}^n \setminus G$  is continuous and  $u = 0$  on  $\partial\Omega \setminus G$ . In particular, this implies that the set

$$V := \{x \in \mathbf{R}^n \setminus G : u(x) < \varepsilon\}$$

is relatively open in  $\mathbf{R}^n \setminus G$  and  $\partial\Omega \subset G \cup V$ . We can also find  $w_\varepsilon \in W^{1,p(\cdot)}(\mathbf{R}^n)$  such that  $w_\varepsilon = 1$  on  $G$ ,  $0 \leq w_\varepsilon \leq 1$  in  $\mathbf{R}^n$  and

$$\int_{\mathbf{R}^n} (|w_\varepsilon|^{p(x)} + |\nabla w_\varepsilon|^{p(x)}) dx < \varepsilon.$$

As  $G \cup V$  is open and contains  $\partial\Omega$ , it follows that the function  $u_\varepsilon := (1 - w_\varepsilon)(u - \varepsilon)_+ \chi_\Omega$  has compact support in  $\Omega$  and it is shown as in the proof of Theorem 11.2.6 in [13] that  $\|u - u_\varepsilon\|_{W^{1,p(\cdot)}(\Omega)} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , i.e.  $u \in W_0^{1,p(\cdot)}(\Omega)$ .  $\square$

**Proposition 2.5.** *Assume that  $u$  is quasicontinuous in  $\Omega$ . Then  $u \in W_0^{1,p(\cdot)}(\Omega)$  if and only if*

$$\tilde{u} := \begin{cases} u & \text{in } \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

*is quasicontinuous and belongs to  $W^{1,p(\cdot)}(\mathbf{R}^n)$ .*

**Proof.** Assume first that  $u \in W_0^{1,p(\cdot)}(\Omega)$ . By Corollary 11.2.5 in [13], there is a quasicontinuous function  $v \in W^{1,p(\cdot)}(\mathbf{R}^n)$  such that  $v = u$  a.e. in  $\Omega$  and  $v = 0$  q.e. outside  $\Omega$ . By Lemma 2.3,  $v = u$  q.e. in  $\Omega$ , and thus  $\tilde{u} = v$  q.e. in  $\mathbf{R}^n$ . Hence  $\tilde{u} \in W^{1,p(\cdot)}(\mathbf{R}^n)$  and  $\tilde{u}$  is quasicontinuous. The converse follows directly from Lemma 2.4.  $\square$

The following “squeezing lemma” is useful when proving that certain functions belong to  $W_0^{1,p(\cdot)}(\Omega)$ .

**Lemma 2.6.** *Let  $u \in W^{1,p(\cdot)}(\Omega)$  and  $u_1, u_2 \in W_0^{1,p(\cdot)}(\Omega)$  be such that  $u_1 \leq u \leq u_2$  a.e. in  $\Omega$ . Then  $u \in W_0^{1,p(\cdot)}(\Omega)$ .*

We let  $B(x, r)$  be the open ball with centre  $x$  and radius  $r$ .

**Proof.** Replacing each function  $v = u_1, u_2, u$  by its quasicontinuous representative

$$\limsup_{r \rightarrow 0} \int_{B(x,r)} v \, dx$$

(provided by Theorem 11.4.4 in Diening, Harjulehto, Hästö, and Růžička [13]), we can assume that  $u_1, u_2$  and  $u$  are quasicontinuous in  $\Omega$  and  $u_1 \leq u \leq u_2$  everywhere in  $\Omega$ .

To be able to apply Lemma 2.4, we need to show that the zero extension of  $u$  to  $\mathbf{R}^n \setminus \Omega$  is quasicontinuous in  $\mathbf{R}^n$ . To this end, Proposition 2.5 implies that both  $u_1$  and  $u_2$  can be extended by zero outside  $\Omega$  to obtain quasicontinuous functions on  $\mathbf{R}^n$ . In other words, given  $\varepsilon > 0$ , there exists an open set  $G$  with  $C_{p(\cdot)}(G) < \varepsilon$  such that the restrictions  $u_1|_{\mathbf{R}^n \setminus G}$  and  $u_2|_{\mathbf{R}^n \setminus G}$  are continuous. Since  $u|_{\mathbf{R}^n \setminus G}$  lies between  $u_1|_{\mathbf{R}^n \setminus G}$  and  $u_2|_{\mathbf{R}^n \setminus G}$ , and  $u = 0$  on  $\partial\Omega$ , we conclude that  $u|_{\mathbf{R}^n \setminus G}$  is continuous at all  $x \in \partial\Omega \setminus G$ . It is clearly continuous in  $\mathbf{R}^n \setminus \overline{\Omega}$  and quasicontinuous in  $\Omega$ . Thus,  $u$  is quasicontinuous in  $\mathbf{R}^n$ . Lemma 2.4 then shows that  $u \in W_0^{1,p(\cdot)}(\Omega)$ .  $\square$

### 3. Supersolutions and obstacle problems

In this section we include several auxiliary results about supersolutions and obstacle problems. In particular, we discuss relations between these two notions, existence and uniqueness of the solutions, their interior regularity and a comparison principle. We shall consider the following type of obstacle problem.

**Definition 3.1.** Let  $f \in W^{1,p(\cdot)}(\Omega)$  and  $\psi : \Omega \rightarrow [-\infty, \infty]$ . Then we define

$$\mathcal{K}_{\psi,f} = \{v \in W^{1,p(\cdot)}(\Omega) : v - f \in W_0^{1,p(\cdot)}(\Omega) \text{ and } v \geq \psi \text{ a.e. in } \Omega\}.$$

A function  $u \in \mathcal{K}_{\psi,f}$  is a *solution of the  $\mathcal{K}_{\psi,f}$ -obstacle problem* if

$$\int_{\Omega} |\nabla u|^{p(x)} \, dx \leq \int_{\Omega} |\nabla v|^{p(x)} \, dx \quad \text{for all } v \in \mathcal{K}_{\psi,f}.$$

The following equivalent definition of obstacle problems is given in Harjulehto, Hästö, Koskenoja, Lukkari, and Marola [16], p. 3427. The result in [16] is obtained for a bounded  $\Omega$ , but the proof is valid also for unbounded sets. In this paper, however, we will need it only for bounded sets.

**Proposition 3.2.** *The function  $u$  is a solution of the  $\mathcal{K}_{\psi,f}$ -obstacle problem if and only if*

$$\int_{\Omega} p(x) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla(v - u) \, dx \geq 0 \quad \text{for all } v \in \mathcal{K}_{\psi,f}.$$

**Definition 3.3.** A function  $u \in W_{\text{loc}}^{1,p(\cdot)}(\Omega)$  is a *(super)solution of the  $p(\cdot)$ -Laplace equation* if

$$\int_{\varphi \neq 0} |\nabla u|^{p(x)} \, dx \leq \int_{\varphi \neq 0} |\nabla(u + \varphi)|^{p(x)} \, dx$$

for all (nonnegative)  $\varphi \in C_0^\infty(\Omega)$ . A  *$p(\cdot)$ -harmonic function* is a continuous solution.

Clearly,  $u$  is a solution if and only if it is both a supersolution and a subsolution (i.e.  $-u$  is a supersolution). It is also immediate that a solution of an obstacle problem is a supersolution. Conversely, if  $u$  is a supersolution in  $\Omega$  and  $\Omega' \Subset \Omega$  is open then by the density of  $C_0^\infty(\Omega')$  in  $W_0^{1,p(\cdot)}(\Omega')$  we see that  $u$  is a solution of the obstacle problem in  $\Omega'$  with  $u$  as the obstacle and the boundary values. (Recall that  $A \Subset \Omega$  if the closure of  $A$  is a compact subset of  $\Omega$ .) The following characterization of (super)solutions then follows from Proposition 3.2, cf. Harjulehto, Hästö, Koskenoja, Lukkari, and Marola [16], p. 3427. By the density of  $C_0^\infty(\Omega)$  again, it is equivalent to require that (3.1) holds for all (nonnegative)  $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ .

**Proposition 3.4.** A function  $u \in W_{\text{loc}}^{1,p(\cdot)}(\Omega)$  is a (super)solution if and only if

$$\int_{\Omega} p(x)|\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx \geq 0 \tag{3.1}$$

for all (nonnegative)  $\varphi \in C_0^\infty(\Omega)$ .

For a function  $u : \Omega \rightarrow \mathbf{R}$ , let

$$u^*(x) = \text{ess lim inf}_{y \rightarrow x} u(y), \quad x \in \Omega.$$

It is easy to see that  $u^*$  is always lower semicontinuous, see the proof of Theorem 8.22 in Björn and Björn [9]. We call  $u^*$  the *lsc-regularization* of  $u$ , and also say that  $u$  is *lsc-regularized* if  $u = u^*$ .

**Theorem 3.5.** Assume that  $u$  is a supersolution in  $\Omega$ . Then  $u^*$  is a quasicontinuous supersolution in  $\Omega$  and  $u^* = u$  a.e. in  $\Omega$ . Moreover, if  $u$  is quasicontinuous, then  $u^* = u$  q.e. in  $\Omega$ .

**Proof.** By Theorem 4.1 (and Remark 4.2) in Harjulehto, Kinnunen, and Lukkari [19],  $u^* = u$  a.e., and thus also  $u^*$  is a supersolution. By Theorem 6.1 in Harjulehto, Hästö, Koskenoja, Lukkari, and Marola [16],  $u^*$  is superharmonic (see Section 4 below for the definition of superharmonic functions). It then follows from Theorem 6.7 in Harjulehto and Latvala [20], that  $u^*$  is quasicontinuous.

Moreover, if  $u$  is quasicontinuous, then  $u^* = u$  q.e. in  $\Omega$ , by Lemma 2.3.  $\square$

In the rest of this section we assume that  $\Omega$  is a bounded nonempty open set.

**Theorem 3.6.** If  $\psi$  is bounded from above,  $f$  is bounded, and  $\mathcal{K}_{\psi,f} \neq \emptyset$ , then there exists a solution  $u$  of the  $\mathcal{K}_{\psi,f}$ -obstacle problem, and the solution is unique up to sets of measure zero. Moreover,  $u^*$  is the unique lsc-regularized solution, and  $u^*$  is bounded.

**Proof.** The existence is proved as in Appendix I in Heinonen, Kilpeläinen, and Martio [22], namely by showing the monotonicity, coercivity and weak continuity for the operator

$$\mathcal{L}_p(\cdot) : \{ \nabla v : v \in \mathcal{K}_{\psi,f} \} \rightarrow L^{p'(\cdot)}(\Omega, \mathbf{R}^n), \quad \text{where } 1/p(x) + 1/p'(x) \equiv 1,$$

defined by

$$\langle \mathcal{L}_p(\cdot) \mathbf{v}, \mathbf{u} \rangle := \int_{\Omega} p(x)|\mathbf{v}(x)|^{p(x)-2} \mathbf{v}(x) \cdot \mathbf{u}(x) \, dx.$$

These properties are for the variable exponent verified in the same way as in the constant exponent case, cf. Appendix I in [22] and p. 3427 in Harjulehto, Hästö, Koskenoja, Lukkari, and Marola [16].

The uniqueness follows from Theorem 3.2 in [16]. Indeed, if  $u$  and  $v$  are solutions of the obstacle problem, then both are supersolutions and  $\min\{u, v\} \in \mathcal{K}_{\psi,f}$ . Theorem 3.2 in [16] then implies that  $u \leq v$  and  $v \leq u$  a.e.

As for the last part,  $u^* = u$  a.e. by Theorem 3.5, and thus  $u^*$  is also a solution of the  $\mathcal{K}_{\psi,f}$ -obstacle problem. Since  $u^*$  is independent of which solution  $u$  we choose of the  $\mathcal{K}_{\psi,f}$ -obstacle problem, we see that it is the unique lsc-regularized solution.

Let  $M = \max\{\sup|f|, \sup\psi\}$ . Then the truncation  $v := \max\{\min\{u, M\}, -M\}$  of  $u$  at  $\pm M$  is also a solution, and by the uniqueness we see that  $|u^*| \leq M$ .  $\square$

**Theorem 3.7.** Assume that  $\psi : \Omega \rightarrow [-\infty, \infty)$  is continuous (as an extended real-valued function) and bounded from above, that  $f$  is bounded, and that  $\mathcal{K}_{\psi,f} \neq \emptyset$ . Then the lsc-regularized solution of the  $\mathcal{K}_{\psi,f}$ -obstacle problem is continuous.

**Proof.** See Theorem 4.11 in [16].  $\square$

**Remark 3.8.** A direct consequence is that if  $u$  is a locally bounded solution, in the sense of Definition 3.3, then  $u^*$  is continuous. Indeed, if  $u$  is a solution, then it is locally a solution of an unrestricted obstacle problem with itself as boundary values. Hence  $u^*$  is locally continuous, i.e. continuous.

**Definition 3.9.** Let  $f \in W^{1,p(\cdot)}(\Omega)$  be bounded. Then we define the Sobolev solution  $Hf$  of the Dirichlet problem with boundary values  $f$  to be the continuous solution of the  $\mathcal{K}_{-\infty, f}$ -obstacle problem.

Note that  $Hf$  depends also on  $p(\cdot)$ . Since  $u = Hf$  is a solution of the unrestricted obstacle problem, i.e. with obstacle  $-\infty$ , it follows that

$$\int_{\Omega} |\nabla u|^{p(x)} dx \leq \int_{\Omega} |\nabla(u + \varphi)|^{p(x)} dx \tag{3.2}$$

for all  $\varphi \in W_0^{1,p(\cdot)}(\Omega)$  and in particular for all  $\varphi \in C^\infty(\Omega)$ . Subtracting

$$\int_A |\nabla u|^{p(x)} dx = \int_A |\nabla(u + \varphi)|^{p(x)} dx < \infty,$$

where  $A = \{x \in \Omega: \varphi(x) = 0\}$ , from both sides of (3.2) shows that  $u$  is a continuous solution in the sense of Definition 3.3, i.e. a  $p(\cdot)$ -harmonic function.

The following comparison principle will be important for us.

**Lemma 3.10 (Comparison principle).** *If  $f_1, f_2 \in W^{1,p(\cdot)}(\Omega)$  are bounded and  $(f_1 - f_2)_+ \in W_0^{1,p(\cdot)}(\Omega)$ , then  $Hf_1 \leq Hf_2$  in  $\Omega$ .*

It follows that if  $f_1, f_2 \in \text{Lip}(\overline{\Omega})$  and  $f_1 = f_2$  on  $\partial\Omega$ , then  $Hf_1 = Hf_2$ . We can therefore define  $Hf$  for  $f \in \text{Lip}(\partial\Omega)$  to be  $H\tilde{f}$  for any extension  $\tilde{f} \in \text{Lip}(\overline{\Omega})$  such that  $\tilde{f} = f$  on  $\partial\Omega$ . Among such extensions are the so-called McShane extensions, see e.g. Theorem 6.2 in Heinonen [21].

In view of Lemma 2.4,  $(f_1 - f_2)_+ \in W_0^{1,p(\cdot)}(\Omega)$  whenever  $f_1, f_2 \in W^{1,p(\cdot)}(\Omega)$  are quasicontinuous in  $\mathbf{R}^n$  and  $f_1 \leq f_2$  q.e. on  $\partial\Omega$ .

The following generalization of the comparison principle above is sometimes useful. Even though we will not use it in this paper, we have chosen to include it here since the proof of it is not more involved than a direct proof of Lemma 3.10.

**Lemma 3.11 (Comparison principle for obstacle problems).** *Let  $\psi_j : \Omega \rightarrow [-\infty, \infty)$  be bounded from above and  $f_j \in W^{1,p(\cdot)}(\Omega)$  be bounded and such that  $\mathcal{K}_{\psi_j, f_j} \neq \emptyset$ . Let further  $u_j$  be a solution of the  $\mathcal{K}_{\psi_j, f_j}$ -obstacle problem,  $j = 1, 2$ . If  $\psi_1 \leq \psi_2$  a.e. in  $\Omega$  and  $(f_1 - f_2)_+ \in W_0^{1,p(\cdot)}(\Omega)$ , then  $u_1 \leq u_2$  a.e. in  $\Omega$ .*

Moreover, the lsc-regularizations satisfy  $u_1^* \leq u_2^*$  everywhere in  $\Omega$ .

**Proof of Lemma 3.10.** Let  $\psi_1 = \psi_2 \equiv -\infty$ . After noting that  $u_1^* = Hf_1$  and  $u_2^* = Hf_2$  the result follows from (the last part of) Lemma 3.11.  $\square$

**Proof of Lemma 3.11.** Let  $u = \min\{u_1, u_2\}$ . Then

$$\begin{aligned} W_0^{1,p(\cdot)}(\Omega) \ni u_1 - f_1 &\geq u - f_1 = \min\{u_1 - f_1, u_2 - f_1\} \\ &\geq \min\{u_1 - f_1, u_2 - f_2 - (f_1 - f_2)_+\} \in W_0^{1,p(\cdot)}(\Omega). \end{aligned}$$

Lemma 2.6 implies that  $u - f_1 \in W_0^{1,p(\cdot)}(\Omega)$ . As  $u \geq \psi_1$  a.e. in  $\Omega$ , we get that  $u \in \mathcal{K}_{\psi_1, f_1}$ . Similarly  $v = \max\{u_1, u_2\} \in \mathcal{K}_{\psi_2, f_2}$ .

Let  $A = \{x \in \Omega: u_1(x) > u_2(x)\}$ . Since  $u_2$  is a solution of the  $\mathcal{K}_{\psi_2, f_2}$ -obstacle problem, we have that

$$\int_{\Omega} |\nabla u_2(x)|^{p(x)} dx \leq \int_{\Omega} |\nabla v(x)|^{p(x)} dx = \int_A |\nabla u_1(x)|^{p(x)} dx + \int_{\Omega \setminus A} |\nabla u_2(x)|^{p(x)} dx.$$



Thus

$$\int_A |\nabla u_2(x)|^{p(x)} dx \leq \int_A |\nabla u_1(x)|^{p(x)} dx.$$

It follows that

$$\int_{\Omega} |\nabla u(x)|^{p(x)} dx = \int_A |\nabla u_2(x)|^{p(x)} dx + \int_{\Omega \setminus A} |\nabla u_1(x)|^{p(x)} dx \leq \int_{\Omega} |\nabla u_1(x)|^{p(x)} dx.$$

As  $u_1$  is a solution of the  $\mathcal{K}_{\psi_1, f_1}$ -obstacle problem, so is  $u$ . By the uniqueness in [Theorem 3.6](#), we have

$$u_1 = u = \min\{u_1, u_2\} \quad \text{a.e. in } \Omega,$$

and thus  $u_1 \leq u_2$  a.e. in  $\Omega$ .

The pointwise comparison of the lsc-regularizations follows directly from their definitions and the above a.e.-inequality.  $\square$

#### 4. Superharmonic functions

In this section we consider superharmonic functions and show that they are lsc-regularized. This in turn leads to the characterization of bounded superharmonic functions advertised in [Theorem 1.4](#), and to another characterization of general superharmonic functions. These superharmonic functions are often called  $p(\cdot)$ -superharmonic, but to simplify the terminology we have here refrained from making the dependence on  $p(\cdot)$  explicit.

**Definition 4.1.** A function  $u : \Omega \rightarrow (-\infty, \infty]$  is *superharmonic* in  $\Omega$  if

- (i)  $u$  is lower semicontinuous;
- (ii)  $u$  is finite almost everywhere;
- (iii) for every nonempty open set  $\Omega' \Subset \Omega$  and all functions  $v \in C(\overline{\Omega}')$  which are  $p(\cdot)$ -harmonic in  $\Omega'$  and satisfy  $v \leq u$  on  $\partial\Omega'$ , it is true that  $v \leq u$  in  $\Omega'$ .

A function  $u : \Omega \rightarrow [-\infty, \infty)$  is *subharmonic* if  $-u$  is superharmonic.

In the variable exponent literature superharmonic functions are often assumed to belong to  $L^t_{\text{loc}}(\Omega)$  for some  $t > 0$ , see e.g. Latvala, Lukkari, and Toivanen [\[25\]](#). For our purposes the more general definition above is sufficient. In the constant  $p$  case condition (ii) is usually replaced by the equivalent condition

- (ii')  $u \not\equiv \infty$  in every component of  $\Omega$ .

Whether this equivalence is true also for variable exponents is not known. However, for the results in this paper we could as well have replaced (ii) by (ii') and required that  $u$  in [Theorem 4.4](#) satisfies (ii').

The following lemma is well known and easily proved directly from the definition.

**Lemma 4.2.** *If  $u$  and  $v$  are superharmonic, then so is  $\min\{u, v\}$ .*

The following result is well known for constant  $p$ , but seems to be new in the variable exponent setting.

**Theorem 4.3.** *If a function is superharmonic, then it is lsc-regularized.*

**Proof.** Let  $u$  be a superharmonic function and  $x_0 \in \Omega$  be arbitrary. Since  $u$  is lower semicontinuous,

$$u(x_0) \leq \liminf_{y \rightarrow x_0} u(y) \leq \text{ess lim inf}_{y \rightarrow x_0} u(y) =: u^*(x_0).$$

In order to obtain the converse inequality we assume first that  $u$  is bounded from above. Without loss of generality we can assume that  $u(x_0) > 0$ . Let  $0 < \delta \leq u(x_0)$  be arbitrary. By the lower semicontinuity of  $u$ , we can find a ball  $B \ni x_0$  such that  $2B \subseteq \Omega$  and  $u \geq u(x_0) - \delta$  in  $2B$ . Then  $v = u - (u(x_0) - \delta)$  is a bounded nonnegative superharmonic function in  $2B$ .

Theorem 6.5 in Harjulehto, Hästö, Koskenoja, Lukkari, and Marola [16] provides us with an increasing sequence of continuous supersolutions  $v_j$  in  $B$  such that  $v_j \nearrow v$  everywhere in  $B$ . Theorem 3.7 and Remark 3.8 in Harjulehto, Kinnunen, and Lukkari [19] imply the following weak Harnack inequality for sufficiently small  $R > 0$  and some  $q > 0$ ,

$$\int_{B(x_0, 2R)} v_j^q dx \leq C \left( \operatorname{ess\,inf}_{B(x_0, R)} v_j + R \right)^q, \tag{4.1}$$

where the constants  $q$  and  $C$  depend on the bound for  $v$ , but not on  $R$ . Indeed, the proof of Lemma 3.6 in [19] reveals that for a bounded  $v$ , the  $L^s(B(x_0, 4R))$ -norm in (3.33) in [19] can be substituted by the  $L^\infty(\Omega)$ -norm, which gives the independence of  $q$  on  $R$ . We can clearly assume that  $q < 1$ . Since  $v_j$  is continuous, the right-hand side in (4.1) is majorized by

$$C(v_j(x_0) + R)^q \leq C(v(x_0) + R)^q = C(\delta + R)^q \leq C(\delta^q + R^q).$$

Inserting this into (4.1) and letting  $j \rightarrow \infty$  gives

$$C(\delta^q + R^q) \geq \int_{B(x_0, 2R)} (u - (u(x_0) - \delta))^q dx \geq \int_{B(x_0, 2R)} u^q dx - (u(x_0) - \delta)^q.$$

Hence

$$(u(x_0) - \delta)^q + C\delta^q \geq \int_{B(x_0, 2R)} u^q dx - CR^q \geq \left( \operatorname{ess\,inf}_{B(x_0, 2R)} u \right)^q - CR^q \rightarrow u^*(x_0)^q,$$

as  $R \rightarrow 0$ . Since  $\delta$  was arbitrary, we conclude that  $u(x_0) \geq u^*(x_0)$  if  $u$  is bounded from above.

Let us now consider the case when  $u$  is unbounded. Let  $a < u^*(x_0)$  be real. Then  $u_a := \min\{u, a\}$  is superharmonic, by Lemma 4.2, and thus  $u_a$  is lsc-regularized by the first part of the proof. Hence

$$u(x_0) \geq u_a(x_0) = \operatorname{ess\,lim\,inf}_{y \rightarrow x_0} u_a(y) = \min \left\{ a, \operatorname{ess\,lim\,inf}_{y \rightarrow x_0} u(y) \right\} = \min \{ a, u^*(x_0) \} = a.$$

As  $a$  was arbitrary we see that  $u(x_0) \geq u^*(x_0)$ .  $\square$

We are now ready to obtain the characterization of superharmonic functions in Theorem 1.4, i.e. that a function locally bounded from above is superharmonic if and only if it is an lsc-regularized supersolution.

**Proof of Theorem 1.4.** Assume first that  $u$  is superharmonic. Then  $u$  is lsc-regularized by Theorem 4.3. That  $u$  is locally bounded from below follows directly from the lower semicontinuity (and the fact that  $u$  does not take the value  $-\infty$ ). Hence  $u$  is locally bounded and Corollary 6.6 in Harjulehto, Hästö, Koskenoja, Lukkari, and Marola [16] shows that  $u$  is a supersolution.

The converse follows directly from Theorem 6.1 in [16].  $\square$

For unbounded functions the characterization is (necessarily) a bit more involved.

**Theorem 4.4.** *Let  $u : \Omega \rightarrow (-\infty, \infty]$  be a function which is finite a.e. Then the following are equivalent:*

- (a)  $u$  is superharmonic in  $\Omega$ ;
- (b)  $\min\{u, k\}$  is superharmonic in  $\Omega$  for all  $k = 1, 2, \dots$ ;
- (c)  $u$  is lsc-regularized, and  $\min\{u, k\}$  is a supersolution in  $\Omega$  for all  $k = 1, 2, \dots$ ;
- (d)  $\min\{u, k\}$  is an lsc-regularized supersolution in  $\Omega$  for all  $k = 1, 2, \dots$ .

**Proof.** (a)  $\Rightarrow$  (b) This follows from Lemma 4.2.

(b)  $\Rightarrow$  (a) That  $u$  is lower semicontinuous follows directly from the fact that  $\min\{u, k\}$ ,  $k = 1, 2, \dots$ , are lower semicontinuous. Let next  $\Omega' \Subset \Omega$  be a nonempty open set and  $v \in C(\overline{\Omega}')$  be  $p(\cdot)$ -harmonic in  $\Omega'$  satisfying  $v \leq u$  on  $\partial\Omega'$ . Let  $m = \sup_{\overline{\Omega}'} v < \infty$  and let  $k > m$  be a positive integer. Then  $v \leq \min\{u, k\}$  on  $\partial\Omega'$ . Since  $\min\{u, k\}$  is superharmonic it follows that  $v \leq \min\{u, k\} \leq u$  in  $\Omega'$ . Thus  $u$  is superharmonic.

(b)  $\Leftrightarrow$  (d) This follows from Theorem 1.4.

(a)  $\Rightarrow$  (c) That  $u$  is lsc-regularized follows from Theorem 4.3. That  $\min\{u, k\}$  is a supersolution follows from the already shown implication (a)  $\Rightarrow$  (d).

(c)  $\Rightarrow$  (d) It is enough to show that  $\min\{u, k\}$  is lsc-regularized, but this follows directly from the fact that  $u$  is lsc-regularized.  $\square$

### 5. The Kellogg property

From now on we assume that  $\Omega$  is a bounded nonempty open set.

In this section we extend the definition of Sobolev solutions of the Dirichlet problem (Definition 3.9) to continuous boundary data and show that the solutions are  $p(\cdot)$ -harmonic. We also introduce regular and irregular boundary points and prove the Kellogg property.

**Definition 5.1.** Given  $f \in C(\partial\Omega)$ , define  $Hf : \Omega \rightarrow \mathbf{R}$  by

$$Hf(x) = \sup_{\text{Lip}(\partial\Omega) \ni \varphi \leq f} H\varphi(x), \quad x \in \Omega.$$

Here we abuse notation, since if  $f \in W^{1,p(\cdot)}(\Omega)$ , then  $Hf$  has already been defined by Definition 3.9. However, as continuous functions can be uniformly approximated by Lipschitz functions, the comparison principle (Lemma 3.10), together with the fact that  $H(f + a) = Hf + a$  for  $a \in \mathbf{R}$ , shows that the two definitions of  $Hf$  coincide in this case.

The comparison principle (Lemma 3.10) extends immediately to functions in  $C(\partial\Omega)$  in the following way.

**Lemma 5.2 (Comparison principle).** If  $f_1, f_2 \in C(\partial\Omega)$  and  $f_1 \leq f_2$  q.e. on  $\partial\Omega$ , then  $Hf_1 \leq Hf_2$  in  $\Omega$ .

Let us next show that  $Hf$  is indeed  $p(\cdot)$ -harmonic even for  $f \in C(\partial\Omega)$ .

**Lemma 5.3.** Let  $f \in C(\partial\Omega)$ . Then  $Hf$  is  $p(\cdot)$ -harmonic in  $\Omega$  and

$$Hf(x) = \inf_{\text{Lip}(\partial\Omega) \ni \varphi \geq f} H\varphi(x) = \lim_{j \rightarrow \infty} Hf_j(x), \quad x \in \Omega,$$

for every sequence  $\{f_j\}_{j=1}^\infty$  of functions in  $\text{Lip}(\partial\Omega)$  converging uniformly to  $f$ .

**Proof.** Let  $f_j \in \text{Lip}(\partial\Omega)$  be such that  $\sup_{\partial\Omega} |f - f_j| < 1/j$ ,  $j = 1, 2, \dots$ . Then  $\sup_{\partial\Omega} |f_{j'} - f_{j''}| \leq 2/j$  whenever  $j', j'' \geq j$ , and the comparison principle implies that for all  $x \in \Omega$ ,

$$Hf_{j'}(x) - \frac{2}{j} \leq Hf_{j''}(x) \leq Hf_{j'}(x) + \frac{2}{j},$$

i.e. the sequence  $\{Hf_j(x)\}_{j=1}^\infty$  is a Cauchy sequence. Hence, the limit  $h(x) := \lim_{j \rightarrow \infty} Hf_j(x)$  exists, and is a  $p(\cdot)$ -harmonic function in  $\Omega$ , by the uniform convergence result in Corollary 5.3 in Harjulehto, Hästö, Koskenoja, Lukkari, and Marola [16]. Using the comparison principle again, it follows that

$$\begin{aligned} h(x) &= \lim_{j \rightarrow \infty} H(f_j - 1/j)(x) \leq \sup_{\text{Lip}(\partial\Omega) \ni \varphi \leq f} H\varphi(x) \\ &\leq \inf_{\text{Lip}(\partial\Omega) \ni \varphi \geq f} H\varphi(x) \leq \lim_{j \rightarrow \infty} H(f_j + 1/j)(x) = h(x). \quad \square \end{aligned}$$

**Definition 5.4.** Let  $x_0 \in \partial\Omega$ . Then  $x_0$  is *regular* if

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) = f(x_0) \quad \text{for all } f \in C(\partial\Omega).$$

We also say that  $x_0$  is *irregular* if it is not regular.

See [Theorem 7.1](#) below for characterizations of regular boundary points.

Next we establish the Kellogg property ([Theorem 1.1](#)), which says that q.e. boundary point is regular. The proof is based on the following pasting lemma, which may be of independent interest.

**Lemma 5.5.** Let  $x \in \partial\Omega$  and  $B = B(x, r)$ . Let  $f \in \text{Lip}(\partial\Omega)$  be such that  $f = M$  on  $B \cap \partial\Omega$ , where  $M := \sup_{\partial\Omega} f$ . Let further

$$u = \begin{cases} Hf & \text{in } \Omega, \\ M & \text{in } B \setminus \Omega. \end{cases}$$

Then  $u$  is a quasicontinuous supersolution in  $B$ .

**Proof.** Extend  $f$  to a Lipschitz function on  $\overline{\Omega}$  and let  $f = M$  on  $B \setminus \overline{\Omega}$ . Then  $f \in \text{Lip}(B) \subset W^{1,p(\cdot)}(B)$ . Let

$$v = \begin{cases} u - f & \text{in } B \cup \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $v = 0$  in  $\mathbf{R}^n \setminus \Omega$  and  $v = Hf - f \in W_0^{1,p(\cdot)}(\Omega)$ . As  $v$  is continuous in  $\Omega$ , [Proposition 2.5](#) shows that  $v \in W^{1,p(\cdot)}(B)$  and that  $v$  is quasicontinuous. Thus  $u \in W^{1,p(\cdot)}(B)$  and  $u$  is quasicontinuous in  $B$ . By the comparison principle ([Lemma 5.2](#)),  $u \leq M$  in  $B$ .

To show that  $u$  is a supersolution in  $B$ , let  $\varphi \in C_0^\infty(B)$  be nonnegative. We shall prove the inequality

$$\int_{\varphi \neq 0} |\nabla u|^{p(x)} dx \leq \int_{\varphi \neq 0} |\nabla(u + \varphi)|^{p(x)} dx.$$

Let  $\varphi' := \min\{\varphi, M - u\} \in W_0^{1,p(\cdot)}(B)$ , which is quasicontinuous and nonnegative in  $B$ . Then  $\varphi' = 0$  in  $B \setminus \Omega$  and hence  $\varphi' \in W_0^{1,p(\cdot)}(B \cap \Omega)$ , by [Proposition 2.5](#). Since  $u$  is  $p(\cdot)$ -harmonic in  $B \cap \Omega$ , we have that

$$\int_{\varphi' \neq 0} |\nabla u|^{p(x)} dx \leq \int_{\varphi' \neq 0} |\nabla(u + \varphi')|^{p(x)} dx.$$

Note that  $\varphi' = 0 \neq \varphi$  if and only if  $u = M$ , in which case  $\nabla u = 0$  a.e. Thus

$$\int_{\varphi \neq 0} |\nabla u|^{p(x)} dx = \int_{\varphi' \neq 0} |\nabla u|^{p(x)} dx \leq \int_{\varphi' \neq 0} |\nabla(u + \varphi')|^{p(x)} dx.$$

As  $u + \varphi' = \min\{u + \varphi, M\}$  we have  $|\nabla(u + \varphi')| \leq |\nabla(u + \varphi)|$ . Since  $\varphi \neq 0$  whenever  $\varphi' \neq 0$ , this finishes the proof.  $\square$

**Proof of Theorem 1.1.** For each  $j = 1, 2, \dots$ , we can cover  $\partial\Omega$  by a finite number of balls  $B_{j,k} = B(x_{j,k}, 1/j)$ ,  $1 \leq k \leq N_j$ . Let  $\varphi_{j,k}$  be a Lipschitz function with support in  $3B_{j,k}$  such that  $0 \leq \varphi_{j,k} \leq 1$  and  $\varphi_{j,k} = 1$  on  $2B_{j,k}$ . Let further  $\varphi_{j,k,q} = q\varphi_{j,k}$  for  $0 < q \in \mathbf{Q}$ . Consider the sets

$$I_{j,k,q} = \left\{ x \in \overline{B}_{j,k} \cap \partial\Omega : \liminf_{\Omega \ni y \rightarrow x} H\varphi_{j,k,q}(y) < \varphi_{j,k,q}(x) = q \right\}.$$

Note that  $I_{j,k,q}$  contains only irregular points. Let further

$$u_{j,k,q} = \begin{cases} H\varphi_{j,k,q} & \text{in } \Omega, \\ q & \text{in } 2B_{j,k} \setminus \Omega, \end{cases}$$

which is a quasicontinuous supersolution in  $2B_{j,k}$  by Lemma 5.5. As  $u_{j,k,q}$  is continuous in  $\Omega$ , we have  $u_{j,k,q}^* = H\varphi_{j,k,q}$  in  $\Omega$ . By Theorem 3.5,  $u_{j,k,q}^* = u_{j,k,q}$  q.e. in  $2B_{j,k}$  and hence

$$q = u_{j,k,q}(x) = u_{j,k,q}^*(x) = \liminf_{\Omega \ni y \rightarrow x} u_{j,k,q}^*(y) = \liminf_{\Omega \ni y \rightarrow x} H\varphi_{j,k,q}(y)$$

for q.e.  $x \in \bar{B}_{j,k} \cap \partial\Omega$ . Thus  $C_{p(\cdot)}(I_{j,k,q}) = 0$ .

Now consider a function  $\varphi \in C(\partial\Omega)$  and assume that we do not have

$$\lim_{\Omega \ni y \rightarrow x} H\varphi(y) = \varphi(x)$$

for some  $x \in \partial\Omega$ . By considering  $-\varphi$  if necessary, and adding a constant, we can assume that  $\varphi \geq 0$  and that  $\liminf_{\Omega \ni y \rightarrow x} H\varphi(y) < \varphi(x)$ .

Since  $\varphi$  is continuous we can find a ball  $B_{j,k}$  containing the point  $x$  so that

$$M := \inf_{3B_{j,k} \cap \partial\Omega} \varphi > \liminf_{\Omega \ni y \rightarrow x} H\varphi(y) \geq 0.$$

We can then also find a rational  $q$  such that  $M > q > \liminf_{\Omega \ni y \rightarrow x} H\varphi(y)$ .

Thus,  $\varphi_{j,k,q} \leq \varphi$  on  $\partial\Omega$ , and hence, by the comparison principle (Lemma 5.2),

$$\liminf_{\Omega \ni y \rightarrow x} H\varphi_{j,k,q}(y) \leq \liminf_{\Omega \ni y \rightarrow x} H\varphi(y) < q = \varphi_{j,k,q}(x),$$

i.e.  $x \in I_{j,k,q}$ . Thus the set of irregular points

$$I_{p(\cdot)} = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{N_j} \bigcup_{\substack{q \in \mathbf{Q} \\ q > 0}} I_{j,k,q}, \tag{5.1}$$

is a countable union of sets of zero  $p(\cdot)$ -capacity, and hence itself of zero  $p(\cdot)$ -capacity.  $\square$

**Remark 5.6.** It is easy to see that

$$I_{j,k,q} = \bigcup_{l=1}^{\infty} (\bar{B}_{j,k} \cap \partial\Omega \cap \overline{\{y \in \Omega : H\varphi_{j,k,q}(y) < q - 1/l\}}),$$

is a countable union of compact sets. Together with the identity (5.1) this shows that  $I_{p(\cdot)}$  is an  $F_\sigma$  set.

### 6. Removable singularities

In this section we are going to prove Theorem 1.3. Let us first state it in a slightly more precise form.

**Theorem 6.1.** *Let  $F \subset \Omega$  be relatively closed and such that  $C_{p(\cdot)}(F) = 0$ . Let  $u$  be a bounded  $p(\cdot)$ -harmonic function in  $\Omega \setminus F$ . Then  $u$  has a unique  $p(\cdot)$ -harmonic extension to  $\Omega$  given by*

$$U(x) = \text{ess } \liminf_{\Omega \setminus F \ni y \rightarrow x} u(y), \quad x \in \Omega.$$

*If moreover  $u \in W^{1,p(\cdot)}(\Omega \setminus F)$ , then  $U \in W^{1,p(\cdot)}(\Omega)$  and  $\|U\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{W^{1,p(\cdot)}(\Omega \setminus F)}$ .*

Note that the boundedness assumption cannot be omitted even in the constant  $p$  case, as shown by the function  $u(x) = -|x|^{(p-n)/(p-1)}$ , which is  $p$ -harmonic in  $B(0, 1) \setminus \{0\} \subset \mathbf{R}^n$  but not in  $B(0, 1)$ . It also shows that the assumption that  $u$  be bounded from below cannot be dropped from Theorem 6.2 below either.

Theorem 6.1 follows directly from Proposition 6.4 below and the following removability result for bounded superharmonic functions.

**Theorem 6.2.** *Let  $F \subset \Omega$  be relatively closed and such that  $C_{p(\cdot)}(F) = 0$ . Let  $u$  be a superharmonic function in  $\Omega \setminus F$  which is bounded from below. Then  $u$  has a unique superharmonic extension  $U$  to  $\Omega$  given by*

$$U(x) = \operatorname{ess\,lim\,inf}_{\Omega \setminus F \ni y \rightarrow x} u(y), \quad x \in \Omega.$$

*If moreover  $u \in W^{1,p(\cdot)}(\Omega \setminus F)$ , then  $U \in W^{1,p(\cdot)}(\Omega)$  and  $\|U\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{W^{1,p(\cdot)}(\Omega \setminus F)}$ .*

To prove [Theorem 6.2](#) we need the following lemma. It is similar to Lemma 3.1 in Lukkari [28], but since one also needs that  $0 \leq \varphi_j \leq 1$ , we provide the short proof and clarify this point.

**Lemma 6.3.** *Let  $K$  be a compact set. If  $C_{p(\cdot)}(K) = 0$ , then there exists a sequence  $\{\varphi_j\}_{j=1}^\infty$  of  $C^\infty(\mathbf{R}^n)$  functions with the following properties:*

- (a)  $0 \leq \varphi_j \leq 1$  in  $\mathbf{R}^n$  and  $\varphi_j \equiv 0$  in a neighbourhood of  $K$ ;
- (b)  $\lim_{j \rightarrow \infty} \int_\Omega |\nabla \varphi_j|^{p(x)} dx = 0$ ;
- (c)  $\lim_{j \rightarrow \infty} \varphi_j = 1$  and  $\lim_{j \rightarrow \infty} \nabla \varphi_j = 0$  a.e. in  $\mathbf{R}^n$ .

**Proof.** By Lemma 10.1.9 in Diening, Harjulehto, Hästö, and Růžička [13], the infimum in the definition of  $C_{p(\cdot)}(K)$  can be taken over all nonnegative  $u \in C^\infty(\mathbf{R}^n)$  such that  $u \geq 1$  in a neighbourhood of  $K$ . In fact, it follows from the proof (which implicitly uses the standard mollification through Theorem 9.1.6 in [13]) that one can also assume that  $0 \leq u \leq 1$ . Thus, there are  $u_j \in C^\infty(\mathbf{R}^n)$  such that  $0 \leq u_j \leq 1$  in  $\mathbf{R}^n$ ,  $u_j = 1$  in a neighbourhood of  $K$  and

$$\int_{\mathbf{R}^n} (u_j^{p(x)} + |\nabla u_j|^{p(x)}) dx \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

Letting  $\varphi_j = 1 - u_j$  and passing to a subsequence then finishes the proof.  $\square$

In what follows the Lebesgue measure of a set in  $\mathbf{R}^n$  is denoted by  $|\cdot|$ .

**Proof of Theorem 6.2.** We first show the uniqueness. Let  $V$  be any superharmonic extension of  $u$ . Since  $V$  is lsc-regularized, by [Theorem 4.3](#), and  $|F| = 0$ , we see that

$$V(x) = \operatorname{ess\,lim\,inf}_{\Omega \ni y \rightarrow x} V(y) = \operatorname{ess\,lim\,inf}_{\Omega \setminus F \ni y \rightarrow x} u(y) = U(x), \quad x \in \Omega,$$

which shows the uniqueness.

Let us now turn to the existence. Assume to begin with that  $u$  is bounded. By [Theorem 1.4](#),  $u$  is an lsc-regularized supersolution in  $\Omega \setminus F$ . It is straightforward that  $U$  is bounded and lsc-regularized in  $\Omega$  and that  $U = u$  in  $\Omega \setminus F$ . We shall show that  $U$  is a supersolution in  $\Omega$ , and thus a bounded superharmonic extension of  $u$ , by [Theorem 1.4](#) again, as required.

First, we show that  $U \in W_{\text{loc}}^{1,p(\cdot)}(\Omega)$ . Let  $B \Subset \Omega$  be a ball and  $\eta \in C_0^\infty(B)$  be such that  $0 \leq \eta \leq 1$  and  $\eta = 1$  in  $\frac{1}{2}B$ . Let  $\{\varphi_j\}_{j=1}^\infty$  be as in [Lemma 6.3](#), with  $K = F \cap \operatorname{supp} \eta$ , and consider  $\eta_j = \eta \varphi_j$ . Since  $u$  is bounded, we may assume that  $u \leq 0$ . Then  $-u \eta_j^{p^+} \in W_0^{1,p(\cdot)}(\Omega \setminus F)$  is nonnegative and compactly supported in  $\Omega \setminus F$ . Thus we have

$$\int_\Omega p(x) |\nabla u|^{p(x)-2} \nabla u \cdot (-\eta_j^{p^+} \nabla u - p^+ u \eta_j^{p^+-1} \nabla \eta_j) dx \geq 0.$$

Hence,

$$\int_\Omega p(x) |\nabla u|^{p(x)} \eta_j^{p^+} dx \leq p^+ \int_\Omega p(x) |\nabla u|^{p(x)-1} |u| \eta_j^{p^+-1} |\nabla \eta_j| dx. \tag{6.1}$$

The last integrand can be estimated for every  $0 < \varepsilon < 1$  and  $x \in \Omega$  using the Young inequality as

$$\frac{|u||\nabla\eta_j|}{\varepsilon}(\varepsilon|\nabla u|^{p(x)-1}\eta_j^{p^+-1}) \leq \frac{(|u||\nabla\eta_j|)^{p(x)}}{p(x)\varepsilon^{p(x)}} + \frac{\varepsilon^{p'(x)}}{p'(x)}|\nabla u|^{p(x)}\eta_j^{(p^+-1)p'(x)}. \tag{6.2}$$

Since  $p'(x) \geq (p^+)' = p^+/(p^+ - 1)$  and  $1/p'(x) < 1$ , inserting this into (6.1) yields

$$\int_{\Omega} p(x)|\nabla u|^{p(x)}\eta_j^{p^+} dx \leq \frac{p^+}{\varepsilon^{p^+}} \int_{\Omega} |u|^{p(x)}|\nabla\eta_j|^{p(x)} dx + p^+\varepsilon^{(p^+)'} \int_{\Omega} p(x)|\nabla u|^{p(x)}\eta_j^{p^+} dx. \tag{6.3}$$

By choosing  $\varepsilon$  small enough we can include the last integral in the left-hand side. (Note that it is finite.) As a consequence, we have for every  $j = 1, 2, \dots$ ,

$$\begin{aligned} \int_{\frac{1}{2}B} |\nabla u|^{p(x)}\varphi_j^{p^+} dx &\leq \int_{\Omega} |\nabla u|^{p(x)}\eta_j^{p^+} dx \leq C(p^+) \int_{\Omega} |u|^{p(x)}|\nabla\eta_j|^{p(x)} dx \\ &\leq C(p^+, u) \int_{\Omega} (|\nabla\varphi_j| + |\nabla\eta|)^{p(x)} dx, \end{aligned} \tag{6.4}$$

since  $u$  is bounded. By Lemma 6.3(b), the last integral remains bounded as  $j \rightarrow \infty$ . Thus we get from Lemma 6.3(c) and dominated convergence that  $\nabla u \in L^{p(\cdot)}(\frac{1}{2}B)$ . Since  $B \Subset \Omega$  was arbitrary,  $\nabla u \in L^{p(\cdot)}_{loc}(\Omega)$ .

To conclude that  $U \in W^{1,p(\cdot)}_{loc}(\Omega)$  it remains to show that  $\nabla u$  is the distributional gradient of  $U$  in  $\Omega$ . To this end, let  $\eta \in C^\infty_0(\Omega)$  be arbitrary and let  $\{\varphi_j\}_{j=1}^\infty$  be as in Lemma 6.3 with  $K = F \cap \text{supp } \eta$ . Then  $\eta\varphi_j \in C^\infty_0(\Omega \setminus F)$ . Since  $\nabla u$  is the distributional gradient of  $u$  in  $\Omega \setminus F$ , we have

$$0 = \int_{\Omega \setminus F} (u\nabla(\eta\varphi_j) + \eta\varphi_j\nabla u) dx = \int_{\Omega} u\eta\nabla\varphi_j dx + \int_{\Omega} \varphi_j(U\nabla\eta + \eta\nabla u) dx.$$

The first integral in the right-hand side tends to zero by Lemma 6.3(b), (2.3) and the Hölder inequality. Since  $0 \leq \varphi_j \leq 1$  and  $|U\nabla\eta + \eta\nabla u| \in L^1(\Omega)$ , the last integral tends to

$$\int_{\Omega} (U\nabla\eta + \eta\nabla u) dx$$

by Lemma 6.3(c) and dominated convergence. Thus,  $\nabla u$  is the distributional gradient of  $U$  in  $\Omega$ , and  $U \in W^{1,p(\cdot)}_{loc}(\Omega)$ .

It remains to be proven that  $U$  is a supersolution in the whole of  $\Omega$ . Let  $0 \leq \eta \in C^\infty_0(\Omega)$  be arbitrary. As above,  $\eta\varphi_j \in C^\infty_0(\Omega \setminus F)$  is an admissible test function, where  $\{\varphi_j\}_{j=1}^\infty$  again are given by Lemma 6.3 with  $K = F \cap \text{supp } \eta$ . Since  $u$  is a supersolution in  $\Omega \setminus F$ , it holds that

$$\int_{\Omega} p(x)|\nabla u|^{p(x)-2}(\nabla u \cdot \nabla\varphi_j)\eta dx + \int_{\Omega} p(x)|\nabla u|^{p(x)-2}(\nabla u \cdot \nabla\eta)\varphi_j dx \geq 0. \tag{6.5}$$

As  $|\nabla u|^{p(\cdot)-1} \in L^{p'(\cdot)}_{loc}(\Omega)$ , the Hölder inequality (2.4) implies that the first term in (6.5) is majorized by

$$2p^+ \max_{\Omega} |\eta| \| |\nabla u|^{p(\cdot)-1} \|_{L^{p'(\cdot)}(\text{supp } \eta)} \| \nabla\varphi_j \|_{L^{p(\cdot)}(\text{supp } \eta)},$$

which tends to zero as  $j \rightarrow \infty$ , by Lemma 6.3(b) together with (2.3).

As for the second term in (6.5), the Young inequality shows that  $|\nabla u|^{p(x)-1} \in L^1(\text{supp } \eta)$ . Hence the second term in (6.5) converges by dominated convergence. Letting  $j \rightarrow \infty$  in (6.5) then shows that

$$\int_{\Omega} p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\eta dx \geq 0. \tag{6.6}$$

Thus  $U$  is a supersolution in  $\Omega$ .

Finally, consider the case when  $u$  is unbounded. By Lemma 4.2,  $u_k := \min\{u, k\}$  is a bounded superharmonic function in  $\Omega \setminus F$  which, by the above, has  $U_k := \min\{U, k\}$  as a bounded superharmonic extension to  $\Omega$ . By Theorem 4.4,  $U$  is superharmonic in  $\Omega$ .

If moreover,  $u \in W^{1,p(\cdot)}(\Omega \setminus F)$ , then  $\{u_k\}_{k=1}^\infty$  is a Cauchy sequence in  $W^{1,p(\cdot)}(\Omega \setminus F)$ . By the above,  $\nabla u_k$  is the distributional gradient of  $U_k$  in  $\Omega$ . Since  $|F| = 0$  it follows that  $\|U_k\|_{W^{1,p(\cdot)}(\Omega)} = \|u_k\|_{W^{1,p(\cdot)}(\Omega \setminus F)}$ . Hence  $\{U_k\}_{k=1}^\infty$  is a Cauchy sequence in  $W^{1,p(\cdot)}(\Omega)$  with limit  $U$ , and thus  $\|U\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{W^{1,p(\cdot)}(\Omega \setminus F)}$ .  $\square$

**Proposition 6.4.** *Assume that  $F \subset \Omega$  is relatively closed and  $|F| = 0$ . Let  $u$  be a bounded  $p(\cdot)$ -harmonic function in  $\Omega \setminus F$ , which has a superharmonic extension  $U$  and a subharmonic extension  $V$  to  $\Omega$ . Then both  $U$  and  $V$  are unique and  $U = V$  is  $p(\cdot)$ -harmonic in  $\Omega$ .*

**Proof.** Since  $U$  is lsc-regularized and  $|F| = 0$ , we have that

$$U(x) = \operatorname{ess\,lim\,inf}_{\Omega \ni y \rightarrow x} U(y) = \operatorname{ess\,lim\,inf}_{\Omega \setminus F \ni y \rightarrow x} u(y), \quad x \in \Omega,$$

and thus  $U$  is unique. Moreover,  $U$  is bounded, as  $u$  is bounded. By Theorem 1.4,  $U$  is an lsc-regularized supersolution and since  $U = V$  a.e.,  $U$  is also a subsolution. Thus,  $U$  is a solution. Since  $U$  is lsc-regularized, it follows from Remark 3.8 that  $U$  is continuous in  $\Omega$ , and thus  $p(\cdot)$ -harmonic in  $\Omega$ . Similarly  $V$  is continuous in  $\Omega$ , and as  $U = V$  a.e. in  $\Omega$  it follows that  $U = V$  everywhere in  $\Omega$ .  $\square$

The following two lemmas will be needed in the next section to prove the trichotomy (Theorem 1.2). We state them already here to avoid a digression later on.

**Lemma 6.5.** *Assume that  $G \subset \mathbf{R}^n$  is open and connected. If  $F \subset G$  is relatively closed with  $C_{p(\cdot)}(F) = 0$ , then  $G \setminus F$  is connected.*

**Proof.** Proposition 10.1.10 in Diening, Harjulehto, Hästö, and Růžička [13] gives us that  $C_{p^-}(F) = 0$ . A simple modification of Lemma 2.46 in Heinonen, Kilpeläinen, and Martio [22] implies that  $G \setminus F$  is connected.  $\square$

**Lemma 6.6.** *Assume that  $G \subset \mathbf{R}^n$  is open and connected. If  $F \subsetneq G$  is relatively closed, then  $C_{p(\cdot)}(F) = 0$  if and only if  $C_{p(\cdot)}(\partial F \cap G) = 0$ .*

**Proof.** The necessity follows immediately from  $C_{p(\cdot)}(\partial F \cap G) \leq C_{p(\cdot)}(F) = 0$ .

In order to show the converse implication, assume that  $C_{p(\cdot)}(\partial F \cap G) = 0$ . Then, by Lemma 6.5,  $G \setminus (\partial F \cap G)$  is connected and so  $\operatorname{int} F = \emptyset$ . Hence,  $F = \partial F \cap G$ , and thus  $C_{p(\cdot)}(F) = 0$ .  $\square$

In the setting of metric spaces Lemma 6.6 can be found as Lemma 4.5 in Björn and Björn [9] for the constant  $p$  case. Therein, the use of Newtonian spaces simplifies the argument.

### 7. Boundary regularity and trichotomy

In this section we prove one of the main results of this paper, namely the trichotomy (Theorem 1.2) between regular, semiregular and strongly irregular boundary points.

Recall that an irregular boundary point  $x_0 \in \partial\Omega$  is semiregular if the limit

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) \text{ exists for all } f \in C(\partial\Omega); \tag{7.1}$$

and strongly irregular if for all  $f \in C(\partial\Omega)$  there is a sequence  $\{y_j\}_{j=1}^\infty$  such that

$$\Omega \ni y_j \rightarrow x_0 \text{ and } Hf(y_j) \rightarrow f(x_0), \text{ as } j \rightarrow \infty. \tag{7.2}$$



**Proof of Theorem 1.2.** *Case 1.* There is  $r > 0$  such that  $C_{p(\cdot)}(B \cap \partial\Omega) = 0$ , where  $B = B(x_0, r)$ .

By Lemma 6.6,  $C_{p(\cdot)}(B \setminus \Omega) = 0$  and thus  $B \subset \overline{\Omega}$ . Let  $f \in C(\partial\Omega)$ . By Theorem 6.1, the  $p(\cdot)$ -harmonic function  $Hf$  has a  $p(\cdot)$ -harmonic extension  $U$  to  $\Omega \cup B$ . Since  $U$  is continuous we have

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) = U(x_0),$$

i.e. (7.1) holds and  $x_0$  is either regular or semiregular.

*Case 2.* The capacity  $C_{p(\cdot)}(B(x_0, r) \cap \partial\Omega) > 0$  for all  $r > 0$ . (Note that this is complementary to Case 1.)

For every  $j = 1, 2, \dots$ , we thus have  $C_{p(\cdot)}(B(x_0, 1/j) \cap \partial\Omega) > 0$ , and by the Kellogg property (Theorem 1.1) there is a regular boundary point  $x_j \in B(x_0, 1/j) \cap \partial\Omega$ . (We do not require the  $x_j$  to be distinct.)

As  $x_j$  is regular, we can find  $y_j \in B(x_j, 1/j) \cap \Omega$  so that  $|Hf(y_j) - f(x_j)| < 1/j$ . It follows directly that  $y_j \rightarrow x_0$  and  $Hf(y_j) \rightarrow f(x_0)$ , as  $j \rightarrow \infty$ , i.e. (7.2) holds, and thus  $x_0$  is either regular or strongly irregular.  $\square$

We finish this section by characterizing regular boundary points in several ways. Semiregular boundary points will be characterized in Section 8. In view of the trichotomy result this indirectly characterizes the strongly irregular points as well.

**Theorem 7.1.** Let  $x_0 \in \partial\Omega$  and  $d(x) := d(x, x_0)$ . Then the following are equivalent:

- (a) The point  $x_0$  is a regular boundary point.
- (b) It is true that

$$\lim_{\Omega \ni y \rightarrow x_0} H(jd)(y) = 0 \quad \text{for all } j = 1, 2, \dots$$

- (c) It is true that

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) = f(x_0)$$

for all bounded  $f \in W^{1,p(\cdot)}(\Omega)$  such that  $f(x_0) := \lim_{\Omega \ni y \rightarrow x_0} f(y)$  exists.

- (d) It is true that

$$\limsup_{\Omega \ni y \rightarrow x_0} Hf(y) \leq \limsup_{\Omega \ni y \rightarrow x_0} f(y)$$

for all bounded  $f \in W^{1,p(\cdot)}(\Omega)$ .

In the constant  $p$  case it is enough if (b) holds for  $j = 1$ , which is easily seen since  $H(jd) = jHd$  in this case. In the variable exponent case this latter fact is not true, but it is not known whether it suffices that (b) holds for  $j = 1$  also in this case. In fact, the situation is similar for  $p$ -parabolic equations in the sense that if  $u$  is a  $p$ -parabolic function and  $a \in \mathbf{R}$ , then  $u + a$  is  $p$ -parabolic, but  $au$  is in general not  $p$ -parabolic. In the  $p$ -parabolic case a similar characterization of boundary regularity to the one above was obtained by Björn, Björn, Gianazza, and Parviainen [10]. Therein a characterization of boundary regularity in terms of the existence of a family of barriers was also obtained. It would be interesting to obtain a similar characterization in our variable exponent elliptic case. Whether one barrier could suffice for boundary regularity in the  $p$ -parabolic case or in the variable exponent elliptic case is an open question.

**Proof.** (a)  $\Rightarrow$  (b) This follows directly from Definition 5.4 by taking  $f = jd$  for  $j = 1, 2, \dots$

(b)  $\Rightarrow$  (d) Let  $A > \limsup_{\Omega \ni y \rightarrow x_0} f(y)$  be real and  $M = \sup_{\partial\Omega} (f - A)_+$ . Let further  $r > 0$  be such that  $f(x) < A$  for  $x \in B(x_0, r) \cap \partial\Omega$ , and let  $j > M/r$  be an integer. Then  $f \leq A + Md/r < A + jd$  on  $\partial\Omega$ . It follows from the comparison principle in Lemma 3.10 that

$$\limsup_{\Omega \ni y \rightarrow x_0} Hf(y) \leq A + \lim_{\Omega \ni y \rightarrow x_0} H(jd)(y) = A.$$

Letting  $A \rightarrow \limsup_{\Omega \ni y \rightarrow x_0} f(y)$  gives  $\limsup_{\Omega \ni y \rightarrow x_0} Hf(y) \leq \limsup_{\Omega \ni y \rightarrow x_0} f(y)$ .

(d)  $\Rightarrow$  (c) Applying (d) to  $-f$  yields

$$\liminf_{\Omega \ni y \rightarrow x_0} Hf(y) = - \limsup_{\Omega \ni y \rightarrow x_0} H(-f)(y) \geq -(-f(x_0)) = f(x_0).$$

Together with (d) this gives the desired conclusion.

(c)  $\Rightarrow$  (a) Let  $f \in C(\partial\Omega)$ . By the comparison principle together with uniform approximation by Lipschitz functions we may as well assume that  $f \in \text{Lip}(\partial\Omega)$ . We find an extension  $\tilde{f} \in \text{Lip}(\overline{\Omega})$  such that  $\tilde{f} = f$  on  $\partial\Omega$  (e.g. a McShane extension). Then by definition and (c),

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) = \lim_{\Omega \ni y \rightarrow x_0} H\tilde{f}(y) = f(x_0). \quad \square$$

### 8. Characterizations of semiregular points

Similarly to regular points, semiregular points can be characterized by a number of equivalent conditions. This will be done in [Theorem 8.4](#), but before that we obtain the following characterizations of relatively open sets of semiregular points.

**Theorem 8.1.** *Let  $V \subset \partial\Omega$  be relatively open. Then the following are equivalent:*

- (a') *The set  $V$  consists entirely of semiregular points.*
- (b') *The set  $V$  does not contain any regular point.*
- (c') *It is true that  $C_{p(\cdot)}(V) = 0$ .*
- (d') *The set  $\Omega \cup V$  is open in  $\mathbf{R}^n$ , and every bounded  $p(\cdot)$ -harmonic function in  $\Omega$  has a  $p(\cdot)$ -harmonic extension to  $\Omega \cup V$ .*
- (e') *The set  $\Omega \cup V$  is open in  $\mathbf{R}^n$ ,  $|V| = 0$ , and every bounded superharmonic function in  $\Omega$  has a superharmonic extension to  $\Omega \cup V$ .*
- (f') *For every  $f \in C(\partial\Omega)$ , the  $p(\cdot)$ -harmonic extension  $Hf$  depends only on  $f|_{\partial\Omega \setminus V}$  (i.e. if  $f, h \in C(\partial\Omega)$  and  $f = h$  on  $\partial\Omega \setminus V$ , then  $Hf \equiv Hh$ ).*

Together with the implication (a)  $\Rightarrow$  (e) in [Theorem 8.4](#) this theorem shows that the set  $S$  of all semiregular boundary points can be characterized as the largest relatively open subset of  $\partial\Omega$  having any of the properties above. Equivalently, it can be written e.g. as

$$S = \bigcup \{V \subset \partial\Omega : C_{p(\cdot)}(V) = 0 \text{ and } V \text{ is relatively open}\}. \tag{8.1}$$

By (d') we also see that  $S$  is contained in the interior of  $\overline{\Omega}$ , i.e.  $S \subset \partial\Omega \setminus \partial\overline{\Omega}$ . Note however that it can happen that  $S \neq \partial\Omega \setminus \partial\overline{\Omega}$ , as the following examples show.

**Example 8.2.** Let  $C_{p(\cdot)}(\{x\}) > 0$  and  $G \ni x$ , where  $G$  is a bounded open set, and let  $\Omega := G \setminus \{x\}$ . Then  $x$  is regular with respect to  $\Omega$ , by the Kellogg property, but  $x \in \partial\Omega \setminus \partial\overline{\Omega}$ , cf. [Example 8.6](#).

**Example 8.3.** Let  $n = 2$  and  $p \equiv 2$ . Let further  $\Omega$  be the slit disc  $B((0, 0), 1) \setminus ((-1, 0] \times \{0\})$ . It is well known that  $\Omega$  is regular, and hence  $S = \emptyset$ . However,  $\partial\Omega \setminus \partial\overline{\Omega} = (-1, 0] \times \{0\}$ .

The strong minimum principle says that if  $\Omega$  is connected,  $u$  is superharmonic in  $\Omega$  and  $u$  attains its minimum in  $\Omega$ , then  $u$  is constant in  $\Omega$ . The proof of the implication (d')  $\Rightarrow$  (a') is considerably easier when the strong minimum principle is available, but it is not known if it holds in our generality. The strong minimum principle for the variable exponent case was obtained by Fan, Zhao, and Zhang [15] under the assumption that  $p \in C^1(\overline{\Omega})$ . Theorem 5.3 in Harjulehto, Hästö, Latvala, and Toivanen [17] shows that the strong minimum principle holds also under the weaker assumption that  $p$  satisfies a Dini-type condition, see (5.1) in [17].

**Proof of Theorem 8.1.** (a') ⇒ (b') This is trivial.

(b') ⇒ (c') This follows directly from the Kellogg property (Theorem 1.1).

(c') ⇒ (e') Let  $x \in V$  and let  $G$  be a connected neighbourhood of  $x$ , such that  $G \cap \partial\Omega \subset V$ . By Lemma 6.5 sets of zero  $p(\cdot)$ -capacity cannot separate space, and hence  $G \setminus \partial\Omega$  must be connected. Since  $G \setminus \partial\Omega = (G \cap \Omega) \cup (G \setminus \overline{\Omega})$  and  $G \cap \Omega \neq \emptyset$ , we get that  $G \subset \overline{\Omega}$ . As  $G \cap \partial\Omega \subset V$ , this implies that  $G \subset \Omega \cup V$ . Since  $x \in V$  was arbitrary, we conclude that  $\Omega \cup V$  is open. That  $|V| = 0$  follows directly from the fact that  $C_{p(\cdot)}(V) = 0$ . The extension is now provided by Theorem 6.2.

(e') ⇒ (d') Let  $u$  be a bounded  $p(\cdot)$ -harmonic function on  $\Omega$ . By assumption,  $u$  has a superharmonic extension  $U$  to  $\Omega \cup V$ . Also  $-u$  has a superharmonic extension  $W$  to  $\Omega \cup V$ . Thus  $-W$  is a subharmonic extension of  $u$  to  $\Omega \cup V$ . By Proposition 6.4,  $U = -W$  is  $p(\cdot)$ -harmonic.

(c') ⇒ (f') Let  $f, h \in C(\partial\Omega)$  with  $f = h$  on  $\partial\Omega \setminus V$ . Then  $\eta := h - f \in C(\partial\Omega)$  and  $\eta = 0$  on  $\partial\Omega \setminus V$ .

Let  $f_j \in \text{Lip}_c(\mathbf{R}^n) \subset W^{1,p(\cdot)}(\mathbf{R}^n)$  converge uniformly to  $f$  on  $\partial\Omega$ . (Here  $\text{Lip}_c(\mathbf{R}^n)$  consists of Lipschitz functions on  $\mathbf{R}^n$  with compact support.) Let also  $\eta'_j \in \text{Lip}_c(\mathbf{R}^n)$  be such that  $|\eta'_j - \eta| < 1/j$  on  $\partial\Omega$ . Letting  $\eta_j = (\eta'_j - 1/j)_+ - (\eta'_j + 1/j)_-$  we see that  $\eta_j = 0$  on  $\partial\Omega \setminus V$  and  $\eta_j \rightarrow \eta$  uniformly on  $\partial\Omega$ .

As  $C_{p(\cdot)}(V) = 0$ , Lemma 2.4 shows that  $\eta_j \in W_0^{1,p(\cdot)}(\Omega)$  and thus  $Hf_j = H(f_j + \eta_j)$ . As  $f_j + \eta_j \rightarrow f + \eta = h$  and  $f_j \rightarrow f$  uniformly on  $\partial\Omega$ , Lemma 5.3 implies  $H(f_j + \eta_j) \rightarrow Hh$  and  $Hf_j \rightarrow Hf$  in  $\Omega$ , i.e.  $Hh = Hf$ .

(f') ⇒ (b') Let  $x_0 \in V$ . As  $V$  is relatively open in  $\partial\Omega$ , there exists  $r > 0$  such that  $B(x_0, r) \cap \partial\Omega \subset V$ . Define  $f(x) = (1 - d(x, x_0)/r)_+$ . Then  $f \in \text{Lip}_c(\mathbf{R}^n)$  and  $f = 0$  on  $\partial\Omega \setminus V$ . Using (f') we conclude that  $Hf = H0 = 0$  in  $\Omega$ . Since  $f(x_0) = 1$ , this shows that  $x_0$  is not regular.

(d') ⇒ (a') Let  $x_0 \in V$  and  $f \in C(\partial\Omega)$ . Since  $Hf$  has a  $p(\cdot)$ -harmonic extension  $u$  to  $\Omega \cup V$ , it follows that

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) = \lim_{\Omega \ni y \rightarrow x_0} u(y) = u(x_0), \tag{8.2}$$

and thus the limit in the left-hand side always exists. It remains to show that  $x_0$  is irregular.

As  $V$  is relatively open in  $\partial\Omega$ , there exists  $r > 0$  such that  $B(x_0, r) \cap \partial\Omega \subset V$ . Define  $h(x) = (1 - d(x, x_0)/r)_+$ . By assumption,  $Hh$  has a  $p(\cdot)$ -harmonic extension  $U$  to  $\Omega \cup V$ . We shall show that  $U \equiv 0$  in  $\Omega \cup V$ , as then

$$\lim_{\Omega \ni y \rightarrow x_0} Hh(y) = \lim_{\Omega \ni y \rightarrow x_0} U(y) = 0 \neq 1 = h(x_0),$$

i.e.  $x_0$  is irregular.

As  $0 \leq h \leq 1$ , we see that  $0 \leq Hh \leq 1$  and also  $0 \leq U \leq 1$ . By the Kellogg property (Theorem 1.1),

$$\lim_{\Omega \ni y \rightarrow x} U(y) = \lim_{\Omega \ni y \rightarrow x} Hh(y) = h(x) = 0$$

q.e. in  $\partial(\Omega \cup V) = \partial\Omega \setminus V$ , i.e. for all  $x \in \partial(\Omega \cup V) \setminus E$ , where  $C_{p(\cdot)}(E) = 0$ . Moreover, (8.2) applied to  $h$  and  $y_j \in V$  instead of  $f$  and  $x_0$  implies that

$$0 \leq \limsup_{j \rightarrow \infty} U(y_j) \leq \limsup_{\Omega \ni y \rightarrow x} U(y) = 0,$$

whenever  $y_j \in V$  converge to some  $x \in \partial(\Omega \cup V) \setminus E$ . Hence,

$$\lim_{\Omega \cup V \ni y \rightarrow x} U(y) = h(x) = 0$$

for  $x \in \partial(\Omega \cup V) \setminus E$ . Note that we cannot use the comparison principle (Lemma 3.10) directly to prove that  $U \equiv 0$  in  $\Omega \cup V$ , since we do not know that  $U \in W_0^{1,p(\cdot)}(\Omega \cup V)$ . (We know that  $U \in W_{\text{loc}}^{1,p(\cdot)}(\Omega \cup V)$  and that  $U \in W^{1,p(\cdot)}(\Omega)$ , but since it could a priori happen that  $|V \setminus \Omega| > 0$ , we cannot, at this point, even deduce that  $U \in W^{1,p(\cdot)}(\Omega \cup V)$ .)

Let  $0 < \varepsilon < 1$  and find an open set  $G \supset E$  such that  $C_{p(\cdot)}(G) < \varepsilon$ . Let also  $\varphi \in W^{1,p(\cdot)}(\mathbf{R}^n)$  be such that  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  on  $G$  and

$$\int_{\mathbf{R}^n} (\varphi^{p(x)} + |\nabla\varphi|^{p(x)}) dx \leq \varepsilon. \tag{8.3}$$

For every  $x \in \partial(\Omega \cup V) \setminus E$  there exists a ball  $B_x \ni x$  such that  $0 \leq U < \varepsilon$  in  $2B_x \cap (\Omega \cup V)$ . Exhaust  $\Omega \cup V$  by open sets

$$\Omega_1 \subset \Omega_2 \subset \cdots \in \bigcup_{j=1}^{\infty} \Omega_j = \Omega \cup V.$$

Then

$$\overline{\Omega \cup V} \subset \bigcup_{j=1}^{\infty} \Omega_j \cup G \cup \bigcup_{x \in \partial(\Omega \cup V) \setminus E} B_x.$$

By compactness, there exists  $j > 1/\varepsilon$  such that

$$\partial\Omega_j \subset G \cup \bigcup_{x \in \partial(\Omega \cup V) \setminus E} B_x.$$

Then  $0 \leq U \leq \varepsilon$  on  $\partial\Omega_j \setminus G$  and as  $\varphi \geq \chi_G$ , we obtain  $U \leq \varepsilon + \varphi$  on  $\partial\Omega_j$ . Since  $U \in W^{1,p(\cdot)}(\Omega_j)$ , it is its own  $p(\cdot)$ -harmonic extension in  $\Omega_j$ , i.e.  $U = H_{\Omega_j}U$ . If we let  $v = H_{\Omega_j}\varphi$ , then  $U \leq \varepsilon + v$  in  $\Omega_j$ , by the comparison principle (Lemma 3.10). The Poincaré inequality (Theorem 8.2.4 in Diening, Harjulehto, Hästö, and Růžička [13]), applied to  $v - \varphi \in W_0^{1,p(\cdot)}(\Omega_j)$  and some ball  $B \supset \Omega_j$ , yields

$$\|v - \varphi\|_{L^{p(\cdot)}(B)} \leq C_B \|\nabla(v - \varphi)\|_{L^{p(\cdot)}(B)} \leq C_B (\|\nabla v\|_{L^{p(\cdot)}(B)} + \|\nabla\varphi\|_{L^{p(\cdot)}(B)}). \quad (8.4)$$

Since  $v = H_{\Omega_j}\varphi$ , we conclude from (2.2) and (8.3) that

$$\|\nabla v\|_{L^{p(\cdot)}(B)} \leq \left( \int_B |\nabla v(x)|^{p(x)} dx \right)^{1/p^+} \leq \left( \int_B |\nabla\varphi(x)|^{p(x)} dx \right)^{1/p^+}.$$

Inserting this into (8.4), together with (2.2) again and (8.3), gives

$$\begin{aligned} \|v\|_{L^{p(\cdot)}(\Omega_j)} &\leq \|\varphi\|_{L^{p(\cdot)}(B)} + \|v - \varphi\|_{L^{p(\cdot)}(B)} \\ &\leq \left( \int_B |\varphi(x)|^{p(x)} dx \right)^{1/p^+} + 2C_B \left( \int_B |\nabla\varphi(x)|^{p(x)} dx \right)^{1/p^+} \leq 3C_B \varepsilon^{1/p^+}. \end{aligned}$$

Here we assume that  $C_B \geq 1$ . It follows that

$$\|U\|_{L^{p(\cdot)}(\Omega_j)} \leq \|\varepsilon + v\|_{L^{p(\cdot)}(\Omega_j)} \leq \varepsilon \|1\|_{L^{p(\cdot)}(\Omega)} + 3C_B \varepsilon^{1/p^+}.$$

Letting  $\varepsilon \rightarrow 0$  (and thus  $j \rightarrow \infty$ ) implies  $\|U\|_{L^{p(\cdot)}(\Omega \cup V)} = 0$ , and hence  $U \equiv 0$  in  $\Omega \cup V$ .  $\square$

We are now ready to characterize semiregular boundary points in several different ways. Note that (b) below shows that semiregularity is a local property, even though we have not shown that regularity is a local property. The latter however follows from the Wiener criterion, whose usage we have avoided in this paper. It thus also follows that strong irregularity is a local property. It would be nice to have a simpler and more direct proof (without appealing to the Wiener criterion) that regularity is a local property. Such proofs are available in the constant  $p$  case, using barrier characterizations, see Theorem 9.8 and Proposition 9.9 in Heinonen, Kilpeläinen, and Martio [22] for the weighted  $\mathbf{R}^n$  case, and Theorem 6.1 in Björn and Björn [8] (or [9, Theorem 11.11]) for metric spaces.

**Theorem 8.4.** *Let  $x_0 \in \partial\Omega$ ,  $\delta > 0$  and  $d(y) = d(y, x_0)$ . Then the following are equivalent:*

- (a) *The point  $x_0$  is semiregular.*
- (b) *The point  $x_0$  is semiregular with respect to  $G := \Omega \cap B(x_0, \delta)$ .*
- (c) *There is no sequence  $\{y_j\}_{j=1}^{\infty}$  such that  $\Omega \ni y_j \rightarrow x_0$ , as  $j \rightarrow \infty$ , and*

$$\lim_{j \rightarrow \infty} Hf(y_j) = f(x_0) \quad \text{for all } f \in C(\partial\Omega).$$

- (d) *It is true that  $x_0 \notin \overline{\{x \in \partial\Omega: x \text{ is regular}\}}$ .*
- (e) *There is a neighbourhood  $V$  of  $x_0$  such that  $C_{p(\cdot)}(V \cap \partial\Omega) = 0$ .*

- (f) There is a neighbourhood  $V$  of  $x_0$  such that  $C_{p(\cdot)}(V \setminus \Omega) = 0$ .
- (g) The point  $x_0$  is irregular and there is a neighbourhood  $V$  of  $x_0$  such that every bounded  $p(\cdot)$ -harmonic function in  $\Omega$  has a  $p(\cdot)$ -harmonic extension to  $\Omega \cup V$ .
- (h) There is a neighbourhood  $V$  of  $x_0$  such that  $V \subset \overline{\Omega}$  and every bounded  $p(\cdot)$ -harmonic function in  $\Omega$  has a  $p(\cdot)$ -harmonic extension to  $\Omega \cup V$ .
- (i) There is a neighbourhood  $V$  of  $x_0$  such that  $|V \setminus \Omega| = 0$  and every bounded superharmonic function in  $\Omega$  has a superharmonic extension to  $\Omega \cup V$ .
- (j) There is a neighbourhood  $V$  of  $x_0$  such that for every  $f \in C(\partial\Omega)$ , the  $p(\cdot)$ -harmonic extension  $Hf$  depends only on  $f|_{\partial\Omega \setminus V}$  (i.e. if  $f, h \in C(\partial\Omega)$  and  $f = h$  on  $\partial\Omega \setminus V$ , then  $Hf \equiv Hh$ ).
- (k) It is true that for some positive integer  $j$ ,

$$\lim_{\Omega \ni y \rightarrow x_0} H(jd)(y) > 0.$$

- (l) It is true that for some positive integer  $j$ ,

$$\liminf_{\Omega \ni y \rightarrow x_0} H(jd)(y) > 0.$$

**Remark 8.5.** Note that (c) says that if  $x_0$  is strongly irregular, then the sequence  $\{y_j\}_{j=1}^\infty$  occurring in (1.2) can be chosen independently of  $f \in C(\partial\Omega)$ .

**Proof.** (d)  $\Leftrightarrow$  (e)  $\Leftrightarrow$  (j)  $\Rightarrow$  (a) This follows directly from Theorem 8.1, with  $V$  in Theorem 8.1 corresponding to  $V \cap \partial\Omega$  here.

(a)  $\Rightarrow$  (k) The limit

$$c_j := \lim_{\Omega \ni y \rightarrow x_0} H(jd)(y)$$

exists for  $j = 1, 2, \dots$ . If  $c_j$  were 0 for  $j = 1, 2, \dots$ , then  $x_0$  would be regular, by Theorem 7.1, a contradiction. Thus  $c_j > 0$  for some positive integer  $j$ .

(k)  $\Rightarrow$  (l)  $\Rightarrow$  (c) This is trivial.

$\neg$ (d)  $\Rightarrow$   $\neg$ (c) For each  $j \geq 1$ ,  $B(x_0, 1/j^2) \cap \partial\Omega$  contains a regular boundary point  $x_j$ . Let  $f_j = jd \in C(\partial\Omega)$ . Then we can find  $y_j \in B(x_j, 1/j) \cap \Omega$  so that

$$\frac{1}{j} > |f_j(x_j) - Hf_j(y_j)|.$$

Since  $0 \leq f_j(x_j) \leq 1/j$ , we have  $Hf_j(y_j) \leq 2/j$ . Moreover,  $y_j \rightarrow x_0$  as  $j \rightarrow \infty$ .

Let now  $f \in C(\partial\Omega)$ . Without loss of generality we may assume that  $|f| \leq M < \infty$  and that  $f(x_0) = 0$ . Let  $\varepsilon > 0$ . Then we can find  $k$  such that

$$|f| \leq \varepsilon \quad \text{on } B(x_0, 1/k) \cap \partial\Omega.$$

For  $j \geq Mk$  we have  $f_j \geq M$  on  $\partial\Omega \setminus B(x_0, 1/k)$  and hence  $|f| \leq \varepsilon + f_j$  on  $\partial\Omega$ . It follows that for  $j \geq Mk$ ,

$$Hf(y_j) \leq \varepsilon + Hf_j(y_j) \leq \varepsilon + \frac{2}{j} \rightarrow \varepsilon, \quad \text{as } j \rightarrow \infty$$

and

$$Hf(y_j) \geq -\varepsilon - Hf_j(y_j) \geq -\varepsilon - \frac{2}{j} \rightarrow -\varepsilon, \quad \text{as } j \rightarrow \infty.$$

Letting  $\varepsilon \rightarrow 0$  gives  $\lim_{j \rightarrow \infty} Hf(y_j) = 0$ , i.e. (c) fails.

(e)  $\Leftrightarrow$  (b) Note first that (e) is equivalent to the existence of a neighbourhood  $W$  of  $x_0$  with  $C_{p(\cdot)}(W \cap \partial G) = 0$ . But this is equivalent to (b), by the already proved equivalence (e)  $\Leftrightarrow$  (a) applied to  $G$  instead of  $\Omega$ .

(e)  $\Rightarrow$  (f) By Theorem 8.1, (c')  $\Rightarrow$  (e'), the set  $\Omega \cup (V \cap \partial\Omega)$  is open, and we can use this as our set  $V$  in (f).

(f)  $\Rightarrow$  (e) This is trivial.

(f)  $\Leftrightarrow$  (h)  $\Leftrightarrow$  (i) In all three statements it follows directly that  $V \subset \overline{\Omega}$ . Thus their equivalence follows directly from [Theorem 8.1](#), with  $V$  in [Theorem 8.1](#) corresponding to  $V \cap \partial\Omega$  here.

(h)  $\Rightarrow$  (g) The first part is obvious and we only need to show that  $x_0$  is irregular, but this follows from the already proved implication (h)  $\Rightarrow$  (a).

(g)  $\Rightarrow$  (a) Let  $f \in C(\partial\Omega)$ . Then  $Hf$  has a  $p(\cdot)$ -harmonic extension  $U$  to  $\Omega \cup V$  for some neighbourhood  $V$  of  $x_0$ . It follows that

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) = U(x_0),$$

and thus the limit in the left-hand side always exists. Since  $x_0$  is irregular it follows that  $x_0$  must be semiregular.  $\square$

We end this paper with some examples of semiregular and strongly irregular boundary points.

**Example 8.6** (*The punctured ball*). Let  $\Omega = B(x_0, r) \setminus \{x_0\}$ . Then  $x_0$  is semiregular if  $C_{p(\cdot)}(\{x_0\}) = 0$ , and regular otherwise. Indeed, when  $C_{p(\cdot)}(\{x_0\}) = 0$  this follows from [Theorem 8.1](#), and is also a special case of [Proposition 8.7](#) below, while if  $C_{p(\cdot)}(\{x_0\}) > 0$  it follows from the Kellogg property ([Theorem 1.1](#)). The remaining boundary points are all regular by the sufficiency part of the Wiener criterion, or some weaker version of it.

**Proposition 8.7.** *Let  $K$  be a compact set with  $C_{p(\cdot)}(K) = 0$ . Then there is a domain  $\Omega$  such that  $K$  is the set of semiregular boundary points and all other boundary points are regular.*

**Proof.** Let  $B$  be an open ball containing  $K$  and let  $\Omega = B \setminus K$ . Then  $K$  is an open subset of  $\partial\Omega$  and as  $C_{p(\cdot)}(K) = 0$  it follows from [Theorem 8.1](#) that  $K$  consists entirely of semiregular points. By the sufficiency part of the Wiener criterion, or some weaker version of it, all other boundary points are regular.  $\square$

**Proposition 8.8.** *Assume that  $p^+ \leq n$ . Let  $K_1$  and  $K_2$  be two disjoint compact subsets of  $\mathbf{R}^n$  with  $C_{p(\cdot)}(K_1) = C_{p(\cdot)}(K_2) = 0$ . Then there is a domain  $\Omega$  such that  $K_1$  is the set of semiregular boundary points,  $K_2$  is the set of strongly irregular boundary points, and all other boundary points are regular.*

The proof of this is very similar to the proof of the corresponding result for the constant  $p$  case, as given for [Theorem 4.1](#) in A. Björn [7], and we leave it to the interested reader to verify. Here we need to use the Wiener criterion. An essential fact also used in the proof is that points have zero capacity, which is the reason for the requirement  $p^+ \leq n$ . Whether the result is true without this condition is not clear, see [Section 5](#) in [7].

## Acknowledgements

A.B. and J.B. were supported by the Swedish Research Council. The research was done during T.A. appointment at Linköping University. He would like to express the gratitude to Department of Mathematics for hospitality.

## References

- [1] E. Acerbi, G. Mingione, Regularity results for stationary electro-rheological fluids, *Arch. Ration. Mech. Anal.* 164 (2002) 213–259.
- [2] E. Acerbi, G. Mingione, Gradient estimates for the  $p(x)$ -Laplacean system, *J. Reine Angew. Math.* 584 (2005) 117–148.
- [3] T. Adamowicz, P. Hästö, Mappings of finite distortion and PDE with nonstandard growth, *Int. Math. Res. Not.* 2010 (2010) 1940–1965.
- [4] T. Adamowicz, P. Hästö, Harnack’s inequality and the strong  $p(\cdot)$ -Laplacian, *J. Differ. Equ.* 250 (2011) 1631–1649.
- [5] Yu.A. Alkhutov, O.V. Krasheninnikova, Continuity at boundary points of solutions of quasilinear elliptic equations with a nonstandard growth condition, *Izv. Ross. Akad. Nauk Ser. Mat.* 68 (6) (2004) 3–60 (in Russian). English transl.: *Izv. Math.* 68 (2004) 1063–1117.
- [6] D.H. Armitage, S.J. Gardiner, *Classical Potential Theory*, Springer, London, 2001.
- [7] A. Björn, A regularity classification of boundary points for  $p$ -harmonic functions and quasiminimizers, *J. Math. Anal. Appl.* 338 (2008) 39–47.
- [8] A. Björn, J. Björn, Boundary regularity for  $p$ -harmonic functions and solutions of the obstacle problem, *J. Math. Soc. Jpn.* 58 (2006) 1211–1232.
- [9] A. Björn, J. Björn, *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts Math., vol. 17, European Math. Soc., Zürich, 2011.
- [10] A. Björn, J. Björn, U. Gianazza, M. Parviainen, Boundary regularity for degenerate and singular parabolic equations, in preparation.

- [11] A. Björn, J. Björn, N. Shanmugalingam, The Dirichlet problem for  $p$ -harmonic functions on metric spaces, *J. Reine Angew. Math.* 556 (2003) 173–203.
- [12] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* 66 (4) (2006) 1383–1406.
- [13] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lect. Notes Math., vol. 2017, Springer, Berlin, Heidelberg, 2011.
- [14] X.-L. Fan, Global  $C^{1,\alpha}$  regularity for variable exponent elliptic equations in divergence form, *J. Differ. Equ.* 235 (2007) 397–417.
- [15] X.-L. Fan, Y.Z. Zhao, Q.-H. Zhang, A strong maximum principle for  $p(x)$ -Laplace equations, *Chin. Ann. Math., Ser. A* 24 (2003) 495–500 (in Chinese). English transl.: *Chin. J. Contemp. Math.* 24 (2003) 277–282.
- [16] P. Harjulehto, P. Hästö, M. Koskenoja, T. Lukkari, N. Marola, An obstacle problem and superharmonic functions with nonstandard growth, *Nonlinear Anal.* 67 (2007) 3424–3440.
- [17] P. Harjulehto, P. Hästö, V. Latvala, O. Toivanen, The strong minimum principle for quasisuperminimizers of non-standard growth, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 28 (2011) 731–742.
- [18] P. Harjulehto, P. Hästö, Ú.V. Lê, M. Nuortio, Overview of differential equations with non-standard growth, *Nonlinear Anal.* 72 (2010) 4551–4574.
- [19] P. Harjulehto, J. Kinnunen, T. Lukkari, Unbounded supersolutions of nonlinear equations with nonstandard growth, *Bound. Value Probl.* 2007 (2007), Art. ID 48348, 20 pp.
- [20] P. Harjulehto, V. Latvala, Fine topology of variable exponent energy superminimizers, *Ann. Acad. Sci. Fenn. Math.* 33 (2008) 491–510.
- [21] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer, New York, 2001.
- [22] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, 2nd ed., Dover, Mineola, NY, 2006.
- [23] E. Henriques, Regularity for the porous medium equation with variable exponent: the singular case, *J. Differ. Equ.* 244 (2008) 2578–2601.
- [24] T. Kilpeläinen, A remark on the uniqueness of quasi continuous functions, *Ann. Acad. Sci. Fenn. Math.* 23 (1) (1998) 261–262.
- [25] V. Latvala, T. Lukkari, O. Toivanen, The fundamental convergence theorem for  $p(\cdot)$ -superharmonic functions, *Potential Anal.* 35 (2011) 329–351.
- [26] H. Lebesgue, Sur des cas d’impossibilité du problème de Dirichlet ordinaire, in *Vie de la société* (in the part C. R. Séances Soc. Math. France (1912)), p. 17, *Bull. Soc. Math. Fr.* 41 (1913) 1–62 (supplément spécial).
- [27] J. Lukeš, J. Malý, On the boundary behaviour of the Perron generalized solution, *Math. Ann.* 257 (1981) 355–366.
- [28] T. Lukkari, Singular solutions of elliptic equations with nonstandard growth, *Math. Nachr.* 282 (2009) 1770–1787.
- [29] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lect. Notes Math., vol. 1748, Springer, Berlin, 2000.
- [30] S. Zaremba, Sur le principe de Dirichlet, *Acta Math.* 34 (1911) 293–316.