



# On some results of Moser and of Bangert

## Sur quelques résultats de Moser et Bangert

P.H. Rabinowitz<sup>a,1,\*</sup>, E. Stredulinsky<sup>b</sup>

<sup>a</sup> *Department of Mathematics University of Wisconsin-Madison, Madison, WI 53706, USA*

<sup>b</sup> *Department of Mathematics University of Wisconsin-Richland, Richland Center, WI 53581, USA*

Received 5 May 2003; accepted 17 October 2003

Available online 13 February 2004

---

### Abstract

A new proof is given of results in (V. Bangert, *AIHP Anal. Nonlin.* 6 (1989) 95) on the existence of minimal (in the sense of Giaquinta and Guisti) heteroclinic solutions of a nonlinear elliptic PDE. Bangert's work is based on an earlier paper of Moser (*AIHP Anal. Nonlin.* 3 (1986) 229). Unlike (V. Bangert, *AIHP Anal. Nonlin.* 6 (1989) 95), the proof here is variational in nature, and involves the minimization of a 'renormalized' functional. It is meant to be the first step towards finding locally vs. globally minimal solutions of the PDE.

© 2004 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved

### Résumé

Nous donnons une démonstration nouvelle des résultats de Bangert sur l'existence d'une solution minimale (au sens de Giaquinta et Guisti) hétéroclinique d'un EDP elliptique non linéaire. Le travail de Bangert est basé sur un article de Moser (*AIHP Anal. Nonlin.* 3 (1986) 229). Contrairement à (V. Bangert, *AIHP Anal. Nonlin.* 6 (1989) 95), la démonstration est variationnelle en nature, et utilise la minimisation d'une fonctionnelle « renormalisée ». C'est une tentative de premier pas pour trouver des solutions minimales localement, plutôt que globalement de l'EDP.

© 2004 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved

*MSC:* 35J20; 58E15

*Keywords:* Monotone twist maps; Minimal heteroclinic; Aubry–Mather theory; Renormalized functional

---

---

\* Corresponding author.

*E-mail addresses:* [rabinowi@math.wisc.edu](mailto:rabinowi@math.wisc.edu) (P.H. Rabinowitz), [estredul@richland.uwc.edu](mailto:estredul@richland.uwc.edu) (E. Stredulinsky).

<sup>1</sup> This research was sponsored in part by the National Science Foundation under grant #MCS-8110556.

## 1. Introduction

Motivated by work of Aubry [3] and of Mather [7] on monotone twist maps, Moser [8] made the first steps towards finding analogues of their theory in the setting of quasilinear elliptic partial differential equations on  $\mathbb{R}^n$ . He studied the equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \mathcal{F}_{p_i}(x, u, Du) - \mathcal{F}_u(x, u, Du) = 0, \quad (1.1)$$

which arises formally as the Euler equation of the functional

$$\int_{\mathbb{R}^n} \mathcal{F}(x, u, Du) dx, \quad (1.2)$$

where  $\mathcal{F}(x, u, p)$  is e.g.  $C^3$  in its arguments and 1-periodic in the components of  $x \in \mathbb{R}^n$  and in  $u$ .  $\mathcal{F}$  further satisfies structural conditions that imply weak solutions of (1.1) are actually classical solutions of the equation [8]. Let  $\mathcal{N}_\alpha$  denote the set of solutions of (1.1) that have a prescribed  $\alpha \in \mathbb{R}^n$  as rotation vector, that are minimal in the sense of Giaquinta and Guisti [6], and whose graphs viewed on  $T^{n+1}$  have no self intersections. The notion of rotation vector here is the extension to  $\mathbb{R}^n$  of the usual rotation number used in dynamical systems. Likewise the minimal and non-self intersection properties are the analogues for solutions of (1.1) of properties of monotone twist maps. Among other things, Moser showed  $\mathcal{N}_\alpha \neq \emptyset$  for all  $\alpha \in \mathbb{R}^n$  and obtained various qualitative and quantitative properties for the members of  $\mathcal{N}_\alpha$ . E.g. for  $\mathcal{N}_0$ , he obtained solutions that are 1-periodic in  $x_1, \dots, x_n$ . Letting  $\mathcal{M}_\alpha$  denote the subset of  $\mathcal{N}_\alpha$  whose existence Moser established, he further proved  $\mathcal{M}_\alpha$  is an ordered set, i.e.  $v, w \in \mathcal{M}_\alpha$  implies  $v \equiv w$ ,  $v < w$ , or  $v > w$ .

Bangert carried Moser's analysis further in various ways. Among other things, he showed that whenever  $\alpha \in \mathbb{Q}^n$  and  $\mathcal{M}_\alpha$  possesses a gap, i.e. there are adjacent  $v_0 < w_0$  in  $\mathcal{M}_\alpha$ , then there is a  $U_1 \in \mathcal{N}_\alpha$  which is heteroclinic in an appropriate sense from  $v_0$  to  $w_0$  and likewise another solution of (1.1) heteroclinic from  $w_0$  to  $v_0$ . E.g. if  $\alpha = 0$  so  $v_0$  and  $w_0$  are adjacent members of  $\mathcal{M}_0$ , there is a  $U_1 \in \mathcal{N}_0$  heteroclinic in  $x_1$  from  $v_0$  to  $w_0$ . (Similarly there are members of  $\mathcal{N}_0$  heteroclinic in  $x_i$  from  $v_0$  to  $w_0$ ,  $2 \leq i \leq n$ .) Moreover if  $\mathcal{M}_0^1$  denotes the set of such heteroclinic solutions,  $\mathcal{M}_0^1$  is ordered. If there is a gap  $v_1 < w_1$  in  $\mathcal{M}_0^1$ , there is a  $U_2 \in \mathcal{M}_0^1$  which is also heteroclinic in  $x_2$  from  $v_1$  to  $w_1$ . Further gap conditions lead to yet more complicated heteroclinics.

There are analogues of  $U_i$  in the Aubry–Mather Theory: gaps between periodic invariant sets lead to the existence of heteroclinic invariant curves joining the periodic sets. Moreover given any formal chain of such heteroclinic invariant curves, there are actual invariant curves shadowing the chain. Thus it is natural to seek such shadowing solutions of (1.1). In recent years, variational methods have been devised to carry out such constructions in dynamical systems or PDE settings, see e.g. [10,5,1,2,9]. These methods require a variational characterization of the basic solutions such as  $U_1$  above. However Bangert's clever existence argument in [4] is not variational in nature. Therefore as a first step towards constructing more complex solutions of (1.1), in this paper we provide a variational approach to find the type of solutions Bangert discovered. This will only be done for the special but still significant setting of  $\alpha = 0$  and  $\mathcal{F}(x, u, p) = \frac{1}{2}|p|^2 + F(x, u)$  where

(F<sub>1</sub>)  $F \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ ,

(F<sub>2</sub>)  $F(x, u)$  is 1-periodic in  $x_1, \dots, x_n$  and  $u$ , and

(F<sub>3</sub>)  $F$  is even in  $x_1, \dots, x_n$ .

Thus (1.1) becomes

$$-\Delta u + F_u(x, u) = 0. \quad (\text{PDE})$$

A serious difficulty that has to be overcome in the variational approach to (PDE) is that the associated functionals defined on the classes of functions of the heteroclinic types of  $U_1, U_2$ , etc. are infinite. Thus to obtain  $U_1$  requires a renormalization, i.e. subtracting an infinite term from the natural functional. Likewise obtaining  $U_2$  requires a second renormalization, etc.

The existence of  $U_1$  will be carried out in Section 2 and its (Giaquinta–Guisti) minimality will be established in Section 3. An inductive argument will then be given in Section 4 to treat the general case.

## 2. The simplest heteroclinics

In this section, it will be shown how to obtain solutions of (PDE) that are heteroclinic in  $x_1$  from  $v_0$  to  $w_0$  where  $v_0 < w_0$  are an adjacent pair of solutions of (PDE) that are 1-periodic in  $x_1, \dots, x_n$ .

For ease of notation, set

$$L(u) = \frac{1}{2} |\nabla u|^2 + F(x, u)$$

with  $F$  satisfying (F<sub>1</sub>)–(F<sub>3</sub>). Let

$$\Gamma_0 = \{u \in W_{loc}^{1,2}(\mathbb{R}^n) \mid u \text{ is 1-periodic in } x_1, \dots, x_n\}.$$

For  $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{Z}^n$ , set  $T(\ell) = [\ell_1, \ell_1 + 1] \times \dots \times [\ell_n, \ell_n + 1]$ . Define

$$J_0(u) = \int_{T(0)} L(u) \, dx.$$

Finally define

$$c_0 = \inf_{u \in \Gamma_0} J_0(u). \tag{2.1}$$

In [3], Moser showed:

**Proposition 2.2.** *If  $\mathcal{M}_0 \equiv \{u \in \Gamma_0 \mid J_0(u) = c_0\}$ , then*

- 1°  $\mathcal{M}_0 \neq \emptyset$  and if  $u \in \mathcal{M}_0$ ,  $u$  is a classical solution of (PDE).
- 2°  $\mathcal{M}_0$  is an ordered set.

A further property of  $\mathcal{M}_0$  is:

**Corollary 2.3.** *If  $u \in \mathcal{M}_0$ ,  $u$  is even in  $x_1, \dots, x_n$ .*

**Proof.** Set  $\zeta_i(x) = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n)$ ,  $1 \leq i \leq n$ . If  $u \in \mathcal{M}_0$ , then  $u_i(x) = u(\zeta_i(x)) \in \mathcal{M}_0$  via (F<sub>3</sub>). If  $u_i \neq u$ , by 2° of Proposition 2.2, either (i)  $u_i(x) > u(x)$  or (ii)  $u_i(x) < u(x)$  for all  $x \in \mathbb{R}^n$ . If (i) occurs,

$$u_i(\zeta_i(x)) = u(x) < u_i(x) = u(\zeta_i(x)), \tag{2.4}$$

a contradiction. Similarly (ii) cannot hold. Thus  $u_i \equiv u$ ,  $1 \leq i \leq n$ .  $\square$

To continue, assume there is a gap in  $\mathcal{M}_0$ :

(\*)<sub>0</sub> There are adjacent  $v_0, w_0 \in \mathcal{M}_0$  with  $v_0 < w_0$ .

Set

$$\widehat{\Gamma}_1 = \{u \in W_{\text{loc}}^{1,2}(\mathbb{R}^n) \mid v_0 \leq u \leq w_0 \text{ a.e. and } u \text{ is 1-periodic in } x_2, \dots, x_n\}.$$

Now the renormalized functional  $J_1(u)$  for this setting can be introduced. For  $j \in \mathbb{Z}$  and  $k \in \{1, \dots, n\}$ , set

$$\tau_{-j}^k u(x) = u(x_1, \dots, x_k + j, \dots, x_n)$$

and for  $i \in \mathbb{Z}$ , define

$$J_{1,i}(u) = J_0(\tau_{-i}^1 u) - c_0. \quad (2.5)$$

Alternatively set  $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ , the usual basis in  $\mathbb{R}^n$ . Then

$$J_{1,i}(u) = \int_{T(i e_1)} L(u) \, dx - c_0.$$

Now define

$$J_1(u) = \sum_{i \in \mathbb{Z}} J_{1,i}(u).$$

The next proposition provides the properties of  $J_1$  that will be required here. Hypothesis  $(F_3)$  plays its main role in  $1^\circ$ – $2^\circ$ .

**Proposition 2.6.** For  $u \in \widehat{\Gamma}_1$ ,

$1^\circ$   $J_{1,i}(u) \geq 0$  for all  $i \in \mathbb{Z}$ .

$2^\circ$   $J_1(u) \geq 0$ .

$3^\circ$   $\int_{T(i e_1)} L(u) \, dx \leq J_1(u) + c_0$  for any  $i \in \mathbb{Z}$ .

$4^\circ$   $J_1$  is weakly lower semicontinuous (lsc) (with respect to  $W_{\text{loc}}^{1,2}(\mathbb{R}^n)$ ) on  $\widehat{\Gamma}_1$ .

**Proof.** For  $u \in \widehat{\Gamma}_1$  and  $x_1 \in [i + \frac{1}{2}, i + 1]$ , set

$$\varphi_i^+(u) = \tau_{-i}^1 u \quad (2.7)$$

and for  $x_1 \in [i, i + \frac{1}{2}]$ , set

$$\varphi_i^-(u) = \tau_{-i}^1 u. \quad (2.8)$$

Extend  $\varphi_i^\pm(u)$  first as even functions about  $x_1 = i + \frac{1}{2}$  and then 1-periodically in  $x_1$ . Continuing to denote these extensions by  $\varphi_i^\pm(u)$ , their definition implies  $\varphi_i^\pm(u) \in \Gamma_0$  and

$$J_{1,i}(u) = \frac{1}{2}(J_{1,i}(\varphi_i^+(u)) + J_{1,i}(\varphi_i^-(u))) \geq 0. \quad (2.9)$$

Thus  $1^\circ$  and hence  $2^\circ$  hold. By  $1^\circ$ ,

$$J_{1,i}(u) = \int_{T(i e_1)} L(u) \, dx - c_0 \leq J_1(u) \quad (2.10)$$

so  $3^\circ$  is valid. To prove  $4^\circ$ , note first that  $J_0$  is weakly lsc on  $\Gamma_0$ . Let  $(u_k)$  be a sequence in  $\widehat{\Gamma}_1$  and  $u_k \rightarrow u$  weakly in  $W_{\text{loc}}^{1,2}(\mathbb{R}^n)$  as  $k \rightarrow \infty$ . Set

$$J_{1;p,q}(u) \equiv \sum_p^q J_{1,i}(u).$$

Then by the weak lsc of  $J_0$ , (2.5), (2.9) and 1<sup>o</sup>,

$$\begin{aligned}
 J_{1;p,q}(u) &= \frac{1}{2} \sum_p^q (J_{1,i}(\varphi_i^+(u)) + J_{1,i}(\varphi_i^-(u))) \\
 &\leq \frac{1}{2} \sum_p^q (\liminf_{k \rightarrow \infty} J_{1,i}(\varphi_i^+(u_k)) + \liminf_{k \rightarrow \infty} J_{1,i}(\varphi_i^-(u_k))) \\
 &\leq \frac{1}{2} \liminf_{k \rightarrow \infty} \sum_p^q (J_{1,i}(\varphi_i^+(u_k)) + J_{1,i}(\varphi_i^-(u_k))) \\
 &= \liminf_{k \rightarrow \infty} J_{1;p,q}(u_k) \leq \liminf_{k \rightarrow \infty} J_1(u_k).
 \end{aligned}
 \tag{2.11}$$

Since (2.11) is valid for all  $p \leq q \in \mathbb{Z}$ ,

$$J_1(u) \leq \liminf_{k \rightarrow \infty} J_1(u_k)
 \tag{2.12}$$

and 4<sup>o</sup> holds.  $\square$

**Remark 2.13.** The argument used to prove 1<sup>o</sup> shows

$$c_0 = \inf_{u \in W^{1,2}(T(0))} J_0(u)$$

and if  $u \in W^{1,2}(T(0))$  with  $J(u) = c_0$ , then  $u \in \mathcal{M}_0$ . See e.g. [9] for a similar argument.

Next the class of functions that will be used to find a solution of (PDE) heteroclinic in  $x_1$  from  $v_0$  to  $w_0$  can be introduced. Set

$$\Gamma_1 = \{u \in \widehat{\Gamma}_1 \mid u \leq \tau_{-1}^1 u \text{ and } v_0 \neq u \neq w_0\}.$$

The members of  $\Gamma_1$  automatically are heteroclinic in  $x_1$  from  $v_0$  to  $w_0$  (in a weak sense) as the next result shows.

**Proposition 2.14.** *If  $u \in \Gamma_1$  and  $J_1(u) < \infty$ , then  $\tau_j^1 u \rightarrow v_0$  and  $\tau_{-j}^1 u \rightarrow w_0$  weakly in  $W^{1,2}(T(0))$  as  $j \rightarrow \infty$ .*

**Proof.** Since  $v_0 \leq \tau_j^1(u) \leq w_0$  for all  $j \in \mathbb{Z}$ , 3<sup>o</sup> of Proposition 2.6 shows  $(\tau_j^1 u)$  is bounded in  $W^{1,2}(T(0))$ . This with the monotonicity property,  $u \leq \tau_{-1}^1 u$ , shows there is a unique  $v \in W^{1,2}(T(0))$  such that  $\tau_j^1 u \rightarrow v$  weakly in  $W^{1,2}(T(0))$  and strongly in  $L^2(T(0))$  as  $j \rightarrow \infty$  and

$$v_0 \leq v \leq u \leq w_0.
 \tag{2.15}$$

Since  $J_1(u) < \infty$ ,  $J_0(\tau_j^1 u) \rightarrow c_0$  as  $|j| \rightarrow \infty$ . But  $J_0$  is weakly lsc so  $J_0(v) \leq c_0$ . Now Remark 2.13 shows  $J_0(v) = c_0$  and  $v \in \mathcal{M}_0$ . Hence (2.15), (\*), and  $u \neq w_0$  imply  $v = v_0$ . Similarly  $\tau_j^1 u \rightarrow w_0$  as  $j \rightarrow -\infty$  weakly in  $W^{1,2}(T(0))$ .  $\square$

**Remark 2.16.** In fact, by arguments in [9], the convergence of  $\tau_j^1 u$  is in  $W^{1,2}(T(0))$ .

Now to obtain a solution of (PDE) heteroclinic in  $x_1$  from  $v_0$  to  $w_0$ , set

$$c_1 = \inf_{u \in \Gamma_1} J_1(u).
 \tag{2.17}$$

**Theorem 2.18.** Let  $F$  satisfy (F<sub>1</sub>)–(F<sub>3</sub>) and let  $(*)_0$  hold. Then

- 1° There is a  $U_1 \in \Gamma_1$  such that  $J_1(U) = c_1$ .
- 2° Any such  $U_1$  is a classical solution of (PDE).
- 3°  $U_1$  is heteroclinic (in  $C^2$ ) from  $v_0$  to  $w_0$ , i.e.  $\|U_1 - v_0\|_{C^2(T(je_1))} \rightarrow 0$  as  $j \rightarrow -\infty$  and  $\|U_1 - w_0\|_{C^2(T(je_1))} \rightarrow 0$  as  $j \rightarrow \infty$ .
- 4°  $v_0 < U_1 < \tau_{-1}^1 U_1 < w_0$ .
- 5°  $\mathcal{M}_1 \equiv \{u \in \Gamma_1 \mid J_1(u) = c_1\}$  is an ordered set.
- 6° If  $u \in \mathcal{M}_1$ ,  $u$  is even in  $x_2, \dots, x_n$ .

**Remark 2.19.** In Section 3, it will be further shown that  $U_1$  is minimal in the sense of Giaquinta and Guisti.

**Proof of Theorem 2.18.** Let  $(u_k) \subset \Gamma_1$  be a minimizing sequence for (2.17). Normalize  $(u_k)$  so that for  $i < 0$ ,

$$\int_{T(je_1)} u_k \, dx \leq \frac{1}{2} \int_{T(0)} (v_0 + w_0) \, dx < \int_{T(0)} u_k \, dx. \quad (2.20)$$

That this can be done follows from Proposition 2.14. By 3° of Proposition 2.6,  $(u_k)$  is bounded in  $W_{\text{loc}}^{1,2}(\mathbb{R}^n)$ . Therefore there is a  $U_1 \in W_{\text{loc}}^{1,2}(\mathbb{R}^n)$ , 1-periodic in  $x_2, \dots, x_k$  such that along a subsequence,  $u_k \rightarrow U_1$  weakly in  $W_{\text{loc}}^{1,2}(\mathbb{R}^n)$  and strongly in  $L_{\text{loc}}^2(\mathbb{R}^n)$ . Hence by the properties of  $u_k$ ,  $v_0 \leq U_1 \leq \tau_{-1} U_1 \leq w_0$  and by (2.20),  $U_1 \in \Gamma_1$ . Thus

$$J_1(U_1) \geq c_1. \quad (2.21)$$

On the other hand, 4° of Proposition 2.6 and  $U_1 \in \Gamma_1$  imply

$$J_1(U_1) \leq c_1. \quad (2.22)$$

Combining (2.21)–(2.22) gives 1° of Theorem 2.18.

To verify 2°–5°, modifications of arguments from [9] will be employed. To prove that  $U_1$  satisfies (PDE), let  $z \in \mathbb{R}^n$  and let  $B_r(z)$  be a ball of radius  $r$  about  $z$  with  $r < \frac{1}{2}$ . For  $j \in \mathbb{Z}^n$ , set  $z_j = z + j$ . Let  $U^* = U_1$  in  $\mathbb{R}^n \setminus \bigcup_{j \in \mathbb{Z}^n} B_r(z_j)$  and  $U^* = u_j^*$  in  $B_r(z_j)$  where  $u_j^*$  minimizes

$$\int_{B_r(z_j)} L(\varphi) \, dx \quad (2.23)$$

over

$$\mathcal{S}_r(z_j) = \{\varphi \in W_{\text{loc}}^{1,2}(\mathbb{R}^n) \mid \varphi = U_1 \text{ in } \mathbb{R}^n \setminus B_r(z_j)\}. \quad (2.24)$$

As in Lemma 2.4 of [9], there is a  $u_j \in \mathcal{S}_r(z_j)$  minimizing (2.23) and any such minimizer of (2.23) is a solution of (PDE) in  $B_r(z_j)$ . A priori, there may not be a unique minimizer but as in Lemma 2.5 of [9], the set of minimizers is ordered and there is a unique smallest one which is chosen to be  $u_j^*$ . Note that

$$\int_{B_r(z_j)} L(U_1) \, dx \geq \int_{B_r(z_j)} L(u_j^*) \, dx \quad (2.25)$$

for all  $j \in \mathbb{Z}^n$ . Hence

$$J_1(U_1) \geq J_1(U^*). \quad (2.26)$$

We claim  $U^* \in \Gamma_1$  and therefore

$$J_1(U^*) \geq J_1(U_1). \quad (2.27)$$

But then by (2.25), for all  $j \in \mathbb{Z}^n$ ,

$$\int_{B_r(z_j)} L(U_1) \, dx = \int_{B_r(z_j)} L(u_j^*) \, dx.$$

Thus  $U_1$  is a minimizer of (2.23) in  $\mathcal{S}_r(z_j)$  and hence a solution of (PDE).

To verify that  $U^* \in \Gamma_1$ , observe first that the 1-periodicity of  $U_1$  in  $x_2, \dots, x_n$  implies the same for  $U^*$ . Thus  $U^* \in \Gamma_1$  if (a)  $v_0 \leq U^* \leq w_0$ , (b)  $U^* \leq \tau_{-1}^1 U^*$ , and (c)  $v_0 \not\equiv U^* \not\equiv w_0$ . To prove (a), since  $v_0 \leq U_1 \leq w_0$ , it suffices to show  $v_0 \leq u_j \leq w_0$  in  $B_r(z_j)$ . Suppose e.g.  $u_j(y) < v_0(y)$  for some  $y \in B_r(z_j)$ . Set  $\psi_j = \min(v_0, u_j)$ . Then  $\psi_j = v_0$  in  $\mathbb{R}^n \setminus B_r(z_j)$ . Let  $S$  be a unit  $n$ -cube centered at  $z_j$ . Then  $\psi_j = v_0$  in  $S \setminus B_r(z_j)$ ,  $\psi_j = u_j$  near  $y$ , and  $\psi_j|_S$  extends naturally to  $\mathbb{R}^n$  as an element of  $\Gamma_0$ . Continuing to denote this extension by  $\psi_j$ ,

$$\begin{aligned} c_0 &= J_0(v_0) \leq J_0(\psi_j) \\ &= \int_{S \setminus B_r(z_j)} L(v_0) \, dx + \int_{B_r(z_j) \cap \{v_0 \leq u_j\}} L(v_0) \, dx + \int_{B_r(z_j) \cap \{v_0 > u_j\}} L(u_j) \, dx. \end{aligned} \tag{2.28}$$

Hence

$$\int_{B_r(z_j) \cap \{v_0 > u_j\}} L(v_0) \, dx \leq \int_{B_r(z_j) \cap \{v_0 > u_j\}} L(u_j) \, dx. \tag{2.29}$$

Set  $\chi_j = \max(v_0, u_j)$  so  $\chi_j \in \mathcal{S}_r(z_j)$  and

$$\int_{B_r(z_j)} L(u_j) \, dx \leq \int_{B_r(z_j)} L(\chi_j) \, dx = \int_{B_r(z_j) \cap \{v_0 > u_j\}} L(v_0) \, dx + \int_{B_r(z_j) \cap \{v_0 \leq u_j\}} L(u_j) \, dx$$

so

$$\int_{B_r(z_j) \cap \{v_0 > u_j\}} L(u_j) \, dx \leq \int_{B_r(z_j) \cap \{v_0 > u_j\}} L(v_0) \, dx. \tag{2.30}$$

Combining (2.29) and (2.30) yields equality in these expressions and returning to (2.28) shows

$$c_0 = J_0(v_0) \leq J_0(\psi_j) = J_0(v_0). \tag{2.31}$$

Hence  $\psi_j \in \mathcal{M}_0$ . But by Proposition 2.2,  $\psi_j$  cannot both =  $v_0$  in  $S \setminus B_r(z_j)$  and =  $u_j$  near  $y$ . Thus the assumption that  $u_j(y) < v_0(y)$  is not tenable and  $v_0 \leq u_j$ . Similarly  $u_j \leq w_0$  and (a) has been proved.

To obtain (b), suppose not. Then as in the proof of Proposition 2.3 of [9], for some  $j$ , there is an  $x_0 \in B_r(z_j)$  such that

$$u_{j+e_1}^*(x_0 + e_1) < u_j^*(x_0). \tag{2.32}$$

For  $x \in B_{1/2}(z_j)$ , set  $\psi(x) = u_{j+e_1}^*(x + e_1)$ ,  $\chi = \max(u_j^*, \psi)$ ,  $\zeta = \min(u_j^*, \psi)$ . Then  $\zeta = u_j^* = U_1 \leq \tau_{-1}^1 U_1 = \psi = \chi$  on  $B_{1/2}(z_j) \setminus B_r(z_j)$ . Thus

$$\int_{B_r(z_j)} (L(\zeta) + L(\chi)) \, dx = \int_{B_r(z_j)} (L(u_j^*) + L(\psi)) \, dx \tag{2.33}$$

which shows that  $\zeta$  minimizes (2.23) over  $\mathcal{S}_r(z_j)$  and  $\tau_1 \chi$  minimizes (2.23) over  $\mathcal{S}_r(z_j + e_1)$ . Therefore by the definitions of  $U^*$  and  $\zeta$ ,

$$u_j^*(x_0) \leq \zeta(x_0) \leq \psi(x_0) = u_{j+e_1}^*(x_0 + e_1) \tag{2.34}$$

contrary to (2.32). Thus (b) is verified. To prove that (c) is valid, suppose not. If  $U^* \equiv w_0$ , then  $U_1 = w_0$  in  $\mathbb{R}^n \setminus \bigcup_{j \in \mathbb{Z}^n} (z_j)$  so  $\tau_j^1 U_1 \not\rightarrow v_0$  in  $L^2(T(0))$  as  $j \rightarrow \infty$ , contrary to Proposition 2.14 for  $U_1$ . Similarly  $U^* \not\equiv v_0$ . Thus (a), (b), and (c) hold and  $U_1$  satisfies (PDE). Since any  $U \in \mathcal{M}_1$  is trivially the limit of a minimizing sequence for (2.17), the argument just given shows  $U$  satisfies (PDE) and 2° of the theorem is satisfied.

To prove 3° of Theorem 2.18, observe that  $\|U_1 - v_0\|_{L^2(T(i e_1))} \rightarrow 0$  as  $i \rightarrow -\infty$  by Proposition 2.14. Since  $U_1$  and  $v_0$  are bounded in  $L^\infty(\mathbb{R}^n)$ , local Schauder estimates show  $U_1 - v_0$  is bounded in  $C_{loc}^{2,\alpha}(\mathbb{R}^n)$  for any  $\alpha \in (0, 1)$ . Standard interpolation inequalities then yield 3° for  $U_1 - v_0$  and a similar argument applies to  $U_1 - w_0$ .

To establish 4°–5°, we begin with 5°. Let  $V, W \in \mathcal{M}_1$ . If 5° is false, setting  $\varphi = \max(V, W)$  and  $\psi = \min(V, W)$ , then  $\varphi(z) = \psi(z)$  for some  $z \in \mathbb{R}^n$ . Suppose for the moment that  $\varphi, \psi \in \Gamma_1$ . Arguing as in the proof of Proposition 2.2 [8], for all  $i \in \mathbb{Z}$ ,

$$\int_{T(i e_1)} L(\varphi) \, dx + \int_{T(i e_1)} L(\psi) \, dx = \int_{T(i e_1)} L(V) \, dx + \int_{T(i e_1)} L(W) \, dx \tag{2.35}$$

and this implies

$$2c_1 \leq J_1(\varphi) + J_1(\psi) = J_1(V) + J_1(W) = 2c_1. \tag{2.36}$$

Thus  $J_1(\varphi), J_1(\psi) = c_1$  and by 1°–2°,  $\varphi$  and  $\psi$  are solutions of (PDE). But  $\varphi - \psi \geq 0$ ,  $\varphi(z) = \psi(z)$ , and  $\varphi - \psi$  is a solution of the linear elliptic partial differential equation

$$-\Delta \Phi + A(x)\Phi = 0 \tag{2.37}$$

where

$$\begin{aligned} A(x) &= \frac{F_u(x, \varphi(x)) - F_u(x, \psi(x))}{\varphi(x) - \psi(x)}, \quad \varphi(x) > \psi(x) \\ &= F_{uu}(x, \varphi(x)), \quad \varphi(x) = \psi(x). \end{aligned}$$

Further writing (2.37) as

$$-\Delta \Phi + \max(A, 0)\Phi = -\min(A, 0)\Phi \geq 0, \tag{2.38}$$

the maximum principle implies  $\varphi \equiv \psi$ , a contradiction.

To verify that  $\varphi, \psi \in \Gamma_1$ , it suffices to prove that

$$\tau_{-1}^1 \chi \geq \chi \tag{2.39}$$

for  $\chi = \varphi, \psi$ . First for  $\varphi$ , note that

$$\tau_{-1}^1 \varphi(x) = \varphi(x_1 + 1, x_2, \dots) = \max(V(x_1 + 1, x_2, \dots), W(x_1 + 1, x_2, \dots)). \tag{2.40}$$

If  $\tau_{-1}^1 \varphi(x) = \tau_{-1}^1 V(x)$ , since  $\tau_{-1}^1 V(x) \geq V(x)$ , then by (2.40),

$$\tau_{-1}^1 V(x) \geq \tau_{-1}^1 W(x) \geq W(x).$$

A similar argument applies if  $\tau_{-1}^1 \varphi(x) = \tau_{-1}^1 W(x)$ . Hence (2.39) holds for  $\varphi$ .

Next to prove (2.39) for  $\psi$ , if  $\tau_{-1}^1 \psi(x) = \tau_{-1}^1 V(x)$  and  $\psi(x) = V(x)$ , (2.39) is valid while if  $\tau_{-1}^1 \psi = \tau_{-1}^1 V(x)$  and  $\psi(x) = W(x)$ ,

$$\tau_{-1}^1 \psi(x) = \tau_{-1}^1 V(x) \geq V(x) \geq W(x) = \psi(x).$$

A similar argument obtains if the roles of  $V$  and  $W$  are reversed. Thus  $\varphi, \psi \in \Gamma_1$  and 5° is proved.

To get 4°, note that

$$v_0 \leq U_1 \leq \tau_{-1}^1 U_1 \leq w_0. \tag{2.41}$$



Now the maximum principle can be used exactly as in (2.37)–(2.38) to get strict inequalities in (2.41). Lastly 6° follows from 5° and the argument of Corollary 2.3.

The proof of Theorem 2.18 is complete. □

### 3. Minimality of $U_1$

As was mentioned in the introduction, Moser studied solutions of (1.1) that were minimal in the sense of Giaquinta and Guisti. In this section it will be shown that  $U_1$  is a minimal solution of (PDE) in this sense.

Following [6],  $U_1$  is a minimal solution of (PDE) if for any bounded domain  $\Omega \subset \mathbb{R}^n$  with a smooth boundary,

$$\int_{\Omega} L(u) \, dx \geq \int_{\Omega} L(U_1) \, dx \tag{3.1}$$

for any  $u \in W_{loc}^{1,2}(\mathbb{R}^n)$  with  $u = U_1$  in  $\mathbb{R}^n \setminus \Omega$ . In other words  $U$  minimizes  $\int_{\Omega} L(\cdot) \, dx$  over the class of  $W^{1,2}(\Omega)$  functions having  $U_1$  as boundary values. The proof of Theorem 2.18 shows that  $U_1$  satisfies (3.1) when  $\Omega$  is any ball of radius  $r < \frac{1}{2}$ . To extend this property to the more general class of bounded  $\Omega$ 's with a smooth boundary requires showing that  $c_0$  and  $c_1$  can be characterized as minimizers of functionals in broader classes of functions.

To begin, let  $p = (p_1, \dots, p_n) \in \mathbb{N}^n$  and set

$$\begin{aligned} \Gamma_0(p) &= \{u \in W_{loc}^{1,2}(\mathbb{R}^n) \mid u \text{ is } p_i \text{ periodic in } x_i, 1 \leq i \leq n\}, \\ I_p(u) &= \int_0^{p_1} \dots \int_0^{p_n} L(u) \, dx, \\ c_0(p) &= \inf_{u \in \Gamma_0(p)} I_p(u), \end{aligned}$$

and

$$\mathcal{M}_0(p) = \{u \in \Gamma_0(p) \mid I_p(u) = c_0(p)\}.$$

The proof of Proposition 2.2 shows that  $\mathcal{M}_0(p) \neq \emptyset$  and is an ordered set.

**Lemma 3.2.**  $\mathcal{M}_0(p) = \mathcal{M}_0$  (and therefore  $c_0(p) = (\prod_1^n p_i) c_0$ ).

**Proof.** It suffices to show that  $\tau_{-1}^i u = u$  for  $1 \leq i \leq n$  and any  $u \in \mathcal{M}_0(p)$ . If not, since  $\mathcal{M}_0(p)$  is ordered, either (i)  $\tau_{-1}^i u > u$  or (ii)  $\tau_{-1}^i u < u$ . If e.g. (i) occurs,

$$u < \tau_{-1}^i u < \dots < \tau_{-p_i}^i u = u,$$

a contradiction. Similarly (ii) cannot occur and the lemma follows. □

Next it will be shown that there is an analogue of Lemma 3.2 in the setting of Theorem 2.18. Let  $\ell = (\ell_2, \dots, \ell_n) \in \mathbb{N}^{n-1}$  and define

$$\widehat{\Gamma}_1(\ell) = \{u \in W_{loc}^{1,2}(\mathbb{R}^n) \mid v_0 \leq u \leq w_0 \text{ and } u \text{ is } \ell_i \text{ periodic in } x_i, 2 \leq i \leq n\}.$$

Let  $p \in \mathbb{N}^n$  with  $p = (p_1, \ell)$  and let  $i \in \mathbb{Z}$ . For  $u \in \widehat{\Gamma}_1(\ell)$ , define

$$J_{1,i}^p(u) = \sum_{k_2=0}^{\ell_2-1} \dots \sum_{k_n=0}^{\ell_n-1} \left( \int_{T((ip_1, k))} L(u) \, dx - c_0 \right)$$

and

$$J_1^p(u) = \sum_{i \in \mathbb{Z}} J_{1,i}^p(u).$$

Observe that with slight modifications in the proof,  $J_1^p$  has the same properties on  $\widehat{\Gamma}_1(\ell)$  as does  $J_1$  on  $\widehat{\Gamma}_1$  given by Proposition 2.6. Set

$$\Gamma_1(p) = \{u \in \widehat{\Gamma}_1(\ell) \mid u \leq \tau_{-p_1}^1 u \text{ and } v_0 \neq u \neq w_0\}.$$

Replacing  $\Gamma_1, J_1,$  and  $T(0)$  by  $\Gamma_1(p), J_1^p,$  and  $\bigcup_{0 \leq k_r \leq p_r} T(k_1 e_1 + \dots + k_n e_n)$ , the proof of Proposition 2.14 carries over to the current setting.

Now define

$$c_1(p) = \inf_{u \in \Gamma_1(p)} J_1^p(u).$$

By the above observations, the argument of Theorem 2.18 (with  $r < \frac{1}{2} \min_{1 \leq i \leq n} p_i$  now permitted) applies here so

$$\mathcal{M}_1(p) \equiv \{u \in \Gamma_1(p) \mid J_1^p(u) = c_1(p)\}$$

is a nonempty ordered set of solutions of (PDE). The analogue of Lemma 3.2 in this setting is:

**Lemma 3.3.**  $\mathcal{M}_1(p) = \mathcal{M}_1$  and  $c_1(p) = (\prod_{i=1}^n p_i) c_1$ .

**Proof.** It suffices to show that whenever  $u \in \mathcal{M}_1(p)$ : (i)  $\tau_{-1}^i u = u, 2 \leq i \leq n,$  and (ii)  $\tau_{-1}^1 u \geq u$ . The proof of (i) is the same as that of Lemma 3.2. For (ii), observe that  $\tau_{-1}^1 u \in \mathcal{M}_1(p)$  which is ordered. Hence if (ii) fails,  $u > \tau_{-1}^1 u$  so by the definition of  $\Gamma_1(p)$ ,

$$u \leq \tau_{-p_1}^1 u < \tau_{-p_1+1}^1 u < \dots < u,$$

a contradiction.  $\square$

**Remark 3.4.** By (F<sub>2</sub>), the replacement of  $[ip_1, ip_1 + 1]$  in  $T(ip_1, k)$  by  $[ip_1 + j, ip_1 + j + 1]$  for any  $j \in \mathbb{Z}$  does not effect the above arguments. The same is true if  $\ell$  is replaced by  $\ell + q$  for any  $q \in \mathbb{R}^{n-1}$ .

**Theorem 3.5.** Any  $U \in \mathcal{M}_1$  is a minimal solution of (PDE) in the sense of Giaquinta and Guisti.

**Proof.** To show that (3.1) is satisfied, let  $z \in \mathbb{R}^n$  and  $r > 0$  such that  $\Omega \subset B_r(z)$ . Set

$$\mathcal{S}_r(z) = \{u \in W_{loc}^{1,2}(\mathbb{R}^n) \mid u = U \text{ in } \mathbb{R}^n \setminus B_r(z)\}.$$

It suffices to prove that

$$\int_{B_r(z)} L(u) \, dx \geq \int_{B_r(z)} L(U) \, dx \tag{3.6}$$

for any  $u \in \mathcal{S}_r(z)$ . By Lemma 3.3,  $\mathcal{M}_1 = \mathcal{M}_1(p)$  for any  $p \in \mathbb{N}^n$ . Choose  $p$  so that  $\min_{1 \leq i \leq n} p_i > 2r$ . Further exploiting Remark 3.4, it can be assumed that  $\Omega \subset B_r(z) \subset T(p)$ . Hence by the proof of Theorem 2.18,  $U$  minimizes  $\int_{B_r(z)} L(\cdot) \, dx$  over  $\mathcal{S}_r(z)$  and the proof is complete.  $\square$

An immediate consequence of Theorem 3.5 is

**Corollary 3.7.**  $U_1$  is the unique minimizer of  $\int_{B_r(z)} L(\cdot) \, dx$  in  $\mathcal{S}_r(z)$ .

**Proof.** Suppose  $u \in S_r(z)$  so that

$$\int_{B_r(z)} L(u) \, dx = \int_{B_r(z)} L(U_1) \, dx. \tag{3.8}$$

Let  $r^* > r$ . Then by (3.8),

$$\int_{B_{r^*}(z)} L(u) \, dx = \int_{B_{r^*}(z) \setminus \overline{B}_r(z)} L(U_1) \, dx + \int_{B_r(z)} L(u) \, dx = \int_{B_{r^*}(z)} L(U_1) \, dx. \tag{3.9}$$

Hence  $U_1 \in S_{r^*}(z)$  and minimizes  $\int_{B_{r^*}(z)} L(\cdot) \, dx$  over  $S_{r^*}(z)$ . Again as in Lemma 2.5 of [9], the set of such minimizers is ordered. Since  $u$  and  $U_1$  belong to this set and  $u = U_1$  in  $B_{r^*}(z) \setminus \overline{B}_r(z)$ ,  $u \equiv U_1$ . The proof is complete.  $\square$

#### 4. The general case

The goal of this section is to show how the results of Sections 1–2 together with induction and minimization arguments can be used to obtain more complex heteroclinic solutions of (PDE) corresponding to those obtained by Bangert [4] via his nonvariational approach.

To give an idea of the inductive procedure at level two, suppose  $\mathcal{M}_1$  as obtained in Theorem 1.18 satisfies a gap condition:

(\*)<sub>1</sub> There are adjacent  $v_1 < w_1$  in  $\mathcal{M}_1$ .

Define the set of functions  $\widehat{\Gamma}_2$  via

$$\widehat{\Gamma}_2 = \{u \in W_{\text{loc}}^{1,2}(\mathbb{R}^n) \mid v_1 \leq u \leq \tau_{-1}^1 u \leq w_1 \text{ and } u \text{ is 1-periodic in } x_3, \dots, x_n\}.$$

The renormalized functional,  $J_2$  on  $\widehat{\Gamma}_2$  is defined by

$$J_2(u) = \sum_{i \in \mathbb{Z}} J_{2,i}(u)$$

where

$$J_{2,i}(u) = J_1(\tau_{-i}^2 u) - c_1.$$

Suppose that  $J_2$  on  $\widehat{\Gamma}_2$  has the analogues of the properties of  $J_1$  on  $\widehat{\Gamma}_1$  as given by Proposition 2.6 with  $3^\circ$  replaced by

$$\int_{T(\ell_1 e_1 + \ell_2 e_2)} L(u) \, dx \leq J_2(u) + c_0 + c_1$$

for any  $\ell_1, \ell_2 \in \mathbb{Z}$ . Suppose also that Proposition 2.14 is valid with appropriate changes of sub- or superscript 1's to 2's. Setting

$$\Gamma_2 = \{u \in \widehat{\Gamma}_2 \mid u \leq \tau_{-1}^2 u \text{ and } v_1 \not\equiv u \not\equiv w_1\}$$

and

$$c_2 = \inf_{u \in \Gamma_2} J_2(u),$$

the analogue of Theorem 2.18 holds here providing a solution of (PDE) that is heteroclinic in  $x_1$  from  $v_0$  to  $w_0$  and heteroclinic in  $x_2$  from  $v_1$  to  $w_1$ . Moreover as in Theorem 3.5,  $U_2$  is a minimal solution of (PDE) in the sense of Giaquinta and Guisti.

Now setting

$$\mathcal{M}_2 = \{u \in \Gamma_2 \mid J_2(u) = c_2\},$$

a gap condition  $(*)_2$  can be introduced and the process continues. To carry out the induction argument properly, let  $m < n$  and assume the gap condition:

$(*)_i$  There are adjacent  $v_i < w_i$  in  $\mathcal{M}_i$  holds for  $i = 0, \dots, m - 1$ . Let

$$\widehat{\Gamma}_i = \{u \in W_{\text{loc}}^{1,2}(\mathbb{R}^n) \mid v_{i-1} \leq u \leq \tau_{-1}^j u \leq w_{i-1}, 1 \leq j < i \text{ and } u \text{ is } 1\text{-periodic in } x_{i+1}, \dots, x_n\} \quad (4.1)_i$$

for  $1 \leq i \leq m$ . The  $i$ th renormalized functional,  $J_i(u)$ , is given by

$$J_i(u) = \sum_{p \in \mathbb{Z}} J_{i,p}(u) \quad (4.2)_i$$

where

$$J_{i,p}(u) = J_{i-1}(\tau_{-p}^i u) - c_{i-1}. \quad (4.3)_i$$

Suppose that  $J_i$  on  $\widehat{\Gamma}_i$  possesses the following properties for  $1 \leq i \leq m$ :

**Proposition 4.4<sub>i</sub>.** For  $u \in \widehat{\Gamma}_i$ ,

- 1°  $J_{i,p}(u) \geq 0$  for all  $p \in \mathbb{Z}$ .
- 2°  $J_i(u) \geq 0$ .
- 3°  $\int_{T(\sum_1^i \ell_q e_q)} L(u) \, dx \leq J_i(u) + \sum_0^{i-1} c_q$ .
- 4°  $J_i$  is weakly lsc (in  $W_{\text{loc}}^{1,2}(\mathbb{R}^n)$ ) on  $\widehat{\Gamma}_i$ .

For  $1 \leq i \leq m$ , set

$$\Gamma_i = \{u \in \widehat{\Gamma}_i \mid u \leq \tau_1^i u \text{ and } v_{i-1} \neq u \neq w_{i-1}\} \quad (4.5)_i$$

and assume (with the understanding that  $\sum_1^0 \ell_q e_q = 0$ ):

**Proposition 4.6<sub>i</sub>.** If  $u \in \Gamma_i$  and  $J_i(u) < \infty$ , then as  $j \rightarrow \infty$ ,  $\tau_j^i u \rightarrow v_{i-1}$  weakly in  $W^{1,2}(T(\sum_1^{i-1} \ell_q e_q))$  for all  $\ell_1, \dots, \ell_{i-1} \in \mathbb{Z}$  and as  $j \rightarrow -\infty$ ,  $\tau_j^i u \rightarrow w_{i-1}$  weakly in  $W^{1,2}(T(\sum_1^{i-1} \ell_q e_q))$  for all  $\ell_1, \dots, \ell_{i-1} \in \mathbb{Z}$ .

Finally define

$$c_i = \inf_{u \in \Gamma_i} J_i(u), \quad 1 \leq i \leq m, \quad (4.7)_i$$

and assume:

**Theorem 4.8<sub>i</sub>.** Let  $F$  satisfy (F<sub>1</sub>)–(F<sub>3</sub>) and let  $(*)_i$  hold. Then

- 1° There is a  $U_i \in \Gamma_i$  such that  $J_i(U_i) = c_i$ .
- 2° Any such  $U_i$  is a classical solution of (PDE).

3°  $U_i$  is heteroclinic from  $v_{i-1}$  to  $w_{i-1}$ :

$$\begin{aligned} \|U_i - v_{i-1}\|_{C^2(T(\sum_1^i \ell_k e_k))} &\rightarrow 0 \quad \text{as } \ell_i \rightarrow -\infty \quad \text{and} \\ \|U_i - w_{i-1}\|_{C^2(T(\sum_1^i \ell_k e_k))} &\rightarrow 0 \quad \text{as } \ell_i \rightarrow \infty. \end{aligned}$$

4°  $v_{i-1} < U_i < \tau_{-1}^j U_i < w_{i-1}$ ,  $1 \leq j \leq i$ .

5°  $\mathcal{M}_i = \{u \in \Gamma_i \mid J_i(u) = c_i\}$  is an ordered set.

6° If  $u \in \mathcal{M}_i$ ,  $u$  is even in  $x_{i+1}, \dots, x_n$ .

Finally suppose that for  $1 \leq i \leq m$ .

**Theorem 4.9.** Any  $U \in \mathcal{M}_i$  is a minimal solution of (PDE) in the sense of Giaquinta and Guisti.

**Corollary 4.10.**  $U_i$  is the unique minimizer of  $\int_{B_r(z)} L(\cdot) dx$  over  $\mathcal{S}_r(z) = \{\varphi \in W_{\text{loc}}^{1,2}(\mathbb{R}^n) \mid \varphi = U_i \text{ in } \mathbb{R}^n \setminus B_r(z)\}$ .

With these inductive facts at hand, the results can be extended to level  $m + 1$ . To do so, begin by assuming there is a gap in  $\mathcal{M}_m$ :

(\*)<sub>m</sub> There are adjacent  $v_m < w_m$  in  $\mathcal{M}_m$ .

Then with  $\widehat{\Gamma}_{m+1}$  defined in (4.1)<sub>m+1</sub> and  $J_{m+1}$  from (4.2)<sub>m+1</sub>, we can give the

**Proof of Proposition 4.4**<sub>m+1</sub>. To verify  $1_{m+1}^o - 2_{m+1}^o$ , let  $u \in \widehat{\Gamma}_{m+1}$  and for  $x_{m+1} \in [p + \frac{1}{2}, p + 1]$ , set

$$\varphi_p^+(u) = \tau_{-p}^{m+1} u \tag{4.11}$$

and for  $x_{m+1} \in [p, p + \frac{1}{2}]$ , set

$$\varphi_p^-(u) = \tau_{-p}^{m+1} u. \tag{4.12}$$

Extend these functions to 1-periodic functions in  $x_{m+1}$  as in the proof of Proposition 2.6. Then  $\varphi_p^\pm(u) \in \widehat{\Gamma}_m$  and either (i)  $\varphi_p^\pm(u) \in \Gamma_m$  or (ii)  $\varphi_p^\pm(u) \in \{v_m, w_m\}$ . If (i) holds,

$$J_{m+1,p}(\varphi_p^\pm(u)) = J_m(\varphi_p^\pm(u)) - c_m \geq 0$$

so

$$J_{m+1,p}(u) = \frac{1}{2}(J_{m+1,p}(\varphi_p^+(u)) + J_{m+1,p}(\varphi_p^-(u))) \geq 0$$

and  $1_{m+1}^o$  is valid while if (ii) holds, (4.3)<sub>m+1</sub> and (4.7)<sub>m</sub> yield  $1_{m+1}^o$  with equality. Now  $2_{m+1}^o$  is immediate. Arguing as in the  $m = 0$  case, using  $\varphi_p^\pm$ ,

$$J_m(\tau_{-\ell_{m+1}}^{m+1} u) \leq J_{m+1}(u) + c_m \tag{4.13}$$

follows from  $1_{m+1}^o - 2_{m+1}^o$ . Now applying (4.3)<sub>m</sub> and 3° of Proposition 4.4<sub>m</sub> gives  $3_{m+1}^o$ . Lastly 4° of Proposition 4.4<sub>m</sub> and the analogue here of the argument centered around (2.11) yield  $4_{m+1}^o$ . □

Next with  $\Gamma_{m+1}$  as provided by (4.5)<sub>m+1</sub>, we have the

**Proof of Proposition 4.6**<sub>m+1</sub>. Since  $J_{m+1}(u) < \infty$ , by 3° of Proposition 4.4<sub>m</sub>, the sequence  $(\tau_\ell^{m+1} u)_{\ell \in \mathbb{Z}}$  is bounded in  $W^{1,2}(T(\sum_1^m \ell_i e_i))$  for each  $\ell_1, \dots, \ell_m \in \mathbb{Z}$ . Therefore there is a  $v \in W_{\text{loc}}^{1,2}(\mathbb{R}^m \times [0, 1] \times \mathbb{R}^{n-(m+1)})$ ,

1-periodic in  $x_{m+2}, \dots, x_n$  such that along a subsequence,  $\tau_\ell^{m+1}u$  converges weakly to  $v$  in  $W^{1,2}(T(\sum_1^m \ell_i e_i))$  as  $\ell \rightarrow \infty$  for each  $\ell_1, \dots, \ell_m \in \mathbb{Z}$ . Moreover  $u \leq \tau_{-1}^{m+1}u$  shows the sequence converges monotonically to  $v$ . Hence as  $\ell \rightarrow \infty$ ,  $\tau_{\ell-1}^{m+1}u \rightarrow \tau_{-1}^{m+1}v = v$ , i.e.  $v$  is 1-periodic in  $x_{m+1}$  so  $v \in W_{loc}^{1,2}(\mathbb{R}^n)$ . Moreover the corresponding properties for  $u$  imply

$$v_m \leq v \leq \tau_{-1}^i v \leq w_m < w_{m-1}, \quad 1 \leq j \leq m. \tag{4.14}$$

Therefore  $v \in \Gamma_m$ .

By (4.2)<sub>m+1</sub>–(4.3)<sub>m+1</sub>, 1<sup>o</sup> of Proposition 4.4<sub>m+1</sub> and  $J_{m+1}(u) < \infty$ , as  $|p| \rightarrow \infty$ ,

$$J_m(\tau_{-p}^{m+1}u) \rightarrow c_m. \tag{4.15}$$

Observe that if  $\varphi_p^\pm(u)$  are as in the proof of Proposition 4.4<sub>m+1</sub>,  $\varphi_p^\pm(u) \rightarrow \varphi_0^\pm(v)$  as  $p \rightarrow \infty$  weakly in  $W^{1,2}(T(\sum_1^m \ell_i e_i))$  for each  $\ell_1, \dots, \ell_m \in \mathbb{Z}$ . The functions  $\varphi_p^\pm(u), \varphi_0^\pm(v)$  belong to  $\widehat{\Gamma}_m$ . Hence

$$c_m \leq J_m(\varphi_0^\pm(v)) \tag{4.16}$$

and

$$\begin{aligned} c_m &\leq J_m(v) = \frac{1}{2}J_m(\varphi_0^+(v)) + \frac{1}{2}J_m(\varphi_0^-(v)) \\ &\leq \frac{1}{2} \liminf_{p \rightarrow \infty} J_m(\varphi_p^+(u)) + \frac{1}{2} \liminf_{p \rightarrow \infty} J_m(\varphi_p^-(u)) \\ &\leq \frac{1}{2} \liminf_{p \rightarrow \infty} (J_m(\varphi_p^+(u)) + J_m(\varphi_p^-(u))) \\ &= \liminf_{p \rightarrow \infty} J_m(\tau_{-p}^{m+1}u) = c_m \end{aligned} \tag{4.17}$$

via (4.15). Consequently by (4.17),  $J_m(v) = c_m$ . Therefore (4.14) and  $(*)_m$  show  $v \in \{v_m, w_m\}$ . But  $u \in \Gamma_{m+1}$  so  $v$  being the monotone limit of  $(\tau_{-p}^{m+1}u)$  as  $p \rightarrow \infty$  implies  $v = v_m$ . Similarly  $\tau_{-p}^{m+1}u \rightarrow w_m$  as  $p \rightarrow -\infty$  and the proof of Proposition 4.6<sub>m+1</sub> is complete.  $\square$

Finally defining  $c_{m+1}$  via (4.7)<sub>m+1</sub> brings us the

**Proof of Theorem 4.8<sub>m+1</sub>.** Let  $(u_m) \subset \Gamma_{m+1}$  be a minimizing sequence for (4.7)<sub>m+1</sub>. Then there is an  $M > 0$  such that

$$J_{m+1}(u_k) \leq M, \quad k \in \mathbb{N}. \tag{4.18}$$

By Proposition 4.6<sub>m+1</sub>,  $u_k$  can be normalized so that for  $\ell < 0$ ,

$$\int_{T(\ell e_{m+1})} u_k \, dx \leq \frac{1}{2} \int_{T(0)} (v_m + w_m) \, dx < \int_{T(0)} u_k \, dx. \tag{4.19}$$

By (4.18) and 3<sup>o</sup> of Proposition 4.4<sub>m+1</sub>,  $(u_k)$  is bounded in  $W_{loc}^{1,2}(\mathbb{R}^n)$ . Therefore there is a  $U = U_{m+1} \in W_{loc}^{1,2}(\mathbb{R}^n)$  such that, along a subsequence,  $u_k \rightarrow U$  weakly in  $W_{loc}^{1,2}(\mathbb{R}^n)$ , strongly in  $L_{loc}^2(\mathbb{R}^n)$ , and pointwise a.e. as  $k \rightarrow \infty$ . Hence

$$v_m \leq U \leq \tau_{-1}^j U \leq w_m, \quad 1 \leq j \leq m + 1 \tag{4.20}$$

and  $U$  is 1-periodic in  $x_{m+2}, \dots, x_n$ . The normalization (4.19) implies

$$\int_{T(\ell e_{m+1})} U \, dx \leq \frac{1}{2} \int_{T(0)} (v_m + w_m) \, dx \leq \int_{T(0)} U \, dx \tag{4.21}$$

and therefore  $v_m \neq U \neq w_m$ . Hence  $U \in \Gamma_{m+1}$  and

$$J_{m+1}(U) \geq c_{m+1}. \tag{4.22}$$

Since  $(u_k)$  is a minimizing sequence, by 4° of Proposition 4.4<sub>m+1</sub>,

$$J_{m+1}(U) \leq c_{m+1}. \tag{4.23}$$

Thus 1° of Theorem 4.8<sub>m+1</sub> is valid.

Assuming 2° of Theorem 4.8<sub>m+1</sub> for the moment, the  $W_{\text{loc}}^{1,2}(\mathbb{R}^n)$  bounds for  $U$  given by (4.13) together with Proposition 4.6<sub>m+1</sub> yields 3° of Theorem 4.8<sub>m+1</sub> with  $C^2$  replaced by  $L^2$ . But then the argument of Theorem 2.18 gives convergence in  $C^2$ . Likewise, replacing  $\tau_{-1}^1$  by  $\tau_{-1}^i$ ,  $1 \leq i \leq m + 1$ , in (2.32)–(2.33) and following sentences shows  $\varphi, \psi \in \Gamma_{m+1}$ . Then replacing  $T(ie_1)$  in (2.27) by  $T(\sum_1^{m+1} \ell_i e_i)$  shows  $c_{m+1} = J_{m+1}(\varphi) = J_{m+1}(\psi)$  and the reasoning following this implies 4° of Theorem 4.8<sub>m+1</sub>. Then 5°–6° also follow as earlier.

Lastly to verify 2° of Theorem 4.8<sub>m+1</sub>, the proof of 2° of Theorem 2.18 can be applied here provided that (a)  $v_m \leq \varphi_j \leq w_m$  for any minimizer  $\varphi_j$  of (2.23) over

$$\mathcal{S}_{r,m+1}(z_j) = \{ \varphi \in W_{\text{loc}}^{1,2}(\mathbb{R}^n) \mid \varphi = U_{m+1} \text{ in } \mathbb{R}^n \setminus B_r(z_j) \}.$$

(b)  $U^* \leq \tau_{-1}^i U^*$ ,  $1 \leq i \leq m + 1$ , and (c)  $v_m \neq U^* \neq w_m$ . To prove (a), note that  $v_m \leq U_{m+1} \leq w_m$ . Therefore  $\psi = \min(\varphi_j, v_m) \in \mathcal{S}_{r,m}(z_j)$  (with  $U_m = v_m$ ) and  $\chi = \max(\varphi_j, v_m) \in \mathcal{S}_{r,m+1}(z_j)$ . Hence by Theorem 4.9<sub>m</sub>,

$$\int_{B_r(z_j)} L(\psi) \, dx \geq \int_{B_r(z_j)} L(v_m) \, dx \tag{4.24}$$

and by the definition of  $\varphi_j$ ,

$$\int_{B_r(z_j)} L(\chi) \, dx \geq \int_{B_r(z_j)} L(\varphi_j) \, dx. \tag{4.25}$$

Adding (4.24)–(4.25) shows:

$$\begin{aligned} \int_{B_r(z_j)} L(v_m) \, dx + \int_{B_r(z_j)} L(\varphi_j) \, dx &= \int_{B_r(z_j)} L(\psi) \, dx + \int_{B_r(z_j)} L(\chi) \, dx \\ &\geq \int_{B_r(z_j)} L(v_m) \, dx + \int_{B_r(z_j)} L(\varphi_j) \, dx. \end{aligned} \tag{4.26}$$

Hence

$$\int_{B_r(z_j)} L(\psi) \, dx = \int_{B_r(z_j)} L(v_m) \, dx \tag{4.27}$$

and

$$\int_{B_r(z_j)} L(\chi) \, dx = \int_{B_r(z_j)} L(\varphi_j) \, dx.$$

But (4.27) and Corollary 4.10<sub>m</sub> imply  $\psi \equiv v_m$ , i.e.  $\varphi_j \geq v_m$ . Similarly  $\varphi_j \leq w_m$  and (a) is proved.

To check that (b) holds, we argue exactly as in the proof of the analogous situation in Theorem 2.18 – see (2.32)–(2.34) with  $e_1$  replaced by  $e_\ell$ ,  $1 \leq \ell \leq m + 1$ . Lastly (c) follows the same lines as its analogue in the proof of Theorem 2.18. This completes the proof of Theorem 4.8<sub>m+1</sub>.  $\square$

Next to prove Theorem 4.9 $_{m+1}$  requires the extension of Lemmas 3.2 and 3.3 from level  $m$  to level  $m + 1$  and is carried out exactly as earlier. Likewise Corollary 4.10 $_{m+1}$  is proved exactly as in Corollary 3.7 and the induction process is complete.

## References

- [1] S. Alama, Y. Li, On “multibump” bound states for certain semilinear elliptic equations, *Indiana Univ. Math. J.* 41 (1992) 983–1026.
- [2] F. Alessio, L. Jeanjean, P. Montecchiari, Existence of infinitely many stationary layered solutions in  $\mathbb{R}^2$  for a class of periodic Allen–Cahn equations, *Comm. PDE* 27 (2002) 1537–1574.
- [3] S. Aubry, P.Y. LeDaeron, The discrete Frenkel–Kantorova model and its extensions I-Exact results for the ground states, *Physica D* 8 (1983) 381–422.
- [4] V. Bangert, On minimal laminations of a torus, *AIHP Anal. Nonlin.* 6 (1989) 95–138.
- [5] V. Coti Zelati, P.H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on  $\mathbb{R}^n$ , *Comm. Pure Appl. Math.* 45 (1992) 1217–1269.
- [6] M. Giaquinta, E. Guisti, On the regularity of the minima of variational integrals, *Acta Math.* 148 (1982) 31–46.
- [7] J.N. Mather, Existence of quasi-periodic orbits for twist homeomorphisms of the annulus, *Topology* 21 (1982) 457–467.
- [8] J. Moser, Minimal solutions of variational problems on a torus, *AIHP Anal. Nonlin.* 3 (1986) 229–272.
- [9] P.H. Rabinowitz, E. Stredulinsky, Mixed states for an Allen–Cahn type equation, *Comm. Pure Appl. Math.* 56 (2003) 1078–1134.
- [10] E. Séré, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.* 209 (1992) 27–42.