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Bang-bang property for time optimal control of semilinear heat equation

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Abstract

This paper studies the bang-bang property for time optimal controls governed by semilinear heat equation in a bounded domain with control acting locally in a subset. Also, we present the null controllability cost for semilinear heat equation and an observability estimate from a positive measurable set in time for the linear heat equation with potential.

Keywords: Semilinear heat equation; Time optimal control; Bang-bang property; Observability estimate from measurable sets

1. Introduction and main result

This paper continues the investigations carried out in [14]. Our main result deals with the bang-bang property for time optimal controls governed by semilinear heat equations with control acting locally. We complete the result in [14] in two directions: the nonlinearity of the equation; the geometry on which the equation takes place.

Let Ω be a bounded connected open set of $\mathbb{R}^n$, $n \geqslant 1$, with boundary $\partial\Omega$ of class C^2. Let ω be an open and non-empty subset of Ω and denote $1_{|\cdot}$ for the characteristic function of a set in the place where $\cdot$ stays. Let $y_0 \in L^2(\Omega)$ and $v \in L^\infty(0,+\infty; L^2(\Omega))$. Consider the following semilinear heat equation with initial data y_0 and external force v:

$$\begin{cases} \partial_t y - \Delta y + f(y) = 1_{|\omega} v & \text{in } \Omega \times (0,+\infty), \\ y = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ y(\cdot,0) = y_0 & \text{in } \Omega. \end{cases}$$

Existence and uniqueness of the solution y is ensured with the following assumptions: $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz and satisfies the "good-sign" condition $f(s)s \geqslant 0$ for all $s \in \mathbb{R}$ (and consequently, $f(0) = 0$). In such a case, for any $T > 0$, the solution y is in $C([0,T]; L^2(\Omega))$ and the above equation holds in the sense of distributions in $\Omega \times (0,T)$.

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Our motivation is a null control problem for semilinear heat equations which means that our goal consists in finding $v \in L^\infty(0,+\infty; L^2(\Omega))$ such that $y(\cdot,T)=0$ in Ω.

The first natural null control problem solved in the literature is the following.

Question 1. What are the assumptions on f in order that the property

$$\begin{cases} \forall y_0 \in L^2(\Omega),\ \forall T>0,\ \exists M>0,\ \exists v \in L^\infty\big(0,+\infty; L^2(\Omega)\big), \\ \quad \text{such that } y(\cdot,T)=0 \text{ in } \Omega \text{ and } \|v\|_{L^\infty(0,+\infty;L^2(\Omega))} \leqslant M \end{cases}$$

holds? Notice that the existence of a null control v gives the one of the bound M. This property is intensively studied in the literature (see e.g. [2,8,7]) and is called null controllability for semilinear heat equation. It holds for any nonlinear terms which are locally Lipschitz and slightly superlinear. Precisely, it is enough for f to satisfy $f(0)=0$ and

$$\lim_{|s|\to\infty} \frac{|f(s)|}{|s|\ln^{3/2}(1+|s|)} = 0.$$

In particular, if we assume that f is globally Lipschitz with $f(0)=0$, then null controllability for the corresponding semilinear heat equation holds.

However, we can formulate another type of null control problem as follows.

Question 2. What are the assumptions on f in order that the property

$$\begin{cases} \forall y_0 \in L^2(\Omega),\ \forall M>0,\ \exists T>0,\ \exists v \in L^\infty\big(0,+\infty; L^2(\Omega)\big), \\ \quad \text{such that } y(\cdot,T)=0 \text{ in } \Omega \text{ and } \|v\|_{L^\infty(0,+\infty;L^2(\Omega))} \leqslant M \end{cases}$$

holds? In this article, we will prove the existence of T and v under the assumption that f is globally Lipschitz and satisfies the “good-sign” condition. Once existence of a couple (y,v) is established for $y_0 \in L^2(\Omega)\backslash\{0\}$ and $M>0$ given, via suitable assumption on f, we introduce the following admissible set of controls

$$\begin{aligned} \mathcal{V}_M = \Big\{ & v \in L^\infty\big(0,+\infty; L^2(\Omega)\big);\ \|v\|_{L^\infty(0,+\infty;L^2(\Omega))} \leqslant M \text{ and the solution } y \\ & \text{corresponding to } v \text{ satisfies } y(\cdot,T)=0 \text{ in } \Omega \text{ for some } T>0\Big\}. \end{aligned}$$

Among all the control functions $v \in \mathcal{V}_M$, we select the infimum of all such time:

$$T^* = \inf\{T;\ v \in \mathcal{V}_M\},$$

i.e., the minimal time needed to drive the system to rest with control functions in $\mathcal{V}_M$.

A control v^* such that the corresponding solution y satisfies $y(\cdot,T^*)=0$ in Ω is called time optimal control. In this article, we shall prove the existence of a time optimal control v^* under the assumption that f is globally Lipschitz and satisfies the “good-sign” condition.

Now, we are able to state our main result.

Theorem 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a globally Lipschitz function satisfying $f(s)s \geqslant 0$ for all $s \in \mathbb{R}$. Then for any $y_0 \in L^2(\Omega)\backslash\{0\}$ and any $M>0$, any time optimal control v^* satisfies the bang-bang property: $\|v^*(\cdot,t)\|_{L^2(\Omega)} = M$ for a.e. $t \in (0,T^*)$.*

Clearly, bang-bang property is of high importance in optimal control theory as mentioned in [6] and [11]. In particular, the bang-bang property for certain time optimal controls governed by parabolic equations can be provided by making use of Pontryagin’s maximum principle (see [9,10,17]). Another approach to get bang-bang property for linear heat equation consists in following a strategy based on null controllability with control functions acting on measurable set in time variable as in [12] and [16]. Recently, the authors in [1] established an observability inequality for the linear heat equation, where the observation is a subset of positive measure in space and time, and from which they obtained another kind of bang-bang property of time optimal problem for the linear heat equation with bounded controls in space and time. Naturally, the extension of this strategy for nonlinear parabolic equations requires a fixed point argument and an observability inequality for heat equations with space- and time-dependent potentials.

This paper is organized as follows. Section 2 is devoted to the null controllability for semilinear heat equation with control functions acting on $\omega \times E$ where $|E| > 0$. We present (see Theorem 2) and prove an estimate of the cost of the control functions when f is globally Lipschitz. Before giving the proof of Theorem 2, we recall the linear case and the observability estimate needed (see Theorem 4). In Section 3, applying Theorem 2 in a very special case, we prove the existence for admissible control (see Theorem 5) when f is globally Lipschitz and satisfies the "good-sign" condition. Next we deduce the existence of time optimal (see Theorem 6). The proof of our main result, Theorem 1, concerning the bang-bang property for time optimal controls governed by semilinear heat equation with local control is given in Section 4. Finally, in Section 5, we prove the observability estimate of Theorem 4.

2. Null controllability for semilinear heat equation

The goal of this section is to present the null controllability for semilinear heat equation with control functions acting on $\omega \times E$ where $|E| > 0$. A particular attention is given to the cost estimate.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz function. Let $0 \leqslant T_0 < T_1 < T_2$ and $E \subset (T_1, T_2)$ with $|E| > 0$. Then for any $\phi \in C([T_0, T_2], L^2(\Omega))$ and any $w_0 \in L^2(\Omega)$, there are a constant $\kappa > 0$ and a function $v_1 \in L^\infty(0, +\infty; L^2(\Omega))$ such that*

$$\|v_1\|_{L^\infty(0,+\infty;L^2(\Omega))} \leqslant \kappa \|w_0\|_{L^2(\Omega)}$$

and the solution $w = w(x,t)$ of

$$\begin{cases} \partial_t w - \Delta w + f(\phi + w) - f(\phi) = 1_{|\omega \times E} v_1 & \text{in } \Omega \times (T_0, T_2), \\ w = 0 & \text{on } \partial\Omega \times (T_0, T_2), \\ w(\cdot, T_0) = w_0 & \text{in } \Omega, \end{cases}$$

satisfies $w(\cdot, T_2) = 0$ in $L^2(\Omega)$. Further,

$$\kappa = e^{\widetilde{K}} e^{c(T_1 - T_0)} e^{K(1 + c + c(T_2 - T_1))}.$$

Here, $c = c(f)$, $K = K(\Omega, \omega) > 1$ and $\widetilde{K} = \widetilde{K}(\Omega, \omega, E)$ are positive constants which do not depend on T_0.

Remark 1. When $E = (T_1, T_2)$, then $\kappa = e^{c(T_1 - T_0)} e^{K(1 + \frac{1}{T_2 - T_1} + c + c(T_2 - T_1))}$.

2.1. Linear case

In this section, we treat the case $f(\phi + w) = aw + f(\phi)$ that is the linear heat equation with potential.

Theorem 3. *Let $0 \leqslant T_0 < T_1 < T_2$ and $E \subset (T_1, T_2)$ with $|E| > 0$. Let $a \in L^\infty(\Omega \times (T_0, T_2))$. Then for any $z_0 \in L^2(\Omega)$, there is a function $v_0 \in L^\infty(\Omega \times (0, +\infty))$ such that the solution $z = z(x,t)$ of*

$$\begin{cases} \partial_t z - \Delta z + az = 1_{|\omega \times E} v_0 & \text{in } \Omega \times (T_0, T_2), \\ z = 0 & \text{on } \partial\Omega \times (T_0, T_2), \\ z(\cdot, T_0) = z_0 & \text{in } \Omega, \end{cases}$$

satisfies $z(\cdot, T_2) = 0$ in $L^2(\Omega)$. Further,

$$\|v_0\|_{L^\infty(\Omega \times (0,+\infty))} \leqslant e^{\widetilde{K}} e^{(T_1 - T_0)\|a\|_{L^\infty(\Omega \times (T_0, T_1))}} \times e^{K(1 + (T_2 - T_1)\|a\|_{L^\infty(\Omega \times (T_1, T_2))} + \|a\|^{2/3}_{L^\infty(\Omega \times (T_1, T_2))})} \|z_0\|_{L^2(\Omega)}.$$

Here, $K = K(\Omega, \omega) > 1$ and $\widetilde{K} = \widetilde{K}(\Omega, \omega, E)$ are positive constants which do not depend on T_0.

Remark 2. When $E = (T_1, T_2)$, then $\widetilde{K} = K \frac{1}{T_2 - T_1}$.

Proof of Theorem 3. We divide its proof into three steps. In the first step, we start to solve

$$\begin{cases} \partial_t z - \Delta z + az = 0 & \text{in } \Omega \times (T_0, T_1), \\ z = 0 & \text{on } \partial\Omega \times (T_0, T_1), \\ z(\cdot, T_0) = z_0 & \text{in } \Omega. \end{cases}$$

Therefore, $z(\cdot, T_1) \in L^2(\Omega)$ and it is well-known that

$$\left\| z(\cdot, T_1) \right\|_{L^2(\Omega)} \leqslant e^{(T_1 - T_0)\|a\|_{L^\infty(\Omega\times(T_0,T_1))}} \|z_0\|_{L^2(\Omega)}.$$

The second step consists in establishing the existence of a function $v \in L^\infty(\Omega \times (0, +\infty))$ such that the solution $\tilde{z} = \tilde{z}(x, t)$ of

$$\begin{cases} \partial_t \tilde{z} - \Delta \tilde{z} + a\tilde{z} = 1_{|\omega\times E} v & \text{in } \Omega \times (T_1, T_2), \\ \tilde{z} = 0 & \text{on } \partial\Omega \times (T_1, T_2), \\ \tilde{z}(\cdot, T_1) = z(\cdot, T_1) & \text{in } \Omega, \end{cases}$$

satisfies $\tilde{z}(\cdot, T_2) = 0$ in $L^2(\Omega)$. Further,

$$\|v\|_{L^\infty(\Omega\times(T_1,T_2))} \leqslant e^{\widetilde{K}} e^{K(1+(T_2-T_1)\|a\|_{L^\infty(\Omega\times(T_1,T_2))} + \|a\|^{2/3}_{L^\infty(\Omega\times(T_1,T_2))})} \left\| z(\cdot, T_1) \right\|_{L^2(\Omega)}.$$

Here, $K = K(\Omega, \omega) > 1$ and $\widetilde{K} = \widetilde{K}(\Omega, \omega, E)$ are positive constants which do not depend on T_0. Finally, in the last step, we choose

$$v_0(\cdot, t) = \begin{cases} 0 & \text{if } t \in (0, T_1) \cup [T_2, +\infty), \\ v(\cdot, t) & \text{if } t \in [T_1, T_2). \end{cases}$$

Since

$$\|v_0\|_{L^\infty(\Omega\times(0,+\infty))} = \|v\|_{L^\infty(\Omega\times(T_1,T_2))},$$

the desired result holds. It is standard to get the existence of the above function v from an observability estimate. More precisely, we apply the following result. Its proof is provided in Section 5. □

Theorem 4 *(Observability estimate). Let ω be an open and non-empty subset of Ω. Let $T > 0$ and E be a subset of positive measure in $(0, T)$. Then there are two constants $K = K(\Omega, \omega)$ and $\widetilde{K} = \widetilde{K}(\Omega, \omega, E) > 0$ such that for any $a = a(x, t) \in L^\infty(\Omega \times (0, T))$ and any $\varphi_0 \in L^2(\Omega)$, the solution $\varphi = \varphi(x, t)$ of*

$$\begin{cases} \partial_t \varphi - \Delta\varphi + a\varphi = 0 & \text{in } \Omega \times (0, T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T), \\ \varphi(\cdot, 0) = \varphi_0 & \text{in } \Omega, \end{cases}$$

satisfies

$$\left\| \varphi(\cdot, T) \right\|_{L^2(\Omega)} \leqslant e^{\widetilde{K}} e^{K(1+T\|a\|_{L^\infty(\Omega\times(0,T))} + \|a\|^{2/3}_{L^\infty(\Omega\times(0,T))})} \int_{\omega\times E} \left| \varphi(x, t) \right| dx\, dt.$$

Remark 3. This is a refined observability estimate. When $E = (0, T)$, then the observability constant becomes

$$e^{K(1+\frac{1}{T}+T\|a\|_{L^\infty(\Omega\times(0,T))} + \|a\|^{2/3}_{L^\infty(\Omega\times(0,T))})}.$$

This is in accordance with the work of [4]. When E is a positive measurable set with 0 its Lebesgue point, then the observability constant becomes, for some $\ell_1 \in E \cap (0, T)$,

$$e^{K(1+\frac{1}{\ell_1}+\ell_1\|a\|_{L^\infty(\Omega\times(0,T))} + \|a\|^{2/3}_{L^\infty(\Omega\times(0,T))})}.$$

2.2. Nonlinear case with Kakutani's fixed point

In this section, we prove Theorem 2. Let $0 \leqslant T_0 < T_1 < T_2$ and $E \subset (T_1, T_2)$ with $|E| > 0$. Let $\phi \in C([T_0, T_2], L^2(\Omega))$ and $w_0 \in L^2(\Omega)$.

By a classical density argument, we may assume that $f \in C^1$. We shall use Kakutani's fixed point theorem to prove the result. First, define for any $(x,t) \in \Omega \times (T_0, T_2)$,

$$a(x,t,r) = \begin{cases} \frac{f(\phi(x,t)+r)-f(\phi(x,t))}{r} & \text{if } r \neq 0, \\ f'(\phi(x,t)) & \text{if } r = 0. \end{cases}$$

And consider

$$\mathcal{K} = \left\{\xi \in L^2\big(\Omega \times (T_0, T_2)\big);\ \|\xi\|_{L^2(T_0,T_2;H_0^1(\Omega))\cap H^1(T_0,T_2;H^{-1}(\Omega))} \leqslant \hat{\kappa}\right\}$$

where $\hat{\kappa} > 0$ will be determined later. Since $f : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function, we have that for a.e. $(x,t) \in \Omega \times (T_0, T_2)$ and any $r \in \mathbb{R}$

$$\big|a(x,t,r)\big| \leqslant L(f)$$

where $L(f) > 0$ is the Lipschitz constant of the function f.

Next, using the fact $L^\infty(\Omega \times (0,+\infty)) \subset L^\infty(0,+\infty; L^2(\Omega))$, we know by Theorem 3 that for any $\xi \in L^2(\Omega \times (T_0, T_2))$, there are a function $v_0 \in L^\infty(0,+\infty; L^2(\Omega))$ and a corresponding solution $z = z(x,t)$ of

$$\begin{cases} \partial_t z - \Delta z + a\big(\cdot,\cdot,\xi(\cdot,\cdot)\big)z = 1_{|\omega\times E} v_0 & \text{in } \Omega \times (T_0, T_2), \\ z = 0 & \text{on } \partial\Omega \times (T_0, T_2), \\ z(\cdot, T_0) = w_0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

such that

$$z(\cdot, T_2) = 0 \quad \text{in } L^2(\Omega) \tag{2.2}$$

and

$$\|v_0\|_{L^\infty(0,+\infty;L^2(\Omega))} \leqslant \widehat{K}\|w_0\|_{L^2(\Omega)}. \tag{2.3}$$

Here and throughout the proof of Theorem 2,

$$\widehat{K} = e^{\widetilde{K}} e^{(T_1-T_0)L(f)} e^{K(1+(T_2-T_1)L(f)+L(f)^{2/3})}$$

where $K = K(\Omega, \omega) > 1$ and $\widetilde{K} = \widetilde{K}(\Omega, \omega, E)$ are positive constants which do not depend on T_0. Therefore, we can define the map

$$\begin{aligned} \Lambda : \mathcal{K} &\to L^2\big(\Omega \times (T_0, T_2)\big), \\ \xi &\mapsto z \end{aligned}$$

where (2.1), (2.2), (2.3) hold.

Now, we check that Kakutani's fixed point theorem is applicable. For convenience, let us state this result (see e.g. [3]).

Theorem *(Kakutani's fixed point). Let $\mathcal{Z}$ be a Banach space and Π be a non-empty convex compact subset of $\mathcal{Z}$. Let $\Lambda : \Pi \to \mathcal{Z}$ be a set-valued mapping satisfying the following assumptions*:

i) *$\Lambda(\xi)$ is a non-empty convex set of $\mathcal{Z}$ for every $\xi \in \Pi$.*
ii) *$\Lambda(\Pi) \subset \Pi$.*
iii) *$\Lambda : \Pi \to \mathcal{Z}$ is upper semicontinuous in $\mathcal{Z}$.*

Then Λ possesses a fixed point in the set Π.

Here $\mathcal{Z} = L^2(\Omega \times (T_0, T_2))$ and $\Pi = \mathcal{K}$ with an adequate choice of $\hat{\kappa}$ given below. Clearly, $\mathcal{K}$ is a non-empty convex compact set in $L^2(\Omega \times (T_0, T_2))$. Further, from the above arguments, $\Lambda(\xi)$ is a non-empty convex set in $L^2(\Omega \times (T_0, T_2))$. Thus i) holds.

Let us prove that ii) holds with an adequate choice of $\hat{\kappa}$. By a standard energy method, using the fact that $|a| \leqslant L(f)$ and (2.1), (2.2), (2.3), there exists $C > 0$ such that

$$\|z\|^2_{C([T_0,T_2];L^2(\Omega))} + \int_{T_0}^{T_2} \|z(\cdot,t)\|^2_{H_0^1(\Omega)}\,dt \leqslant C\|w_0\|^2_{L^2(\Omega)}.$$

Combining the latter with the fact that $|a| \leqslant L(f)$, we deduce that the solution z satisfies

$$\|z\|_{L^2(T_0,T_2;H_0^1(\Omega))\cap H^1(T_0,T_2;H^{-1}(\Omega))} \leqslant C\|w_0\|_{L^2(\Omega)},$$

for some $C = C(\Omega, \omega, E, T_2, L(f))$ which is a positive constant which does not depend on T_0. Hence, if we take $\hat{\kappa}$ as follows

$$\hat{\kappa} = C\|w_0\|_{L^2(\Omega)}$$

then $\Lambda(\mathcal{K}) \subset \mathcal{K}$.

Let us finally prove the upper semicontinuity of $\Lambda : \mathcal{K} \to L^2(\Omega \times (T_0, T_2))$. We need to prove that if $\xi_m \in \mathcal{K} \to \xi$ strongly in $L^2(\Omega \times (T_0, T_2))$ and if $p_m \in \Lambda(\xi_m) \to p$ strongly in $L^2(\Omega \times (T_0, T_2))$, then $p \in \Lambda(\xi)$. To this end, firstly, we claim that there exists a subsequence of $(m)_{m\geqslant 1}$, still denoted in the same manner, such that

$$a\big(\cdot,\cdot,\xi_m(\cdot,\cdot)\big)p_m \to a\big(\cdot,\cdot,\xi(\cdot,\cdot)\big)p \quad \text{strongly in } L^2\big(\Omega \times (T_0, T_2)\big). \tag{2.4}$$

Indeed, since $\xi_m \to \xi$ strongly in $L^2(\Omega \times (T_0, T_2))$, we have that there exists a subsequence of $(m)_{m\geqslant 1}$, still denoted by itself, such that

$$\xi_m(x,t) \to \xi(x,t) \quad \text{for a.e. } (x,t) \in \Omega \times (T_0, T_2).$$

On one hand, for (x,t) with $\xi(x,t) \neq 0$, by the above, there exists a positive integer m_0 depending on (x,t) such that

$$\xi_m(x,t) \neq 0 \quad \forall m \geqslant m_0,$$

which implies by the definition of a,

$$a\big(x,t,\xi_m(x,t)\big) \to a\big(x,t,\xi(x,t)\big) \quad \text{as } m \to +\infty. \tag{2.5}$$

On the other hand, for any (x,t) such that $\xi(x,t) = 0$, by the definition of a, we have that $a(x,t,\xi(x,t)) = f'(\phi(x,t))$. Since

$$a\big(x,t,\xi_m(x,t)\big) = \begin{cases} \frac{f(\phi(x,t)+\xi_m(x,t)) - f(\phi(x,t))}{\xi_m(x,t)} & \text{if } \xi_m(x,t) \neq 0, \\ f'(\phi(x,t)) & \text{if } \xi_m(x,t) = 0, \end{cases}$$

it gives

$$a\big(x,t,\xi_m(x,t)\big) \to a\big(x,t,\xi(x,t)\big) \quad \text{as } m \to +\infty.$$

This, combined with (2.5), implies

$$a\big(x,t,\xi_m(x,t)\big) \to a\big(x,t,\xi(x,t)\big) \quad \text{for a.e. } (x,t) \in \Omega \times (T_0, T_2).$$

From the latter, the fact that $|a| \leqslant L(f)$ and the Lebesgue dominated convergence theorem it follows that

$$\begin{aligned}
&\big\|a\big(\cdot,\cdot,\xi_m(\cdot,\cdot)\big)p_m - a\big(\cdot,\cdot,\xi(\cdot,\cdot)\big)p\big\|^2_{L^2(\Omega\times(T_0,T_2))} \\
&\quad \leqslant 2\big\|a\big(\cdot,\cdot,\xi_m(\cdot,\cdot)\big)(p_m - p)\big\|^2_{L^2(\Omega\times(T_0,T_2))} + 2\big\|\big(a\big(\cdot,\cdot,\xi_m(\cdot,\cdot)\big) - a\big(\cdot,\cdot,\xi(\cdot,\cdot)\big)\big)p\big\|^2_{L^2(\Omega\times(T_0,T_2))} \\
&\quad \leqslant 2L(f)^2\|p_m - p\|^2_{L^2(\Omega\times(T_0,T_2))} + 2\big\|\big(a\big(\cdot,\cdot,\xi_m(\cdot,\cdot)\big) - a\big(\cdot,\cdot,\xi(\cdot,\cdot)\big)\big)p\big\|^2_{L^2(\Omega\times(T_0,T_2))} \\
&\quad \to 0.
\end{aligned}$$

This completes the proof of (2.4). Secondly, since $p_m \in \Lambda(\xi_m)$, there exists $(v_m)_{m\geqslant 1}$ satisfying

$$\begin{cases} (p_m)_t - \Delta p_m + a(\cdot,\cdot,\xi_m(\cdot,\cdot))p_m = 1_{|\omega\times E}v_m & \text{in } \Omega\times(T_0,T_2), \\ p_m = 0 & \text{on } \partial\Omega\times(T_0,T_2), \\ p_m(\cdot,T_0) = w_0 & \text{in } \Omega, \\ p_m(\cdot,T_2) = 0 & \text{in } \Omega, \end{cases} \tag{2.6}$$
$$\|p_m\|_{L^2(T_0,T_2;H_0^1(\Omega))\cap H^1(T_0,T_2;H^{-1}(\Omega))} + \|p_m\|_{L^2(T_1,T_2;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(T_1,T_2;L^2(\Omega))} \leqslant C,$$

where $C > 0$ is a constant independent of m, and

$$\|v_m\|_{L^\infty(0,+\infty;L^2(\Omega))} \leqslant \widehat{K}\|w_0\|_{L^2(\Omega)}. \tag{2.7}$$

Thus, we deduce the existence of v and subsequences $(v_{m'})_{m'\geqslant 1}$ and $(p_{m'})_{m'\geqslant 1}$ such that

$$v_{m'} \to v \quad \text{weakly star in } L^\infty(0,+\infty;L^2(\Omega)), \tag{2.8}$$
$$p_{m'} \to p \quad \text{weakly in } L^2(T_0,T_2;H_0^1(\Omega)) \cap H^1(T_0,T_2;H^{-1}(\Omega)), \tag{2.9}$$
$$p_{m'}(\cdot,T_2) \to p(\cdot,T_2) \quad \text{strongly in } L^2(\Omega). \tag{2.10}$$

Finally, passing to the limit for $m' \to +\infty$ in (2.6) and (2.7), by (2.4) and (2.8), (2.9), (2.10), we obtain that $p \in \Lambda(\xi)$.

By Kakutani's fixed point theorem, we conclude that there exists $w \in \mathcal{K}$ with an adequate choice of $\hat{\kappa}$ such that $w \in \Lambda(w)$, i.e., there is a control $v_1 \in L^\infty(0,+\infty;L^2(\Omega))$ satisfying

$$\|v_1\|_{L^\infty(0,+\infty;L^2(\Omega))} \leqslant \widehat{K}\|w_0\|_{L^2(\Omega)},$$

with the same $\widehat{K}$ given in (2.3), and the corresponding solution $w = w(x,t)$ solves

$$\begin{cases} \partial_t w - \Delta w + a(\cdot,\cdot,w(\cdot,\cdot))w = 1_{|\omega\times E}v_1 & \text{in } \Omega\times(T_0,T_2), \\ w = 0 & \text{on } \partial\Omega\times(T_0,T_2), \\ w(\cdot,T_0) = w_0 & \text{in } \Omega, \\ w(\cdot,T_2) = 0 & \text{in } \Omega. \end{cases}$$

Since for any $(x,t) \in \Omega\times(T_0,T_2)$,

$$a(x,t,w(x,t))w(x,t) = f(\phi(x,t)+w(x,t)) - f(\phi(x,t)),$$

we finally get

$$\begin{cases} \partial_t w - \Delta w + f(\phi+w) - f(\phi) = 1_{|\omega\times E}v_1 & \text{in } \Omega\times(T_0,T_2), \\ w = 0 & \text{on } \partial\Omega\times(T_0,T_2), \\ w(\cdot,T_0) = w_0 & \text{in } \Omega, \\ w(\cdot,T_2) = 0 & \text{in } \Omega. \end{cases}$$

This completes the proof of Theorem 2.

3. Existence of time optimal control

In this section, we start to prove the existence of admissible controls (see e.g. [15]). In other words we prove that

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz function satisfying $f(s)s \geqslant 0$ for all $s \in \mathbb{R}$. Then for any $y_0 \in L^2(\Omega)\backslash\{0\}$ and any $M > 0$, there are a time $T > 0$ and an admissible control $v \in L^\infty(0,+\infty;L^2(\Omega))$ such that $\|v\|_{L^\infty(0,+\infty;L^2(\Omega))} \leqslant M$ and the solution y corresponding to v satisfies $y(\cdot,T) = 0$ in Ω.*

Proof. We divide its proof into three steps.

Step 1. We consider the following equation

$$\begin{cases} \partial_t y - \Delta y + f(y) = 0 & \text{in } \Omega \times (0, T_0), \\ y = 0 & \text{on } \partial\Omega \times (0, T_0), \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

where $T_0 > 0$ will be determined later. By a standard energy method, using the fact that $f(s)s \geqslant 0$, we have

$$\left\| y(\cdot, T_0) \right\|_{L^2(\Omega)} \leqslant e^{-\lambda_1 T_0} \| y_0 \|_{L^2(\Omega)},$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition.

Step 2. We apply Theorem 2 with $T_1 = T_0 + 1$, $T_2 = T_0 + 2$, $E = (T_1, T_2)$, $\phi = 0$ and $w_0 = y(\cdot, T_0)$ in order that there are a constant $\kappa > 0$ and a function $\tilde{v} \in L^\infty(0, +\infty; L^2(\Omega))$ such that the solution $w = w(x, t)$ of

$$\begin{cases} \partial_t w - \Delta w + f(w) = 1_{|\omega \times E} \tilde{v} & \text{in } \Omega \times (T_0, T_0 + 2), \\ w = 0 & \text{on } \partial\Omega \times (T_0, T_0 + 2), \\ w(\cdot, T_0) = y(\cdot, T_0) & \text{in } \Omega, \end{cases}$$

satisfies $w(\cdot, T_0 + 2) = 0$ in $L^2(\Omega)$. Further,

$$\|\tilde{v}\|_{L^\infty(0,+\infty; L^2(\Omega))} \leqslant \kappa \left\| y(\cdot, T_0) \right\|_{L^2(\Omega)},$$

and κ does not depend on T_0.

Step 3. We can easily check that the function

$$v(\cdot, t) = \begin{cases} 0 & \text{if } t \in (0, T_0 + 1] \cup [T_0 + 2, +\infty), \\ \tilde{v}(\cdot, t) & \text{if } t \in (T_0 + 1, T_0 + 2), \end{cases}$$

is an admissible control with $T = T_0 + 2$ when $T_0 > 0$ is taken such that

$$T_0 = \frac{1}{\lambda_1} \ln\left(1 + \frac{\kappa \| y_0 \|_{L^2(\Omega)}}{M}\right)$$

in order that

$$\|v\|_{L^\infty(0,+\infty; L^2(\Omega))} = \|\tilde{v}\|_{L^\infty(T_0+1, T_0+2; L^2(\Omega))} \leqslant \kappa e^{-\lambda_1 T_0} \| y_0 \|_{L^2(\Omega)} \leqslant M.$$

This completes the proof of Theorem 5. □

Now, we establish the existence of time optimal controls (see e.g. [15]). In other words, we shall prove that

Theorem 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz function satisfying $f(s)s \geqslant 0$ for all $s \in \mathbb{R}$. Then for any $y_0 \in L^2(\Omega) \backslash \{0\}$ and any $M > 0$, there is a time optimal control $v^* \in L^\infty(0, +\infty; L^2(\Omega))$ such that $\|v^*\|_{L^\infty(0,+\infty; L^2(\Omega))} \leqslant M$ and the solution y corresponding to v^* satisfies $y(\cdot, T^*) = 0$ in Ω where $T^* = \inf\{T; v \in \mathcal{V}_M\}$.*

Proof. By Theorem 5 and the definition of T^*, $0 \leqslant T^* < \overline{T}$ for some $\overline{T} > 0$. Therefore, there exist sequences $(T_m)_{m \geqslant 1}$ of positive real number and $(v_m)_{m \geqslant 1}$ of function in $L^\infty(0, +\infty; L^2(\Omega))$ such that $T^* = \lim_{m \to \infty} T_m$, $\|v_m\|_{L^\infty(0,+\infty; L^2(\Omega))} \leqslant M$ and the solution $y_m = y_m(x, t)$ corresponding to v_m satisfies

$$\begin{cases} \partial_t y_m - \Delta y_m + f(y_m) = 1_{|\omega} v_m & \text{in } \Omega \times (0, \overline{T}), \\ y_m = 0 & \text{on } \partial\Omega \times (0, \overline{T}), \\ y_m(\cdot, 0) = y_0 & \text{in } \Omega, \\ y_m(\cdot, T_m) = 0 & \text{in } \Omega. \end{cases}$$

We have by a standard energy method, using the bound M on v_m and the “good-sign” condition on f,

$$\|y_m\|^2_{C([0,\overline{T}]; L^2(\Omega))} + \int_0^{\overline{T}} \left\| y_m(\cdot, t) \right\|^2_{H_0^1(\Omega)} dt \leqslant C.$$

Here and throughout the proof, C denotes a generic constant independent of m. Since f is globally Lipschitz and $f(0)=0$, the above inequality implies

$$\|f(y_m)\|_{L^2(\Omega\times(0,\overline{T}))} = \|f(y_m)-f(0)\|_{L^2(\Omega\times(0,\overline{T}))} \leqslant C.$$

Therefore, from the boundedness of $-f(y_m)+1_{|\omega}v_m$, the sequence $(y_m)_{m\geqslant 1}$ is bounded in $H^1(0,\overline{T};H^{-1}(\Omega))$.

Now, we deduce the existence of $v^* \in L^\infty(0,+\infty;L^2(\Omega))$ and subsequences $(v_{m'})_{m'\geqslant 1}$ and $(y_{m'})_{m'\geqslant 1}$ such that

$$v_{m'} \to v^* \quad \text{weakly star in } L^\infty\big(0,+\infty;L^2(\Omega)\big) \text{ with } \|v^*\|_{L^\infty(0,+\infty;L^2(\Omega))} \leqslant M,$$
$$y_{m'} \to y^* \quad \text{weakly in } L^2\big(0,\overline{T};H_0^1(\Omega)\big)\cap H^1\big(0,\overline{T};H^{-1}(\Omega)\big),\ \text{strongly in } C\big([0,\overline{T}];L^2(\Omega)\big).$$

Further,

$$\begin{cases} \partial_t y^* - \Delta y^* + f\big(y^*\big) = 1_{|\omega}v^* & \text{in } \Omega\times(0,\overline{T}),\\ y^*=0 & \text{on } \partial\Omega\times(0,\overline{T}),\\ y^*(\cdot,0)=y_0 & \text{in } \Omega, \end{cases}$$

and

$$\begin{aligned}\big\|y^*\big(\cdot,T^*\big)\big\|_{L^2(\Omega)} &\leqslant \big\|y^*\big(\cdot,T^*\big)-y^*(\cdot,T_{m'})\big\|_{L^2(\Omega)} + \big\|y^*(\cdot,T_{m'})-y_{m'}(\cdot,T_{m'})\big\|_{L^2(\Omega)}\\ &\to 0 \quad \text{when } m'\to\infty.\end{aligned}$$

This gives $y^*(\cdot,T^*)=0$ in Ω and consequently, v^* is a time optimal control. This completes the proof. □

4. Bang-bang property for time optimal control (proof of Theorem 1)

We want to prove that if v^* is a time optimal control corresponding to the optimal time $T^*=\inf\{T;\, v\in\mathcal{V}_M\}$, then $\|v^*(\cdot,t)\|_{L^2(\Omega)}=M$ for a.e. $t\in(0,T^*)$. To prove this, we work by contradiction. Suppose that there are $\varepsilon\in(0,M)$ and a positive measurable subset $E^*\subset(0,T^*)$ such that

$$\big\|v^*(\cdot,t)\big\|_{L^2(\Omega)} \leqslant M-\varepsilon \quad \forall t\in E^*$$

and the solution $y^*=y^*(x,t)$ corresponding to v^* satisfies

$$\begin{cases} \partial_t y^* - \Delta y^* + f\big(y^*\big) = 1_{|\omega}v^* & \text{in } \Omega\times\big(0,T^*\big),\\ y^*=0 & \text{on } \partial\Omega\times\big(0,T^*\big),\\ y^*(\cdot,0)=y_0 & \text{in } \Omega,\\ y^*(\cdot,T^*)=0 & \text{in } \Omega. \end{cases}$$

We claim that there exist a real number $\delta\in(0,T^*)$ and a couple (y,v) such that

$$\begin{cases} \partial_t y - \Delta y + f(y) = 1_{|\omega}v & \text{in } \Omega\times\big(0,T^*-\delta\big),\\ y=0 & \text{on } \partial\Omega\times\big(0,T^*-\delta\big),\\ y(\cdot,0)=y_0 & \text{in } \Omega,\\ y(\cdot,T^*-\delta)=0 & \text{in } \Omega, \end{cases}$$

and $v\in L^\infty(0,+\infty;L^2(\Omega))$ with $\|v\|_{L^\infty(0,+\infty;L^2(\Omega))}\leqslant M$. This is clearly a contradiction with the time optimal assumption $T^*=\inf\{T;\, v\in\mathcal{V}_M\}$.

Now, we prove our claim. We divide its proof into four steps.

Step 1. $T^*>0$ and $0<|E^*|\leqslant T^*$ being given, let $\delta_0=|E^*|/2$ and denote

$$E=E^*\cap\big(\delta_0,T^*\big).$$

Then

$$|E|>0.$$

Indeed, $|E^*\cap(\delta_0,T^*)|\geqslant|E^*|-\delta_0\geqslant|E^*|/2$.

Step 2. We apply Theorem 2 with $0 < T_0 < T_1 < T_2$, $E \subset (T_1, T_2)$ with $|E| > 0$ and $\phi = y^*$, in order that there are a constant $\kappa > 0$ and a function $v_1 \in L^\infty(0, +\infty; L^2(\Omega))$ such that

$$\begin{cases} \partial_t w - \Delta w + f\big(y^* + w\big) - f\big(y^*\big) = 1_{|\omega \times E} v_1 & \text{in } \Omega \times (T_0, T_2), \\ w = 0 & \text{on } \partial\Omega \times (T_0, T_2), \\ w(\cdot, T_0) = w_0 & \text{in } \Omega, \\ w(\cdot, T_2) = 0 & \text{in } \Omega, \\ \|v_1\|_{L^\infty(0,+\infty;L^2(\Omega))} \leqslant \kappa \|w_0\|_{L^2(\Omega)}, \end{cases}$$

and further κ does not depend on T_0.

Step 3. We apply step 2 with $T_0 = \delta$, $T_1 = \delta_0$, $T_2 = T^*$, $w_0 = y_0 - y^*(\cdot, \delta)$, in order that $z = y^* + w$ solves

$$\begin{cases} \partial_t z - \Delta z + f(z) = 1_{|\omega}\big(v^* + 1_{|E} v_1\big) & \text{in } \Omega \times \big(\delta, T^*\big), \\ z = 0 & \text{on } \partial\Omega \times \big(\delta, T^*\big), \\ z(\cdot, \delta) = y_0 & \text{in } \Omega, \\ z(\cdot, T^*) = 0 & \text{in } \Omega. \end{cases}$$

Denote $v_2 = v^* + 1_{|E} v_1$. On one hand, if $t \in (0, +\infty)\backslash E$, then $\|v_2(\cdot, t)\|_{L^2(\Omega)} = \|v^*(\cdot, t)\|_{L^2(\Omega)} \leqslant M$. On the other hand, if $t \in E$, then

$$\begin{aligned} \big\|v_2(\cdot, t)\big\|_{L^2(\Omega)} &\leqslant \big\|v^*(\cdot, t)\big\|_{L^2(\Omega)} + \big\|v_1(\cdot, t)\big\|_{L^2(\Omega)} \\ &\leqslant M - \varepsilon + \kappa \big\|y^*(\cdot, 0) - y^*(\cdot, \delta)\big\|_{L^2(\Omega)}. \end{aligned}$$

Now, we choose δ sufficiently closed to 0 in order that

$$\big\|y^*(\cdot, 0) - y^*(\cdot, \delta)\big\|_{L^2(\Omega)} \leqslant \varepsilon/\kappa.$$

This is possible because $y^* \in C([0, T^*], L^2(\Omega))$. Consequently, $\|v_2(\cdot, t)\|_{L^2(\Omega)} \leqslant M$ for a.e. $t \in (0, +\infty)$.

Step 4. Let $v(\cdot, t) = v_2(\cdot, t + \delta)$. Then $v \in L^\infty(0, +\infty; L^2(\Omega))$ and further we can check that $\|v(\cdot, t)\|_{L^2(\Omega)} \leqslant M$ for a.e. $t \in (0, +\infty)$. Let $y(x, t) = z(x, t + \delta)$. Then it solves

$$\begin{cases} \partial_t y - \Delta y + f(y) = 1_{|\omega} v & \text{in } \Omega \times \big(0, T^* - \delta\big), \\ y = 0 & \text{on } \partial\Omega \times \big(0, T^* - \delta\big), \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \\ y\big(\cdot, T^* - \delta\big) = 0 & \text{in } \Omega. \end{cases}$$

This is the desired claim.

5. The heat equation with potential (proof of Theorem 4)

The proof of Theorem 4 is based on five lemmas. From now, φ denotes the solution of

$$\begin{cases} \partial_t \varphi - \Delta\varphi + a\varphi = 0 & \text{in } \Omega \times (0, T), \\ \varphi = 0 & \text{on } \partial\Omega \times (0, T), \\ \varphi(\cdot, 0) = \varphi_0 & \text{in } \Omega, \end{cases}$$

where $a = a(x, t) \in L^\infty(\Omega \times (0, T))$. We also denote $\|a\|_\infty = \|a\|_{L^\infty(\Omega\times(0,T))}$.

Lemma 1. *For any $\varphi_0 \in L^2(\Omega)$, the solution φ satisfies the two following estimates for any $t \in (0, T]$,*

$$\int_\Omega \big|\varphi(x, t)\big|^2 dx \leqslant e^{2t\|a\|_\infty} \int_\Omega \big|\varphi_0(x)\big|^2 dx \quad \text{and} \quad \int_\Omega \big|\nabla\varphi(x, t)\big|^2 dx \leqslant \frac{e^{3t\|a\|_\infty}}{t} \int_\Omega \big|\varphi_0(x)\big|^2 dx.$$

This result is deduced by energy estimate and is standard. Its proof is omitted here.

Let $x_0 \in \Omega$. Denote by $B_R = B(x_0, R)$ the ball of center x_0 and radius R.

Lemma 2. *Let $R_0 > 0$ and $\lambda > 0$. Introduce for $t \in [0, T]$ and $x_0 \in \Omega$,*

$$G_\lambda(x,t) = \frac{1}{(T-t+\lambda)^{n/2}} e^{-\frac{|x-x_0|^2}{4(T-t+\lambda)}}.$$

Define for $u \in H^1(0,T; L^2(\Omega \cap B_{R_0})) \cap L^2(0,T; H^2 \cap H_0^1(\Omega \cap B_{R_0}))$ and $t \in (0,T]$,

$$N_\lambda(t) = \frac{\int_{\Omega\cap B_{R_0}} |\nabla u(x,t)|^2 G_\lambda(x,t)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx}, \quad \textit{whenever} \int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 dx \neq 0.$$

The following two properties hold.

i)

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx + \int_{\Omega\cap B_{R_0}} \left|\nabla u(x,t)\right|^2 G_\lambda(x,t)\,dx$$
$$= \int_{\Omega\cap B_{R_0}} u(x,t)(\partial_t - \Delta)u(x,t)G_\lambda(x,t)\,dx. \tag{5.1}$$

ii) *When $\Omega \cap B_{R_0}$ is star-shaped with respect to x_0,*

$$\frac{d}{dt}N_\lambda(t) \leqslant \frac{1}{T-t+\lambda}N_\lambda(t) + \frac{\int_{\Omega\cap B_{R_0}} |(\partial_t - \Delta)u(x,t)|^2 G_\lambda(x,t)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx}. \tag{5.2}$$

Proof. The identity follows from some direct computations. The proof of the second one is the same as that in [13, pp. 1240–1245] or [5, Lemma 2]. □

Lemma 3. *Let $R > 0$ and $\delta \in (0, 1]$. Then there are two constants $C_1, C_2 > 0$, only dependent on (R, δ) such that for any $\varphi_0 \in L^2(\Omega)$ with $\varphi_0 \neq 0$, the quantity*

$$h_0 = \frac{C_1}{\ln\left((1+C_2)\left(e^{1+\frac{2C_1}{T}+3T\|a\|_\infty+\|a\|_\infty^{2/3}}\right)\frac{\int_\Omega |\varphi_0(x)|^2\,dx}{\int_{\Omega\cap B_R} |\varphi(x,T)|^2\,dx}\right)} \tag{5.3}$$

has the following two properties.

i)

$$0 < \left(1 + \frac{2C_1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}\right)h_0 < C_1. \tag{5.4}$$

ii) *For any $t \in [T - h_0, T]$, it holds*

$$e^{3T\|a\|_\infty}\int_\Omega \left|\varphi_0(x)\right|^2 dx \leqslant e^{1+C_3\frac{1}{h_0}} \int_{\Omega\cap B_{(1+\delta)R}} \left|\varphi(x,t)\right|^2 dx \tag{5.5}$$

for some $C_3 > C_1$ only dependent on (R, δ).

Remark 4. By the strong unique continuation property for parabolic equations with zero Dirichlet boundary condition, it is impossible to have $\int_{\Omega\cap B_R} |\varphi(x,T)|^2\,dx = 0$ if $\varphi_0 \in L^2(\Omega)$ with $\varphi_0 \neq 0$.

Remark 5. From (5.4), we have $h_0 < T/2$ and therefore $T/2 < T - h_0 < T$. Here, (5.5) says that for any t sufficiently closed to T, the following Hölder interpolation estimate holds.

$$\int_{\Omega\cap B_R} \left|\varphi(x,T)\right|^2 dx \leqslant \left((1+C_2)e^{1+\frac{2C_1}{T}+\|a\|_\infty^{2/3}}\right)\left(e^{3T\|a\|_\infty}\int_\Omega \left|\varphi_0(x)\right|^2 dx\right)^{\frac{C_3-C_1}{C_3}} \times \left(e\int_{\Omega\cap B_{(1+\delta)R}} \left|\varphi(x,t)\right|^2 dx\right)^{\frac{C_1}{C_3}}.$$

This can be compared with [5, Lemma 1].

Proof of Lemma 3. The property (5.4) is clearly true because the following inequality

$$\int_{\Omega\cap B_R} \left|\varphi(\cdot,T)\right|^2 dx \leqslant e^{2T\|a\|_\infty}\int_\Omega |\varphi_0|^2 dx$$

holds by Lemma 1. We prove (5.5) as follows. Let $h>0$, $\rho(x)=|x-x_0|^2$, $\chi\in C_0^\infty(B(x_0,(1+\delta)R))$ be such that $0\leqslant\chi\leqslant 1$, $\chi=1$ on $\{x; |x-x_0|\leqslant(1+3\delta/4)R\}$. We multiply the equation $\partial_t\varphi-\Delta\varphi+a\varphi=0$ by $e^{-\rho/h}\chi^2\varphi$ and integrate over $\Omega\cap B_{(1+\delta)R}$. We get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}|\chi\varphi|^2 dx + \int_{\Omega\cap B_{(1+\delta)R}} \nabla\varphi\nabla\left(e^{-\rho/h}\chi^2\varphi\right) dx = -\int_{\Omega\cap B_{(1+\delta)R}} a e^{-\rho/h}|\chi\varphi|^2 dx.$$

But, $\nabla(e^{-\rho/h}\chi^2\varphi)=\frac{-1}{h}\nabla\rho e^{-\rho/h}\chi^2\varphi+2e^{-\rho/h}\chi\nabla\chi\varphi+e^{-\rho/h}\chi^2\nabla\varphi$. Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}|\chi\varphi|^2 dx + \int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}|\chi\nabla\varphi|^2 dx\\ &\leqslant \int_{\Omega\cap B_{(1+\delta)R}} \left(e^{-\rho/(2h)}|\chi\nabla\varphi|\right)\left(\frac{2}{h}|x-x_0|e^{-\rho/(2h)}\chi|\varphi|+2|\nabla\chi|e^{-\rho/(2h)}|\varphi|\right) dx\\ &\quad+\|a\|_\infty\int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}|\chi\varphi|^2 dx\end{aligned}$$

which gives by Cauchy–Schwarz inequality

$$\begin{aligned}\frac{d}{dt}\int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}|\chi\varphi|^2 dx &\leqslant \left(\frac{4((1+\delta)R)^2}{h^2}+2\|a\|_\infty\right)\int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}|\chi\varphi|^2 dx\\ &\quad+4\int_{\Omega\cap\{x;(1+3\delta/4)R\leqslant\sqrt{\rho(x)}\leqslant(1+\delta)R\}} |\nabla\chi|^2 e^{-\rho/h}|\varphi|^2 dx.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}\left(e^{-(\frac{4((1+\delta)R)^2}{h^2}+2\|a\|_\infty)t}\int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}|\chi\varphi|^2 dx\right)\\ &\leqslant 4\|\nabla\chi\|_{L^\infty}^2 e^{-(\frac{4((1+\delta)R)^2}{h^2}+2\|a\|_\infty)t} e^{-\frac{((1+3\delta/4)R)^2}{h}} e^{2t\|a\|_\infty}\int_\Omega |\varphi_0|^2 dx\end{aligned}$$

which gives by integration between t and T,

$$\int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}\left|\chi\varphi(\cdot,T)\right|^2 dx \leqslant e^{(\frac{4((1+\delta)R)^2}{h^2}+2\|a\|_\infty)(T-t)} \int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}\left|\chi\varphi(\cdot,t)\right|^2 dx$$
$$+ e^{(\frac{4((1+\delta)R)^2}{h^2}+2\|a\|_\infty)T} \int_t^T e^{-\frac{4((1+\delta)R)^2}{h^2}s}\, ds\, 4\|\nabla\chi\|^2_{L^\infty} e^{-\frac{((1+3\delta/4)R)^2}{h}} \int_\Omega |\varphi_0|^2\, dx.$$

Therefore,

$$\int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}\left|\chi\varphi(\cdot,T)\right|^2 dx \leqslant e^{\frac{c_1R^2}{h^2}(T-t)} e^{2T\|a\|_\infty} \int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}\left|\chi\varphi(\cdot,t)\right|^2 dx$$
$$+ e^{\frac{c_1R^2}{h^2}(T-t)}(T-t)4\|\nabla\chi\|^2_{L^\infty} e^{2T\|a\|_\infty} e^{-\frac{c_2R^2}{h}} \int_\Omega |\varphi_0|^2\, dx,$$

with $c_1 = 4(1+\delta)^2$ and $c_2 = (1+3\delta/4)^2$. Set $c_3 = (1+\delta/2)^2$. In particular, $1 < c_3 < c_2$. Recall that $t \leqslant T$. Now suppose that the positive real number h is such that

$$0 < T - \frac{c_2-c_3}{c_1} h \leqslant t,$$

then $\frac{c_1}{h^2}(T-t) \leqslant \frac{c_2-c_3}{h}$ and

$$\int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}\left|\chi\varphi(\cdot,T)\right|^2 dx \leqslant e^{\frac{(c_2-c_3)R^2}{h}} e^{2T\|a\|_\infty} \int_{\Omega\cap B_{(1+\delta)R}} e^{-\rho/h}\left|\chi\varphi(\cdot,t)\right|^2 dx$$
$$+ 4\|\nabla\chi\|^2_{L^\infty} e^{2T\|a\|_\infty} \frac{c_2-c_3}{c_1} h e^{-\frac{c_3R^2}{h}} \int_\Omega |\varphi_0|^2\, dx.$$

Since $\chi = 1$ on $\{x; |x-x_0| \leqslant R\}$, the above estimate yields

$$\int_{\Omega\cap B_R} \left|\varphi(x,T)\right|^2 dx \leqslant e^{\frac{(c_2-c_3+1)R^2}{h}} e^{2T\|a\|_\infty} \int_{\Omega\cap B_{(1+\delta)R}} \left|\varphi(x,t)\right|^2 dx$$
$$+ 4e^{2T\|a\|_\infty}\|\nabla\chi\|^2_{L^\infty} \frac{c_2-c_3}{c_1} h e^{-\frac{(c_3-1)R^2}{h}} \int_\Omega \left|\varphi_0(x)\right|^2 dx, \tag{5.6}$$

whenever $0 < T - \frac{c_2-c_3}{c_1}h \leqslant t$ and $t \leqslant T$. Recall that $h_0 < T$ from (5.4). Now, choose $h \in (0, \frac{c_1}{c_2-c_3}T)$ as follows.

$$h = \frac{c_1}{c_2-c_3} h_0 = \frac{c_1}{c_2-c_3} \frac{C_1}{\ln\left(\left(e^{1+\frac{2C_1}{T}}\right)\frac{(1+C_2)e^{3T\|a\|_\infty+\|a\|_\infty^{2/3}} \int_\Omega |\varphi_0(x)|^2\, dx}{\int_{\Omega\cap B_R} |\varphi(x,T)|^2\, dx}\right)}$$

with $C_1 = \frac{(c_2-c_3)(c_3-1)R^2}{c_1}$ and $C_2 = 4\|\nabla\chi\|^2_{L^\infty} C_1$, in order that for any $T - \frac{c_2-c_3}{c_1}h \leqslant t \leqslant T$,

$$4e^{2T\|a\|_\infty}\|\nabla\chi\|^2_{L^\infty} \frac{c_2-c_3}{c_1} h e^{-\frac{(c_3-1)R^2}{h}} \int_\Omega \left|\varphi_0(x)\right|^2 dx$$
$$= e^{2T\|a\|_\infty} C_2 \frac{h_0}{C_1} e^{-\frac{(c_3-1)R^2}{h}} \int_\Omega \left|\varphi_0(x)\right|^2 dx$$
$$\leqslant \frac{h_0}{C_1} e^{-\frac{(c_3-1)R^2}{h}} (1+C_2) e^{3T\|a\|_\infty+\|a\|_\infty^{2/3}} \int_\Omega |\varphi_0|^2\, dx$$

$$\leqslant e^{-\frac{(c_3-1)R^2}{h}} \frac{1}{e^{1+\frac{2C_1}{T}}} e^{\frac{(c_3-1)R^2}{h}} \int_{\Omega\cap B_R} \left|\varphi(x,T)\right|^2 dx$$

$$\leqslant \frac{1}{e} \int_{\Omega\cap B_R} \left|\varphi(x,T)\right|^2 dx \tag{5.7}$$

where we have used in the third line the fact that $h_0 \leqslant C_1$ from (5.4). The definition of h along with (5.3) was applied in the fourth line. Since $t \in [T-h_0, T)$, it yields that $0 < T - \frac{c_2-c_3}{c_1}h \leqslant t \leqslant T$ and further, combining (5.6) and (5.7) we have

$$\int_{\Omega\cap B_R} \left|\varphi(x,T)\right|^2 dx \leqslant e^{\frac{(c_2-c_3+1)R^2}{h}} e^{2T\|a\|_\infty} \int_{\Omega\cap B_{(1+\delta)R}} \left|\varphi(x,t)\right|^2 dx + \frac{1}{e} \int_{\Omega\cap B_R} \left|\varphi(x,T)\right|^2 dx$$

which gives

$$\left(1-\frac{1}{e}\right) \int_{\Omega\cap B_R} \left|\varphi(x,T)\right|^2 dx \leqslant e^{\frac{(c_2-c_3+1)(c_2-c_3)R^2}{c_1}\frac{1}{h_0}} e^{2T\|a\|_\infty} \int_{\Omega\cap B_{(1+\delta)R}} \left|\varphi(x,t)\right|^2 dx.$$

On the other hand, by the definition of h_0 and the fact that $T\|a\|_\infty h_0 \leqslant C_1$ from (5.4),

$$e^{3T\|a\|_\infty} \int_\Omega \left|\varphi_0(x)\right|^2 dx \leqslant e^{3T\|a\|_\infty} e^{C_1\frac{1}{h_0}} \int_{\Omega\cap B_R} \left|\varphi(x,T)\right|^2 dx$$

$$\leqslant e^{4C_1\frac{1}{h_0}} \int_{\Omega\cap B_R} \left|\varphi(x,T)\right|^2 dx.$$

We conclude that

$$\left(1-\frac{1}{e}\right) e^{3T\|a\|_\infty} \int_\Omega \left|\varphi_0(x)\right|^2 dx \leqslant e^{C_3\frac{1}{h_0}} \int_{\Omega\cap B_{(1+\delta)R}} \left|\varphi(x,t)\right|^2 dx$$

with $C_3 = \frac{(c_2-c_3+1)(c_2-c_3)R^2}{c_1} + 6C_1$. This completes the proof. □

Lemma 4. *Let $0 < r < R$. Suppose that $B_r \subset \Omega$ and $\Omega \cap B_{(1+2\delta)R}$ is star-shaped with respect to x_0 for some $\delta \in (0,1]$. Then there are $C_1, C_2 > 0$ and $\beta \in (0,1)$ such that for any $T > 0$ and $\varphi_0 \in L^2(\Omega)$,*

$$\int_{\Omega\cap B_R} \left|\varphi(x,T)\right|^2 dx \leqslant \left((1+C_2)e^{1+\frac{2C_1}{T}+3T\|a\|_\infty+\|a\|_\infty^{2/3}} \int_\Omega \left|\varphi_0(x)\right|^2 dx\right)^\beta \left(2\int_{B_r} \left|\varphi(x,T)\right|^2 dx\right)^{1-\beta}.$$

Here $C_1, C_2 > 0$ are only dependent on (R,δ). β only depends on (n,r,R,δ).

Proof. There is no loss of generality in assuming that $\varphi_0 \neq 0$. Let $0 < r < R$ and $R_0 = (1+2\delta)R$. Let $\chi \in C_0^\infty(B_{R_0})$, $0 \leqslant \chi \leqslant 1$, $\chi = 1$ on $\{x; |x-x_0| \leqslant (1+3\delta/2)R\}$. We will apply Lemma 2 with $u = \chi\varphi$. First, $(\partial_t - \Delta)u = -au - 2\nabla\chi\nabla\varphi - \Delta\chi\varphi$. Next, define $g = -2\nabla\chi\nabla\varphi - \Delta\chi\varphi$.

Step 1. Notice that g is supported on $\{x; (1+3\delta/2)R \leqslant |x-x_0| \leqslant R_0\}$. Recall the fact that $\chi = 1$ on $\{x; |x-x_0| \leqslant (1+\delta)R\}$. Then there is $C = C(R,\delta) > 0$ such that we have

$$\frac{\int_{\Omega\cap B_{R_0}} u(x,t)g(x,t)G_\lambda(x,t)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx} \leqslant \frac{C(1+t^{-1/2})e^{3t\|a\|_\infty}\int_\Omega |\varphi_0(x)|^2 dx}{\int_{\Omega\cap B_{(1+\delta)R}} |\varphi(x,t)|^2 dx} e^{-\frac{C_4}{T-t+\lambda}}$$

and

$$\int_t^T \frac{\int_{\Omega\cap B_{R_0}} |g(x,s)|^2 G_\lambda(x,s)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,s)|^2 G_\lambda(x,s)\,dx} ds \leqslant \int_t^T \frac{C(1+s^{-1})e^{3s\|a\|_\infty}\int_\Omega |\varphi_0(x)|^2 dx}{\int_{\Omega\cap B_{(1+\delta)R}} |\varphi(x,s)|^2 dx} e^{-\frac{C_4}{T-s+\lambda}} ds$$

with $C_4 = -\frac{((1+\delta)R)^2}{4} + \frac{((1+3\delta/2)R)^2}{4} > 0$. Then we have the existence of $c = c(R,\delta) > 0$ such that for any $t \in [T - h_0, T)$,

$$\frac{\int_{\Omega\cap B_{R_0}} u(x,t)g(x,t)G_\lambda(x,t)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx} \leqslant c\left(1+\frac{1}{\sqrt{T}}\right)e^{C_3\frac{1}{h_0}}e^{-\frac{C_4}{T-t+\lambda}}$$

and

$$\int_t^T \frac{\int_{\Omega\cap B_{R_0}} |g(x,s)|^2 G_\lambda(x,s)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,s)|^2 G_\lambda(x,s)\,dx}\,ds \leqslant c e^{C_3\frac{1}{h_0}}e^{-\frac{C_4}{T-t+\lambda}}$$

by using (5.4) that gives the two inequalities $h_0 < C_1$, $T/2 < T - h_0 \leqslant t < T$ and Lemma 3 saying that

$$\frac{e^{3T\|a\|_\infty}\int_\Omega |\varphi_0(x)|^2\,dx}{\int_{\Omega\cap B_{(1+\delta)R}} |\varphi(x,t)|^2\,dx} \leqslant e^{1+C_3\frac{1}{h_0}} \quad \text{if } T - h_0 \leqslant t < T.$$

Step 2. Now, our plan is to bound $\lambda N_\lambda(T)$. We apply Lemma 2 as follows. First of all, by (5.2)

$$\frac{d}{dt}N_\lambda(t) \leqslant \frac{1}{T-t+\lambda}N_\lambda(t) + \frac{\int_{\Omega\cap B_{R_0}} |(-au+g)(x,t)|^2 G_\lambda(x,t)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx}$$

becomes

$$\frac{d}{dt}\big[(T-t+\lambda)N_\lambda(t)\big] \leqslant (T-t+\lambda)\frac{\int_{\Omega\cap B_{R_0}} |(-au+g)(x,t)|^2 G_\lambda(x,t)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx}.$$

Thus, it holds

$$\frac{d}{dt}\big[(T-t+\lambda)N_\lambda(t)\big] \leqslant 2(T-t+\lambda)\left(\|a\|_\infty^2 + \frac{\int_{\Omega\cap B_{R_0}} |g(x,t)|^2 G_\lambda(x,t)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx}\right)$$

which gives

$$\lambda N_\lambda(T) \leqslant (T-t+\lambda)N_\lambda(t) + 2\|a\|_\infty^2\int_t^T (T-s+\lambda)\,ds + 2\int_t^T (T-s+\lambda)\frac{\int_{\Omega\cap B_{R_0}} |g(x,s)|^2 G_\lambda(x,s)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,s)|^2 G_\lambda(x,s)\,dx}\,ds.$$

Therefore, for any $0 < T-\varepsilon \leqslant t < T$ (where $\varepsilon \in (0, h_0]$ will be determined later)

$$\frac{1}{\varepsilon+\lambda}\lambda N_\lambda(T) \leqslant N_\lambda(t) + 2\varepsilon\|a\|_\infty^2 + 2\int_t^T \frac{\int_{\Omega\cap B_{R_0}} |g(x,s)|^2 G_\lambda(x,s)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,s)|^2 G_\lambda(x,s)\,dx}\,ds. \tag{5.8}$$

Secondly, by (5.1),

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx + \int_{\Omega\cap B_{R_0}} \left|\nabla u(x,t)\right|^2 G_\lambda(x,t)\,dx$$
$$= \int_{\Omega\cap B_{R_0}} u(x,t)(-au+g)(x,t)G_\lambda(x,t)\,dx$$

becomes

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx + N_\lambda(t)\int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx$$

$$= -\int_{\Omega\cap B_{R_0}} a(x,t)\left|u(x,t)\right|^2 G_\lambda(x,t)\,dx$$
$$+ \frac{\int_{\Omega\cap B_{R_0}} u(x,t)g(x,t)G_\lambda(x,t)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx} \int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx. \tag{5.9}$$

Therefore, combining (5.8) and (5.9), we obtain that for any $0 < T-\varepsilon \leqslant t < T$

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx + \frac{1}{\varepsilon+\lambda}\lambda N_\lambda(T)\int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx$$
$$\leqslant \left(\|a\|_\infty + 2\varepsilon\|a\|_\infty^2\right)\int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx + \int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx$$
$$\times \left(\frac{\int_{\Omega\cap B_{R_0}} u(x,t)g(x,t)G_\lambda(x,t)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx} + 2\int_t^T \frac{\int_{\Omega\cap B_{R_0}} |g(x,s)|^2 G_\lambda(x,s)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,s)|^2 G_\lambda(x,s)\,dx}\,ds\right).$$

Now, define, for any $\varepsilon \in (0, h_0]$,

$$Q_{h_0,\varepsilon,\lambda} = c\left(3 + \frac{1}{\sqrt{T}}\right)e^{(C_3+C_4)\frac{1}{h_0}}\,e^{-\frac{C_4}{\varepsilon+\lambda}} \tag{5.10}$$

given from the fact that using step 1,

$$\frac{\int_{\Omega\cap B_{R_0}} u(x,t)g(x,t)G_\lambda(x,t)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,t)|^2 G_\lambda(x,t)\,dx} + 2\int_t^T \frac{\int_{\Omega\cap B_{R_0}} |g(x,s)|^2 G_\lambda(x,s)\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,s)|^2 G_\lambda(x,s)d\,x}\,ds$$
$$\leqslant Q_{h_0,\varepsilon,\lambda} \quad \text{for any } 0 < T-\varepsilon \leqslant t < T \text{ with } \varepsilon \in (0, h_0].$$

Then, it holds

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx$$
$$\leqslant -\left(\frac{1}{\varepsilon+\lambda}\lambda N_\lambda(T) - \|a\|_\infty - 2\varepsilon\|a\|_\infty^2 - Q_{h_0,\varepsilon,\lambda}\right)\int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx$$

which implies

$$\frac{d}{dt}\left(e^{2(\frac{1}{\varepsilon+\lambda}\lambda N_\lambda(T) - \|a\|_\infty - 2\varepsilon\|a\|_\infty^2 - Q_{h_0,\varepsilon,\lambda})t}\int_{\Omega\cap B_{R_0}} \left|u(x,t)\right|^2 G_\lambda(x,t)\,dx\right) \leqslant 0$$

for $0 < T-\varepsilon \leqslant t$. Integrating over $(T-\varepsilon, T-\varepsilon/2)$, we get

$$e^{\frac{\varepsilon}{\varepsilon+\lambda}\lambda N_\lambda(T)}\int_{\Omega\cap B_{R_0}} \left|u(x, T-\varepsilon/2)\right|^2 G_\lambda(x, T-\varepsilon/2)\,dx$$
$$\leqslant e^{\varepsilon\|a\|_\infty + 2\varepsilon^2\|a\|_\infty^2} e^{\varepsilon Q_{h_0,\varepsilon,\lambda}}\int_{\Omega\cap B_{R_0}} \left|u(x, T-\varepsilon)\right|^2 G_\lambda(x, T-\varepsilon)\,dx$$

that is

$$e^{\frac{\varepsilon}{\varepsilon+\lambda}\lambda N_\lambda(T)} \int\limits_{\Omega\cap B_{R_0}} \left|u(x,T-\varepsilon/2)\right|^2 \frac{1}{(\varepsilon/2+\lambda)^{n/2}} e^{-\frac{|x-x_0|^2}{4(\varepsilon/2+\lambda)}}\,dx$$
$$\leqslant e^{\varepsilon\|a\|_\infty+2\varepsilon^2\|a\|_\infty^2} e^{\varepsilon Q_{h_0,\varepsilon,\lambda}} \int\limits_{\Omega\cap B_{R_0}} \left|u(x,T-\varepsilon)\right|^2 \frac{1}{(\varepsilon+\lambda)^{n/2}} e^{-\frac{|x-x_0|^2}{4(\varepsilon+\lambda)}}\,dx.$$

Thus,

$$e^{\frac{\varepsilon}{\varepsilon+\lambda}\lambda N_\lambda(T)} \leqslant e^{\varepsilon\|a\|_\infty+2\varepsilon^2\|a\|_\infty^2} e^{\varepsilon Q_{h_0,\varepsilon,\lambda}} \frac{\int_{\Omega\cap B_{R_0}} |u(x,T-\varepsilon)|^2 e^{-\frac{|x-x_0|^2}{4(\varepsilon+\lambda)}}\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,T-\varepsilon/2)|^2 e^{-\frac{|x-x_0|^2}{4(\varepsilon/2+\lambda)}}\,dx}.$$

Now, since $\frac{\varepsilon}{2}\in(0,h_0]$,

$$\frac{\int_{\Omega\cap B_{R_0}} |u(x,T-\varepsilon)|^2 e^{-\frac{|x-x_0|^2}{4(\varepsilon+\lambda)}}\,dx}{\int_{\Omega\cap B_{R_0}} |u(x,T-\varepsilon/2)|^2 e^{-\frac{|x-x_0|^2}{4(\varepsilon/2+\lambda)}}\,dx} \leqslant \frac{e^{\frac{((1+\delta)R)^2}{2\varepsilon}} e^{3T\|a\|_\infty} \int_\Omega |\varphi_0(x)|^2\,dx}{\int_{\Omega\cap B_{(1+\delta)R}} |\varphi(x,T-\varepsilon/2)|^2\,dx}$$
$$\leqslant e^{\frac{((1+\delta)R)^2}{2\varepsilon}} e^{1+C_3\frac{1}{h_0}}.$$

Indeed, by Lemma 3, we know that $\frac{e^{3T\|a\|_\infty}\int_\Omega |\varphi_0(x)|^2\,dx}{\int_{\Omega\cap B_{(1+\delta)R}} |\varphi(x,t)|^2\,dx} \leqslant e^{1+C_3\frac{1}{h_0}}$ if $T-h_0\leqslant t<T$. Therefore, for any $\varepsilon\in(0,h_0]$,

$$\lambda N_\lambda(T) \leqslant \frac{\varepsilon+\lambda}{\varepsilon} \ln\left(e^{\varepsilon\|a\|_\infty+2\varepsilon^2\|a\|_\infty^2} e^{\varepsilon Q_{h_0,\varepsilon,\lambda}} e^{\frac{((1+\delta)R)^2}{2\varepsilon}} e^{1+C_3\frac{1}{h_0}}\right)$$
$$\leqslant \frac{\varepsilon+\lambda}{\varepsilon}\left(\frac{((1+\delta)R)^2}{2\varepsilon} + \varepsilon\|a\|_\infty + 2\varepsilon^2\|a\|_\infty^2 + \varepsilon Q_{h_0,\varepsilon,\lambda} + 1 + C_3\frac{1}{h_0}\right). \tag{5.11}$$

Step 3. Now, we choose $\lambda=\mu\varepsilon$ with $\mu\in(0,1)$ which will be determined later and

$$\varepsilon = \frac{C_4}{2(C_3+C_4)}h_0$$

in order that $Q_{h_0,\varepsilon,\lambda}$ given by (5.10) satisfies the following bound

$$Q_{h_0,\varepsilon,\lambda} \leqslant c\left(3+\frac{1}{\sqrt{T}}\right) e^{\frac{C_3+C_4}{h_0}(1-\frac{2}{1+\mu})} = c\left(3+\frac{1}{\sqrt{T}}\right) e^{\frac{C_3+C_4}{h_0}(\frac{\mu-1}{1+\mu})} \leqslant c\left(3+\frac{1}{\sqrt{T}}\right)$$

and further, using the fact that $\varepsilon\leqslant h_0$, (5.11) becomes

$$\lambda N_\lambda(T) \leqslant 2\left(1+h_0\|a\|_\infty + 2h_0^2\|a\|_\infty^2 + (C_3+C_4)\left(1+\frac{(1+\delta)^2R^2}{C_4}\right)\frac{1}{h_0} + c\left(3+\frac{1}{\sqrt{T}}\right)h_0\right).$$

Next, we deduce that

$$\varepsilon\lambda N_\lambda(T) \leqslant 2\left(\varepsilon+\varepsilon h_0\|a\|_\infty + 2\varepsilon h_0^2\|a\|_\infty^2 + \varepsilon(C_3+C_4)\left(1+\frac{(1+\delta)^2R^2}{C_4}\right)\frac{1}{h_0} + \varepsilon c\left(3+\frac{1}{\sqrt{T}}\right)h_0\right)$$
$$\leqslant 2\left(h_0 + h_0T\|a\|_\infty + 2h_0^3\|a\|_\infty^2 + \frac{1}{2}\left(C_4+(1+\delta)^2R^2\right) + c\left(3h_0^2+\sqrt{\frac{h_0}{T}}h_0^{3/2}\right)\right)$$
$$\leqslant 2\left(2C_1+2C_1^3+\frac{1}{2}\left(C_4+(1+\delta)^2R^2\right)+c\left(3C_1^2+C_1^{3/2}\right)\right)$$

where in the last line, we used the following four inequalities $h_0<C_1$, $h_0<T$, $h_0T\|a\|_\infty<C_1$ and $h_0^3\|a\|_\infty^2<C_1^3$ obtained in (5.4) of Lemma 3. Therefore, we conclude from the above bound of $\varepsilon\lambda N_\lambda(T)$ that

$$\frac{16\lambda}{r^2}\left(\frac{n}{4}+\lambda N_\lambda(T)\right) \leqslant \frac{16}{r^2}\mu\left(\frac{n}{4}C_1+\varepsilon\lambda N_\lambda(T)\right) \leqslant \mu(1+C_0) \tag{5.12}$$

for some $C_0>0$ only depending on (n,r,R,δ).

Step 4. Now, we are able to bound $\int_{\Omega\cap B_{R_0}}|u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx$ as follows. Since $\Omega\cap B_{R_0}$ is star-shaped with respect to x_0, we have (see, for example, [13, p. 1238] or [14, Lemma 2.5], [5, Lemma 3]),

$$\frac{1}{16\lambda}\int_{\Omega\cap B_{R_0}}|x-x_0|^2\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx \leqslant \frac{n}{4}\int_{\Omega\cap B_{R_0}}\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx+\lambda\int_{\Omega\cap B_{R_0}}\left|\nabla u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx$$

which implies

$$\begin{aligned}
\int_{\Omega\cap B_{R_0}}\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx
&\leqslant \int_{B_r}\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx+\int_{(\Omega\cap B_{R_0})\setminus B_r}\frac{|x-x_0|^2}{r^2}\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx\\
&\leqslant \int_{B_r}\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx+\frac{1}{r^2}\left[4\lambda n\int_{\Omega\cap B_{R_0}}\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx+\int_{\Omega\cap B_{R_0}}16\lambda^2\left|\nabla u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx\right]\\
&\leqslant \int_{B_r}\left|\varphi(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx+\frac{16\lambda}{r^2}\left[\frac{n}{4}+\lambda N_\lambda(T)\right]\int_{\Omega\cap B_{R_0}}\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx,
\end{aligned}$$

where in the last line we used the definition of $N_\lambda(T)$ and the fact that $u=\varphi$ in B_r. Combining the above inequality and (5.12), we deduce that

$$\int_{\Omega\cap B_{R_0}}\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx \leqslant \int_{B_r}\left|\varphi(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx+\mu(1+C_0)\int_{\Omega\cap B_{R_0}}\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx. \tag{5.13}$$

Step 5. Now, we choose $\mu\in(0,1)$ as follows.

$$\mu=\frac{1}{2}\frac{1}{(1+C_0)}.$$

Then, $\lambda=\mu\varepsilon=\mu\frac{C_4}{2(C_3+C_4)}h_0=\frac{1}{4}\frac{C_4}{(1+C_0)(C_3+C_4)}h_0$ and by using the definition of h_0, we have

$$\begin{aligned}
\int_{\Omega\cap B_R}\left|\varphi(x,T)\right|^2 dx &\leqslant e^{\frac{R^2}{4\lambda}}\int_{\Omega\cap B_{R_0}}\left|u(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx\\
&\leqslant 2e^{\frac{R^2}{4\lambda}}\int_{B_r}\left|\varphi(x,T)\right|^2 e^{-\frac{|x-x_0|^2}{4\lambda}}\,dx \quad \text{by (5.13)},\\
&\leqslant 2e^{\frac{(1+C_0)(C_3+C_4)R^2}{C_4}\frac{1}{h_0}}\int_{B_r}\left|\varphi(x,T)\right|^2 dx
\end{aligned}$$

$$\leqslant 2\left((1+C_2)\frac{e^{1+\frac{2C_1}{T}+e^{3T\|a\|_\infty+\|a\|_\infty^{2/3}}}\int_\Omega|\varphi_0(x)|^2\,dx}{\int_{\Omega\cap B_R}|\varphi(x,T)|^2\,dx}\right)^{\frac{(1+C_0)(C_3+C_4)R^2}{C_1C_4}}\int_{B_r}\left|\varphi(x,T)\right|^2dx.$$

We conclude that the desired estimate of Lemma 4 holds with $\beta=\frac{(1+C_0)(C_3+C_4)R^2}{C_1C_4+(1+C_0)(C_3+C_4)R^2}\in(0,1)$. This completes the proof of Lemma 4. □

Lemma 5. *Let $\tilde\omega$ be a non-empty open set of Ω. Then there are $C=C(\tilde\omega,\Omega)>0$ and $\tilde\beta=\tilde\beta(\tilde\omega,\Omega)\in(0,1)$ such that for any $T>0$ and $\varphi_0\in L^2(\Omega)$,*

$$\int_\Omega\left|\varphi(x,T)\right|^2dx\leqslant e^{C(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3})}\left(\int_\Omega\left|\varphi_0(x)\right|^2dx\right)^{\tilde\beta}\left(\int_{\tilde\omega}\left|\varphi(x,T)\right|^2dx\right)^{1-\tilde\beta}.$$

Proof. Firstly, by Lemma 4 and constructing a sequence of balls chained along a curve, we claim that, for any compact sets Θ_1 and Θ_2 with non-empty interior in Ω, there are constants $C=C(\Theta_1,\Theta_2,\Omega)>0$ and $\alpha_1=\alpha_1(\Theta_1,\Theta_2,\Omega)\in(0,1)$ such that

$$\int_{\Theta_1}\left|\varphi(x,T)\right|^2dx\leqslant e^{C(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3})}\left(\int_\Omega\left|\varphi_0(x)\right|^2dx\right)^{\alpha_1}\left(\int_{\Theta_2}\left|\varphi(x,T)\right|^2dx\right)^{1-\alpha_1}. \tag{5.14}$$

Indeed, since Θ_1 is a compact set in Ω, there are $R>0$ and finitely many points $x_1,\dots,x_M$ such that $\Theta_1\subset\bigcup_{i=1,\dots,M}B(x_i,R)$ and $B(x_i,3R)\subset\Omega$. Next, for each $i\in\{1,\dots,M\}$, we choose $\rho\in(0,R)$ and finitely many points $q_0,\dots,q_m$ with the following properties:

$$\begin{cases}x_i=q_m,\\ \Theta_2\supset B(q_0,\rho),\\ B(q_{j+1},\rho/2)\subset B(q_j,\rho)\quad\forall j=0,\dots,m-1,\\ B(q_j,3\rho)\subset\Omega\quad\forall j=0,\dots,m.\end{cases}$$

Thanks to Lemma 4, there exist $\sigma,\sigma_1,\alpha_1\in(0,1)$, such that

$$\begin{aligned}\int_{B(x_i,R)}\left|\varphi(x,T)\right|^2dx&\leqslant e^{C(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3})}\left(\int_\Omega\left|\varphi_0(x)\right|^2dx\right)^{\sigma}\left(\int_{B(x_i,\rho/2)}\left|\varphi(x,T)\right|^2dx\right)^{1-\sigma}\\
&=e^{C(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3})}\left(\int_\Omega\left|\varphi_0(x)\right|^2dx\right)^{\sigma}\left(\int_{B(q_m,\rho/2)}\left|\varphi(x,T)\right|^2dx\right)^{1-\sigma}\\
&\leqslant e^{C(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3})}\left(\int_\Omega\left|\varphi_0(x)\right|^2dx\right)^{\sigma}\left(\int_{B(q_{m-1},\rho)}\left|\varphi(x,T)\right|^2dx\right)^{1-\sigma}\\
&\leqslant e^{C(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3})}\left(\int_\Omega\left|\varphi_0(x)\right|^2dx\right)^{\sigma}\\
&\quad\times\left(e^{C(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3})}\left(\int_\Omega\left|\varphi_0(x)\right|^2dx\right)^{\sigma_1}\left(\int_{B(q_{m-1},\rho/2)}\left|\varphi(x,T)\right|^2dx\right)^{1-\sigma_1}\right)^{1-\sigma}\\
&\leqslant\cdots\\
&\leqslant e^{C(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3})}\left(\int_\Omega\left|\varphi_0(x)\right|^2dx\right)^{\alpha_1}\left(\int_{B(q_0,\rho)}\left|\varphi(x,T)\right|^2dx\right)^{1-\alpha_1},\end{aligned}$$

where $C>0$ may change value from line to line. This implies the desired inequality (5.14).

Secondly, since Ω is bounded with a C^2 boundary, there is a finite set of triplet $(q_j, R_j, \delta_j) \in \Omega \times \mathbb{R}_+^* \times (0,1]$, $j = 1, \dots, m$, such that

$$\partial\Omega \subset \bigcup_{j=1,\dots,m} B\big(q_j, (1+2\delta_j)R_j\big)$$

and $\Omega \cap B(q_j, (1+2\delta_j)R_j)$ is star-shaped with center q_j for some δ_j. Then we apply Lemma 4 with $\Omega \cap B(q_j, (1+2\delta_j)R_j)$ for $j = 1, \dots, m$, and the same arguments as above to get that, when ϑ is a neighborhood of $\partial\Omega$ and Θ_3 is a compact set with non-empty interior in Ω, there are constants $C = C(\vartheta, \Theta_3, \Omega) > 0$ and $\alpha_2 = \alpha_2(\vartheta, \Theta_3, \Omega) \in (0,1)$ such that

$$\int_\vartheta \big|\varphi(x,T)\big|^2\,dx \leqslant e^{C(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3})}\left(\int_\Omega |\varphi_0|^2\,dx\right)^{\alpha_2}\left(\int_{\Theta_3}\big|\varphi(x,T)\big|^2\,dx\right)^{1-\alpha_2}.$$

Finally, we derive the desired estimate from the previous two statements with $\Omega \subset (\vartheta \cup \Theta_1)$ and $(\Theta_2 \cup \Theta_3) \subset \tilde\omega$. This completes the proof. □

Now, we are able to present the proof of the observability estimate of Theorem 4.

Proof of Theorem 4. We start with the following interpolation estimate deduced by Lemma 5 and the Young inequality. For any $0 \leqslant t_1 < t_2 \leqslant T$,

$$\big\|\varphi(\cdot,t_2)\big\|_{L^2(\Omega)} \leqslant \frac{K_1}{\varepsilon^\alpha} e^{\frac{K_2}{t_2-t_1}}\big\|\varphi(\cdot,t_2)\big\|_{L^2(\tilde\omega)} + \varepsilon\big\|\varphi(\cdot,t_1)\big\|_{L^2(\Omega)} \quad \forall \varepsilon > 0.$$

Here, $\tilde\omega \Subset \omega \subset \Omega$, $K_1 = e^{\frac{C}{2(1-\tilde\beta)}(1+T\|a\|_\infty+\|a\|_\infty^{2/3})}$ and $K_2 = \frac{C}{2(1-\tilde\beta)}$ in Lemma 5, $\alpha = \frac{\tilde\beta}{1-\tilde\beta}$ in Lemma 5. By Nash inequality and Poincaré inequality,

$$\big\|\varphi(\cdot,t_2)\big\|_{L^2(\tilde\omega)} \leqslant \frac{K_3}{\delta^{n/2}}\big\|\varphi(\cdot,t_2)\big\|_{L^1(\omega)} + \delta\big\|\nabla\varphi(\cdot,t_2)\big\|_{L^2(\Omega)} \quad \forall \delta > 0.$$

Here $K_3 > 0$ only depends on $(\tilde\omega, \omega, \Omega)$. By Lemma 1, we know that

$$\big\|\nabla\varphi(\cdot,t_2)\big\|_{L^2(\Omega)} \leqslant \frac{K_4}{(t_2-t_1)^{1/2}}\big\|\varphi(\cdot,t_1)\big\|_{L^2(\Omega)}$$

with $K_4 = e^{2T\|a\|_\infty}$. Therefore, from the above three estimates with $\frac{K_1}{\varepsilon^\alpha} e^{\frac{K_2}{t_2-t_1}} \delta \frac{K_4}{(t_2-t_1)^{1/2}} = \varepsilon$, we get

$$\begin{aligned}
\big\|\varphi(\cdot,t_2)\big\|_{L^2(\Omega)} &\leqslant \frac{K_1}{\varepsilon^\alpha} e^{\frac{K_2}{t_2-t_1}}\left(\frac{K_3}{\delta^{n/2}}\big\|\varphi(\cdot,t_2)\big\|_{L^1(\omega)} + \delta\frac{K_4}{(t_2-t_1)^{1/2}}\big\|\varphi(\cdot,t_1)\big\|_{L^2(\Omega)}\right) + \varepsilon\big\|\varphi(\cdot,t_1)\big\|_{L^2(\Omega)} \\
&\leqslant \frac{K_1K_3}{\varepsilon^\alpha} e^{\frac{K_2}{t_2-t_1}}\left(\frac{1}{\varepsilon^{\alpha+1}} e^{\frac{K_2}{t_2-t_1}}\frac{K_1K_4}{(t_2-t_1)^{1/2}}\right)^{n/2}\big\|\varphi(\cdot,t_2)\big\|_{L^1(\omega)} + 2\varepsilon\big\|\varphi(\cdot,t_1)\big\|_{L^2(\Omega)} \\
&\leqslant \frac{K_1K_3}{\varepsilon^{\alpha+(\alpha+1)n/2}} e^{(\frac{3n}{4}+1)\frac{K_2}{t_2-t_1}}\left(\frac{K_1K_4}{\sqrt{K_2}}\right)^{n/2}\big\|\varphi(\cdot,t_2)\big\|_{L^1(\omega)} + 2\varepsilon\big\|\varphi(\cdot,t_1)\big\|_{L^2(\Omega)} \\
&\leqslant \frac{K_5}{(2\varepsilon)^\gamma} e^{\frac{K_6}{t_2-t_1}}\big\|\varphi(\cdot,t_2)\big\|_{L^1(\omega)} + 2\varepsilon\big\|\varphi(\cdot,t_1)\big\|_{L^2(\Omega)} \quad \forall \varepsilon > 0,
\end{aligned}$$

denoting $\gamma = \alpha(1+\frac{n}{2}) + \frac{n}{2}$, $K_5 = 2^{\alpha+(\alpha+1)n/2}K_1K_3(\frac{K_1K_4}{\sqrt{K_2}})^{n/2}$ and $K_6 = (\frac{3n}{4}+1)K_2$.

On another hand, let E be a subset of positive measure in $(0,T)$. Let ℓ be a density point of E. Using [14, Proposition 2.1], for each $\tau > 1$, there exists $\ell_1 \in (\ell, T)$, depending on τ and E, such that the sequence $\{\ell_m\}_{m\geqslant 1}$, given by

$$\ell_{m+1} = \ell + \frac{1}{\tau^m}(\ell_1 - \ell),$$

satisfies

$$\ell_m - \ell_{m+1} \leqslant 3\big|E \cap (\ell_{m+1}, \ell_m)\big|.$$

Next, let $0<\ell_{m+2}<\ell_{m+1}\leqslant t<\ell_m<\ell_1<T$. We apply the above interpolation inequality to get

$$\|\varphi(\cdot,t)\|_{L^2(\Omega)}\leqslant \frac{K_5}{\varepsilon^\gamma}e^{\frac{K_6}{t-\ell_{m+2}}}\|\varphi(\cdot,t)\|_{L^1(\omega)}+\varepsilon\|\varphi(\cdot,\ell_{m+2})\|_{L^2(\Omega)}\quad \forall\varepsilon>0.$$

Recall that by Lemma 1

$$\|\varphi(\cdot,\ell_m)\|_{L^2(\Omega)}\leqslant K_4\|\varphi(\cdot,t)\|_{L^2(\Omega)}.$$

Therefore,

$$\|\varphi(\cdot,\ell_m)\|_{L^2(\Omega)}\leqslant K_4\left(\frac{K_5}{\varepsilon^\gamma}e^{\frac{K_6}{t-\ell_{m+2}}}\|\varphi(\cdot,t)\|_{L^1(\omega)}+\varepsilon\|\varphi(\cdot,\ell_{m+2})\|_{L^2(\Omega)}\right)\quad \forall\varepsilon>0.$$

Finally, with $K_7=(K_4)^{1+\gamma}K_5$,

$$\|\varphi(\cdot,\ell_m)\|_{L^2(\Omega)}\leqslant \frac{K_7}{\varepsilon^\gamma}e^{\frac{K_6}{t-\ell_{m+2}}}\|\varphi(\cdot,t)\|_{L^1(\omega)}+\varepsilon\|\varphi(\cdot,\ell_{m+2})\|_{L^2(\Omega)}\quad \forall\varepsilon>0.$$

Integrating it over $t\in E\cap(\ell_{m+1},\ell_m)$, it yields that

$$\begin{aligned}\left|E\cap(\ell_{m+1},\ell_m)\right|\|\varphi(\cdot,\ell_m)\|_{L^2(\Omega)}\leqslant{}& \frac{K_7}{\varepsilon^\gamma}e^{\frac{K_6}{\ell_{m+1}-\ell_{m+2}}}\int_{\ell_{m+1}}^{\ell_m}1_E\|\varphi(\cdot,t)\|_{L^1(\omega)}\,dt\\ &+\varepsilon\left|E\cap(\ell_{m+1},\ell_m)\right|\|\varphi(\cdot,\ell_{m+2})\|_{L^2(\Omega)}\quad \forall\varepsilon>0.\end{aligned}$$

That is, using the fact that $\ell_m-\ell_{m+1}=\frac{1}{\tau^m}(\tau-1)(\ell_1-\ell)$,

$$\begin{aligned}\|\varphi(\cdot,\ell_m)\|_{L^2(\Omega)}&\leqslant\frac{1}{|E\cap(\ell_{m+1},\ell_m)|}\frac{K_7}{\varepsilon^\gamma}e^{K_6[\frac{1}{\ell_1-\ell}\frac{\tau^{m+1}}{\tau-1}]}\int_{\ell_{m+1}}^{\ell_m}1_E\|\varphi(\cdot,t)\|_{L^1(\omega)}\,dt+\varepsilon\|\varphi(\cdot,\ell_{m+2})\|_{L^2(\Omega)}\\ &\leqslant 3\left[\frac{1}{\ell_1-\ell}\frac{\tau^m}{\tau-1}\right]\frac{K_7}{\varepsilon^\gamma}e^{K_6[\frac{1}{\ell_1-\ell}\frac{\tau^{m+1}}{\tau-1}]}\int_{\ell_{m+1}}^{\ell_m}1_E\|\varphi(\cdot,t)\|_{L^1(\omega)}\,dt\\ &\quad+\varepsilon\|\varphi(\cdot,\ell_{m+2})\|_{L^2(\Omega)}\quad \forall\varepsilon>0.\end{aligned}$$

Therefore,

$$\|\varphi(\cdot,\ell_m)\|_{L^2(\Omega)}\leqslant\frac{1}{\varepsilon^\gamma}\frac{3}{\tau}\frac{K_7}{K_6}e^{2K_6[\frac{1}{\ell_1-\ell}\frac{\tau^{m+1}}{\tau-1}]}\int_{\ell_{m+1}}^{\ell_m}1_E\|\varphi(\cdot,t)\|_{L^1(\omega)}\,dt+\varepsilon\|\varphi(\cdot,\ell_{m+2})\|_{L^2(\Omega)}\quad \forall\varepsilon>0.$$

Take $d=2K_6[\frac{1}{\ell_1-\ell}\frac{1}{\tau(\tau-1)}]$. It guarantees that

$$\varepsilon^\gamma e^{-d\tau^{m+2}}\|\varphi(\cdot,\ell_m)\|_{L^2(\Omega)}-\varepsilon^{1+\gamma}e^{-d\tau^{m+2}}\|\varphi(\cdot,\ell_{m+2})\|_{L^2(\Omega)}\leqslant\frac{3}{\tau}\frac{K_7}{K_6}\int_{\ell_{m+1}}^{\ell_m}1_E\|\varphi(\cdot,t)\|_{L^1(\omega)}\,dt\quad \forall\varepsilon>0.$$

Take $\varepsilon=e^{-d\tau^{m+2}}$, then

$$e^{-(\gamma+1)d\tau^{m+2}}\|\varphi(\cdot,\ell_m)\|_{L^2(\Omega)}-e^{-(2+\gamma)d\tau^{m+2}}\|\varphi(\cdot,\ell_{m+2})\|_{L^2(\Omega)}\leqslant\frac{3}{\tau}\frac{K_7}{K_6}\int_{\ell_{m+1}}^{\ell_m}1_E\|\varphi(\cdot,t)\|_{L^1(\omega)}\,dt.$$

Take $\tau=\sqrt{\frac{\gamma+2}{\gamma+1}}$, then

$$e^{-(2+\gamma)d\tau^{m}}\|\varphi(\cdot,\ell_m)\|_{L^2(\Omega)}-e^{-(2+\gamma)d\tau^{m+2}}\|\varphi(\cdot,\ell_{m+2})\|_{L^2(\Omega)}\leqslant\frac{3}{\tau}\frac{K_7}{K_6}\int_{\ell_{m+1}}^{\ell_m}1_E\|\varphi(\cdot,t)\|_{L^1(\omega)}\,dt.$$

Changing m to $2m'$ and summing the above from $m'=1$ to infinity give the desired result. Indeed,

$$\begin{aligned}\frac{1}{K_4}e^{-(2+\gamma)d\tau^2}\|\varphi(\cdot,T)\|_{L^2(\Omega)} &\leqslant e^{-(2+\gamma)d\tau^2}\|\varphi(\cdot,\ell_2)\|_{L^2(\Omega)}\\ &\leqslant \sum_{m'=1}^{+\infty}\left(e^{-(2+\gamma)d\tau^{2m'}}\|\varphi(\cdot,\ell_{2m'})\|_{L^2(\Omega)} - e^{-(2+\gamma)d\tau^{2m'+2}}\|\varphi(\cdot,\ell_{2m'+2})\|_{L^2(\Omega)}\right)\\ &\leqslant \frac{3}{\tau}\frac{K_7}{K_6}\sum_{m'=1}^{+\infty}\int_{\ell_{2m'+1}}^{\ell_{2m'}} 1_E\|\varphi(\cdot,t)\|_{L^1(\omega)}\,dt\\ &\leqslant \frac{3}{\tau}\frac{K_7}{K_6}\int_0^T 1_E\|\varphi(\cdot,t)\|_{L^1(\omega)}\,dt.\end{aligned}$$

This concludes the proof of Theorem 4. □

Remark 6. When $E=(0,T)$, then we can take the sequence $\{\ell_m\}_{m\geqslant 1}$, as follows

$$\ell_{m+1}=\frac{T}{\tau^m},$$

so that the observability constant becomes

$$e^{K(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3})}.$$

When E is a positive measurable set with 0 its Lebesgue point, then we can take the sequence $\{\ell_m\}_{m\geqslant 1}$, as follows

$$\ell_{m+1}=\frac{\ell_1}{\tau^m},$$

where the existence of ℓ_1 comes from [14, Proposition 2.1], so that the observability constant becomes

$$e^{K(1+\frac{1}{\ell_1}+\ell_1\|a\|_\infty+\|a\|_\infty^{2/3})}.$$

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