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Ann. I. H. Poincaré – AN 21 (2004) 639–656



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Positive solutions of slightly supercritical elliptic equations in symmetric domains

Solutions positives pour l'équation $\Delta u + u^{\frac{n+2}{n-2}+\varepsilon} = 0$ en domaines symétriques

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Received 14 April 2003; received in revised form 24 July 2003; accepted 22 September 2003

Available online 13 February 2004

Abstract

This paper deals with existence and multiplicity of solutions for problem $P(\varepsilon, \Omega)$ below, which concentrate and blow-up at a finite number of points as $\varepsilon \rightarrow 0$. We give sufficient conditions on Ω which guarantee that the following property holds: there exists $\bar{k}(\Omega)$ such that, for each $k \geq \bar{k}(\Omega)$, problem $P(\varepsilon, \Omega)$, for $\varepsilon > 0$ small enough, has at least one solution blowing up as $\varepsilon \rightarrow 0$ at exactly k points. Exploiting the properties of the Green and Robin functions, we also prove that the blow up points approach the boundary of Ω as $k \rightarrow \infty$. Moreover we present some examples which show that $P(\varepsilon, \Omega)$ may have k -spike solutions of this type also when Ω is a contractible domain, not necessarily close to domains with nontrivial topology and, for $\varepsilon > 0$ small and k large enough, even when it is very close to star-shaped domains.

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Résumé

Nous démontrons que, si le domaine Ω satisfait certaines conditions, le problème $P(\varepsilon, \Omega)$ ci-dessous, pour $\varepsilon > 0$ suffisamment petit et k grand, admet des solutions qui pour $\varepsilon \rightarrow 0$ se concentrent et explosent exactement en k points. Nous prouvons aussi que le point de concentration s'approche du bord de Ω quand $k \rightarrow \infty$ et que le nombre de solutions est arbitrairement grand pourvu que ε soit suffisamment petit. La méthode de démonstration repose sur les propriétés des fonctions de Green et de Robin du laplacien sur Ω . De plus nous donnons des exemples qui montrent que parmi les ouverts bornés Ω qui satisfont nos conditions, il y en a aussi de contractiles, qui ne sont pas de perturbations d'ouverts non contractiles et peuvent même être arbitrairement proches de domaines étoilés.

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MSC: 35J20; 35J60; 35J65

Keywords: Supercritical problems; Multi-spike solutions; Contractible domains

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1. Introduction

Let us consider the following problem

$$P(\varepsilon, \Omega) \quad \begin{cases} \Delta u + u^{\frac{n+2}{n-2}+\varepsilon} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of \mathbb{R}^n , $n \geq 3$, and ε is a small positive parameter.

It is well known that semilinear elliptic problems of this form have at least one solution in any bounded domain Ω when the nonlinear term has superlinear and subcritical growth (i.e. $\varepsilon < 0$). On the contrary, when $\varepsilon \geq 0$ (i.e. the nonlinearity is critical or supercritical from the point of view of Sobolev embedding) the existence of solutions to $P(\varepsilon, \Omega)$ depends strongly on the geometrical properties of the domain Ω . Indeed, as a consequence of the well known Pohozaev's identity (see [21]), $P(\varepsilon, \Omega)$ cannot have any solution if $\varepsilon \geq 0$ and Ω is star-shaped while, on the other hand, it is easy to see that it has solution for all $\varepsilon \geq 0$ if Ω is for example an annulus, as pointed out by Kazdan and Warner in [12]. Hence many researches have been devoted to study the effect of the domain shape on the solvability of this problem when $\varepsilon \geq 0$.

For $\varepsilon = 0$, an existence result is proved by Coron in [6] for domains with a small hole (see also [22] for a multiplicity result in presence of several holes). In [2] Bahri and Coron proved a general result (answering in particular a question raised by Nirenberg) which guarantees the existence of a solution for $P(0, \Omega)$ when Ω has "nontrivial topology" (in the sense that suitable homology groups of Ω are nontrivial). Notice that this nontriviality condition (which covers a large class of domains) is only sufficient for the solvability but not necessary, as shown by some examples of contractible domains Ω where $P(0, \Omega)$ has solutions; these examples (which answer a question posed by Brezis) have been found by Dancer in [7], Ding in [10] and the second author in [17].

After Bahri and Coron result [2] the natural question arises whether the nontriviality of Ω in the sense of [2] is a sufficient condition for the existence of a solution to $P(\varepsilon, \Omega)$ even when $\varepsilon > 0$ (this question was posed by Rabinowitz, as reported by Brezis in [4]). The results proved in [18], [19] and [20] show that for $\varepsilon > 0$ this condition is neither sufficient, nor necessary. In fact, for $\varepsilon > 0$ large enough, nonexistence results hold also in some domains with nontrivial topology in the sense of [2] (see [18] and [19]) while, on the other hand, existence and multiplicity results hold for all $\varepsilon > 0$ in the same contractible domains considered in [17] (see [20]).

When $\varepsilon \rightarrow 0$ the problem presents some concentration phenomena, which have been first investigated in the subcritical case, i.e. when $\varepsilon \rightarrow 0$ from below: see Atkinson and Peletier [1], Brezis and Peletier [5], Rey [23–25], Han [11] and Bahri, Li and Rey [3]. In particular, in [3] Bahri, Li and Rey obtained multipeak solutions blowing-up as $\varepsilon \rightarrow 0$ from below at some points which are characterized as critical points of suitable functions defined in terms of the Green and Robin functions in Ω . In [9] similar phenomena are described in the supercritical case and, in domains with small holes, for $\varepsilon > 0$ small enough, it is proved the existence of solutions blowing-up at some pairs of points localized near the holes.

In this paper our aim is to analyse the effect of the domain shape on the existence and the multiplicity of solutions which blow-up at an arbitrarily large number of points. To this end we consider domains having radial symmetry with respect to a pair of variables (see condition (2.1)) and we prove that, under suitable assumptions on Ω , there exists $\bar{k}(\Omega)$ such that, for all $k \geq \bar{k}(\Omega)$, problem $P(\varepsilon, \Omega)$, for $\varepsilon > 0$ small enough, has solutions blowing-up as $\varepsilon \rightarrow 0$ at exactly k points, regularly placed around circles, whose distance from the boundary of Ω tends to zero as $k \rightarrow \infty$. Thus, in particular, we obtain that the number of geometrically distinct solutions tends to infinity as $\varepsilon \rightarrow 0$ from above and that the solutions may have an arbitrarily large number of blow-up points. Notice that, on the contrary, Bahri, Li and Rey proved in [3] that, when $\varepsilon \rightarrow 0$ from below, the blow-up points remain uniformly away from the boundary of Ω and that, for k large enough, there is no solution which blows up at k points as $\varepsilon \rightarrow 0$ from below.

It is worth pointing out that the existence and multiplicity results we prove in this paper (which hold also in domains with holes non necessarily small) do not require that the domain Ω has nontrivial topology or is a

perturbation of domains having different topological properties (as in [6,22]). In fact, some examples we present below show that these results hold also in contractible domains which (unlike the cases considered in [7,10,17]) are not required to be close to nontrivial domains. Indeed, $P(\varepsilon, \Omega)$ may have solutions for $\varepsilon > 0$ small enough even if Ω is very close to star-shaped domains (see Remark 2.9); on the contrary, when $\varepsilon > 0$ is large, a nonexistence result of Dancer and Zhang (see [8]) holds, which extends Pohozaev result to “nearly star-shaped” domains.

Let us remark that also in [20] the existence of solutions to $P(\varepsilon, \Omega)$ in some contractible domains Ω is proved for $\varepsilon > 0$ (indeed, for all $\varepsilon > 0$, not necessarily small); however the solutions obtained in [20] do not concentrate and blow up as $\varepsilon \rightarrow 0$, but converge to solutions of $P(0, \Omega)$. On the contrary, the solutions of $P(\varepsilon, \Omega)$ we obtain in this paper do not converge to solutions of $P(0, \Omega)$ (even if $P(0, \Omega)$ has solutions), since they vanish as $\varepsilon \rightarrow 0$; indeed, it is possible that $P(\varepsilon, \Omega)$ has solutions for $\varepsilon > 0$ small while $P(0, \Omega)$ has no solution (what happens, for example, if Ω is sufficiently close to a star-shaped domain).

The results we present here have been first announced in [13].

The paper is organized as follows: in Section 2 we state the main results (Theorems 2.1 and 2.3) and give some examples of domains, even contractible and close to star-shaped domains, where these theorems apply. If both theorems apply, we indicate a possible way to recognize that they give actually distinct solutions. In Section 3 we describe, under the symmetry conditions (2.1) and (2.2), the finite dimensional reduction method we use in the proofs. Finally, in Section 4 we use variational-topological arguments to find critical points of the finite dimensional energy functional and prove the main results.

2. Statement of the main theorems and examples

Let Ω be a smooth bounded domain of \mathbb{R}^n satisfying the following symmetry conditions:

$$(x_1, x_2, \dots, x_n) \in \Omega \Leftrightarrow (\sqrt{x_1^2 + x_2^2}, 0, x_3, \dots, x_n) \in \Omega, \tag{2.1}$$

$$(x_1, \dots, x_i, \dots, x_n) \in \Omega \Leftrightarrow (x_1, \dots, -x_i, \dots, x_n) \in \Omega, \quad \text{for } i = 3, \dots, n - 1. \tag{2.2}$$

Exploiting these symmetry properties, we look for solutions to problem $P(\varepsilon, \Omega)$ of the form

$$u_{k,\varepsilon}(x) = [n(n-2)]^{\frac{n-2}{4}} \sum_{i=1}^k \left[\frac{(\varepsilon \lambda_{k,\varepsilon}^2)^{1/(n-2)}}{(\varepsilon \lambda_{k,\varepsilon}^2)^{2/(n-2)} + |x - \xi_{i,k,\varepsilon}|^2} \right]^{\frac{n-2}{2}} + \theta_{k,\varepsilon}(x), \tag{2.3}$$

where $\theta_{k,\varepsilon} \rightarrow 0$ uniformly as $\varepsilon \rightarrow 0$, $\lambda_{k,\varepsilon}$ is a concentration parameter and the concentration points $\xi_{i,k,\varepsilon}$ belong to Ω and have the form

$$\xi_{i,k,\varepsilon} = \left(\rho_{k,\varepsilon} \cos \frac{2\pi}{k} i, \rho_{k,\varepsilon} \sin \frac{2\pi}{k} i, 0, \dots, 0, \tau_{k,\varepsilon} \right) \quad \text{for } i = 1, \dots, k. \tag{2.4}$$

More precisely, the method we use allows us to say that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \|\theta_{k,\varepsilon}\|_{L^\infty(\Omega)} < +\infty \tag{2.5}$$

and, for all $x = (x_1, \dots, x_n) = (\rho \cos \theta, \rho \sin \theta, x_3, \dots, x_n) \in \Omega$,

$$\theta_{k,\varepsilon}(x_1, x_2, \dots, x_i, \dots, x_n) = \theta_{k,\varepsilon}(x_1, x_2, \dots, -x_i, \dots, x_n) \quad \text{for } i = 2, \dots, n - 1, \tag{2.6}$$

$$\theta_{k,\varepsilon} \left(\rho \cos \left(\theta + \frac{2\pi}{k} \right), \rho \sin \left(\theta + \frac{2\pi}{k} \right), x_3, \dots, x_n \right) = \theta_{k,\varepsilon}(\rho \cos \theta, \rho \sin \theta, x_3, \dots, x_n). \tag{2.7}$$

Theorem 2.1. *Let Ω be a smooth bounded domain of \mathbb{R}^n , $n \geq 3$, satisfying symmetry conditions (2.1) and (2.2). Let us set*

$$S(\Omega) = \{(\rho, \tau) \in \mathbb{R}^2: \rho > 0, (\rho, 0, \dots, 0, \tau) \in \Omega\} \tag{2.8}$$

and consider the function $\Pi_\Omega : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\Pi_\Omega(\rho, \tau) = \begin{cases} \rho & \text{if } (\rho, \tau) \in \overline{S(\Omega)}, \\ +\infty & \text{otherwise.} \end{cases}$$

Assume that there exists in \mathbb{R}^2 an open subset A such that

$$0 < \inf_A \Pi_\Omega < \inf_{\partial A} \Pi_\Omega. \quad (2.9)$$

Then there exist $\bar{k} = \bar{k}(\Omega)$ and a sequence $(\varepsilon_k)_k$, $\varepsilon_k > 0$ for all $k \geq \bar{k}$, such that, for all $k \geq \bar{k}$ and $\varepsilon \in]0, \varepsilon_k]$, $P(\varepsilon, \Omega)$ has at least one solution $u_{k,\varepsilon}$ of the form (2.3) with $\theta_{k,\varepsilon}$ satisfying properties (2.5)–(2.7).

The concentration parameters $\lambda_{k,\varepsilon}$ behave as follows:

$$\lim_{\varepsilon \rightarrow 0} \lambda_{k,\varepsilon} > 0 \quad \forall k \geq \bar{k} \quad \text{and} \quad \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \lambda_{k,\varepsilon} = 0. \quad (2.10)$$

Moreover, if we set

$$M_A = \{(\rho, \tau) \in A : \Pi_\Omega(\rho, \tau) = \min_A \Pi_\Omega\},$$

the concentration points $\xi_{i,k,\varepsilon}$ satisfy

$$\lim_{k \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \text{dist}[(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}), M_A] = 0. \quad (2.11)$$

Remark 2.2. It is easy to verify that the sets $S(\Omega)$ and M_A introduced in Theorem 2.1 satisfy

$$\{(\rho, 0, \dots, 0, \tau) \in \mathbb{R}^n : \rho > 0, (\rho, \tau) \in \partial S(\Omega)\} \subset \partial \Omega, \quad (2.12)$$

$$M_A \subset \partial S(\Omega), \quad (2.13)$$

$$(\rho, \tau) \in M_A \Rightarrow \rho = \min_A \Pi_\Omega > 0. \quad (2.14)$$

Therefore, taking into account (2.11), we infer that the concentration points $\xi_{i,k,\varepsilon}$ approach the boundary of Ω as $k \rightarrow \infty$.

Theorem 2.3. Let Ω be a smooth bounded domain of \mathbb{R}^n , $n \geq 3$, satisfying symmetry conditions (2.1) and (2.2). Moreover, assume that there exist ρ_1, ρ_2, ρ_3 and τ_1, τ_2, τ_3 in \mathbb{R} such that $\tau_1 < \tau_2 < \tau_3$, $\max\{\rho_1, \rho_3\} < \rho_2$, Ω contains $(\rho_1, 0, \dots, 0, \tau_1)$ and $(\rho_3, 0, \dots, 0, \tau_3)$, while $(\rho_2, 0, \dots, 0, \tau_2) \notin \Omega$. Furthermore, assume that there exists a continuous function $\gamma : [\tau_1, \tau_3] \rightarrow \mathbb{R}^+$ such that $\gamma(\tau_1) = \rho_1$, $\gamma(\tau_3) = \rho_3$, $\gamma(\tau_2) > \rho_2$ and

$$(\gamma(\tau), 0, \dots, 0, \tau) \in \Omega \quad \forall \tau \in [\tau_1, \tau_3].$$

Then there exist $\bar{k} = \bar{k}(\Omega)$ and a sequence $(\varepsilon_k)_k$, $\varepsilon_k > 0 \forall k \geq \bar{k}$, such that, for all $k \geq \bar{k}$ and $\varepsilon \in]0, \varepsilon_k]$, $P(\varepsilon, \Omega)$ has at least one solution $u_{k,\varepsilon}$ of the form (2.3), where $\theta_{k,\varepsilon}$ satisfies properties (2.5)–(2.7).

Moreover, the concentration parameters $\lambda_{k,\varepsilon}$ satisfy

$$\liminf_{\varepsilon \rightarrow 0} \lambda_{k,\varepsilon} > 0 \quad \forall k \geq \bar{k} \quad \text{and} \quad \lim_{k \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \lambda_{k,\varepsilon} = 0, \quad (2.15)$$

while for the concentration points $\xi_{i,k,\varepsilon}$ we have

$$\lim_{k \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \text{dist}(\xi_{i,k,\varepsilon}, \partial \Omega) = 0 \quad \text{for } i = 1, \dots, k. \quad (2.16)$$

Remark 2.4. Let Ω be a smooth bounded domain of \mathbb{R}^n , $n \geq 3$, satisfying conditions (2.1) and (2.2). Assume that there exist $a, b \in \mathbb{R}$, $a < b$, $f : [a, b] \rightarrow \mathbb{R}^+$, $\delta > 0$ such that

$$\{(\rho, 0, \dots, 0, \tau) \in \mathbb{R}^n : \tau \in [a, b], f(\tau) < \rho < f(\tau) + \delta\} \subset \Omega \quad (2.17)$$

while

$$\{(\rho, 0, \dots, 0, \tau) \in \mathbb{R}^n : \tau \in [a, b], f(\tau) - \delta < \rho \leq f(\tau)\} \cap \Omega = \emptyset. \tag{2.18}$$

Then Theorem 2.1 applies for example when

$$0 < \min_{[a,b]} f < \min\{f(a), f(b)\},$$

while the assumptions of Theorem 2.3 are satisfied when

$$\max_{[a,b]} f > \max\{f(a), f(b)\}.$$

When both theorems apply, they give rise to actually distinct solutions, as one can infer from the different asymptotic behaviour of the corresponding critical values (see Propositions 4.2, 4.3 and Remark 4.4).

Notice that the symmetry conditions (2.1) and (2.2) required in Theorems 2.1 and 2.3 are satisfied, in particular, when the domain Ω has radial symmetry with respect to an axis; for example, when

$$(x_1, \dots, x_n) \in \Omega \iff \left(\left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2}, 0, \dots, 0, x_n \right) \in \Omega. \tag{2.19}$$

In particular, Ω satisfies property (2.19) if, for example, there exist a, b in \mathbb{R} and two functions $\rho_1, \rho_2 : [a, b] \rightarrow [0, +\infty[$ such that

$$\overline{\Omega} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : a \leq x_n \leq b, \rho_1^2(x_n) \leq \sum_{i=1}^{n-1} x_i^2 \leq \rho_2^2(x_n) \right\}. \tag{2.20}$$

For domains of this type, any positive local minimum or maximum for the function ρ_1 gives rise to a k -spike solution of problem $P(\varepsilon, \Omega)$ for $\varepsilon > 0$ small and k large enough.

The examples below are concerned with domains of this type.

Example 2.5. Consider a domain Ω of the form $\Omega = B(c_2, r_2) \setminus \overline{B(c_1, r_1)}$, where $B(c_1, r_1)$ and $B(c_2, r_2)$ are two balls of \mathbb{R}^n such that $0 < r_1 < r_2$ (clearly we can assume that the centres of the balls are on the x_n -axis).

If $|c_1 - c_2| < r_2 - r_1$, then Theorem 2.3 guarantees the existence of a k -spike solution of problem $P(\varepsilon, \Omega)$ when $\varepsilon > 0$ is small and k large enough (notice that r_1 is not required to be small).

If $r_2 - r_1 < |c_1 - c_2| < \sqrt{r_2^2 - r_1^2}$, Theorem 2.1 applies too, so we obtain two distinct k -spike solutions of $P(\varepsilon, \Omega)$ for $\varepsilon > 0$ small and k large (notice that in this case Ω is contractible).

Remark 2.6. One could object that $\Omega = B(c_2, r_2) \setminus \overline{B(c_1, r_1)}$ is not a smooth domain (as we require in Theorems 2.1 and 2.3) when $r_2 - r_1 \leq |c_1 - c_2| < r_2 + r_1$. However, it is clear that these theorems can be easily extended in order to cover the case of a domain with piecewise smooth boundary. On the other hand, standard smoothing techniques can be used to obtain a smooth domain preserving its geometrical properties.

The same remark holds for the piecewise smooth domains considered in the examples below.

Example 2.7. For all $\sigma > 0$ and $r > 1$, set

$$\Omega_r^\sigma = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : 1 < |x| < r, \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} > \sigma x_n \right\}. \tag{2.21}$$

Then Ω_r^σ is a contractible domain and we can apply both Theorems 2.1 and 2.3, obtaining two distinct k -spike solutions of $P(\varepsilon, \Omega_r^\sigma)$ for $\varepsilon > 0$ small and k large enough.

Contractible domains of this type have been first considered in [7,10,17] for the critical case (i.e. $\varepsilon = 0$). Arguing as in these papers one can show that for all $r > 1$ there exists $\sigma(r) > 0$ such that $P(0, \Omega_r^\sigma)$ has solution for all $\sigma \in]0, \sigma(r)[$, while it seems natural to expect that it has no solution if σ is large enough. On the contrary, Theorems 2.1 and 2.3 apply for all $\sigma > 0$ and $r > 1$ and give solutions which do not converge to solutions of $P(0, \Omega_r^\sigma)$ since they vanish as $\varepsilon \rightarrow 0$.

Example 2.8. One can give examples of domains Ω (even contractible) where the number of distinct solutions is arbitrarily large. It suffices to consider the same contractible domains already used in [17] and [20], which can be written in the form (2.20) for a function ρ_1 satisfying the following property: there exist $t_0 < t_1 < t_2 < \dots < t_{2h-1} < t_{2h}$ such that

$$\min\{\rho_1(t) : t \geq t_1\} > 0$$

and

$$\rho_1(t_{2i}) < \rho_1(t_{2i+1}), \quad \rho_1(t_{2i+1}) > \rho_1(t_{2i+2}) \quad \text{for } i = 0, \dots, h - 1.$$

Hence Theorems 2.1 and 2.3 apply and guarantee the existence of $2h$ solutions.

Remark 2.9. By using Theorems 2.1 and 2.3, for k large and $\varepsilon > 0$ small enough, we can prove the existence of k -spike solutions of problem $P(\varepsilon, \Omega)$ even in domains Ω which are “nearly starshaped” in the sense we specify below (a different definition of nearly starshaped domain is used in [8] in order to extend Pohožaev result to nonstarshaped domains when $\varepsilon > 0$ is large).

For any smooth bounded domain Ω of \mathbb{R}^n , let us set (as in [14])

$$\text{star}(\Omega) = \sup_{x_0 \in \Omega} \inf \left\{ v(x) \cdot \frac{x - x_0}{|x - x_0|} : x \in \partial\Omega \right\},$$

where $v(x)$ denotes the outward normal to $\partial\Omega$ in x . It is natural to say that Ω is nearly starshaped if $\text{star}(\Omega)^- = \max\{0, -\text{star}(\Omega)\}$ is small.

Our aim is to construct a sequence of smooth bounded domains $(\Omega_j)_j$ such that $\lim_{j \rightarrow \infty} \text{star}(\Omega_j) = 0$ and, for all $j \in \mathbb{N}$, $P(\varepsilon, \Omega_j)$ has k -spike solutions of the form (2.3) for k large and $\varepsilon > 0$ small enough.

To this end, it suffices to consider the above defined domain Ω_r^σ (see (2.21)). Since Ω_r^σ is not smooth, we consider the domain

$$\mathcal{N}_\delta(\Omega_r^\sigma) = \{x \in \mathbb{R}^n : \text{dist}(x, \Omega_r^\sigma) < \delta\}. \tag{2.22}$$

Notice that, if $0 < \delta < \sqrt{2}/2$, then $\mathcal{N}_\delta(\Omega_r^j)$ is a smooth bounded domain for all $j \geq 1$ and

$$\lim_{j \rightarrow \infty} \text{star}(\mathcal{N}_\delta(\Omega_r^j)) = 0.$$

Moreover, for all $j \geq 1$ we can apply both Theorems 2.1 and 2.3 and obtain two distinct k -spike solutions of $P(\varepsilon, \mathcal{N}_\delta(\Omega_r^j))$ for k large and $\varepsilon > 0$ small enough. Thus, for $\Omega_j = \mathcal{N}_\delta(\Omega_r^j)$, our assertion is proved.

Notice that we can also prove that for all positive integer h there exists a sequence of smooth bounded domains $(\Omega_{h,j})_j$ such that $\lim_{j \rightarrow \infty} \text{star}(\Omega_{h,j}) = 0$ and, for j large enough, $P(\varepsilon, \Omega_{h,j})$ has at least $2h$ k -spike solutions for k large and $\varepsilon > 0$ small enough. In fact, let us consider the bounded domain

$$\Omega_{r,h}^\sigma = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : 1 < |x| < r, \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} > \sigma x_n, \text{dist}(x, \mathcal{C}_m^\sigma) > 1 \text{ for } m = 1, \dots, h - 1 \right\}, \tag{2.23}$$

where

$$C_m^\sigma = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n-1} x_i^2 = 9m^2, x_n = \frac{3m}{\sigma} \right\}. \tag{2.24}$$

Now fix $\delta \in]0, \sqrt{2}/2[$ and set

$$\Omega_{h,j} = \mathcal{N}_\delta(\Omega_{j,h}^j) = \{x \in \mathbb{R}^n : \text{dist}(x, \Omega_{j,h}^j) < \delta\}. \tag{2.25}$$

Then $\Omega_{h,j}$ is a smooth bounded domain for all $j \geq 1$ and $\lim_{j \rightarrow \infty} \text{star}(\Omega_{h,j}) = 0$ for all positive integer h . Moreover, Theorems 2.1 and 2.3 apply and (taking also into account Propositions 4.2, 4.3 and Remark 4.4) guarantee that, for j large enough (depending on h), problem $P(\varepsilon, \Omega_{h,j})$ has at least $2h$ distinct k -spike solutions for k large and $\varepsilon > 0$ small enough. Thus the sequence $(\Omega_{h,j})_j$ satisfies the desired properties.

3. Preliminary results

In order to prove Theorems 2.1 and 2.3, we shall use a finite dimensional reduction procedure, introduced in [3] and [24] for subcritical and critical problems, suitably adapted to the supercritical case (see for example the references in [9] and, for a different approach, see also [16,27]).

Let us describe here this procedure. Consider the functions

$$\bar{U}_{\xi,\mu}(x) = [n(n-2)]^{\frac{n-2}{4}} \left(\frac{\mu}{\mu^2 + |x-\xi|^2} \right)^{\frac{n-2}{2}} \quad \forall \xi \in \mathbb{R}^n, \mu > 0. \tag{3.1}$$

It is well known (see [26]) that these functions, extremals for the Sobolev critical embedding, are all the positive solutions of problem

$$\begin{cases} \Delta U + U^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} U(x) = 0. \end{cases}$$

Let us denote by $U_{\xi,\mu}$ the projection of $\bar{U}_{\xi,\mu}$ onto $H_0^1(\Omega)$, namely the solution of problem

$$\begin{cases} -\Delta U_{\xi,\mu} = \bar{U}_{\xi,\mu}^{\frac{n+2}{n-2}} & \text{in } \Omega, \\ U_{\xi,\mu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

In other words $U_{\xi,\mu} = \bar{U}_{\xi,\mu} - \chi_{\xi,\mu}$, where $\chi_{\xi,\mu}$ is the solution of problem

$$\begin{cases} \Delta \chi_{\xi,\mu} = 0 & \text{in } \Omega, \\ \chi_{\xi,\mu} = \bar{U}_{\xi,\mu} & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Arguing as in [3,9,24] and taking also into account the symmetry properties of the domain, one can prove the following lemmas.

Lemma 3.1. *Let Ω be a smooth bounded domain satisfying conditions (2.1) and (2.2). Choose $\delta \in]0, 1[$ small enough in such a way that the set*

$$S_\delta(\Omega) = \{(\rho, \tau) \in S(\Omega) : \text{dist}[(\rho, \tau), \partial S(\Omega)] > \delta\}$$

is not empty ($S(\Omega)$ is introduced in Theorem 2.1, see (2.8)).

Then there exists a sequence $(\varepsilon_k)_k$, $\varepsilon_k > 0 \forall k \in \mathbb{N}$, such that for each $\varepsilon \in]0, \varepsilon_k[$ there exists a smooth map

$$\tilde{\theta}_{k,\varepsilon} : S_\delta(\Omega) \times]\delta, 1/\delta[\rightarrow H_0^1(\Omega) \cap L^\infty(\Omega)$$

having the following property: the function $u_{k,\varepsilon}(\rho, \tau, \lambda) \in H_0^1(\Omega) \cap L^\infty(\Omega)$, defined by

$$u_{k,\varepsilon}(\rho, \tau, \lambda) = \sum_{i=1}^k U_{i,\varepsilon}(\rho, \tau, \lambda) + \tilde{\theta}_{k,\varepsilon}(\rho, \tau, \lambda) \quad \forall (\rho, \tau) \in S_\delta(\Omega), \quad \forall \lambda \in]\delta, 1/\delta[$$

where $U_{i,\varepsilon}(\rho, \tau, \lambda) = U_{\xi_{i,k},(\lambda^{2\varepsilon})^{1/(n-2)}}$ with $\xi_{i,k} = (\rho \cos \frac{2\pi}{k}i, \rho \sin \frac{2\pi}{k}i, 0, \dots, 0, \tau)$, solves problem $P(\varepsilon, \Omega)$ if and only if (ρ, τ, λ) is a critical point for the function

$$F_{k,\varepsilon}(\rho, \tau, \lambda) = J_\varepsilon(u_{k,\varepsilon}(\rho, \tau, \lambda)),$$

where J_ε is the functional defined by

$$J_\varepsilon(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p+1+\varepsilon} \int_{\Omega} |u|^{p+1+\varepsilon} dx, \quad \text{with } p = \frac{n+2}{n-2}.$$

Moreover, the function $\tilde{\theta}_{k,\varepsilon}(\rho, \tau, \lambda)$ satisfies

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1/2} \|\tilde{\theta}_{k,\varepsilon}(\rho, \tau, \lambda)\|_{L^\infty(\Omega)} < +\infty \quad (3.4)$$

uniformly with respect to $(\rho, \tau) \in S_\delta(\Omega)$ and $\lambda \in]\delta, 1/\delta[$.

Furthermore, for all $(\rho, \tau) \in S_\delta(\Omega)$ and $\lambda \in]\delta, 1/\delta[$,

$$\tilde{\theta}_{k,\varepsilon}(\rho, \tau, \lambda) = \tilde{\theta}_{k,\varepsilon}(\rho, \tau, \lambda) \circ T_i \quad \text{for } i = 2, \dots, n-1 \quad (3.5)$$

and

$$\tilde{\theta}_{k,\varepsilon}(\rho, \tau, \lambda) = \tilde{\theta}_{k,\varepsilon}(\rho, \tau, \lambda) \circ \Sigma_k, \quad (3.6)$$

where $T_i, \Sigma_k : \Omega \rightarrow \Omega$ are defined by

$$T_i(x_1, \dots, x_i, \dots, x_n) = (x_1, \dots, -x_i, \dots, x_n),$$

$$\Sigma_k(\rho \cos \theta, \rho \sin \theta, x_3, \dots, x_n) = (\rho \cos(\theta + 2\pi/k), \rho \sin(\theta + 2\pi/k), x_3, \dots, x_n).$$

Remark 3.2. From (3.4)–(3.6) one can easily infer that the function $\theta_{k,\varepsilon}$ verifies (2.5)–(2.7). In fact (see (3.3))

$$\theta_{k,\varepsilon} = \tilde{\theta}_{k,\varepsilon}(\rho, \tau, \lambda) - \sum_{i=1}^k \chi_{\xi_{i,k},(\lambda^{2\varepsilon})^{1/(n-2)}}.$$

Lemma 3.3. Let $F_{k,\varepsilon} : S_\delta(\Omega) \times]\delta, 1/\delta[\rightarrow \mathbb{R}$ be the function introduced in Lemma 3.1. Then

$$F_{k,\varepsilon}(\rho, \tau, \lambda) = kS_n + k\bar{a}_n \varepsilon \lg \varepsilon + \varepsilon [k\bar{b}_n + \bar{c}_n \psi_k(\rho, \tau, \lambda)] + \varphi_{k,\varepsilon}(\rho, \tau, \lambda),$$

where $S_n, \bar{a}_n, \bar{b}_n, \bar{c}_n$ are suitable constants depending only on the dimension n , $\varepsilon^{-1} \varphi_{k,\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$ in $C^1(S_\delta(\Omega) \times]\delta, 1/\delta[)$ and

$$\psi_k(\rho, \tau, \lambda) = \frac{\lambda^2}{2} \bar{p}_n \left[\sum_{i=1}^k H(\xi_{i,k}, \xi_{i,k}) - 2 \sum_{1 \leq i < j \leq k} G(\xi_{i,k}, \xi_{j,k}) \right] + k \lg \lambda, \quad (3.7)$$

where \bar{p}_n is a positive constant depending only on n , $G(x, y)$ denotes the Green function of $-\Delta$ with zero Dirichlet condition on the boundary of Ω and $H(x, y)$ its regular part.

Now let us set

$$\psi_{k,\varepsilon}(\rho, \tau, \lambda) = \frac{1}{\varepsilon \bar{c}_n} [F_{k,\varepsilon}(\rho, \tau, \lambda) - kS_n - k\bar{a}_n \varepsilon \lg \varepsilon - \varepsilon k\bar{b}_n]. \quad (3.8)$$

It is clear that $(\rho, \tau, \lambda) \in S_\delta(\Omega) \times]\delta, 1/\delta[$ is a critical point for $F_{k,\varepsilon}$ if and only if it is a critical point for $\psi_{k,\varepsilon}$. On the other hand

$$\psi_{k,\varepsilon}(\rho, \tau, \lambda) = \psi_k(\rho, \tau, \lambda) + \frac{1}{\varepsilon \bar{c}_n} \varphi_{k,\varepsilon}(\rho, \tau, \lambda), \tag{3.9}$$

so $\psi_{k,\varepsilon} \rightarrow \psi_k$ as $\varepsilon \rightarrow 0$ in $C^1(S_\delta(\Omega) \times]\delta, 1/\delta[)$. Thus, every critical point for ψ_k , which persists with respect to small C^1 perturbations, gives rise to critical points for $\psi_{k,\varepsilon}$ and then to solutions $u_{k,\varepsilon}$ for $P(\varepsilon, \Omega)$ of the form (2.3) with $\theta_{k,\varepsilon}$ satisfying (2.5)–(2.7).

4. Proof of the main theorems

By Lemmas 3.1 and 3.3, our problem reduces to finding critical points of ψ_k , which persist with respect to small C^1 perturbations.

Taking into account the symmetry of Ω , it is clear that

$$\psi_k(\rho, \tau, \lambda) = k \left[\frac{\lambda^2}{2} \Phi_k(\rho, \tau) + \lg \lambda \right] \tag{4.1}$$

with

$$\Phi_k(\rho, \tau) = \bar{p}_n \left[H(\xi_{1,k}, \xi_{1,k}) - \sum_{i=2}^k G(\xi_{1,k}, \xi_{i,k}) \right]. \tag{4.2}$$

Notice that any critical point (ρ, τ, λ) for ψ_k must satisfy condition

$$\lambda^2 = [-\Phi_k(\rho, \tau)]^{-1}, \tag{4.3}$$

which is possible only if $\Phi_k(\rho, \tau) < 0$, and (ρ, τ) must be a critical point for Φ_k ; conversely, if (ρ, τ) is a critical point for Φ_k and $\Phi_k(\rho, \tau) < 0$, then (ρ, τ, λ) , with $\lambda = [-\Phi_k(\rho, \tau)]^{-1/2}$, is a critical point for ψ_k .

Thus, finding critical points for ψ_k is equivalent to finding critical points (ρ, τ) for Φ_k , such that $\Phi_k(\rho, \tau) < 0$. The following lemma is a crucial step in this direction.

Lemma 4.1. *Let Ω be a smooth bounded domain of \mathbb{R}^n , $n \geq 3$, satisfying symmetry conditions (2.1) and (2.2). Then the function Φ_k (see (4.2)) behaves as follows:*

- (a) $\Phi_k(\rho, \tau) \rightarrow +\infty$ as $(\rho, \tau) \rightarrow (\hat{\rho}, \hat{\tau})$ for all $(\hat{\rho}, \hat{\tau}) \in \partial S(\Omega)$ such that $\hat{\rho} > 0$; moreover $\frac{\partial \Phi_k}{\partial \nu}(\rho, \tau) \rightarrow +\infty$ as $(\rho, \tau) \rightarrow (\hat{\rho}, \hat{\tau})$, where ν denotes the outward normal to the boundary of $S_\delta(\Omega)$ for $\delta = \text{dist}[(\rho, \tau), \partial S(\Omega)]$ (notice that $\partial S_\delta(\Omega)$ is smooth near (ρ, τ) if (ρ, τ) lies in a suitable neighbourhood of $(\hat{\rho}, \hat{\tau})$);
- (b) there exists a sequence $(c_k)_k$ in \mathbb{R} , $c_k \rightarrow +\infty$, such that

$$\frac{1}{c_k} \Phi_k(\rho, \tau) \geq -\rho^{2-n} \quad \forall (\rho, \tau) \in S(\Omega), \quad \forall k \in \mathbb{N} \tag{4.4}$$

and

$$\lim_{k \rightarrow \infty} \frac{1}{c_k} \Phi_k(\rho, \tau) = -\rho^{2-n} \quad \forall (\rho, \tau) \in S(\Omega); \tag{4.5}$$

- (c) for all $\tau_0 \in \mathbb{R}$ such that $x_0 = (0, 0, \dots, \tau_0) \in \partial \Omega$, there exists $r_0 > 0$ such that

$$\inf \left\{ \frac{\partial \Phi_k}{\partial \nu_0}(\sqrt{x_1^2 + x_2^2}, x_n): (x_1, \dots, x_n) \in \Omega, 0 < x_1^2 + x_2^2 < r_0^2, |x_n - \tau_0| < r_0 \right\} > 0, \tag{4.6}$$

where ν_0 denotes the outward normal to the boundary of Ω at x_0 .

Proof. Firstly, notice that

$$G(x, y) = \omega_n |x - y|^{2-n} - H(x, y) \quad \forall x, y \in \Omega$$

for a suitable constant $\omega_n > 0$. Therefore

$$\Phi_k(\rho, \tau) = \bar{p}_n \left[\sum_{i=1}^k H(\xi_{1,k}, \xi_{i,k}) - \omega_n \sum_{i=2}^k |\xi_{1,k} - \xi_{i,k}|^{2-n} \right]. \tag{4.7}$$

For the proof of (a) observe that, if $(\rho, \tau) \rightarrow (\hat{\rho}, \hat{\tau}) \in \partial S(\Omega)$ and $\hat{\rho} > 0$, then the points $\xi_{i,k}$ (for $i = 1, \dots, k$) approach the boundary of Ω . Therefore we obtain that $\Phi_k(\rho, \tau) \rightarrow +\infty$ because $H(\xi_{1,k}, \xi_{1,k}) \rightarrow +\infty$ while the other terms (i.e., $H(\xi_{1,k}, \xi_{i,k})$ and $|\xi_{1,k} - \xi_{i,k}|^{2-n}$ for $i = 2, \dots, k$) remain bounded; in analogous way we infer that $\frac{\partial \Phi_k}{\partial \nu}(\rho, \tau) \rightarrow +\infty$ from the fact that $\frac{\partial}{\partial \nu} H(\xi_{1,k}, \xi_{1,k}) \rightarrow +\infty$ while $\frac{\partial}{\partial \nu} H(\xi_{1,k}, \xi_{i,k})$ and $\frac{\partial}{\partial \nu} |\xi_{1,k} - \xi_{i,k}|^{2-n}$ remain bounded for $i = 2, \dots, k$.

In order to prove (b), notice that

$$\Phi_k(\rho, \tau) = \bar{p}_n \left[\sum_{i=1}^k H(\xi_{1,k}, \xi_{i,k}) - \omega_n \rho^{2-n} \sum_{i=2}^k |\bar{\xi}_{1,k} - \bar{\xi}_{i,k}|^{2-n} \right] \tag{4.8}$$

with

$$\bar{\xi}_{i,k} = \left(\cos \frac{2\pi}{k} i, \sin \frac{2\pi}{k} i, 0, \dots, 0 \right) \quad \forall i = 1, \dots, k.$$

Now set $c_k = \bar{p}_n \omega_n \sum_{i=2}^k |\bar{\xi}_{1,k} - \bar{\xi}_{i,k}|^{2-n}$ and observe that, as $k \rightarrow \infty$,

$$\frac{c_k}{k} \rightarrow \frac{\bar{p}_n \omega_n}{2\pi} \int_0^{2\pi} [(1 - \cos t)^2 + \sin^2 t]^{\frac{2-n}{2}} dt = +\infty, \tag{4.9}$$

while

$$\frac{1}{k} \sum_{i=1}^k H(\xi_{1,k}, \xi_{i,k}) \rightarrow \frac{1}{2\pi} \int_0^{2\pi} H(\gamma_{\rho,\tau}(0), \gamma_{\rho,\tau}(t)) dt, \tag{4.10}$$

where $\gamma_{\rho,\tau} : [0, 2\pi] \rightarrow \Omega$ is defined by

$$\gamma_{\rho,\tau}(t) = (\rho \cos t, \rho \sin t, 0, \dots, 0, \tau).$$

Hence assertion (b) follows easily taking into account that H is positive and

$$\int_0^{2\pi} H(\gamma_{\rho,\tau}(0), \gamma_{\rho,\tau}(t)) dt < +\infty \quad \forall (\rho, \tau) \in S(\Omega). \tag{4.11}$$

For the proof of (c), let us first notice that $v_0 = (0, 0, \dots, 0, \pm 1)$ because Ω satisfies symmetry conditions (2.1) and (2.2). If we consider for example the case $v_0 = (0, 0, \dots, 0, 1)$, then (4.6) is equivalent to

$$\inf \left\{ \frac{\partial \Phi_k}{\partial \tau}(\rho, \tau) : (\rho, \tau) \in S(\Omega), \rho < r_0, |\tau - \tau_0| < r_0 \right\} > 0. \tag{4.12}$$

Now observe that $\frac{\partial \Phi_k}{\partial \tau}(\rho, \tau) = \bar{p}_n \sum_{i=1}^k \frac{\partial}{\partial \tau} H(\xi_{1,k}, \xi_{i,k})$; so (4.12) follows because $\sum_{i=1}^k \frac{\partial}{\partial \tau} H(\xi_{1,k}, \xi_{i,k}) \rightarrow +\infty$ as $\rho \rightarrow 0$ and $\tau \rightarrow \tau_0$ ($(\rho, \tau) \in S(\Omega)$). \square

Proof of Theorem 2.1. Let us prove first that there exists \bar{k} such that for all $k \geq \bar{k}$ the infimum $\inf_{A \cap S(\Omega)} \Phi_k$ is achieved and

$$\lim_{k \rightarrow \infty} \min_{A \cap S(\Omega)} \Phi_k = -\infty. \tag{4.13}$$

In fact, because of (2.9) we can choose $(\bar{\rho}, \bar{\tau}) \in A \cap S(\Omega)$ such that $0 < \bar{\rho} < \inf_{\partial A} \Pi_\Omega$. Therefore, by Lemma 4.1,

$$\lim_{k \rightarrow \infty} \frac{1}{c_k} \Phi_k(\bar{\rho}, \bar{\tau}) = -\bar{\rho}^{2-n} < -(\inf_{\partial A} \Pi_\Omega)^{2-n}. \tag{4.14}$$

On the other hand, for all $k \in \mathbb{N}$,

$$\frac{1}{c_k} \Phi_k(\rho, \tau) \geq -\rho^{2-n} \geq -(\inf_{\partial A} \Pi_\Omega)^{2-n} \quad \forall (\rho, \tau) \in \partial A \cap S(\Omega) \tag{4.15}$$

and (since we have assumed $\inf_A \Pi_\Omega > 0$) by (a) of Lemma 4.1 we have

$$\lim_{(\rho, \tau) \rightarrow (\hat{\rho}, \hat{\tau})} \Phi_k(\rho, \tau) = +\infty \quad \forall (\hat{\rho}, \hat{\tau}) \in \bar{A} \cap \partial S(\Omega). \tag{4.16}$$

Therefore $\inf_{A \cap S(\Omega)} \Phi_k$ is achieved for k large enough and

$$\min_{A \cap S(\Omega)} \Phi_k \leq \Phi_k(\bar{\rho}, \bar{\tau}), \tag{4.17}$$

which implies (4.13) by (4.14) because $\lim_{k \rightarrow \infty} c_k = +\infty$.

Now, for k large enough and for all $c \in]\min_{A \cap S(\Omega)} \Phi_k, -c_k(\inf_{\partial A} \Pi_\Omega)^{2-n}[$, let us set

$$\Phi_k^c = \{(\rho, \tau) \in A \cap S(\Omega) : \Phi_k(\rho, \tau) \leq c\}, \tag{4.18}$$

$$V_k^c = \{(\rho, \tau, \lambda) \in \mathbb{R}^3 : (\rho, \tau) \in \Phi_k^c, \lambda = [-\Phi_k(\rho, \tau)]^{-1/2}\}. \tag{4.19}$$

It is easy to verify that (ρ, τ) is a minimum point for Φ_k on $A \cap S(\Omega)$ if and only if $(\rho, \tau, [-\Phi_k(\rho, \tau)]^{-1/2})$ is a minimum point for ψ_k on V_k^c .

Let us fix $\eta \in]0, (-\min_{A \cap S(\Omega)} \Phi_k)^{-1/2}[$ and set

$$S_k^\eta = \{(\rho, \tau, \lambda) \in \mathbb{R}^3 : (\rho, \tau) \in \Phi_k^c, |\lambda - [-\Phi_k(\rho, \tau)]^{-1/2}| \leq \eta\}. \tag{4.20}$$

It is easy to verify that

$$\begin{aligned} & \min\{\psi_k(\rho, \tau, \lambda) : \Phi_k(\rho, \tau) = c, \lambda = (-c)^{-1/2}\} \\ & > \min_{V_k^c} \psi_k = \max\{\psi_k(\rho, \tau, \lambda) : (\rho, \tau, \lambda) \in S_k^\eta, \Phi_k(\rho, \tau) = \min_{A \cap S(\Omega)} \Phi_k\} \\ & > \max\{\psi_k(\rho, \tau, \lambda) : \Phi_k(\rho, \tau) = \min_{A \cap S(\Omega)} \Phi_k, |\lambda - [-\Phi_k(\rho, \tau)]^{-1/2}| = \eta\}. \end{aligned}$$

Then we can choose $c > \min_{A \cap S(\Omega)} \Phi_k$, sufficiently close to $\min_{A \cap S(\Omega)} \Phi_k$, in such a way that

$$\max\{\psi_k(\rho, \tau, \lambda) : (\rho, \tau) \in \Phi_k^c, |\lambda - [-\Phi_k(\rho, \tau)]^{-1/2}| = \eta\} < \min_{V_k^c} \psi_k. \tag{4.21}$$

Moreover, there exists $\bar{\eta} \in]0, \eta[$, small enough, such that

$$\begin{aligned} & \max\{\psi_k(\rho, \tau, \lambda) : (\rho, \tau, \lambda) \in S_k^\eta, \Phi_k(\rho, \tau) = \min_{A \cap S(\Omega)} \Phi_k\} \\ & < \min\{\psi_k(\rho, \tau, \lambda) : \Phi_k(\rho, \tau) = c, \lambda > 0, |\lambda - (-c)^{-1/2}| \leq \bar{\eta}\}. \end{aligned} \tag{4.22}$$

On the other hand, we have

$$\min\left\{\left|\frac{\partial \psi_k}{\partial \lambda}(\rho, \tau, \lambda)\right| : \Phi_k(\rho, \tau) = c, \lambda > 0, |\lambda - (-c)^{-1/2}| \geq \frac{\bar{\eta}}{2}\right\} > 0. \tag{4.23}$$

Since $\psi_{k,\varepsilon} \rightarrow \psi_k$ in $C^1(S_k^\eta)$ as $\varepsilon \rightarrow 0$, then for $\varepsilon > 0$ small enough (4.21)–(4.23) hold with $\psi_{k,\varepsilon}$ in place of ψ_k .

Moreover, notice that there exists no continuous map

$$\Theta : \{(\rho, \tau, \lambda) \in S_k^\eta : \Phi_k(\rho, \tau) = \min_{A \cap S(\Omega)} \Phi_k\} \rightarrow S_k^\eta \setminus V_k^c$$

satisfying $\Theta(\rho, \tau, \lambda) = (\rho, \tau, \lambda)$ for all (ρ, τ, λ) such that $|\lambda - [-\Phi_k(\rho, \tau)]^{-1/2}| = \eta$.

It follows, by standard arguments, that in S_k^η the function $\psi_{k,\varepsilon}$ has at least one critical point $(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon})$ such that

$$\min_{V_k^c} \psi_{k,\varepsilon} \leq \psi_{k,\varepsilon}(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) \leq \max\{\psi_{k,\varepsilon}(\rho, \tau, \lambda) : (\rho, \tau, \lambda) \in S_k^\eta, \Phi_k(\rho, \tau) = \min_{A \cap S(\Omega)} \Phi_k\}.$$

Now, letting $\varepsilon \rightarrow 0$, $c \rightarrow \min_{A \cap S(\Omega)} \Phi_k$ and $\eta \rightarrow 0$, we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \psi_{k,\varepsilon}(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) &= \min_{V_k^c} \psi_k = \max\{\psi_k(\rho, \tau, \lambda) : (\rho, \tau, \lambda) \in S_k^\eta, \Phi_k(\rho, \tau) = \min_{A \cap S(\Omega)} \Phi_k\}, \\ \lim_{\varepsilon \rightarrow 0} \Phi_k(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}) &= \min_{A \cap S(\Omega)} \Phi_k, \end{aligned} \tag{4.24}$$

$$\lim_{\varepsilon \rightarrow 0} \lambda_{k,\varepsilon} = [-\min_{A \cap S(\Omega)} \Phi_k]^{-1/2}. \tag{4.25}$$

In order to describe the behaviour as $k \rightarrow \infty$, let us consider the set

$$A_\sigma = \{(\rho, \tau) \in A : \text{dist}[(\rho, \tau), M_A] < \sigma\}. \tag{4.26}$$

Notice that $\inf_{A \setminus A_\sigma} \Pi_\Omega > \min_A \Pi_\Omega$; therefore, for k large enough,

$$\min_{A \cap S(\Omega)} \frac{1}{c_k} \Phi_k < -[\inf_{A \setminus A_\sigma} \Pi_\Omega]^{2-n} \leq \inf_{A \setminus A_\sigma} \frac{1}{c_k} \Phi_k. \tag{4.27}$$

Thus we infer that, for k large enough, all the minimum points for Φ_k on $A \cap S(\Omega)$ belong to A_σ . Moreover

$$\limsup_{k \rightarrow \infty} \min_{A \cap S(\Omega)} \frac{1}{c_k} \Phi_k \leq -[\inf_{A \setminus A_\sigma} \Pi_\Omega]^{2-n}. \tag{4.28}$$

Letting $\sigma \rightarrow 0$, we obtain (2.11) and

$$\lim_{k \rightarrow \infty} \min_{A \cap S(\Omega)} \frac{1}{c_k} \Phi_k = -[\min_A \Pi_\Omega]^{2-n}, \tag{4.29}$$

which, by (4.25), implies

$$\lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \lambda_{k,\varepsilon} = \lim_{k \rightarrow \infty} [-\min_{A \cap S(\Omega)} \Phi_k]^{-1/2} = 0. \quad \square \tag{4.30}$$

Proof of Theorem 2.3. Notice that

$$\frac{1}{c_k} \Phi_k(\gamma(\tau), \tau) \rightarrow -[\gamma(\tau)]^{2-n} \quad \text{as } k \rightarrow \infty, \tag{4.31}$$

uniformly with respect to τ in $[\tau_1, \tau_3]$, as one can verify arguing as in the proof of Lemma 4.1.

Hence, a direct computation shows that, as $k \rightarrow \infty$,

$$\frac{1}{2} \lg c_k + \frac{1}{k} \sup\{\psi_k(\gamma(\tau), \tau, \lambda) : \tau \in \{\tau_1, \tau_3\}, \lambda > 0\} \rightarrow -\frac{1}{2} + \frac{n-2}{2} \lg \max\{\rho_1, \rho_3\} \tag{4.32}$$

and

$$\frac{1}{2} \lg c_k + \frac{1}{k} \sup\{\psi_k(\gamma(\tau), \tau, \lambda) : \tau \in [\tau_1, \tau_3], \lambda > 0\} \rightarrow -\frac{1}{2} + \frac{n-2}{2} \lg \bar{\gamma}, \tag{4.33}$$

where $\bar{\gamma} = \max_{\tau \in [\tau_1, \tau_3]} \gamma(\tau)$.

Now observe that, since $\Phi_k(\rho, \tau) \geq -c_k \rho^{2-n}$ for each $(\rho, \tau) \in S(\Omega)$ and $k \in \mathbb{N}$, then

$$\begin{aligned} & \frac{1}{2} \lg c_k + \frac{1}{k} \inf \{ \psi_k(\rho, \tau, \lambda) : (\rho, \tau) \in S(\Omega), \rho > \rho_2, \tau = \tau_2, \lambda = (c_k \rho_2^{2-n})^{-1/2} \} \\ & \geq -\frac{1}{2} + \frac{n-2}{2} \lg \rho_2 \quad \forall k \in \mathbb{N}. \end{aligned} \tag{4.34}$$

Let us set

$$Q_k = \left\{ (\gamma(\tau), \tau, \lambda) : \tau \in [\tau_1, \tau_3], \lambda \in \left[\frac{1}{c_k}, 1 \right] \right\} \tag{4.35}$$

(notice that $\frac{1}{c_k} < (c_k \rho_2^{2-n})^{-1/2} < 1$ for k large enough).

A direct computation shows that

$$\frac{1}{2} \lg c_k + \frac{1}{k} \sup \left\{ \psi_k(\gamma(\tau), \tau, \lambda) : \lambda \in \left[\frac{1}{c_k}, 1 \right], \tau \in [\tau_1, \tau_3] \right\} \rightarrow -\infty \quad \text{as } k \rightarrow \infty. \tag{4.36}$$

Thus, from (4.32)–(4.36), since $\rho_2 > \max\{\rho_1, \rho_3\}$, for k large enough we have

$$\begin{aligned} \max_{\partial Q_k} \psi_k & < \inf \{ \psi_k(\rho, \tau, \lambda) : (\rho, \tau) \in S(\Omega), \rho > \rho_2, \tau = \tau_2, \lambda = (c_k \rho_2^{2-n})^{-1/2} \} \\ & \leq \max_{Q_k} \psi_k < \min \{ \psi_k(\rho, \tau, \lambda) : \rho \in [\bar{\gamma}_1, \bar{\gamma}_2], \tau = \tau_2, \lambda = (c_k \rho_2^{2-n})^{-1/2} \}, \end{aligned} \tag{4.37}$$

where $\bar{\gamma}_1, \bar{\gamma}_2$ are suitably chosen in such a way that $\rho_2 < \bar{\gamma}_1 < \gamma(\tau_2) < \bar{\gamma}_2$ and $(\rho, \tau_2) \in S(\Omega) \forall \rho \in [\bar{\gamma}_1, \bar{\gamma}_2]$ (this choice of $\bar{\gamma}_1$ and $\bar{\gamma}_2$ is indeed possible because of (a) of Lemma 4.1). Moreover, notice that (4.31) implies

$$\lim_{k \rightarrow \infty} \max \{ \Phi_k(\gamma(\tau), \tau) : \tau \in [\tau_1, \tau_3] \} = -\infty; \tag{4.38}$$

so k can be chosen large enough such that we have, in addition,

$$\max \{ \Phi_k(\gamma(\tau), \tau) : \tau \in [\tau_1, \tau_3] \} < 0. \tag{4.39}$$

Taking into account that $\psi_{k,\varepsilon} \rightarrow \psi_k$ as $\varepsilon \rightarrow 0$ uniformly on the compact subsets of $S(\Omega) \times \mathbb{R}^+$, we infer that, for $\varepsilon > 0$ small enough,

$$\begin{aligned} \max_{\partial Q_k} \psi_{k,\varepsilon} & < \inf \{ \psi_{k,\varepsilon}(\rho, \tau, \lambda) : \rho \in [\bar{\gamma}_1, \bar{\gamma}_2], \tau = \tau_2, \lambda = (c_k \rho_2^{2-n})^{-1/2} \} \\ & \leq \max_{Q_k} \psi_{k,\varepsilon} < \min \{ \psi_{k,\varepsilon}(\rho, \tau, \lambda) : \rho \in [\bar{\gamma}_1, \bar{\gamma}_2], \tau = \tau_2, \lambda = (c_k \rho_2^{2-n})^{-1/2} \}. \end{aligned} \tag{4.40}$$

Now, set $E_0 = \{ \tau_0 \in \mathbb{R} : (0, \dots, 0, \tau_0) \in \partial \Omega \}$ and observe that, since Ω is a smooth bounded domain satisfying condition (2.1), because of (c) of Lemma 4.1, we can choose $\bar{r} > 0$ small enough such that, for all $\tau_0 \in E_0$, the set

$$A_{\tau_0} = \{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_1^2 + x_2^2 < \bar{r}^2, x_3 = x_4 = \dots = x_{n-1} = 0, |x_n - \tau_0| < \bar{r} \} \tag{4.41}$$

satisfies:

$$\inf \left\{ \frac{\partial \Phi_k}{\partial \nu_0}(\sqrt{x_1^2 + x_2^2}, x_n) : (x_1, \dots, x_n) \in \Omega \cap A_{\tau_0} \right\} > 0 \tag{4.42}$$

and $(\nu(x), \nu_0) > 0$ for all $x \in A_{\tau_0} \cap \partial \Omega$, where $\nu(x)$ denotes the outward normal to the boundary of Ω at x .

Notice that (if $E_0 \neq \emptyset$) we have

$$\lim_{\delta \rightarrow 0} \sup \{ \psi_k(\rho, \tau, \lambda) : (\rho, \tau) \in S(\Omega), \rho = \delta, \lambda > 0, \text{dist}(\tau, E_0) > \bar{r}/2 \} = -\infty, \tag{4.43}$$

as one can obtain by a direct computation taking into account that, if $\bar{\tau} \notin E_0$, then $\Phi_k(\rho, \tau) \rightarrow -\infty$ as $(\rho, \tau) \rightarrow (0, \bar{\tau})$.

Now, let us point out that (a) of Lemma 4.1 implies

$$\liminf_{\delta \rightarrow 0} \left\{ \frac{\partial \Phi_k}{\partial \nu}(\rho, \tau) : (\rho, \tau) \in S(\Omega) \cap \partial S_\delta(\Omega), \rho > \delta, (\rho, 0, \dots, 0, \tau) \notin \bigcup_{\tau_0 \in E_0} A_{\tau_0} \right\} = +\infty, \tag{4.44}$$

where ν denotes the outward normal on $\partial S_\delta(\Omega)$ (notice that, for $\delta > 0$ small enough, $\partial S_\delta(\Omega)$ is smooth near any $(\rho, \tau) \in \partial S_\delta(\Omega)$, such that $\rho > \delta$). In fact, if $(\rho, \tau) \in S(\Omega)$, $\text{dist}[(\rho, \tau), \partial S(\Omega)] = \delta$, $\rho > \delta$ and $(\rho, 0, \dots, 0, \tau) \notin \bigcup_{\tau_0 \in E_0} A_{\tau_0}$, then (up to any subsequence) (ρ, τ) converges as $\delta \rightarrow 0$ to some point $(\hat{\rho}, \hat{\tau}) \in \partial S(\Omega)$, such that $\hat{\rho} > 0$.

Taking into account (4.43) and (4.44), we can now choose $\bar{\delta} > 0$ small enough such that $S_{\bar{\delta}}(\Omega)$ contains the points $(\gamma(\tau), \tau)$ for $\tau \in [\tau_1, \tau_3]$ and (ρ, τ_2) for $\rho \in [\bar{\gamma}_1, \bar{\gamma}_2]$ and, in addition, $\partial S_{\bar{\delta}}(\Omega)$ is smooth near all $(\rho, \tau) \in S_{\bar{\delta}}(\Omega)$ such that $\rho > \bar{\delta}$,

$$\inf \left\{ \frac{\partial \Phi_k}{\partial \nu}(\rho, \tau) : (\rho, \tau) \in S(\Omega) \cap \partial S_{\bar{\delta}}(\Omega), \rho > \bar{\delta}, (\rho, 0, \dots, 0, \tau) \notin \bigcup_{\tau_0 \in E_0} A_{\tau_0} \right\} > 0, \tag{4.45}$$

$$\sup \{ \psi_k(\rho, \tau, \lambda) : (\rho, \tau) \in S(\Omega), \rho = \bar{\delta}, \lambda > 0, \text{dist}(\tau, E_0) > \bar{r}/2 \} < \max_{\partial Q_k} \psi_k. \tag{4.46}$$

Now notice that, taking into account (4.39), we can choose a constant \bar{c} such that $\max\{\Phi_k(\gamma(\tau), \tau) : \tau \in [\tau_1, \tau_3]\} < \bar{c} < 0$. Hence, set $\Phi_k^{\bar{c}} = \{(\rho, \tau) \in S(\Omega) : \Phi_k(\rho, \tau) \leq \bar{c}\}$ and observe that

$$\sup \{ \psi_k(\rho, \tau, \lambda) : (\rho, \tau) \in \Phi_k^{\bar{c}} \cap S_{\bar{\delta}}(\Omega) \} \rightarrow -\infty \text{ as } \lambda \rightarrow 0 \text{ or } \lambda \rightarrow +\infty. \tag{4.47}$$

So there exists $\bar{\lambda}_1, \bar{\lambda}_2$ such that $0 < \bar{\lambda}_1 < 1/c_k, \bar{\lambda}_2 > 1$ and

$$\max \{ \psi_k(\rho, \tau, \lambda) : (\rho, \tau) \in \Phi_k^{\bar{c}} \cap S_{\bar{\delta}}(\Omega) \} < \max_{\partial Q_k} \psi_k \text{ for } \lambda = \bar{\lambda}_1 \text{ or } \lambda = \bar{\lambda}_2. \tag{4.48}$$

Taking into account the choice of \bar{c} , we can fix $\bar{\eta} > 0$ small enough such that

$$\min \{ \psi_k(\rho, \tau, \lambda) : \Phi_k(\rho, \tau) = \bar{c}, \lambda > 0, |\lambda - (-\bar{c})^{-1/2}| \leq \bar{\eta} \} > \max_{Q_k} \psi_k; \tag{4.49}$$

also observe that

$$\min \left\{ \left| \frac{\partial \psi_k}{\partial \lambda}(\rho, \tau, \lambda) \right| : \Phi_k(\rho, \tau) = \bar{c}, \lambda > 0, |\lambda - (-\bar{c})^{-1/2}| \geq \frac{\bar{\eta}}{2} \right\} > 0. \tag{4.50}$$

Notice that (4.45) implies that

$$\inf \left\{ \frac{\partial \psi_k}{\partial \nu}(\rho, \tau, \lambda) : (\rho, \tau) \in S(\Omega) \cap \partial S_{\bar{\delta}}(\Omega), \rho > \bar{\delta}, (\rho, 0, \dots, 0, \tau) \notin \bigcup_{\tau_0 \in E_0} A_{\tau_0}, \lambda \in [\bar{\lambda}_1, \bar{\lambda}_2] \right\} > 0; \tag{4.51}$$

moreover (4.42) yields (if $\bar{\delta} < \bar{r}$)

$$\inf \left\{ \frac{\partial \psi_k}{\partial \nu_0}(\sqrt{x_1^2 + x_2^2}, x_n, \lambda) : (\sqrt{x_1^2 + x_2^2}, x_n) \in S_{\bar{\delta}}(\Omega), (x_1, x_2, 0, \dots, 0, x_n) \in A_{\tau_0}, \lambda \in [\bar{\lambda}_1, \bar{\lambda}_2] \right\} > 0 \quad \forall \tau_0 \in E_0; \tag{4.52}$$

finally (as an immediate consequence of (4.46))

$$\sup \{ \psi_k(\rho, \tau, \lambda) : (\rho, \tau) \in S(\Omega), \rho = \bar{\delta}, \text{dist}(\tau, E_0) > \bar{r}/2, \lambda \in [\bar{\lambda}_1, \bar{\lambda}_2] \} < \max_{\partial Q_k} \psi_k. \tag{4.53}$$

Hence, since $\psi_{k,\varepsilon} \rightarrow \psi_k$ as $\varepsilon \rightarrow 0$ in any compact subset of $S(\Omega) \times \mathbb{R}^+$, for $\varepsilon > 0$ small enough (4.48)–(4.53) hold with $\psi_{k,\varepsilon}$ in place of ψ_k .

It follows that in $S_{\bar{\delta}}(\Omega) \cap \Phi_k^{\bar{c}} \times]\bar{\lambda}_1, \bar{\lambda}_2[$ there exists at least one critical point $(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon})$ for $\psi_{k,\varepsilon}$ such that

$$\inf\{\psi_{k,\varepsilon}(\rho, \tau, \lambda): \rho \in [\bar{\gamma}_1, \bar{\gamma}_2], \tau = \tau_2, \lambda = (c_k \rho_2^{2-n})^{-1/2}\} \leq \psi_{k,\varepsilon}(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) \leq \max_{Q_k} \psi_{k,\varepsilon}. \tag{4.54}$$

In fact, if no such a critical point there exists, one can prove by standard arguments that there exists a continuous map $\Gamma: Q_k \times [0, 1] \rightarrow \bar{S}_{\bar{\delta}}(\Omega) \cap \Phi_k^{\bar{c}} \times [\bar{\lambda}_1, \bar{\lambda}_2]$ such that:

$$\begin{aligned} \Gamma(\rho, \tau, \lambda, 0) &= (\rho, \tau, \lambda) \quad \forall (\rho, \tau, \lambda) \in Q_k, \\ \Gamma(\rho, \tau, \lambda, t) &= (\rho, \tau, \lambda) \quad \forall t \in [0, 1] \text{ if } (\rho, \tau, \lambda) \in \partial Q_k, \\ (\rho, \tau_2, (c_k \rho_2^{2-n})^{-1/2}) &\notin \Gamma(Q_k \times [0, 1]) \quad \text{for } \rho \in \{\bar{\gamma}_1, \bar{\gamma}_2\}, \\ \{\Gamma(\rho, \tau, \lambda, 1): (\rho, \tau, \lambda) \in Q_k\} \cap \{(\rho, \tau, \lambda): \rho \in [\bar{\gamma}_1, \bar{\gamma}_2], \tau = \tau_2, \lambda = (c_k \rho_2^{2-n})^{-1/2}\} &= \emptyset, \end{aligned}$$

which is impossible.

Let us analyse the asymptotic behaviour of $(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon})$ as $\varepsilon \rightarrow 0$. If (up to any subsequence) $(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) \rightarrow (\rho_k, \tau_k, \lambda_k)$, then $(\rho_k, \tau_k, \lambda_k) \in S_{\bar{\delta}}(\Omega) \cap \Phi_k^{\bar{c}} \times]\bar{\lambda}_1, \bar{\lambda}_2[$ and is a critical point for ψ_k (since $\psi_{k,\varepsilon} \rightarrow \psi_k$ in C^1 sense). Moreover, by (4.54),

$$\psi_{k,\varepsilon}(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) \rightarrow \psi_k(\rho_k, \tau_k, \lambda_k) \leq \max_{Q_k} \psi_k. \tag{4.55}$$

It follows that $\lambda_k > 0$ for all $k \geq \bar{k}$ and

$$\lim_{k \rightarrow \infty} \frac{1}{k} \psi_k(\rho_k, \tau_k, \lambda_k) = -\infty,$$

which implies $\lim_{k \rightarrow \infty} \lambda_k = 0$. Thus (2.15) is proved (notice that $\bar{\delta}, \bar{\lambda}_1$ and $\bar{\lambda}_2$ depend on k ; (2.15) and (2.16) imply that $\bar{\lambda}_1 \rightarrow 0$ and $\bar{\delta} \rightarrow 0$ as $k \rightarrow \infty$).

For the proof of (2.16) it suffices to show that if (up to any subsequence) $(\rho_k, \tau_k) \rightarrow (\bar{\rho}, \bar{\tau})$ as $k \rightarrow \infty$, then $(\bar{\rho}, 0, \dots, 0, \bar{\tau}) \in \partial\Omega$.

If $\bar{\rho} > 0$, then $(\bar{\rho}, \bar{\tau}) \in \partial S(\Omega)$ (which implies $(\bar{\rho}, 0, \dots, 0, \bar{\tau}) \in \partial\Omega$ since $\bar{\rho} > 0$). In fact, if $(\bar{\rho}, \bar{\tau}) \notin \partial S(\Omega)$, then, arguing as in the proof of assertion (b) in Lemma 4.1, we should have $\lim_{k \rightarrow \infty} \frac{1}{c_k} \frac{\partial \Phi_k}{\partial \rho}(\rho_k, \tau_k) = (n - 2)\bar{\rho}^{1-n} \neq 0$, which is impossible since $\frac{\partial \Phi_k}{\partial \rho}(\rho_k, \tau_k) = 0$ for all $k \geq \bar{k}$.

On the other hand, if $\bar{\rho} = 0$ and $\bar{\tau} \notin E_0$, then $\lim_{k \rightarrow \infty} \frac{1}{c_k} \Phi_k(\rho_k, \tau_k) = -\infty$ (which can be proved arguing as in the proof of (b) of Lemma 4.1). But this fact gives a contradiction because, letting $\varepsilon \rightarrow 0$ in (4.54) and using (4.4), we obtain $\liminf_{k \rightarrow \infty} \frac{1}{c_k} \Phi_k(\rho_k, \tau_k) \geq -\rho_2^{2-n} > -\infty$. Thus assertion (2.16) is proved too. \square

The following propositions describe the asymptotic behaviour of the critical values corresponding to the solutions given by Theorems 2.1 and 2.3 respectively.

Proposition 4.2. *The solution $u_{k,\varepsilon}$ given by Theorem 2.1 satisfies*

$$\lim_{k \rightarrow \infty} \left[\frac{1}{2} \lg c_k + \frac{1}{k} \lim_{\varepsilon \rightarrow 0} \psi_{k,\varepsilon}(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) \right] = -\frac{1}{2} + \frac{n-2}{2} \lg \min_A \Pi_{\Omega}. \tag{4.56}$$

Proof. The method we used in the proof of Theorem 2.1, in order to find the critical point $(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon})$ for $\psi_{k,\varepsilon}$, shows that (up to a subsequence)

$$(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) \rightarrow (\rho_k, \tau_k, \lambda_k) \quad \text{as } \varepsilon \rightarrow 0, \tag{4.57}$$

where

$$\Phi_k(\rho_k, \tau_k) = \min_{A \cap S(\Omega)} \Phi_k \quad \text{and} \quad \lambda_k = \left[- \min_{A \cap S(\Omega)} \Phi_k \right]^{-1/2}.$$

Moreover

$$\lim_{\varepsilon \rightarrow 0} \psi_{k,\varepsilon}(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) = \psi_k(\rho_k, \tau_k, \lambda_k).$$

Thus (4.56) follows easily from (4.29), taking into account that

$$\frac{1}{k} \psi_k(\rho_k, \tau_k, \lambda_k) = -\frac{1}{2} - \frac{1}{2} \lg[-\Phi_k(\rho_k, \tau_k)]. \quad \square$$

Proposition 4.3. *The solution $u_{k,\varepsilon}$ obtained in Theorem 2.3 satisfies*

$$\liminf_{k \rightarrow \infty} \left[\frac{1}{2} \lg c_k + \frac{1}{k} \liminf_{\varepsilon \rightarrow 0} \psi_{k,\varepsilon}(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) \right] \geq -\frac{1}{2} + \frac{n-2}{2} \lg \rho_2 \tag{4.58}$$

and

$$\limsup_{k \rightarrow \infty} \left[\frac{1}{2} \lg c_k + \frac{1}{k} \limsup_{\varepsilon \rightarrow 0} \psi_{k,\varepsilon}(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) \right] \leq -\frac{1}{2} + \frac{n-2}{2} \lg \left[\max_{\tau \in [\tau_1, \tau_3]} \gamma(\tau) \right]. \tag{4.59}$$

Proof. Let us use the same notations introduced in the proof of Theorem 2.3 and observe that (4.54) implies

$$\liminf_{\varepsilon \rightarrow 0} \psi_{k,\varepsilon}(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) \geq \inf \{ \psi_k(\rho, \tau, \lambda) : \rho \in [\bar{\gamma}_1, \bar{\gamma}_2], \tau = \tau_2, \lambda = (c_k \rho_2^{2-n})^{-1/2} \} \tag{4.60}$$

because $\psi_{k,\varepsilon} \rightarrow \psi_k$ as $\varepsilon \rightarrow 0$ uniformly on the compact subsets of $S(\Omega) \times \mathbb{R}^+$. Hence (4.58) follows easily from (4.34) taking into account that $\bar{\gamma}_1 > \rho_2$.

In analogous way, in order to prove (4.59), we infer from (4.54) that

$$\limsup_{\varepsilon \rightarrow 0} \psi_{k,\varepsilon}(\rho_{k,\varepsilon}, \tau_{k,\varepsilon}, \lambda_{k,\varepsilon}) \leq \max_{Q_k} \psi_k.$$

Thus (4.59) follows from (4.33) taking into account that

$$\max_{Q_k} \psi_k \leq \sup \{ \psi_k(\gamma(\tau), \tau, \lambda) : \tau \in [\tau_1, \tau_3], \lambda > 0 \}. \quad \square$$

Remark 4.4. Propositions 4.2 and 4.3 can be used in order to distinguish the solutions given by Theorems 2.1 and 2.3 when both theorems apply. For example, let Ω be as in Remark 2.4, that is there exist a, b, f and δ satisfying (2.17) and (2.18). If we assume that

$$0 < \min_{[a,b]} f < \min \{ f(a), f(b) \} \quad \text{and} \quad \max_{[a,b]} f > \max \{ f(a), f(b) \},$$

then both theorems apply; Theorem 2.1 guarantees the existence of a solution $u_{1,k,\varepsilon}$ such that

$$\lim_{k \rightarrow \infty} \left[\frac{1}{2} \lg c_k + \frac{1}{k} \lim_{\varepsilon \rightarrow 0} \psi_{k,\varepsilon}(\rho_{1,k,\varepsilon}, \tau_{1,k,\varepsilon}, \lambda_{1,k,\varepsilon}) \right] = -\frac{1}{2} + \frac{n-2}{2} \lg \min_{[a,b]} f,$$

while the solution $u_{2,k,\varepsilon}$ obtained in Theorem 2.3 satisfies

$$\liminf_{k \rightarrow \infty} \left[\frac{1}{2} \lg c_k + \frac{1}{k} \liminf_{\varepsilon \rightarrow 0} \psi_{k,\varepsilon}(\rho_{2,k,\varepsilon}, \tau_{2,k,\varepsilon}, \lambda_{2,k,\varepsilon}) \right] \geq -\frac{1}{2} + \frac{n-2}{2} \lg \max_{[a,b]} f.$$

Therefore, the solutions $u_{1,k,\varepsilon}$ and $u_{2,k,\varepsilon}$ are actually distinct for $\varepsilon > 0$ small and k large enough.

On the other hand, if for example we assume in addition that f has in $[a, b]$ only one maximum point τ_M and only one minimum point τ_m , then

$$\lim_{k \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} |\tau_{1,k,\varepsilon} - \tau_m| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} |\rho_{1,k,\varepsilon} - \min_{[a,b]} f| = 0,$$

while one can prove that

$$\lim_{k \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} |\tau_{2,k,\varepsilon} - \tau_M| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} |\rho_{2,k,\varepsilon} - \max_{[a,b]} f| = 0.$$

Furthermore, notice that Propositions 4.2 and 4.3 can be used to obtain distinct solutions also when Theorems 2.1 and 2.3 apply in different geometric situations (for example, for different choices of the subset A in Theorem 2.1 or of the function γ in Theorem 2.3). It is what happens, for example, in the case of the domain $\Omega_{h,j}$ considered in Remark 2.9. In this case, by suitable choices of the subset A and of the function γ , one can easily show that, for k large and $\varepsilon > 0$ small enough, the function $\psi_{k,\varepsilon}$ for the domain $\Omega_{h,j}$ has, for j large enough, at least $2h$ critical points, corresponding to $2h$ distinct critical values. Thus $P(\varepsilon, \Omega_{h,j})$ has at least $2h$ distinct k -spike solutions of the form (2.3).

Remark 4.5. Notice that, while the domain Ω presents a radial symmetry with respect to the pair of variables (x_1, x_2) (see condition (2.1)), the solutions $u_{k,\varepsilon}$ obtained in Theorems 2.1 and 2.3 do not present the same symmetry but satisfy only, for all $(\rho \cos \theta, \rho \sin \theta, x_3, \dots, x_n) \in \Omega$,

$$u_{k,\varepsilon} \left(\rho \cos \left(\theta + \frac{2\pi}{k} \right), \rho \sin \left(\theta + \frac{2\pi}{k} \right), x_3, \dots, x_n \right) = u_{k,\varepsilon}(\rho \cos \theta, \rho \sin \theta, x_3, \dots, x_n).$$

Moreover, it is obvious that, for all $\bar{\theta} \in [0, 2\pi]$, the function $u_{k,\varepsilon,\bar{\theta}}$ defined by

$$u_{k,\varepsilon,\bar{\theta}}(\rho \cos \theta, \rho \sin \theta, x_3, \dots, x_n) = u_{k,\varepsilon}(\rho \cos(\theta - \bar{\theta}), \rho \sin(\theta - \bar{\theta}), x_3, \dots, x_n) \tag{4.61}$$

is still a solution of problem $P(\varepsilon, \Omega)$.

Taking into account the method used in the proofs, it is clear that solutions of the form (4.61) (for suitable values of $\bar{\theta}$) persist with respect to any small perturbation of Ω , which preserves the symmetry properties (2.2) and

$$\left(\rho \cos \left(\theta + \frac{2\pi}{k} \right), \rho \sin \left(\theta + \frac{2\pi}{k} \right), x_3, \dots, x_n \right) \in \Omega \Leftrightarrow (\rho \cos \theta, \rho \sin \theta, x_3, \dots, x_n) \in \Omega. \tag{4.62}$$

Finally, notice that in [15] we consider problem $P(\varepsilon, \Omega)$ in a different class of domains Ω which do not satisfy condition (2.2) and, for k large and $\varepsilon > 0$ small enough, we obtain analogous existence and multiplicity results for positive solutions blowing up at exactly k points as $\varepsilon \rightarrow 0$.

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