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Multiplicity of positive solutions for an indefinite superlinear elliptic problem on R^N

Multiplicité des solutions positives d'un problème elliptique superlinéaire indéfini dans R^N

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Abstract

We consider the elliptic problem $-\Delta u - \lambda u = a(x)u^p$, with $p > 1$ and $a(x)$ sign-changing. Under suitable conditions on p and $a(x)$, we extend the multiplicity, existence and nonexistence results known to hold for this equation on a bounded domain (with standard homogeneous boundary conditions) to the case that the bounded domain is replaced by the entire space R^N. More precisely, we show that there exists $\Lambda > 0$ such that this equation on R^N has no positive solution for $\lambda > \Lambda$, at least two positive solutions for $\lambda \in (0, \Lambda)$, and at least one positive solution for $\lambda \in (-\infty, 0] \cup \{\Lambda\}$.

Résumé

On considère le problème elliptique $-\Delta u - \lambda u = a(x)u^p$, où $p > 1$ et $a(x)$ change le signe. Sous des conditions adéquates sur p et $a(x)$, nous étendons les résultats connus sur la multiplicité, l'existence et la non-existence de cette équation sur un domaine borné (avec des conditions aux limites homogènes naturelles) où le domaine borné est remplacé par l'espace tout entier. Plus précisement, nous montrons qu'il existe $\Lambda > 0$ tel que cette équation dans R^N n'a aucune solution positive pour $\lambda > \Lambda$, au moins deux solutions positives pour $\lambda \in (0, \Lambda)$, et au moins une solution positive pour $\lambda \in (-\infty, 0] \cup \{\Lambda\}$.

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1. Introduction

The elliptic problem

$$-\Delta u - \lambda u = a(x)u^p, \tag{1.1}$$

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with $p > 1$ and $a(x)$ sign-changing, has attracted extensive studies recently. This is known as an indefinite superlinear problem. The fact that $a(x)$ changes sign in the underlying domain of the differential equation poses extra difficulties from the well studied cases that $a(x)$ is always negative (the sublinear case) and $a(x)$ is always positive (the superlinear case).

When (1.1) is considered on a bounded domain $\Omega \subset R^N$ with standard homogeneous boundary conditions on $\partial\Omega$, it follows from recent results (see, for example, [17,1,4,5,19,3]) that, under suitable conditions on p and on the behaviour of $a(x)$ near its zero set, (1.1) has a positive solution for $\lambda = \lambda_1(\Omega)$ (the first eigenvalue of the Laplacian under the corresponding boundary conditions on $\partial\Omega$) if and only if

$$\int_\Omega a(x)\phi^{p+1}(x)\,dx < 0, \tag{1.2}$$

where ϕ denotes the (normalized) positive eigenfunction corresponding to $\lambda_1(\Omega)$. Moreover, when (1.2) is satisfied, there exists $\Lambda > 0$ such that (1.1) has at least two positive solutions for every $\lambda \in (\lambda_1(\Omega), \Lambda)$, at least one positive solution for $\lambda = \Lambda$ and for $\lambda = \lambda_1(\Omega)$, and no positive solution for $\lambda > \Lambda$. Under less restrictive conditions, (1.1) has at least one positive solution for each $\lambda < \lambda_1(\Omega)$.

The purpose of this paper is to extend these results to the case that Ω is replaced by the entire space R^N. As will become clear, such an extension involves two kinds of difficulties. One is due to the well-known loss of compactness, the other is due to the fact that $\lambda_1(\Omega)$ is no longer a simple eigenvalue when Ω is replaced by R^N.

In a recent work of Costa and Tehrani [7], such an extension was partially achieved through a variational approach. To overcome these difficulties, [7] considered a problem on R^N including (1.1) as a typical case, but with λ replaced by $\lambda h(x)$, where h is a nonnegative function belonging to the space $L^{N/2}(R^N) \cap L^\alpha(R^N)$ for some $\alpha > N/2$. This allows them to regain compactness for the variational approach. Moreover, the eigenvalue problem

$$-\Delta u = \lambda h(x)u, \quad u \in D^{1,2}(R^N)$$

behaves similarly to the finite domain case, with a simple first eigenvalue $\lambda_1(h) > 0$. Under conditions on p and $a(x)$ similar to those for the bounded domain case, and furthermore,

$$\lim_{|x|\to\infty} a(x) = a_\infty < 0, \tag{1.3}$$

it was shown in [7] that the entire space problem has at least one positive solution for $\lambda \leqslant \lambda_1(h)$, and at least two positive solutions for λ in a small right neighbourhood of $\lambda_1(h)$. The existence of a critical $\Lambda > 0$ as in the finite domain case was not considered in [7]. The introduction in [7] contains a fairly detailed account of other studies of entire space problems, and we refer to that and the references therein for the interested reader.

In this paper, to overcome the above mentioned difficulties, we use a bounded domain approximation approach to study (1.1) on R^N. This allows us to avoid replacing λ by $\lambda h(x)$ as in [7], but we have to carefully control the behavior of the solutions as the domain enlarges to R^N; in particular, we need to obtain good a priori bounds for the solutions over bounded sets of R^N and good estimates of the solutions for large $|x|$. Under similar conditions on $p > 1$ and $a(x)$ as in the bounded domain case, and (1.3), we will obtain a complete extension of the bounded domain result, namely, there exists $\Lambda > 0$ such that (1.1) on R^N has no positive solution for $\lambda > \Lambda$, at least two positive solutions for $\lambda \in (0, \Lambda)$, and at least one positive solution for $\lambda \in (-\infty, 0] \cup \{\Lambda\}$. Note that (1.3) implies (1.2) for all "large" enough Ω.

We would like to point out that a variational approach along the lines of [7] does not seem applicable to problem (1.1) on R^N. In fact, by Lemma 4.3 in this paper, condition (1.3) implies that any positive solution $u(x)$ of (1.1) satisfies

$$\lim_{|x|\to\infty} u(x) = \left(\frac{\max\{\lambda, 0\}}{|a_\infty|}\right)^{1/(p-1)}.$$

Therefore no positive solution of (1.1) with $\lambda > 0$ belongs to the space $D^{1,2}(R^N)$. Moreover, this implies that whatever space one chooses to replace $D^{1,2}(R^N)$ in order to apply a variational approach with suitable compactness conditions, the space has to be λ dependent. This would make such an approach extremely complicated, if possible at all. A direct application of the global bifurcation argument parallel to that in the bounded domain case as used in [3] does not seem to apply to (1.1) either, due to the following reasons. Firstly, the bifurcation approach for the bounded domain case relies heavily on the fact that $\lambda_1(\Omega)$ is a simple eigenvalue of the linearized eigenvalue problem at the trivial solution $u = 0$. But the correspondent of $\lambda_1(\Omega)$ for (1.1) on R^N is 0, and it is well known that 0 is not a simple eigenvalue of the corresponding linearized problem (it is not even an isolated point in the spectrum). Secondly, in order to obtain suitable compactness, one faces the problem of working with a λ dependent function space as well.

Let us now briefly explain our approach. The existence of at least one positive solution for $\lambda \in (0, \Lambda)$ requires almost no restriction except that $p > 1$ and $a(x)$ satisfies (1.3) (in fact a less restrictive one, (2.2), is enough). This is proved in Section 2 by some comparison arguments and local bifurcation analysis for solutions on bounded domains.

The existence of a positive solution for $\lambda = \Lambda$ requires a priori bounds for solutions of bounded domain problems such that the bounds are independent of the size of the domain. In Section 3, we adapt the techniques in [3] to establish such bounds. We also use boundary blow-up problems for this purpose.

The central part of this paper is Section 4, where we prove the multiplicity result for $\lambda \in (0, \Lambda)$ and the existence of at least one positive solution for $\lambda \leqslant 0$. Apart from the a priori bounds established in Section 3, we need a crucial new ingredient, which comes from a careful analysis of the global bifurcation branches of positive solutions for bounded domain problems. Roughly speaking, we will show that the global bifurcation branch bifurcating from $(\lambda, u) = (\lambda_1(\Omega), 0)$ can be decomposed into two connected parts, C_0 and C_∞, where C_0 contains all the minimal positive solutions on Ω, and C_∞ is unbounded and contains none of these minimal positive solutions. We will prove that as Ω enlarges to R^N, the positive solutions on Ω chosen from C_∞ will converge to solutions of (1.1) on R^N which are not minimal positive solutions. This will give rise to two positive solutions on R^N for $\lambda \in (0, \Lambda)$. For $\lambda \leqslant 0$, this will guarantee that the solution so obtained is not the trivial solution 0.

For simplicity of presentation, throughout this paper, we have restricted our discussion to elliptic problems of the special form (1.1) (in fact an equivalent form (2.1)), and all the bounded domains are chosen as balls. By suitable modifications of our arguments (without essential difficulties), our results in Sections 2 and 3 can be extended to the case that Δ is replaced by a second order elliptic operator with constant coefficients which can be obtained through a change of variables from Δ (due solely to the proof of Lemma 2.5, otherwise, Δ can be replaced by a rather general second order elliptic operator, not necessarily self-adjoint), and the nonlinearity replaced by $\lambda\alpha(x)u - b(x)f(u)$, with α a continuous function satisfying $m \leqslant \alpha(x) \leqslant M$ for all $x \in R^N$ and some positive constants m and M, with $f(u)$ locally Lipschitz continuous and behaving like u^p near 0 and near ∞. Our main result (Theorem 4.6) holds if we further assume that $\alpha(x) \to \alpha_\infty \in (0, \infty)$ as $|x| \to \infty$, and that $f(u)/u$ is increasing for $u > 0$. Our results on the bounded domain problems (including Proposition 4.1 but possibly except Lemma 2.5) hold with much more generalities, for example, the underlying domain can be rather general with regular boundary, the differential operator and the nonlinearity can also be very general.

2. Existence and nonexistence results

Let us start with an accurate formulation of our problem. We consider the elliptic equation

$$-\Delta u = \lambda u - b(x)u^p, \quad x \in R^N, \tag{2.1}$$

where $p > 1$, λ is a real parameter, b is a continuous function such that

$$b(x) < 0 \quad \text{in a ball } B_{r_0}(x_0), \qquad b(x) \geqslant \sigma > 0 \quad \text{for } |x| \geqslant R_0. \tag{2.2}$$

Here and throughout this paper, $B_{r_0}(x_0)$ denotes the open ball in R^N with center x_0 and radius r_0. Note that in order to match the notations in some of the references that will be frequently used later, we have replaced $a(x)$ in (1.1) by $-b(x)$ in (2.1).

By a positive solution of (2.1) we mean a function $u \in C^1(R^N)$ such that $u > 0$ on R^N and

$$\int_{R^N} \big(\nabla u \cdot \nabla v - \big(\lambda u - b(x)u^p\big)v\big)\,dx = 0, \quad \forall v \in C_0^\infty(R^N).$$

From classical theory on elliptic equations (see [14]) we know that $u \in W^{2,q}_{\mathrm{loc}}(R^N)$ for any $q > 1$, and u is C^2 (hence a classical solution) if further $b(x)$ is Hölder continuous on R^N.

Theorem 2.1. *Under the above assumptions, there exists $\Lambda > 0$ such that* (2.1) *has at least one positive solution for each $\lambda \in (0, \Lambda)$, and it has no positive solution when $\lambda > \Lambda$.*

The proof of Theorem 2.1 will be based mainly on upper and lower solution arguments. We will use some known results for (2.1) on bounded domains, which are proved by a combination of local bifurcation analysis and upper and lower solution techniques.

The first bounded domain result we will use follows easily from a result due to Berestycki, Capuzzo-Dolcetta and Nirenberg [4, Theorem 2].

Proposition 2.2. *Suppose that* (2.2) *is satisfied. Then there exists $R_* > R_0$ such that for any ball $B_R = B_R(0)$ with $R \geqslant R_*$, there exists $\Lambda_R \in (\lambda_1(B_R), \infty)$ such that the problem*

$$-\Delta u = \lambda u - b(x)u^p, \quad x \in B_R,\ u|_{\partial B_R} = 0 \tag{2.3}$$

has at least one positive solution for $\lambda \in (\lambda_1(B_R), \Lambda_R)$, and no positive solution for $\lambda > \Lambda_R$. Here $\lambda_1(B_R)$ denotes the first eigenvalue of $-\Delta$ on B_R under Dirichlet boundary conditions.

Proof. By Theorem 2 in [4], it suffices to show that

$$\int_{B_R} b(x)\phi_R^{p+1}\,dx > 0, \tag{2.4}$$

for all large R, where ϕ_R denotes the normalized (in L^∞) positive eigenfunction corresponding to the first eigenvalue $\lambda_1(B_R)$. We remark that condition (1.5) in [4] is not needed in their proof of Theorem 2.

To show (2.4), we first observe that, through a simple rescaling, $\phi_R(x) = \phi_1(x/R)$. Therefore,

$$\int_{B_R} b(x)\phi_R^{p+1}(x)\,dx = \int_{B_1} b(Ry)\phi_1^{p+1}(y)R^N\,dy$$
$$= R^N \int_{|y| \leqslant R_0/R} b(Ry)\phi_1^{p+1}(y)\,dy + R^N \int_{R_0/R \leqslant |y| \leqslant 1} b(Ry)\phi_1^{p+1}(y)\,dy.$$

As $R \to \infty$,

$$\int_{|y| \leqslant R_0/R} b(Ry)\phi_1^{p+1}(y)\,dy \to 0,$$

while

$$\int_{R_0/R \leqslant |y| \leqslant 1} b(Ry)\phi_1^{p+1}(y)\,dy \geqslant \int_{R_0/R \leqslant |y| \leqslant 1} \sigma\phi_1^{p+1}(y)\,dy \to \sigma\int_{B_1} \phi_1^{p+1}(y)\,dy > 0.$$

Hence (2.4) holds for all large R. □

Clearly $\lambda_1(B_R)$ decreases as R increases. The next result shows that Λ_R is also a decreasing function; it as well gives some properties of the positive solutions of (2.3).

Proposition 2.3. *Under the conditions of Proposition* 2.2, *for each* $\lambda \in (\lambda_1(B_R), \Lambda_R)$, (2.3) *has a minimal positive solution* u_λ^R *in the sense that any positive solution* u *of* (2.3) *satisfies* $u \geqslant u_\lambda^R$ *in* B_R. *Moreover,*

$$\Lambda_{R_1} \geqslant \Lambda_{R_2} \quad \text{whenever } R_* \leqslant R_1 \leqslant R_2, \tag{2.5}$$

and

$$u_{\lambda_1}^{R_1}(x) \leqslant u_{\lambda_2}^{R_2}(x) \quad \text{whenever both sides are defined and } \lambda_1 \leqslant \lambda_2,\ R_1 \leqslant R_2. \tag{2.6}$$

Proof. The arguments below are rather standard. We sketch them here for completeness.

Suppose that $\lambda \in (\lambda_1(B_R), \Lambda_R)$, and u is a positive solution of (2.3). It is easily checked that for all small $\varepsilon > 0$, $\varepsilon\phi_R < u$ in B_R (by making use of the Hopf boundary lemma), and $\varepsilon\phi_R$ is a lower solution to (2.3). Suppose that this is true for all $\varepsilon \in (0, \varepsilon_0]$. Then by a standard iteration procedure starting from $\varepsilon_0\phi_R$ one obtains a minimal solution of (2.3), say v, in the order interval

$$[\varepsilon_0\phi_R, u] := \{v \in C^1(\overline{B}_R):\ \varepsilon_0\phi_R \leqslant v \leqslant u\}.$$

We claim that v is also minimal among all positive solutions of (2.3). Indeed, since $\varepsilon\phi_R$ is a family of (strict) lower solutions that varies continuously in $\varepsilon \in (0, \varepsilon_0]$, and for any positive solution w of (2.3), we can find some $\varepsilon_1 \in (0, \varepsilon_0]$ such that $\varepsilon_1\phi_R < w$ in B_R, by a well known sweeping principle due to Serrin, it follows that $\varepsilon_0\phi_R \leqslant w$. Now the iteration procedure shows immediately that $v \leqslant w$. Hence v is the minimal positive solution of (2.3). We denote $v = u_\lambda^R$.

To show (2.5), we argue indirectly. Suppose that for some $R_* \leqslant R_1 < R_2$, we have $\Lambda_{R_1} < \Lambda_{R_2}$. Then we can choose a λ such that

$$\max\{\lambda_1(B_{R_2}), \Lambda_{R_1}\} < \lambda < \Lambda_{R_2}.$$

For such λ, the minimal positive solution $u_\lambda^{R_2}$ is defined. Since $u_\lambda^{R_2} > 0$ on ∂B_{R_1}, we can use $u_\lambda^{R_2}$ as an upper solution to (2.3) with $R = R_1$. As before, due to $\lambda > \Lambda_{R_1} > \lambda_1(B_{R_1})$, for all small $\varepsilon > 0$, $\varepsilon\phi_{R_1} < u_\lambda^{R_2}$ in B_{R_1} and they are lower solutions of (2.3) with $R = R_1$. This implies that (2.3) with $R = R_1$ has at least one positive solution, contradicting our choice of λ. This proves (2.5).

To prove (2.6), we observe that $u_{\lambda_2}^{R_2}$ can be used as an upper solution for the equation satisfied by $u_{\lambda_1}^{R_1}$. On the other hand, there are arbitrarily small lower solutions given by $\varepsilon\phi_{R_1}$. Hence (2.3) with $(\lambda, R) = (\lambda_1, R_1)$ has at least one positive solution u satisfying $u \leqslant u_{\lambda_2}^{R_2}$. Now (2.6) follows readily as $u_{\lambda_1}^{R_1}$ is the minimal solution. □

Theorem 2 in [4] also covers the case of Neumann boundary conditions. We can use this result to obtain an analogue of Proposition 2.2 for the corresponding Neumann problem of (2.3). Moreover, the argument in the proof of Proposition 2.3 above shows that the Neumann problem has a minimal positive solution whenever there is a positive solution with $\lambda > 0$. These are summarized in the following result.

Proposition 2.4. *Under condition* (2.2), *for all large* R, *there exists* $\tilde{\Lambda}_R > 0$ *such that the problem*

$$-\Delta u = \lambda u - b(x)u^p, \quad x \in B_R,\ \partial_\nu u|_{\partial B_R} = 0 \tag{2.7}$$

has a minimal positive solution for $\lambda \in (0, \tilde{\Lambda}_R)$, *and no positive solution for* $\lambda > \tilde{\Lambda}_R$.

One might wonder whether there are corresponding properties to (2.5) and (2.6) for the Neumann problem (2.7). This, however, is not the case in general.

We are now ready to present a crucial ingredient for the proof of Theorem 2.1.

Lemma 2.5. *Let Λ_R be given by Proposition* 2.2. *Then*

$$\Lambda_* := \lim_{R\to\infty} \Lambda_R > 0.$$

Proof. From (2.2), we find that there exists a continuous radially symmetric function $\tilde{b}(x) = \tilde{b}(r)$, $r = |x|$, such that

$$\tilde{b}(x) \leqslant b(x), \quad \forall x \in B_{R_0}; \qquad \tilde{b}(x) = \sigma, \quad \forall |x| \geqslant R_0. \tag{2.8}$$

Clearly $\tilde{b}$ also satisfies (2.2). By Proposition 2.4, for some large $R_1 > R_0$, there exists $\tilde{\lambda} > 0$ such that (2.7) with b replaced by $\tilde{b}$ and $\lambda = \tilde{\lambda}$ has a minimal positive solution, say $\tilde{u}$, on B_{R_1}. The minimality of $\tilde{u}$ forces it to be radially symmetric, as the equation is invariant under rotations around the origin. The Hopf boundary lemma implies that $\tilde{u} > 0$ on $\overline{B}_{R_1}$.

Let us denote $\xi = \tilde{u}(R_1) > 0$ and let $\lambda_0 \in (0, \tilde{\lambda})$ be such that

$$\lambda_0 \xi - \sigma \xi^p \leqslant 0.$$

We then choose $R_2 > R_1$ so that $\lambda_1(B_R) < \lambda_0$ for all $R \geqslant R_2$. We will show in a moment that $\Lambda_R \geqslant \lambda_0$ for all $R \geqslant R_2$. As by Proposition 2.3, Λ_R decreases as R increases, this would guarantee that $\lim_{R\to\infty} \Lambda_R \geqslant \lambda_0 > 0$, as we wanted.

Let us define

$$u_0(x) = \begin{cases} \tilde{u}(x), & x \in B_{R_1}, \\ \xi, & |x| \geqslant R_1. \end{cases}$$

It is easily checked that u_0 is a weak upper solution of (2.3) with $R > R_1$ and $\lambda = \lambda_0$ (see also [6]), that is,

$$\int_{B_R} \nabla u_0 \cdot \nabla \psi \geqslant \int_{B_R} \bigl(\lambda_0 u_0 - b(x) u_0^p\bigr)\psi, \quad \forall \psi \in C_0^\infty(B_R),\ \psi \geqslant 0.$$

If further $R \geqslant R_2$, then $\lambda_0 > \lambda_1(B_R)$ and for all small $\varepsilon > 0$, $\varepsilon\phi_R < u_0$ in B_R and are lower solutions to (2.3) with $\lambda = \lambda_0$. Hence there is at least one positive solution and so $\Lambda_R \geqslant \lambda_0$, $\forall R \geqslant R_2$. Moreover

$$u_{\lambda_0}^R(x) \leqslant u_0(x), \quad \forall x \in B_R,\ \forall R \geqslant R_2. \tag{2.9}$$

This finishes the proof. □

Proof of Theorem 2.1. Let λ_0, u_0 and R_2 be as in the proof of Lemma 2.5. Then, by (2.9) and (2.6), we find that for arbitrary $x \in R^N$, $U_0(x) = \lim_{R\to\infty} u_{\lambda_0}^R(x)$ exists and $U_0(x) \leqslant u_0(x)$. Through a regularity consideration and a standard compactness argument, we see that U_0 is a solution of (2.1) with $\lambda = \lambda_0$. U_0 is positive since $U_0 \geqslant u_{\lambda_0}^R$ for every $R \geqslant R_2$. Hence (2.1) has a positive solution for $\lambda = \lambda_0 > 0$.

Define

$$\Lambda := \sup\bigl\{\mu > 0\text{: (2.1) has a positive solution for } \lambda = \mu\bigr\}.$$

Clearly $\Lambda \geqslant \lambda_0$. We also have $\Lambda \leqslant \lambda_1(B_{r_0}(x_0))$. Indeed, if $\Lambda > \lambda_1(B_{r_0}(x_0))$, then we can find $\lambda > \lambda_1(B_{r_0}(x_0))$ such that (2.1) has a positive solution u with such λ. On $B_{r_0}(x_0)$, by (2.2), we have

$$-\Delta u = \lambda u - b(x) u^p \geqslant \lambda u.$$

Let ϕ denotes the normalized positive eigenfunction corresponding to $\lambda_1(B_{r_0}(x_0))$. We deduce

$$\lambda \int_{B_{r_0}(x_0)} u\phi \leqslant \int_{B_{r_0}(x_0)} (-\Delta u)\phi = \int_{B_{r_0}(x_0)} (-\Delta\phi)u + \int_{\partial B_{r_0}(x_0)} \partial_\nu \phi u < \lambda_1\bigl(B_{r_0}(x_0)\bigr) \int_{B_{r_0}(x_0)} \phi u.$$

Hence $\lambda < \lambda_1(B_{r_0}(x_0))$, contradicting our assumption that $\lambda > \lambda_1(B_{r_0}(x_0))$.

It remains to show that (2.1) has a positive solution for every $\lambda \in (0, \Lambda)$. Let λ be such fixed. By the definition of Λ, we can find $\lambda^* \in (\lambda, \Lambda]$ such that (2.1) has a positive solution u^* with $\lambda = \lambda^*$. Then u^* is an upper solution of (2.3) with the above fixed λ on any B_R. Let $R^* > 0$ be large enough so that $\lambda_1(B_R) < \lambda$ for all $R > R^*$. Then for any fixed $R > R^*$ and all small $\varepsilon > 0$, $\varepsilon\phi_R < u^*$ in B_R and are lower solutions to (2.3) with these given λ and R. Hence (2.3) has a positive solution and $\Lambda_R \geqslant \lambda$. It follows from Proposition 2.3 that u_λ^R exists for all $R > R^*$, and $u_\lambda^R \leqslant u^*$. Now much as before, $U^* := \lim_{R\to\infty} u_\lambda^R$ is a positive solution of (2.1) with the given λ. The proof is complete. □

3. A priori estimates and further existence result

In this section, we show that under further restrictions on p and on $b(x)$, a priori estimates (independent of R) for positive solutions of (2.3) can be established. This will enable us to show that (2.1) has at least one positive solution for $\lambda = \Lambda$. In the next section, we will show that (2.1) has at least two positive solutions for $\lambda \in (0, \Lambda)$, and at least one positive solution for $\lambda \leqslant 0$. For these, we will need, apart from the a priori estimates in this section, some global bifurcation arguments, where some subtle ordering properties of positive solutions of (2.3) will become crucial.

To establish the a priori estimates, we let (2.2) be satisfied and denote

$$\Omega_- = \bigl\{x \in B_{R_0}\colon b(x) < 0\bigr\}, \qquad \Omega_+ = \bigl\{x \in B_{R_0}\colon b(x) > 0\bigr\}, \qquad b^-(x) = \min\bigl\{b(x), 0\bigr\}.$$

Theorem 3.1. *Suppose that* (2.2) *holds,* Ω_- *and* Ω_+ *are open sets with* C^2 *boundaries, and that there exist* $\alpha : \overline{\Omega}_- \to (-\infty, 0)$ *which is continuous and bounded away from zero in a neighborhood of* $\partial\Omega_-$*, and a constant* $\gamma \geqslant 0$*, such that*

$$b^-(x) = \alpha(x)\bigl[\operatorname{dist}(x, \partial\Omega_-)\bigr]^\gamma, \quad \forall x \in \Omega_-. \tag{3.1}$$

Also suppose that

$$p < (N + 1 + \gamma)/(N - 1) \tag{3.2}$$

and

$$p < (N + 2)/(N - 2) \quad \text{in case } N \geqslant 3. \tag{3.3}$$

Then for any given $M > R_*$ *(as in Proposition* 2.2*), we can find a constant* $C = C(M)$ *such that any positive solution* u *of* (2.3) *with* $R > M$ *and* $\lambda > -M$ *satisfies*

$$\|u\|_{L^\infty(B_R)} \leqslant C. \tag{3.4}$$

Proof. We adapt the techniques of Amann and Lopez-Gomez [3]. If we can show that there exists $C_0 = C_0(M, M_0) > 0$ such that

$$\sup_{\Omega_-} u \leqslant M_0 \quad \text{implies} \quad \sup_{B_R\setminus\Omega_-} u \leqslant C_0, \tag{3.5}$$

for any positive solution u of (2.3) with $\lambda > -M$ and $R > M$, then (3.4) can be proved exactly as in [3], where a standard blow up argument on Ω_- is used to deduce a contradiction to a Liouville theorem in [4] if (3.4) does not hold. Let us note that by Proposition 2.3, we have $\lambda \leqslant \Lambda_M$ whenever (2.3) has a positive solution on B_R, $R > M$.

Therefore, it suffices to establish (3.5). Fix $R_1 \in (R_0, M)$. We first show that there exists $C_1 = C_1(M, R_1)$ such that

$$\sup_{B_R \setminus B_{R_1}} u \leqslant C_1 \tag{3.6}$$

for any positive solution of (2.3) with $\lambda > -M$ and $R > M$. To see this, let u be such a positive solution. Then, on $B_R \setminus B_{R_0}$, we have

$$-\Delta u = \lambda u - b(x)u^p \leqslant \Lambda_M u - \sigma u^p.$$

Let $r = R_1 - R_0$. By [16], the problem

$$-\Delta v = \Lambda_M v - \sigma v^p \quad \text{in } B_r(0),\ v|_{\partial B_r(0)} = \infty$$

has a unique positive solution v. For fixed $x_0 \in B_R \setminus B_{R_1}$, clearly $v(x - x_0)$ satisfies the same equation as $v(x)$ but with $B_r(0)$ replaced by $B_r(x_0)$. Since $B_r(x_0) \cap B_{R_0} = \emptyset$, applying Lemma 1.1 in [16] to compare $u(x)$ and $v(x - x_0)$ over $B_r(x_0) \cap B_R$, we obtain that $u(x) \leqslant v(x - x_0)$ on this set. In particular, $u(x_0) \leqslant v(0)$. Hence

$$u \leqslant C_1 := v(0), \quad \forall x \in B_R \setminus B_{R_1}.$$

This proves (3.6).

Next we consider u on $\Omega := B_{R_1} \setminus \Omega_-$. Denote $b^+(x) = \max\{b(x), 0\}$. We find that $b^+(x) = 0$ if and only if $x \in D := B_{R_0} \setminus \Omega_+$. By the proof of Theorem 2.3 of [3], we necessarily have $\lambda < \lambda_1(D)$.

Consider now the problem

$$-\Delta w = \lambda_1(D)w - b^+(x)w^p \quad \text{in } \Omega,\ w|_{\partial B_{R_1}} = C_1,\ w|_{\partial\Omega_-} = M_0. \tag{3.7}$$

Since $\Omega_0 := \{x \in \Omega\colon b^+(x) = 0\} = D \setminus \Omega_-$, we have $\lambda_1(\Omega_0) > \lambda_1(D)$, and hence, for some small ε-neighborhood Ω_ε of Ω_0, $\lambda_1(\Omega_\varepsilon) > \lambda_1(D)$. Making use of this fact, we can construct an upper solution of the form kw_0, with $k > 0$ a large constant, and $w_0(x) = \phi_{\Omega_\varepsilon}(x)$ on $\Omega_{\varepsilon/2}$ and $w_0(x) > 0$ on $\overline{B}_{R_1} \setminus \Omega_{\varepsilon/2}$, much as in p. 348 of [3]. Clearly 0 is a lower solution of (3.7). It follows that (3.7) has a positive solution. By Lemma 2.1 in [13], we deduce that (3.7) has a unique solution w. Moreover, noticing $b(x) = b^+(x)$ in Ω and $\lambda < \lambda_1(D)$, we have

$$-\Delta u = \lambda u - b(x)u^p < \lambda_1(D)u - b^+(x)u^p, \quad \forall x \in \Omega.$$

As $u|_{\partial\Omega} \leqslant w|_{\partial\Omega}$ when the condition in (3.5) holds, we use Lemma 2.1 in [13] again and conclude $u \leqslant w$ in Ω. Combining this with (3.6), we obtain (3.5). □

Remark 3.2. (i) As in [3, Theorem 5.2], the conclusion of Theorem 3.1 holds when the conditions (3.1) through to (3.3) are replaced by

$$\overline{\Omega}_- \cap \overline{\Omega}_+ = \emptyset \tag{3.8}$$

and

$$p < N/(N-2) \quad \text{in case } N \geqslant 3. \tag{3.9}$$

(ii) In a recent paper [10], we were able to treat the case that (3.2) is relaxed to $p < (N+2)/(N-2)$, by using the observation that we only need (3.4) for solutions obtained by certain mountain pass processes. In a further recent paper [12], by proving some new Liouville type theorems, we were able to show that (3.4) holds for any positive solution under the condition $p < (N+2)/(N-2)$ only.

Theorem 3.3. *Suppose* (2.2) *holds and that either* (3.1) *through to* (3.3) *hold or both* (3.8) *and* (3.9) *hold. Let Λ and Λ_* be as in Theorem* 2.1 *and Lemma* 2.5*, respectively. Then $\Lambda = \Lambda_*$ and* (2.1) *has a positive solution for $\lambda = \Lambda$.*

Proof. From the definition of Λ_*, we know that $\Lambda_R \geqslant \Lambda_*$ for every $R \geqslant R_*$. By Proposition 2.3, for all small $\varepsilon > 0$, $u^R_{\Lambda_*-\varepsilon}$ exists and is increasing as ε decreases. From Theorem 3.1, we infer that $u^R_{\Lambda_*-\varepsilon} \leqslant C$ for some constant C independent of ε. Hence $U^R := \lim_{\varepsilon\to 0} u^R_{\Lambda_*-\varepsilon}$ exists and is a positive solution of (2.3) with $\lambda = \Lambda_*$. It follows that $u^R_{\Lambda^*}$ exists. Moreover, the argument in Proposition 2.3 shows that $u^R_{\Lambda^*}$ is increasing with R for $R \geqslant R_*$. We now apply Theorem 3.1 again and find that $u^R_{\Lambda_*} \leqslant C_1$ for some constant C_1 independent of R. Hence $U := \lim_{R\to\infty} u^R_{\Lambda_*}$ exists and, as before, is a positive solution of (2.1) with $\lambda = \Lambda_*$. This implies that $\Lambda \geqslant \Lambda^*$.

Were $\Lambda > \Lambda_*$, (2.1) would have a positive solution u for some $\lambda \in (\Lambda_*, \Lambda]$. As in the proof of Theorem 2.1, this would imply that for all large R, (2.3) has a positive solution with this λ. Hence $\Lambda_R \geqslant \lambda$ for all large R, and it follows that $\Lambda_* = \lim \Lambda_R \geqslant \lambda > \Lambda_*$, a contradiction. Hence we must have $\Lambda = \Lambda_*$. □

4. Global bifurcation and multiplicity results

In this section, we make use of global bifurcation arguments to show that (2.1) has at least two positive solutions for $\lambda \in (0, \Lambda)$. We also prove that (2.1) has at least one positive solution for $\lambda \leqslant 0$. Again we use the bounded domain problem (2.3) to approximate the entire space problem (2.1). However, it seems that a priori bound alone is not enough for this purpose. Under the conditions of the last section, we know that for each fixed λ, any positive solution u_R of (2.3) on B_R has an L^∞ bound independent of R. From this it is easy to see that by choosing a suitable sequence $R_n \to \infty$, u_{R_n} converges to a solution u of (2.1) with such λ. The problem is that when $\lambda \in (0, \Lambda)$, we want to make sure that such a solution u can be obtained which is different from the one as obtained in the proof of Theorem 2.1, while when $\lambda \leqslant 0$, we want to make sure that u is not the zero solution. It turns out that this goal can be achieved by choosing u_R from a particular part of the global bifurcation branch of (2.3). The crucial point in our proof is an ordering property which comes from a careful analysis of the global bifurcation branch of (2.3).

Let us now describe the global bifurcation branch in more detail. Suppose that (2.2) is satisfied. Then, from the proof of Proposition 2.2, for all large R, (2.4) holds. It follows from a local bifurcation analysis (see in particular Lemma 6.1 in [4], and [8] generally) that near $(\lambda_1(B_R), 0)$ in the space $X := R \times C^1(\overline{B}_R)$, all the solutions (λ, u) of (2.3) with $u > 0$ lie in a smooth curve

$$\Gamma'_R := \bigl\{\bigl(\lambda(t), u(t)\bigr)\colon\ t \in (0, \varepsilon)\bigr\},$$

where $\lambda(0) = \lambda_1(B_R)$, $u(0) = 0$, and $\lambda(t) > \lambda_1(B_R)$ for $t \in (0, \varepsilon)$, $u(t) = t\phi_R + \mathrm{o}(1)$ for small t.

It is well known that the global bifurcation theory of Rabinowitz can be applied to this case (see [18, Theorem 2.12]) to conclude that there exists an unbounded connected set Γ_R in X such that,

(i) $(\lambda, u) \in \Gamma_R$ implies that u is a positive solution of (2.3),
(ii) $\Gamma'_R \subset \Gamma_R$,
(iii) there is a small neighborhood N_δ of $(\lambda_1(B_R), 0)$ in X such that $N_\delta \cap \Gamma_R = N_\delta \cap \Gamma'_R$.

We assume further that the conditions in Theorem 3.3 are satisfied, so that there exists $C = C(M)$ such that any $(\lambda, u) \in \Gamma_R$ with $\lambda \geqslant -M$ satisfies $\|u\|_{L^\infty(B_R)} \leqslant C$. By Proposition 2.2 we know that $(\lambda, u) \in \Gamma_R$ implies $\lambda \leqslant \Lambda_R$. Thus the unbounded connected set Γ_R becomes so only through $\lambda \to -\infty$, that is,

$$\bigl\{\lambda\colon\ (\lambda, u) \in \Gamma_R\bigr\} \supset \bigl(-\infty, \lambda_1(B_R)\bigr). \tag{4.1}$$

We refer to [2,15] for more detailed discussions for problems of a similar nature.

We are now ready to present some further properties of Γ_R, which will play a key role in the proof of our main multiplicity and existence result in this section.

Let us recall that for $\lambda \in (\lambda_1(B_R), \Lambda_R]$, u^R_λ denotes the minimal positive solution of (2.3). It is also convenient to introduce the notation

$$O_\lambda = (-\infty, \lambda] \times [0, u^R_\lambda],$$

where for any $w \in C^1(\overline{B}_R)$ satisfying $w \geqslant 0$ in B_R,

$$[0, w] = \{u \in C^1(\overline{B}_R):\ 0 \leqslant u \leqslant w \text{ in } B_R\}.$$

Proposition 4.1. *Under the conditions of Theorem* 3.3*, the following are true for every large fixed* R.

(i) $(\lambda, u_\lambda^R) \in \Gamma_R,\ \forall \lambda \in (\lambda_1(B_R), \Lambda_R]$.
(ii) $\Gamma_R^c := (\Gamma_R \setminus O_{\Lambda_R}) \cup \{(\Lambda_R, u_{\Lambda_R}^R)\}$ *is connected.*
(iii) $\{\lambda:\ (\lambda, u) \in \Gamma_R^c\} = (-\infty, \Lambda_R]$.

Proof. We will use some ideas in [9]. Due to (4.1), we can find a sequence $(\lambda_n, u_n) \in \Gamma_R$ such that $\lambda_n \to -\infty$. We may assume that $\lambda_n < 0$ for all n.

Let $x_n \in B_R$ be such that $u_n(x_n) = \max_{\overline{B}_R} u_n$. Then it follows from Bona's maximum principle and the equation for u_n that

$$\lambda_n u_n(x_n) - b(x_n) u_n^p(x_n) \geqslant 0.$$

As $\lambda_n < 0$ and $u_n(x_n) > 0$, this is possible only if $b(x_n) < 0$. We obtain

$$u_n(x_n) \geqslant \left[\lambda_n / b(x_n)\right]^{1/(p-1)} \geqslant \left[|\lambda_n| / |\min_{R^N} b|\right]^{1/(p-1)} \to \infty. \tag{4.2}$$

Let us fix $\mu \in (\lambda_1(B_R), \Lambda_R]$. From the properties of Γ_R' we see that for $(\lambda, u) \in \Gamma_R$ close to $(\lambda_1(B_R), 0)$, it holds $(\lambda, u) \in O_\mu$. On the other hand, (4.2) implies that for all large n, (λ_n, u_n) are outside O_μ. As Γ_R is connected, we must have $\Gamma_R \cap \partial O_\mu \neq \emptyset$.

We claim that

$$\Gamma_R \cap \partial O_\mu = \{(\mu, u_\mu^R)\}. \tag{4.3}$$

Clearly (i) is a consequence of this fact.

To prove (4.3), we choose an arbitrary $(\lambda, u) \in \Gamma_R \cap \partial O_\mu$. By the definition of O_μ, we infer $\lambda \leqslant \mu$, $u \leqslant u_\mu^R$. If $\lambda < \mu$, then from $u \leqslant u_\mu^R$ and $u \not\equiv u_\mu^R$ we can conclude, by making use of the differential equations they satisfy and the strong maximum principle together with the Hopf boundary lemma, $v := u_\mu^R - u > 0$ in B_R, $\partial_\nu v < 0$ on ∂B_R. Since u is a positive solution of (2.3), we also have $u > 0$ in B_R and $\partial_\nu u < 0$ on ∂B_R. These facts imply that u is in the interior of the set $[0, u_\mu^R]$, and hence, as $\lambda < \mu$, (λ, u) is in the interior of O_μ. This is a contradiction. So we necessarily have $\lambda = \mu$. But then we must have $u = u_\mu^R$, since u_μ^R is the minimal positive solution of (2.3) and $u \leqslant u_\mu^R$ is also a positive solution of (2.3). Therefore (4.3) is true.

To prove (ii) we argue indirectly. Suppose that Γ_R^c is not connected. Then there exist two nonempty sets Δ_1 and Δ_2 such that $\Gamma_R^c = \Delta_1 \cup \Delta_2$ and Δ_1, Δ_2 are separated (see [20, p. 9]), that is,

$$\overline{\Delta}_1 \cap \Delta_2 = \emptyset, \qquad \Delta_1 \cap \overline{\Delta}_2 = \emptyset.$$

Since $(\Lambda_R, u_{\Lambda_R}^R) \in \Gamma_R^c$, we may assume that this point lies in Δ_1. Then we have

$$\Delta_2 \cap O_{\Lambda_R} = \emptyset. \tag{4.4}$$

We show next that $\overline{\Delta}_2 \cap O_{\Lambda_R} = \emptyset$. Otherwise, we necessarily have $\overline{\Delta}_2 \cap \partial O_{\Lambda_R} \neq \emptyset$. Let us observe that $(\lambda, u) \in \overline{\Delta}_2$ implies that (λ, u) solves (2.3) with $u \geqslant 0$ and $\lambda \leqslant \Lambda_R$. Therefore if there exists $(\lambda, u) \in \overline{\Delta}_2 \cap \partial O_{\Lambda_R}$, then $0 \leqslant u \leqslant u_{\Lambda_R}^R$, $\lambda \leqslant \Lambda_R$ and (λ, u) solves (2.3). If $\lambda = \Lambda_R$, then since $u_{\Lambda_R}^R$ is the minimal positive solution of (2.3), we have either $u = u_{\Lambda_R}^R$ or $u = 0$. The former possibility cannot occur as we have assumed that $(\Lambda_R, u_{\Lambda_R}^R) \in \Delta_1$. So we must have $u = 0$.

If $\lambda < \Lambda_R$, we can also deduce that $u = 0$, for otherwise, u is a positive solution of (2.3) and we can use the same argument used in proving (4.3) to show that (λ, u) is in the interior of O_{Λ_R}.

So we have proved that in either case, $(\lambda, u) = (\lambda, 0)$. It follows that there exists a sequence $(\lambda_n, u_n) \in \Delta_2$ such that $(\lambda_n, u_n) \to (\lambda, 0)$ in X. The fact that $u_n \to 0$ in $C^1(\overline{B}_R)$ implies that u_n lies in $[0, u^R_{\Lambda_R}]$ when n is large. Since (λ_n, u_n) are positive solutions to (2.3), we must have $\lambda_n \leqslant \Lambda_R$. Hence for all large n, $(\lambda_n, u_n) \in O_{\Lambda_R}$, contradicting (4.4). This proves that $\overline{\Delta}_2 \cap O_{\Lambda_R} = \emptyset$. As a consequence, Δ_2 is separated from both Δ_1 and O_{Λ_R} since O_{Λ_R} is closed. Now define $\Delta_3 := \Delta_1 \cup (O_{\Lambda_R} \cap \Gamma_R)$ and we find that Δ_2 and Δ_3 are separated. But $\Gamma_R = \Delta_2 \cup \Delta_3$. So the above conclusion implies that Γ_R is not connected. This contradiction proves that Γ_R^c is connected. Conclusion (ii) is thus proved.

(iii) follows from (ii) and the following three facts: (a) $(\Lambda_R, u^R_{\Lambda_R}) \in \Gamma_R^c$, (b) (2.3) has no positive solution when $\lambda > \Lambda_R$, (c) Γ_R contains a sequence (λ_n, u_n) with $\lambda_n \to -\infty$ and (4.2) holds, and hence $(\lambda_n, u_n) \in \Gamma_R^c$ for all large n. □

Remark 4.2. Let us note that for any $(\lambda, u) \in \Gamma_R^c$ with $\lambda < \Lambda_R$, $u \notin [0, u^R_{\Lambda_R}]$. This ordering property will play a key role in the proof of our main results.

Apart from Proposition 4.1, in proving the main multiplicity result, we also need some auxiliary equations on enlarging balls or annuli.

Lemma 4.3. *Suppose that R_n is an increasing sequence converging to ∞ and $B_n = B_{R_n}(0)$. Let $\lambda > 0$ and $p > 1$ be fixed and ξ_n be a sequence of positive numbers converging to $\xi > 0$ as $n \to \infty$. Then, for all large n, the problem*

$$-\Delta u = \lambda u - \xi_n u^p \quad \text{in } B_n,\ u|_{\partial B_n} = 0 \tag{4.5}$$

and the problem

$$-\Delta v = \lambda v - \xi_n v^p \quad \text{in } B_n,\ v|_{\partial B_n} = \infty \tag{4.6}$$

have unique positive solutions u_n and v_n, respectively. Moreover,

$$u_n(x) \to (\lambda/\xi)^{1/(p-1)}, \qquad v_n(x) \to (\lambda/\xi)^{1/(p-1)}, \tag{4.7}$$

uniformly on any bounded set of R^N as $n \to \infty$.

Here and in what follows, by $v|_{\partial B_n} = \infty$, we mean $v(x) \to \infty$ as $d(x, \partial B_n) \to 0$.

Proof. For any given small $\varepsilon \in (0, \xi)$, we can find n_0 large so that $\xi - \varepsilon < \xi_n < \xi + \varepsilon$ for all $n \geqslant n_0$. By Lemma 2.2 in [13], (4.5) with ξ_n replaced by $\xi - \varepsilon$ has a unique positive solution $\bar{u}_n$ for all large n, and

$$\lim_{n\to\infty} \bar{u}_n(x) = \left[\lambda/(\xi - \varepsilon)\right]^{1/(p-1)} \tag{4.8}$$

uniformly on any bounded set of R^N.

Similarly, (4.5) with ξ_n replaced by $\xi + \varepsilon$ has a unique positive solution $\underline{u}_n$ for all large n, and

$$\lim_{n\to\infty} \underline{u}_n(x) = \left[\lambda/(\xi + \varepsilon)\right]^{1/(p-1)} \tag{4.9}$$

uniformly on any bounded set of R^N.

Let u_n denote the unique positive solution of (4.5) (which exists whenever $\lambda > \lambda_1(B_n)$). By Lemma 2.1 in [13] we have $\underline{u}_n \leqslant u_n \leqslant \bar{u}_n$. Now we see immediately that the first part of (4.7) follows from (4.8), (4.9) and the arbitrariness of ε.

By Lemma 2.3 in [13], we know that (4.6) with ξ_n replaced by $\xi - \varepsilon$ has a unique positive solution $\bar{v}_n$ for each n, and

$$\lim_{n\to\infty} \bar{v}_n(x) = \left[\lambda/(\xi - \varepsilon)\right]^{1/(p-1)}$$

uniformly on any bounded set of R^N.

Similarly, (4.6) with ξ_n replaced by $\xi+\varepsilon$ has a unique positive solution $\underline{v}_n$ for each n, and

$$\lim_{n\to\infty}\underline{v}_n(x)=\left[\lambda/(\xi+\varepsilon)\right]^{1/(p-1)}$$

uniformly on any bounded set of R^N.

Let v_n denote the unique positive solution of (4.6). Then by Lemma 2.1 in [13], we obtain that $\underline{v}_{n+1}\leqslant v_n\leqslant\bar{v}_{n-1}$ on B_{n-1}. The second part of (4.7) then follows. □

Lemma 4.4. *Let $\lambda>0$, $p>1$ be fixed, and R_n, ξ_n as in Lemma* 4.3. *Denote by A_n the annulus $\{x\in R^N\colon\ R_n/2<|x|<R_n\}$. Then for all large n, the problem*

$$-\Delta u=\lambda u-\xi_n u^p\quad \text{in } A_n,\ u|_{\partial A_n}=0 \tag{4.10}$$

and the problem

$$-\Delta v=\lambda v-\xi_n v^p\quad \text{in } A_n,\ v|_{\{|x|=R_n/2\}}=\infty,\ v|_{\{|x|=R_n\}}=0 \tag{4.11}$$

have unique positive solutions u_n and v_n respectively. Moreover, $u_n(x)=u_n(|x|)$, $v_n(x)=v_n(|x|)$, and if we define, for $r\in(-R_n/2,0]$, $U_n(r)=u_n(R_n+r)$ and $V_n(r)=v_n(R_n+r)$, then, as $n\to\infty$,

$$U_n\to\Phi,\ V_n\to\Phi\quad \text{in } C^1([-T,0]),\ \forall T>0, \tag{4.12}$$

where Φ is the unique positive solution to

$$-\Phi''=\lambda\Phi-\xi\Phi^p,\quad \Phi(-\infty)=(\lambda/\xi)^{1/(p-1)},\ \Phi(0)=0. \tag{4.13}$$

Proof. The existence and uniqueness of u_n for $\lambda>\lambda_1(A_n)$ is well known. The existence and uniqueness of v_n follows from [11] (see the arguments in Section 2 and Remark 2.9 there). The radial symmetry of u_n and v_n follows from their uniqueness. It remains to prove (4.12).

We consider U_n first. It satisfies

$$-U_n''-\frac{N-1}{R_n+r}U_n'=\lambda U_n-\xi_n U_n^p\quad \text{in } (-R_n/2,R_n),\ U_n(-R_n/2)=U_n(0)=0.$$

Let $r_n\in(-R_n/2,0)$ be such that $U_n(r_n)=\max_{[-R_n/2,0]}U_n$. Then from the equation for U_n we deduce

$$\lambda U_n(r_n)-\xi_n U_n(r_n)^p\geqslant 0.$$

It follows easily that $\|U_n\|_{L^\infty([R_n/2,0])}\leqslant C$ for all n and some positive constant C independent of n. Now we can use standard elliptic estimates and a diagonal arguments to choose a subsequence of U_n, which we still denote by U_n for simplicity, such that $U_n\to U$ in $C^1([-T,0])$ for any $T>0$, and U satisfies

$$-U''=\lambda U-\xi U^p\quad \text{in } (-\infty,0),\ U(0)=0. \tag{4.14}$$

We claim that U is positive in $(-\infty,0)$. If this is proved, then it follows from a simple phase plane analysis that U is the unique positive solution of (4.14) and it satisfies $U(-\infty)=(\lambda/\xi)^{1/(p-1)}$. We will denote the unique positive solution to (4.14) by Φ.

To show that U is positive on $(-\infty,0)$, for a fixed $r_0>0$, we choose $r_*>r_0$ such that $\lambda_1(B_{r_*})<\lambda$ and then for all large n we choose a ball $B_{r_*}(y_n)$ in A_n such that the ball touches the outer boundary of A_n. Choose $\xi_*>0$ such that $\xi_*>\xi_n$ for all n and let w_* be the unique positive solution of

$$-\Delta w=\lambda w-\xi_* w^p\quad \text{in } B_{r_*}(0),\ w|_{\partial B_{r_*}(0)}=0.$$

We know that w_* is radially symmetric. Clearly $w_n(x)=w_*(x-y_n)$ solves the same differential equation over $B_{r_*}(y_n)$. Using Lemma 2.1 in [13], we deduce that $u_n\geqslant w_n$ on $B_{r_*}(y_n)$. Hence, $u_n(x)\geqslant w_*(x-y_n)$, and in particular, $u_n(R_n-r_0)\geqslant w_*(r_*-r_0)$. It follows that

$$U(-r_0)=\lim U_n(-r_0)\geqslant w_*(r_*-r_0)>0.$$

Thus U is positive in $(-\infty, 0)$. This finishes our proof that a subsequence of U_n converges to the unique positive solution Φ of (4.14). Since this limit is unique, the entire sequence U_n converges to Φ.

Next we consider v_n and V_n. We first claim that there exists a constant C independent of n such that $v_n \leqslant C$ on the annulus $(2/3)R_n < |x| < R_n$ for all large n. Indeed, fix $r^* > 0$ and choose $\xi^* > 0$ such that $\xi^* < \xi_n$ for all n, and then consider the unique positive solution w^* of the problem

$$-\Delta w = \lambda w - \xi^* w^p \quad \text{in } B_{r^*},\ w|_{\partial B_{r^*}} = \infty.$$

We will show that, for all large n, $v_n(x) \leqslant w^*(0)$ on the annulus $(2/3)R_n \leqslant |x| \leqslant R_n$. Indeed, suppose that n is large enough so that $R_n/6 > r^*$. Then for any chosen x_0 satisfying $(2/3)R_n \leqslant |x_0| \leqslant R_n$, we have $\overline{B}_{r^*}(x_0) \cap \overline{B}_{R_n/2}(0) = \emptyset$. Hence we can use Lemma 2.1 of [13] to compare v_n with $w(x) = w^*(x - x_0)$ over $B_{r^*}(x_0) \cap A_n$ to conclude that $v_n \leqslant w$ in this region. In particular, $v_n(x_0) \leqslant w(x_0) = w^*(0)$, as we claimed.

Thus we have $V_n(r) \leqslant C$ for $r \in [-R_n/3, 0]$ for all large n. As before, by elliptic estimates, subject to a subsequence, $V_n \to V$ in $C^1([-T, 0])$ for any $T > 0$ and V solves (4.14). Since $v_n \geqslant u_n$ (by Lemma 2.1 in [13]), we conclude that $V \geqslant U$ and hence V is a positive solution of (4.14). It follows that $V = \Phi$. □

Remark 4.5. If $\lambda \leqslant 0$, then by [11], v_n still exists and is unique. An examination of the above proof for v_n and V_n shows that, in this case, a subsequence of V_n converges in $C^1([-T, 0])$, $\forall T > 0$, to a nonnegative solution V of (4.14). However, since $\lambda \leqslant 0$, it is easily seen that (4.14) has only one nonnegative solution, that is the zero solution. Thus $V = 0$ and the entire sequence V_n converges to 0. This fact will be needed later.

We are now ready to prove our main result.

Theorem 4.6. *Suppose that the conditions of Theorem* 3.3 *are satisfied. Moreover,*

$$\lim_{|x|\to\infty} b(x) = b_\infty \in [\delta, \infty). \tag{4.15}$$

Then, (2.1) *has at least two positive solutions for each* $\lambda \in (0, \Lambda)$, *and it has at least one positive solution for each* $\lambda \leqslant 0$.

Proof. Let us fix $\lambda \in (0, \Lambda)$. Since $\lambda_1(B_R)$ decreases to 0 and Λ_R decreases to Λ as $R \to \infty$, we can find an increasing sequence $R_n \to \infty$ such that $\lambda_1(B_{R_n}) < \lambda < \Lambda_{R_n}$ for every n. We now choose $(\lambda, u_n) \in \Gamma^c_{R_n}$; this is possible due to Proposition 4.1.

By Theorem 3.1 and Remark 3.2, there exists $C > 0$ independent of n such that

$$\|u_n\|_{L^\infty(B_{R_n})} \leqslant C, \quad \forall n \geqslant 1. \tag{4.16}$$

From (4.16) and the equation for u_n we find that for any fixed ball $B \subset R^N$, by the L^p theory of elliptic equations, $\{u_n|_B\}$ is bounded in $W^{2,q}(B)$ for any $q > 1$. It follows from Sobolev imbedding theorems that $\{u_n|_B\}$ is compact in $C^1(B)$. By choosing a sequence of enlarging balls and a standard diagonal argument, we can extract a subsequence from $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \to u$ in $C^1(B)$ for any bounded set B in R^N. It is easily checked that u solves (2.1).

Since $u_n \geqslant u_\lambda^{R_n}$ and $u_\lambda^{R_n}$ increases with n, we find that $u \geqslant u_\lambda^{R_n}$ for every n. Therefore u is a positive solution of (2.1). Denote, for each $\mu \in (0, \Lambda)$, $u_\mu := \lim_{n\to\infty} u_\mu^{R_n}$, we know that u_μ is a positive solution of (2.1) (with λ replaced by μ). (See the proofs of Theorems 2.1 and 3.3; it is easily seen that u_μ is in fact the minimal positive solution.) It remains to show that

$$u \neq u_\lambda. \tag{4.17}$$

Since $(\lambda, u_n) \in \Gamma^c_{R_n}$ and $\lambda < \Lambda \leqslant \Lambda_{R_n}$, we have $u_n \notin [0, u^{R_n}_{\Lambda_{R_n}}]$. Moreover, by Proposition 2.3, we have $u_\Lambda^{R_n} \leqslant u^{R_n}_{\Lambda_{R_n}}$. Therefore, $u_n \notin [0, u_\Lambda^{R_n}]$. It follows that there exists $x_n \in B_{R_n}$ such that

$$u_n(x_n) > u_\Lambda^{R_n}(x_n). \tag{4.18}$$

We claim that $\{|x_n|\}$ is bounded. Arguing indirectly, we assume that this is not true. Then by passing to a subsequence, we may assume that $|x_n| \to \infty$. By passing to a further subsequence we have exactly three possibilities:

(a) $R_n - |x_n| \to \infty$,

(b) $R_n - |x_n| \to \delta > 0$,

(c) $R_n - |x_n| \to 0$.

We will deduce a contradiction for each case.

In case (a), we can find a sequence of balls $B_{r_n}(x_n) \subset B_{R_n}(0)$ with r_n increasing to ∞ and $|x_n| - r_n \to \infty$. In view of (4.15), we can find two sequences $\{\sigma_n\}$ and $\{\sigma_n^*\}$ such that

$$0 < \sigma_n \leqslant b(x) \leqslant \sigma_n^*, \quad \forall x \in B_{r_n}(x_n),$$
$$\sigma_n \to b_\infty, \quad \sigma_n^* \to b_\infty.$$

Let w_n denote the unique positive solution of (4.5) with $B_n = B_{r_n}(0)$ and $\xi_n = \sigma_n^*$, and let v_n denote the unique positive solution of (4.6) with $\xi_n = \sigma_n$ and $B_n = B_{r_n}(0)$. Then by Lemma 4.3, we find that (4.7) holds for both w_n and v_n if we replace ξ by b_∞ in (4.7).

We now use Lemma 2.1 in [13] to compare $u_n(x)$ with $w_n(x - x_n)$ and with $v_n(x - x_n)$ over $B_{r_n}(x_n)$. We easily find that $w_n(x - x_n) \leqslant u_n(x) \leqslant v_n(x - x_n)$ on this ball. In particular, $w_n(0) \leqslant u_n(x_n) \leqslant v_n(0)$. It now follows from (4.7) that

$$u_n(x_n) \to (\lambda/b_\infty)^{1/(p-1)}. \tag{4.19}$$

Applying a similar argument to $u_\Lambda^{R_n}$ we deduce that

$$u_\Lambda^{R_n}(x_n) \to (\Lambda/b_\infty)^{1/(p-1)}.$$

Since $\lambda < \Lambda$, we deduce from this and (4.19) that, for all large n, $u_n(x_n) < u_\Lambda^{R_n}(x_n)$. But this contradicts (4.18). So case (a) leads to a contradiction.

In case (b), we let A_n denote the annulus $\{x \in R^N\colon\ R_n/2 < |x| < R_n\}$. By (4.15), we can find two sequences $\{\sigma_n\}$ and $\{\sigma_n^*\}$ such that

$$0 < \sigma_n \leqslant b(x) \leqslant \sigma_n^*, \quad \forall x \in A_n,$$
$$\sigma_n \to b_\infty, \quad \sigma_n^* \to b_\infty.$$

Let w_n denote the unique positive solution of (4.10) with the above A_n and $\xi_n = \sigma_n^*$, and let v_n denote the unique positive solution of (4.11) with $\xi_n = \sigma_n$ and A_n defined here.

Applying Lemma 2.1 in [13] we easily see that $w_n \leqslant u_n \leqslant v_n$ on A_n. Using Lemma 4.4, we obtain that

$$u_n(x_n) \to \Phi(\delta). \tag{4.20}$$

Similarly,

$$u_\Lambda^{R_n}(x_n) \to \Phi_*(\delta), \tag{4.21}$$

where Φ_* is the unique positive solution of (4.13) but with λ replaced by Λ. Since $\lambda < \Lambda$, we have $\Phi_*(-\infty) > \Phi(-\infty)$. Hence we can use the one dimensional version of Lemma 2.1 in [13] on $[-T, 0]$ with large $T > 0$ to deduce that $\Phi_* > \Phi$ in $(-\infty, 0)$. Therefore, by (4.20) and (4.21), for all large n, $u_n(x_n) < u_\Lambda^{R_n}(x_n)$. A contradiction to (4.18). So case (b) also leads to a contradiction.

Consider now case (c). Let w_n and v_n be defined as in the discussion of case (b) above, and $W_n(r) = w_n(R_n + r)$, $V_n(r) = v_n(R_n + r)$. Then we have

$$W_n\big(|x_n| - R_n\big) \leqslant u_n(x_n) \leqslant V_n\big(|x_n| - R_n\big).$$

Since $W_n \to \Phi$ and $V_n \to \Phi$ in $C^1([-T,0])$ for any $T > 0$, we have the estimates

$$W_n\big(|x_n| - R_n\big) = \Phi'(0)\big(|x_n| - R_n\big) + o\big(R_n - |x_n|\big),$$

and

$$V_n\big(|x_n| - R_n\big) = \Phi'(0)\big(|x_n| - R_n\big) + o\big(R_n - |x_n|\big).$$

Therefore,

$$u_n(x_n) = \Phi'(0)\big(|x_n| - R_n\big) + o\big(R_n - |x_n|\big). \tag{4.22}$$

In parallel, we have

$$u_\Lambda^{R_n}(x_n) = \Phi_*'(0)\big(|x_n| - R_n\big) + o\big(R_n - |x_n|\big). \tag{4.23}$$

Since $\Phi_* > \Phi$, we must have $\Phi_*'(0) < \Phi'(0)$. Therefore, by (4.22) and (4.23), for all large n, $u_n(x_n) < u_\Lambda^{R_n}(x_n)$. Again a contradiction to (4.18). This proves our claim that $\{|x_n|\}$ is bounded. Let us assume that $x_n \in B$ for all n and some finite closed ball B.

We are now ready to prove (4.17). Suppose for contradiction that it is not true. Then $u = u_\lambda$ and so $u_n(x) \to u_\lambda(x)$ uniformly on any bounded set of R^N. Since $u_\lambda^{R_n} \leqslant u_\Lambda^{R_n}$ (by Proposition 2.3), we deduce $u_\lambda \leqslant u_\Lambda$. By the strong maximum principle, we easily deduce $u_\lambda < u_\Lambda$ on R^N. Therefore, we can find $\varepsilon > 0$ such that $u_\lambda(x) \leqslant u_\Lambda(x) - \varepsilon$ on the closed finite ball B. It follows that for all large n,

$$u_n(x) \leqslant u_\Lambda^{R_n}(x) - \varepsilon/2, \quad \forall x \in B.$$

Taking $x = x_n \in B$ in this inequality we reach a contradiction to (4.18). Hence we must have $u \neq u_\lambda$. This finishes the proof that (2.1) has at least two positive solutions for every $\lambda \in (0, \Lambda)$.

It remains to consider the case $\lambda \leqslant 0$. Fix $\lambda \leqslant 0$ and let $(\lambda, u_n) \in \Gamma_{R_n}^c$, where R_n increases to infinity. As before, by passing to a subsequence we may assume that $u_n \to u$ uniformly on any bounded set of R^N, and u is a nonnegative solution of (2.1). We need to show that $u \neq 0$. This follows from a simple modification of our arguments for the case $\lambda \in (0, \Lambda)$. So we will be rather brief. As in the previous case, we can find $x_n \in B_{R_n}$ such that (4.18) holds. Again it suffices to show that $\{|x_n|\}$ is bounded. If this is not true, then we have three possibilities (a), (b) and (c) as in the previous situation. In case (a), we have

$$u_\Lambda^{R_n}(x_n) \to (\Lambda/b_\infty)^{1/(p-1)},$$

by the same proof as before. By comparing u_n with the unique positive solution of (4.6) with suitable ξ_n and B_n but with λ replaced by an arbitrary $\mu > 0$, we deduce that

$$\overline{\lim}\, u_n(x_n) \leqslant (\mu/b_\infty)^{1/(p-1)}.$$

This implies $u_n(x_n) \to 0$ since $\mu > 0$ is arbitrary. Hence we obtain a contradiction to (4.18). In cases (b) and (c), we can use Remark 4.5 to deduce a contradiction to (4.18). Hence (2.1) has at least one positive solution for each $\lambda \leqslant 0$.

We would like to point out that the case $\lambda < 0$ actually has a much simpler proof. Let u_n be as above. Choose $x_n \in B_{R_n}$ such that $u_n(x_n) = \max u_n$. Then from the equation for u_n and $\lambda < 0$ we find, as in (4.2), that

$$b(x_n) < 0, \quad u_n(x_n) \geqslant \big(|\lambda|/\big|\min_{R^N} b(x)\big|\big)^{1/(p-1)}.$$

It follows that $\{x_n\} \subset B_{R_0}$ and $\max_{B_{R_0}} u(x) \geqslant (|\lambda|/|\min_{R^N} b(x)|)^{1/(p-1)}$. Hence $u \neq 0$. □

Remark 4.7. Under the conditions of Theorem 4.6, it can be shown, by using results in [20] and arguments in the proof of Theorem 4.6, that (2.1) has an unbounded branch of positive solutions, Γ, in the space $R \times L^\infty(R^N)$, that bifurcates from $(\lambda, u) = (0, 0)$ and have similar properties to those for Γ_R given in Proposition 4.1. Indeed, Γ can be obtained as the limit of Γ_{R_n} for some sequence $R_n \to \infty$ (in the sense of [20]) in the space $R \times L^\infty(R^N)$, by making use of the fact that positive solutions (λ, u) of (2.1) satisfies, due to (4.15),

$$u(x) \to (\lambda^+/b_\infty)^{1/(p-1)} \quad \text{as } |x| \to \infty$$

uniformly in u for λ in bounded sets, where $\lambda^+ = \max\{\lambda, 0\}$.

Remark 4.8. The alternative method at the end of the proof of Theorem 4.6 for the case $\lambda < 0$ shows that condition (4.15) is not needed for this case. Therefore, under the conditions of Theorem 3.3 alone, (2.1) has at least one positive solution for each $\lambda < 0$.

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