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Hardy inequalities on Riemannian manifolds and applications

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Abstract

We prove a simple sufficient criterion to obtain some Hardy inequalities on Riemannian manifolds related to quasilinear second order differential operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Namely, if ρ is a nonnegative weight such that $-\Delta_p \rho \geqslant 0$, then the Hardy inequality

$$c \int_M \frac{|u|^p}{\rho^p} |\nabla \rho|^p \, dv_g \leqslant \int_M |\nabla u|^p \, dv_g, \quad u \in C_0^\infty(M),$$

holds. We show concrete examples specializing the function ρ.

Our approach allows to obtain a characterization of p-hyperbolic manifolds as well as other inequalities related to Caccioppoli inequalities, weighted Gagliardo–Nirenberg inequalities, uncertain principle and first order Caffarelli–Kohn–Nirenberg interpolation inequality.

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1. Introduction

An N-dimensional generalization of the classical Hardy inequality asserts that for every $p > 1$

$$c \int_\Omega \frac{|u|^p}{w^p} \, dx \leqslant \int_\Omega |\nabla u|^p \, dx, \quad u \in C_0^\infty(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is an open set, and the weight w is, for instance, $w(x) := |x|$ and $p < N$, or $w(x) := dist(x, \partial\Omega)$ and Ω is convex (see for example [5,10,21,27,41–43,45] and references therein).

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The preeminent role of Hardy inequalities and the knowledge of the best constants involved is a well known fact, as the reader can recognize from the wide literature that uses such a tool in Euclidean or in subelliptic setting as well as on manifolds ([4,6,9,10,12,22,37,46] just to cite a few).

On the other hand, the knowledge of the validity of a Hardy or Gagliardo–Nirenberg or Sobolev or Caffarelli–Kohn–Nirenberg inequality on a manifold M and their best constants allows to obtain qualitative properties on the manifold M. For instance in [2,14,50] it was shown that if M is a complete open Riemannian manifold with nonnegative Ricci curvature in which a Hardy- or Gagliardo–Nirenberg- or Caffarelli–Kohn–Nirenberg-type inequality holds, then M is in some suitable sense *close* to the Euclidean space.

One of our aims is to prove some Hardy inequalities on Riemannian manifolds. In 1997, Carron in [16] studies weighted L^2-Hardy inequalities on a Riemannian manifold M under some geometric assumptions on the weight function ρ, obtaining, among other results, the following inequality

$$c\int_M \frac{u^2}{\rho^2}\,dv_g \leqslant \int_M |\nabla u|^2\,dv_g, \quad u\in C_0^\infty(M), \tag{1.1}$$

where ρ is a nonnegative function such that $|\nabla\rho|=1$, $\Delta\rho\geqslant\frac{\gamma}{\rho}$, $\rho^{-1}\{0\}$ is a compact set of zero capacity and $c=(\frac{\gamma-1}{2})^2$. In [16] the author applies this result to several explicit examples of Riemannian manifolds. Under the same hypotheses on the function ρ, Kombe and Özaydin in [38] extend Carron's result to the case $p\neq 2$ for functions in $C_0^\infty(M\setminus\rho^{-1}\{0\})$, and the authors present an application to the punctured manifold $\mathbb{B}^n\setminus\{0\}$ with $\mathbb{B}^n$ the Poincaré ball model of the hyperbolic space and ρ the distance from the point 0 and $p=2$.

Li and Wang in [40] prove that if M is a hyperbolic manifold (i.e. there exists a symmetric positive Green function $G_x(\cdot)$ for the Laplacian with pole at x), then

$$\frac{1}{4}\int_M \frac{|\nabla_y G_x(y)|^2}{G_x^2(y)}u^2(y)\,dv_g \leqslant \int_M |\nabla u|^2(y)\,dv_g, \quad u\in C_0^\infty\big(M\setminus\{x\}\big).$$

We also mention Miklyukov and Vuorinen, which in [44] prove that the inequality

$$\left(\int_M \big|\alpha\big(\epsilon(x)\big)u(x)\big|^q\,dv_g\right)^{1/q} \leqslant \lambda\left(\int_M \big(\beta\big(\epsilon(x)\big)\big|\nabla u(x)\big|\big)^p\,dv_g\right)^{1/p}, \quad u\in W_0^{1,p}(M),$$

holds for $q\geqslant p$ provided some conditions related to the isoperimetric profile of M are satisfied.

In [1], Adimurthi and Sekar use the fundamental solution of a general second order elliptic operator to derive Hardy-type inequalities and then they extend their arguments to Riemannian manifolds using the fundamental solution of p-Laplacian.

Bozhkov and Mitidieri in [7] prove the validity of (1.1) also for $p\neq 2$ $(1<p<N)$, provided there exists on M a C^1 conformal Killing vector field K such that $\operatorname{div} K=\mu$ with μ a positive constant and $\rho=|K|$.

Let $p>1$ and let ρ be a nonnegative function. Our principal result is a simple criterion to establish if there holds a Hardy inequality involving the weight ρ. Namely, if ρ is p-superharmonic in Ω, that is $-\Delta_p\rho\geqslant 0$, then

$$c\int_M \frac{|u|^p}{\rho^p}|\nabla\rho|^p\,dv_g \leqslant \int_M |\nabla u|^p\,dv_g, \quad u\in C_0^\infty(M), \tag{1.2}$$

holds (see Theorem 2.1). Such a kind of criteria is already established in [20] for a quite general class of second order operators containing, among other examples, the subelliptic operators on Carnot groups. For this goal we shall mainly use a technique introduced by Mitidieri in [45] and developed in [18–20] and in [7,8]. The proof is based on the divergence theorem and on the careful choice of a vector field.

Let us point out some interesting outcomes of our approach. A first issue is that, since it is quite general, our approach includes Hardy inequalities already studied in [1,7,16,38,40] in the case $p=2$ as well as their generalization for $p>1$. Indeed, in all these cited papers, the authors assume extra conditions on the function ρ or on the manifold. Furthermore, in concrete cases, our result yields an explicit value of the constant c. Moreover, in several cases, this value is also the best constant (see [20]). To this regard, we discuss if the best constant is achieved or not and, in the latter case, we study the possibility to add a remainder term.

Another aspect of our technique is that it allows to characterize the p-hyperbolic manifolds. We remind that a manifold M is called *p-hyperbolic* if there exists a symmetric positive Green function $G_x(\cdot)$ for the p-Laplacian with pole at x. We prove that M is p-hyperbolic if and only if there exists a nonnegative nontrivial function $f \in L^1_{loc}(M)$ such that

$$\int_M f|u|^p \, dv_g \leqslant \int_M |\nabla u|^p \, dv_g, \quad u \in C_0^\infty(M).$$

Notice that one of the implications of this characterization for $p = 2$ is the result proved in [40]. During the review process of this work, we have received the paper of Devyver, Fraas and Pinchover [23]. In [23] a general linear second order differential operator P in the Euclidean framework is studied. The authors find a profound relation between the existence of positive supersolutions of $Pu = 0$, Hardy-type inequalities involving P and a weight W and the characterization of the spectrum of the weighted operator. We refer the interested reader to [23,24].

We also obtain a generalization of (1.2). Namely, for a nonnegative function ρ, the inequality

$$\left(\frac{|p-1-\alpha|}{p}\right)^p \int_M \frac{|u|^p}{\rho^p} \rho^\alpha |\nabla\rho|^p \, dv_g \leqslant \int_M |\nabla u|^p \rho^\alpha \, dv_g, \quad u \in C_0^\infty(M), \tag{1.3}$$

holds, provided $-(p-1-\alpha)\Delta_p \rho \geqslant 0$ (see Theorem 3.1). The above inequality contains, as special case, the Caccioppoli inequality. Indeed, if ρ is a p-subharmonic function, that is $\Delta_p\rho \geqslant 0$, then (1.3) holds for $\alpha > p-1$ and, in particular, for $\alpha = p$ we have

$$\frac{1}{p^p} \int_M |u|^p |\nabla\rho|^p \, dv_g \leqslant \int_M |\nabla u|^p \rho^p \, dv_g, \quad u \in C_0^\infty(M).$$

This is the so-called Caccioppoli inequality (see for instance [47] and the references therein for the version $p = 2$ on manifolds).

Another advantage of our approach is that it allows to obtain also other new and known results, like wighted Gagliardo–Nirenberg inequalities and the uncertain principle.

Finally we show that if (1.3) and a Sobolev-type inequality (that is $c|u|_{L^{p^*}} \leqslant |\nabla u|_{L^p}$) hold on M, then we obtain an interpolation inequality involving, as weights, ρ and its gradient. As particular case, our results contain inequalities on manifolds related to the celebrated Caffarelli–Kohn–Nirenberg inequality.

The paper is organized as follows. We present the proof of (1.2) in Section 2, where important consequences and observations are derived. In Section 3 we show natural extensions of (1.2), obtaining also Hardy inequality with weights, Caccioppoli-type inequalities, weighted Gagliardo–Nirenberg inequalities and the uncertain principle. Some remarks on the best constant and if it is attained are discussed in Section 4. In Section 5 we present a first order interpolation inequality. Finally Section 6 is devoted to present some concrete examples of Hardy-type inequalities on manifolds.

Notation. In what follows (M, g) is a complete Riemannian N-dimensional manifold, $\Omega \subset M$ is an open set, dv_g is the volume form associated to the metric g, ∇u and $\operatorname{div} h$ stand respectively for the gradient of a function u and the divergence of a vector field h with respect to the metric g (see [3] for further details). Throughout this paper $p > 1$.

2. Hardy inequalities

In order to state Hardy inequalities involving a weight ρ, the basic assumption we made on ρ is that ρ is p-superharmonic in weak sense. Namely, we assume that $\rho \in L^1_{loc}(\Omega)$, $|\nabla\rho| \in L^{p-1}_{loc}(\Omega)$, and $-\Delta_p\rho \geqslant 0$ on Ω in weak sense, that is for every nonnegative $\varphi \in C_0^1(\Omega)$, we have

$$\int_\Omega |\nabla\rho|^{p-2}(\nabla\rho \cdot \nabla\varphi) \, dv_g \geqslant 0. \tag{2.4}$$

The main result on Hardy inequalities is the following:

Theorem 2.1. *Let $\rho \in W^{1,p}_{loc}(\Omega)$ be a nonnegative function on Ω such that*

$$-\Delta_p \rho \geqslant 0 \quad \text{on } \Omega \text{ in weak sense.}$$

Then $\frac{|\nabla\rho|^p}{\rho^p} \in L^1_{loc}(\Omega)$, and the following inequality holds:

$$\left(\frac{p-1}{p}\right)^p \int\limits_\Omega \frac{|u|^p}{\rho^p}|\nabla\rho|^p \, dv_g \leqslant \int\limits_\Omega |\nabla u|^p \, dv_g, \quad u \in C_0^\infty(\Omega). \tag{2.5}$$

Before proving Theorem 2.1, we shall present some immediate consequences and extensions of the main result.

Definition 2.2. Let $\Omega \subset M$ be an open set. We denote by $D^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$|u|_{D^{1,p}} = \left(\int\limits_\Omega |\nabla u|^p \, dv_g\right)^{1/p}.$$

It is possible to extend the validity of (2.5) to function $u \in C_0^\infty(M)$. This extension is based on the inclusion

$$D^{1,p}(M) \subset D^{1,p}(\Omega). \tag{2.6}$$

The above inclusion is satisfied, for instance, when $M \setminus \Omega$ is a compact set of zero p-capacity (see Appendix A).

Corollary 2.3. *Let $\rho \in L^1_{loc}(M)$ be a function satisfying the assumptions of Theorem 2.1. If (2.6) holds, then $\frac{|\nabla\rho|^p}{\rho^p} \in L^1_{loc}(M)$, and the following inequality holds*:

$$\left(\frac{p-1}{p}\right)^p \int\limits_M \frac{|u|^p}{\rho^p}|\nabla\rho|^p \, dv_g \leqslant \int\limits_M |\nabla u|^p \, dv_g, \quad u \in C_0^\infty(M). \tag{2.7}$$

Proof. The inequality (2.5) holds for every $u \in C_0^\infty(\Omega)$, then it holds for every $u \in D^{1,p}(\Omega)$. Since $C_0^\infty(M) \subset D^{1,p}(M)$, by using (2.6) we conclude the proof. □

In order to illustrate further consequences of Theorem 2.1 we give the following:

Definition 2.4. A manifold M is said *p-hyperbolic*[1] if there exists a symmetric positive Green function $G_x(\cdot)$ for the p-Laplacian with pole at x,[2] if it is not the case we call it *p-parabolic*.

Several equivalent definitions of p-parabolic manifolds can be given. For instance in [48] there is the following (see also the literature therein and [35])

Proposition 2.5. *Let $p > 1$. The following statements are equivalent*:

a) *M is p-parabolic*;
b) *there exists a compact set $K \subset M$ with nonempty interior such that $cap_p(K, M) = 0$*;
c) *there is no nonconstant positive p-superharmonic function on M*;
d) *there exists a sequence of functions $u_j \in C_0^\infty(M)$ such that $0 \leqslant u_j \leqslant 1$, $u_j \to 1$ uniformly on every compact subset of M and $\int_M |\nabla u_j|^p \, dv_g \to 0$.*

[1] Many authors call these manifolds non-p-parabolic.
[2] That is $-\Delta_p G_x = \delta_x$ where δ_x is the Dirac measure concentrated at point x.

Other characterizations of p-parabolic manifolds are based on several properties, for instance on the volume growth, on the isoperimetric profile of the manifold, on some properties of some cohomology, on the recurrence of the Brownian motion. See [28–31,39,48] and the references therein.

From Theorem 2.1 we deduce the following characterization of p-hyperbolicity.

Theorem 2.6. *A manifold M is p-hyperbolic if and only if there exists a nonnegative nontrivial function $f \in L^1_{loc}(M)$ such that*

$$\int_M f|u|^p\,dv_g \leqslant \int_M |\nabla u|^p\,dv_g, \qquad u \in C_0^\infty(M). \tag{2.8}$$

Proof. If M is p-hyperbolic then inequality (2.8) holds with $f = (\frac{p-1}{p})^p \frac{|\nabla G_x|^p}{G_x^p}$. Indeed G_x is nonnegative and satisfies the hypotheses of Theorem 2.1 (see Theorem 6.4 for further details).

Conversely, assume that M is p-parabolic and that (2.8) is valid for a function $f \geqslant 0$. Then from d) of Proposition 2.5 there exists a sequence of functions $u_j \in C_0^1(M)$ such that $0 \leqslant u_j \leqslant 1$, $u_j \to 1$ uniformly on every compact subset of M and

$$\int_M |\nabla u_j|^p\,dv_g \to 0 \quad (\text{as } j \to +\infty).$$

It implies that $\int_D f\,dv_g = 0$ for every compact subset D of M and then $f \equiv 0$. This concludes the proof. □

Remark 2.7. Since the p-hyperbolicity of M is equivalent to the existence of a nonconstant positive p-superharmonic function ρ on M, then by Theorem 2.1 we obtain that inequality (2.8) holds with $f = (\frac{p-1}{p})^p \frac{|\nabla\rho|^p}{\rho^p}$.

Remark 2.8. Our Theorem 2.6 implies that if the manifold M admits a C^1 conformal Killing vector field K (see i.e. [7] for the definition) such that $\operatorname{div} K = \mu \neq 0$ with μ constant and $|K|^{-p} \in L^1_{loc}(M)$, then M is p-hyperbolic. This follows combining Theorem 2.6 and Theorem 4 of [7] (see also Remark 2.11 ii) below).

In order to prove Theorem 2.1 we fix some notation. Let $h \in L^1_{loc}(\Omega)$ be a vector field. We remind that the distribution $\operatorname{div} h$ is defined as

$$\int_\Omega \varphi \operatorname{div} h\,dv_g = -\int_\Omega (\nabla\varphi \cdot h)\,dv_g, \tag{2.9}$$

for every $\varphi \in C_0^1(\Omega)$.

Let $h \in L^1_{loc}(\Omega)$ be a vector field and let $A \in L^1_{loc}(\Omega)$ be a function. In what follows we write $A \leqslant \operatorname{div} h$ meaning that the inequality holds in distributional sense, that is for every $\varphi \in C_0^1(\Omega)$ such that $\varphi \geqslant 0$, we have

$$\int_\Omega \varphi A\,dv_g \leqslant \int_\Omega \varphi \operatorname{div} h\,dv_g = -\int_\Omega (\nabla\varphi \cdot h)\,dv_g. \tag{2.10}$$

Remark 2.9. Let $f \in C^1(\mathbb{R})$ be a real function such that $f(0) = 0$. Taking $\varphi = f(u)$ with $u \in C_0^1(\Omega)$ in (2.9), we have

$$\int_\Omega f(u) \operatorname{div} h\,dv_g = -\int_\Omega f'(u)(\nabla u \cdot h)\,dv_g.$$

In particular, choosing $f(u) = |u|^p$, with $p > 1$, we get

$$\int_\Omega |u|^p \operatorname{div} h\,dv_g = -p\int_\Omega |u|^{p-2}u(\nabla u \cdot h)\,dv_g, \qquad u \in C_0^1(\Omega). \tag{2.11}$$

Lemma 2.10. *Let $h \in L^1_{loc}(\Omega)$ be a vector field and let $A_h \in L^1_{loc}(\Omega)$ be a nonnegative function such that*

i) $A_h \leqslant \operatorname{div} h$,
ii) $\frac{|h|^p}{A_h^{p-1}} \in L^1_{loc}(\Omega)$.

Then for every $u \in C^1_0(\Omega)$ we have

$$\int_\Omega |u|^p A_h \, dv_g \leqslant p^p \int_\Omega \frac{|h|^p}{A_h^{p-1}} |\nabla u|^p \, dv_g. \tag{2.12}$$

Proof. We note that the right hand side of (2.12) is finite since $u \in C^1_0(\Omega)$. Using the identity (2.11) and the Hölder inequality we obtain

$$\begin{aligned}\int_\Omega |u|^p A_h \, dv_g &\leqslant \int_\Omega |u|^p \operatorname{div} h \, dv_g \\ &\leqslant p \int_\Omega |u|^{p-1} |h| |\nabla u| \, dv_g \\ &= p \int_\Omega |u|^{p-1} A_h^{(p-1)/p} \frac{|h|}{A_h^{(p-1)/p}} |\nabla u| \, dv_g \\ &\leqslant p \left(\int_\Omega |u|^p A_h \, dv_g \right)^{(p-1)/p} \left(\int_\Omega \frac{|h|^p}{A_h^{p-1}} |\nabla u|^p \, dv_g \right)^{1/p}.\end{aligned}$$

This completes the proof. □

Specializing the vector field h and the function A_h, we shall deduce from (2.12) Hardy-type inequalities on Riemannian manifolds.

Remark 2.11. Letting us to point out a strategy to get Hardy inequalities at least in some special cases. Under the hypotheses of Lemma 2.10, if $A_h = \operatorname{div} h$, then (2.12) reads as

$$\int_\Omega |u|^p \operatorname{div} h \, dv_g \leqslant p^p \int_\Omega \frac{|h|^p}{|\operatorname{div} h|^{p-1}} |\nabla u|^p \, dv_g. \tag{2.13}$$

i) Let V be a function in $L^1_{loc}(\Omega)$ such that its weak partial derivatives of order up to two are in $L^1_{loc}(\Omega)$. If $\Delta V \geqslant 0$, choosing $h = \nabla V$, we obtain $A_h = \operatorname{div} h = \operatorname{div}(\nabla V) = \Delta V \geqslant 0$. Then from (2.13)

$$\int_\Omega |u|^p |\Delta V| \, dv_g \leqslant p^p \int_\Omega \frac{|\nabla V|^p}{|\Delta V|^{p-1}} |\nabla u|^p \, dv_g. \tag{2.14}$$

This kind of inequalities for the Euclidean setting $\Omega = \mathbb{R}^N$ are already found by Davies and Hinz in [21]. At this point, in order to deduce from (2.14) an inequality like

$$c \int_\Omega \frac{|u|^p}{\rho^p} \, dv_g \leqslant \int_\Omega |\nabla u|^p \, dv_g, \tag{2.15}$$

we have to choose a suitable function V. Let us consider the case when ρ is the distance from a point $o \in M$ and $|\nabla \rho| = 1$. A suitable choice for V is $V = \rho^{2-p}$ if $1 < p < 2$, $V = \ln \rho$ if $p = 2$ and $V = -\rho^{2-p}$ if $2 < p < N$. Following [16], if we require that $\Delta \rho \geqslant \frac{\gamma}{\rho}$ the above choices yield the inequality (2.15) with $c = (\frac{\gamma - p + 1}{p})^p$. The success of this strategy is deeply linked to the hypothesis $|\nabla \rho| = 1$. Indeed, it seems that such a strategy does not work even in the subelliptic setting, where the analogous of the hypothesis $|\nabla \rho| = 1$ does not hold.

Furthermore, the fact that the hypothesis $|\nabla\rho| = 1$ is sometimes restrictive even in the Euclidean case, can be seen in the following example. In the Euclidean unit ball $B_1 \subset \mathbb{R}^N$ the inequality

$$c \int_{B_1} \frac{|u|^p}{||x|\ln|x||^p}\,dx \leqslant \int_{B_1} |\nabla u|^p\,dx \tag{2.16}$$

holds for $1 < p \leqslant N$ (see Section 6.3, Section 6.6 and [20]). If we wish to deduce (2.16) from (2.15) we are forced to choose $\rho = -|x|\ln|x|$. However $|\nabla\rho| \neq 1$.

ii) Let $p < N$. Assume that there exists a C^1 conformal Killing vector field K (see i.e. [7] for the definition) such that $\operatorname{div} K = \frac{N}{2}\mu > 0$. Choosing $h = \frac{K}{|K|^p}$, we have $A_h := \operatorname{div} h = \frac{N-p}{2}\frac{\mu}{|K|^p}$ (see Lemma 3 in [7]) and the inequality (2.13) reads as

$$\left(\frac{N-p}{Np}\right)^p \int_\Omega \frac{\operatorname{div} K}{|K|^p}|u|^p\,dv_g \leqslant \int_\Omega (\operatorname{div} K)^{1-p}|\nabla u|^p\,dv_g. \tag{2.17}$$

Therefore, by Lemma 2.10, (2.17) holds for every $u \in C_0^1(\Omega)$ provided $|K|^{-p} \in L^1_{loc}(\Omega)$. This last fact was obtained in [7, Theorem 4].

Proof of Theorem 2.1. Let $0 < \delta < 1$, and $\rho_\delta := \rho + \delta$. In order to apply Lemma 2.10, we define h and A_h as

$$h := -\frac{|\nabla\rho_\delta|^{p-2}\nabla\rho_\delta}{\rho_\delta^{p-1}} \quad \text{and} \quad A_h := (p-1)\frac{|\nabla\rho_\delta|^p}{\rho_\delta^p}. \tag{2.18}$$

Since $\frac{1}{\rho_\delta} \leqslant \frac{1}{\delta}$, the fact that $\rho \in W^{1,p}_{loc}(\Omega)$ implies that $h \in L^1_{loc}(\Omega)$ and $A_h \in L^1_{loc}(\Omega)$. Moreover, by computation we have

$$\frac{|h|^p}{A_h^{p-1}} = \frac{|\nabla\rho_\delta|^{p(p-1)}}{{\rho_\delta}^{p(p-1)}}\frac{{\rho_\delta}^{p(p-1)}}{|\nabla\rho_\delta|^{p(p-1)}}\frac{1}{(p-1)^{(p-1)}} = \frac{1}{(p-1)^{p-1}} \in L^1_{loc}(\Omega),$$

that is ii) of Lemma 2.10 is fulfilled.

The hypothesis i) of Lemma 2.10 is satisfied provided

$$(p-1)\int_\Omega \frac{|\nabla\rho_\delta|^p}{\rho_\delta^p}\varphi\,dv_g \leqslant \int_\Omega \left(\frac{|\nabla\rho_\delta|^{p-2}\nabla\rho_\delta}{\rho_\delta^{p-1}}\cdot\nabla\varphi\right)dv_g \tag{2.19}$$

holds for every nonnegative function $\varphi \in C_0^1(\Omega)$. Then, for a fixed $\varphi \in C_0^1(\Omega)$ nonnegative, we have to prove (2.19). Let $K = supp\,\varphi \subset \Omega$ and let U be a neighborhood of K with compact closure in Ω. We note that both integrals in (2.19) are finite since $\frac{1}{\rho_\delta} \leqslant \frac{1}{\delta}$ and $\rho \in W^{1,p}_{loc}(\Omega)$. Since

$$|\nabla \ln\rho_\delta| = \frac{|\nabla\rho_\delta|}{\rho_\delta} \leqslant \frac{|\nabla\rho|}{\delta} \in L^p_{loc}(\Omega), \tag{2.20}$$

and $\ln\rho_\delta \in L^p_{loc}(\Omega)$, we have that $\ln\rho_\delta \in W^{1,p}(U)$. Thus, for every $n \in \mathbb{N}$ there exists $\phi_n \in C^\infty(U)$ such that $|\phi_n - \ln\rho_\delta|_{W^{1,p}} < 1/n$, $\phi_n \to \ln\rho_\delta$ pointwise a.e. and $\ln\delta \leqslant \phi_n$.[3]

[3] Reminding that the Sobolev space $W^{1,p}(\Omega)$ is the completion of the set

$$\left\{u \in C^\infty(\Omega)\colon \int_\Omega |u|^p\,dv_g < \infty \text{ and } \int_\Omega |\nabla u|^p\,dv_g < \infty\right\}$$

with respect to the norm

$$|u|_{W^{1,p}} = \left(\int_\Omega |u|^p\,dv_g + \int_\Omega |\nabla u|^p\,dv_g\right)^{1/p},$$

the approximation result follows by slight modification of classical arguments that the reader can find, for instance, in [3].

Setting $\psi_n = e^{\phi_n}$ we have that $\psi_n \in C^\infty(U)$, $\delta \leqslant \psi_n$, $\psi_n \to \rho_\delta$ a.e. and

$$\int_K |\ln \psi_n - \ln \rho_\delta|^p \, dv_g \to 0, \qquad \int_K \left| \frac{\nabla \psi_n}{\psi_n} - \frac{\nabla \rho_\delta}{\rho_\delta} \right|^p dv_g \to 0 \quad (\text{as } n \to +\infty). \tag{2.21}$$

For every $n \in \mathbb{N}$, the function φ_n defined as $\varphi_n := \frac{\varphi}{\psi_n^{p-1}}$ belongs to $C_0^1(\Omega)$ and it is nonnegative since $\varphi \in C_0^1(\Omega)$ is nonnegative and $\psi_n > 0$. Using φ_n as a test function in (2.4) we have

$$0 \leqslant \int_\Omega |\nabla \rho|^{p-2} (\nabla \rho \cdot \nabla \varphi_n) \, dv_g = \int_\Omega |\nabla \rho|^{p-2} \left(\nabla \rho \cdot \nabla \left(\frac{\varphi}{\psi_n^{p-1}} \right) \right) dv_g, \tag{2.22}$$

which, since by computation $\nabla(\frac{\varphi}{\psi_n^{p-1}}) = \frac{\nabla \varphi}{\psi_n^{p-1}} - (p-1)\frac{\nabla \psi_n}{\psi_n^p}\varphi$, implies

$$(p-1) \int_\Omega \frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \psi_n}{\psi_n^p} \varphi \, dv_g \leqslant \int_\Omega \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \cdot \nabla \varphi \right) dv_g. \tag{2.23}$$

Now, letting $n \to +\infty$ we obtain by dominated convergence:

$$\int_\Omega \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \cdot \nabla \varphi \right) dv_g \to \int_\Omega \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_\delta^{p-1}} \cdot \nabla \varphi \right) dv_g,$$

because $|\frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \varphi}{\psi_n^{p-1}}| \leqslant \frac{C|\nabla \rho|^{p-1}}{\delta^{p-1}} \in L^1(U)$. Now we claim that

$$\int_\Omega \frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \psi_n}{\psi_n^p} \varphi \, dv_g = \int_\Omega \frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \frac{\nabla \psi_n}{\psi_n} \varphi \, dv_g \to \int_\Omega \frac{|\nabla \rho|^p}{\rho_\delta^p} \varphi \, dv_g.$$

Indeed,

$$\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \to \frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_\delta^{p-1}} \quad \text{pointwise a.e.}$$

and, since $\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \leqslant \frac{|\nabla \rho|^{p-1}}{\delta^{p-1}} \in L^{p'}(U)$, by Lebesgue dominated convergence theorem we have that

$$\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \to \frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_\delta^{p-1}} \quad \text{in } L^{p'}(U).$$

From this and the fact that

$$\frac{\nabla \psi_n}{\psi_n} \to \frac{\nabla \rho}{\rho_\delta} \quad \text{in } L^p(U)$$

we get the claim. Therefore, letting $n \to +\infty$ in (2.23), we have

$$(p-1) \int_\Omega \frac{|\nabla \rho|^p}{\rho_\delta^p} \varphi \, dv_g \leqslant \int_\Omega \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_\delta^{p-1}} \cdot \nabla \varphi \right) dv_g,$$

which is exactly (2.19), since $\nabla \rho_\delta = \nabla \rho$.

An application of Lemma 2.10 gives

$$\left(\frac{p-1}{p} \right)^p \int_\Omega \frac{|u|^p}{\rho_\delta^p} |\nabla \rho_\delta|^p \, dv_g \leqslant \int_\Omega |\nabla u|^p \, dv_g. \tag{2.24}$$

Finally, letting $\delta \to 0$ in (2.24) and using Fatou's Lemma, we conclude the proof. $\square$

3. Further inequalities

In this section we shall present some slight but natural extensions of Theorem 2.1 and Lemma 2.10. As byproducts of these generalizations we shall obtain Hardy inequalities with a weight in the right hand side, Caccioppoli-type inequalities, weighted Gagliardo–Nirenberg inequalities and the uncertain principle.

A first example of a possible generalization of Theorem 2.1 is the following:

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, and let $\rho \in W^{1,p}_{loc}(\Omega)$ be a nonnegative function satisfying the following properties*:

i) $-(p-1-\alpha)\Delta_p \rho \geqslant 0$ *on Ω in weak sense,*
ii) $\frac{|\nabla\rho|^p}{\rho^{p-\alpha}}, \rho^\alpha \in L^1_{loc}(\Omega)$.

Then the following Hardy inequality holds

$$\left(\frac{|p-1-\alpha|}{p}\right)^p \int_\Omega \rho^\alpha \frac{|u|^p}{\rho^p}|\nabla\rho|^p\,dv_g \leqslant \int_\Omega \rho^\alpha |\nabla u|^p\,dv_g, \quad u \in C_0^\infty(\Omega). \tag{3.25}$$

The proof of the above theorem is similar to the one of Theorem 2.1 and it is based on a careful choice of the vector field h and of the function A_h in Lemma 2.10.

Proof of Theorem 3.1. Let $0<\delta<1$, and $\rho_\delta := \rho+\delta$. In order to apply Lemma 2.10 we choose the vector field h and the function A_h as

$$h := -(p-1-\alpha)\frac{|\nabla\rho_\delta|^{p-2}\nabla\rho_\delta}{\rho_\delta^{p-1-\alpha}}, \qquad A_h := (p-1-\alpha)^2\frac{|\nabla\rho_\delta|^p}{\rho_\delta^{p-\alpha}}. \tag{3.26}$$

Arguing as in the proof of Theorem 2.1, we have to show that

$$(p-1-\alpha)^2\int_\Omega \frac{|\nabla\rho_\delta|^p}{\rho_\delta^{p-\alpha}}\varphi\,dv_g \leqslant (p-1-\alpha)\int_\Omega\left(\frac{|\nabla\rho_\delta|^{p-2}\nabla\rho_\delta}{\rho_\delta^{p-1-\alpha}}\cdot\nabla\varphi\right)dv_g, \tag{3.27}$$

for every nonnegative function $\varphi \in C^1_0(\Omega)$. Let $K := supp\,\varphi \subset \Omega$ and let $U \subset\subset \Omega$ be a neighborhood of K. Let $k>\delta$, and define $\rho_{k\delta} := \inf\{\rho_\delta, k\}$. Arguing as in the proof of Theorem 2.1, we have that there exists a sequence $\{\psi_n\}$ such that $\delta \leqslant \psi_n \leqslant k$, and

$$\int_K |\ln\psi_n - \ln\rho_{k\delta}|^p\,dv_g \to 0, \qquad \int_K\left|\frac{\nabla\psi_n}{\psi_n}-\frac{\nabla\rho_{k\delta}}{\rho_{k\delta}}\right|^p dv_g \to 0 \quad (\text{as } n\to+\infty). \tag{3.28}$$

Then we use $\varphi_n := \frac{\varphi}{\psi_n^{p-1-\alpha}}$ as a test function in the hypothesis i), obtaining

$$(p-1-\alpha)^2\int_\Omega \frac{|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\psi_n}{\psi_n^{p-\alpha}}\varphi\,dv_g \leqslant (p-1-\alpha)\int_\Omega\left(\frac{|\nabla\rho|^{p-2}\nabla\rho}{\psi_n^{p-1-\alpha}}\cdot\nabla\varphi\right)dv_g. \tag{3.29}$$

In the case $\alpha < p-1$ we obtain (3.27) from (3.29) by slight modifications of the proof of Theorem 2.1, so we will omit the proof.

Let $\alpha > p-1$. We claim that, letting $n\to+\infty$ in (3.29), and eventually taking a subsequence, we get

$$(p-1-\alpha)^2\int_\Omega \frac{|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\rho_{k\delta}}{\rho_{k\delta}^{p-\alpha}}\varphi\,dv_g \leqslant (p-1-\alpha)\int_\Omega\left(\frac{|\nabla\rho|^{p-2}\nabla\rho}{\rho_{k\delta}^{p-1-\alpha}}\cdot\nabla\varphi\right)dv_g. \tag{3.30}$$

In fact, for the right hand side the limit follows by dominated convergence, since

$$\left|\frac{|\nabla\rho|^{p-2}\nabla\rho}{\psi_n^{p-1-\alpha}}\cdot\nabla\varphi\right| = |\nabla\rho|^{p-2}|\nabla\rho\cdot\nabla\varphi|\psi_n^{\alpha-p+1} \leqslant C|\nabla\rho|^{p-1}k^{\alpha-p+1} \in L^1(U).$$

Dealing with the left hand side of (3.29), we set

$$\frac{|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\psi_n}{\psi_n^{p-\alpha}}=\frac{|\nabla\rho|^{p-2}\nabla\rho}{\psi_n^{p-1}}\cdot\frac{\nabla\psi_n}{\psi_n},\psi_n^\alpha=:f_n\cdot g_n.$$

As in the proof of Theorem 2.1, we have

$$f_n\to\frac{|\nabla\rho|^{p-2}\nabla\rho}{\rho_{k\delta}^{p-1}}\quad\text{in } L^{p'}(U),\tag{3.31}$$

while from the relations

$$|g_n|\leqslant\left|\frac{\nabla\psi_n}{\psi_n}\right|\cdot k^\alpha\to\left|\frac{\nabla\rho_{k\delta}}{\rho_{k\delta}}\right|\cdot k^\alpha\quad\text{in } L^p(U)$$

we obtain that the sequence g_n is bounded in $L^p(U)$. Therefore, up to a subsequence, g_n is weakly convergent in $L^p(U)$. Since

$$g_n\to\frac{\nabla\rho_{k\delta}}{\rho_{k\delta}}\cdot\rho_{k\delta}^\alpha\quad\text{pointwise a.e.},$$

we have that the convergence is in the weak sense. This fact with (3.31) concludes the claim.

Next step is letting $k\to+\infty$ in (3.30). Let us rewrite the integrand in the right hand side as

$$\left|\frac{|\nabla\rho|^{p-2}\nabla\rho}{\rho_{k\delta}^{p-1-\alpha}}\cdot\nabla\varphi\right|=\left||\nabla\rho|^{p-2}\nabla\rho(\rho_{k\delta})^{\frac{\alpha-p}{p'}}(\rho_{k\delta})^{\frac{\alpha-p}{p}+1}\cdot\nabla\varphi\right|\leqslant C|\nabla\rho|^{p-1}(\rho_\delta)^{\frac{\alpha-p}{p'}}(\rho_\delta)^{\frac{\alpha}{p}},$$

which is in $L^1(U)$, since $C|\nabla\rho|^{p-1}(\rho_\delta)^{\frac{\alpha-p}{p'}}\in L^{p'}(U)$ and $(\rho_\delta)^{\frac{\alpha}{p}}\in L^p(U)$ by hypothesis ii). Thus we can use the dominated convergence to obtain the limit for the right hand side of (3.30).

In order to pass to the limit for $k\to+\infty$ in the left hand side of (3.30), we rewrite the integrand as

$$\frac{|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\rho_{k\delta}}{\rho_{k\delta}^{p-\alpha}}\varphi=\frac{|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\rho_\delta}{\rho_{k\delta}^{p-\alpha}}\chi_{\{\rho_\delta\leqslant k\}}\varphi=\frac{|\nabla\rho|^p}{\rho_{k\delta}^{p-\alpha}}\chi_{\{\rho_\delta\leqslant k\}}\varphi,\tag{3.32}$$

where we have used the fact that $\nabla\rho_\delta=\nabla\rho$. Now, if $\alpha\leqslant p$ we apply the dominated convergence, since the term in (3.32) is dominated by the function $C\frac{|\nabla\rho|^p}{\delta^{p-\alpha}}\in L^1(U)$. Whereas, if $\alpha>p$ we use the monotone convergence, since $|\nabla\rho|^p\rho_{k\delta}^{\alpha-p}\chi_{\{|\rho_\delta|\leqslant k\}}\varphi$ is an increasing sequence of nonnegative functions.

Thus, letting $k\to+\infty$ in (3.30), we get

$$(p-1-\alpha)^2\int_\Omega\frac{|\nabla\rho|^p}{\rho_\delta^{p-\alpha}}\varphi\,dv_g\leqslant(p-1-\alpha)\int_\Omega\left(\frac{|\nabla\rho|^{p-2}\nabla\rho}{\rho_\delta^{p-1-\alpha}}\cdot\nabla\varphi\right)dv_g,\tag{3.33}$$

which is exactly (3.27), since $\nabla\rho_\delta=\nabla\rho$.

As in Theorem 2.1, an application of Lemma 2.10 gives

$$\left(\frac{\alpha-p+1}{p}\right)^p\int_\Omega\rho_\delta^\alpha\frac{|u|^p}{\rho_\delta^p}|\nabla\rho_\delta|^p\,dv_g\leqslant\int_\Omega\rho_\delta^\alpha|\nabla u|^p\,dv_g.\tag{3.34}$$

Finally, letting $\delta\to0$ in (3.34), we conclude the proof. Indeed we can use the dominated convergence for the right hand side, since $\rho_\delta^\alpha|\nabla u|^p\leqslant C(\rho+\delta)^\alpha\leqslant C(\rho+1)^\alpha\in L^1_{loc}(U)$, and apply Fatou's Lemma for the left hand side. □

Remark 3.2. If $\alpha\leqslant p$, then the hypothesis ii) in the above Theorem 3.1, can be avoided. Indeed, since $\rho\in W^{1,p}_{loc}(\Omega)$ we get that $\rho^\alpha\in L^1_{loc}(\Omega)$ and from the proof it follows that $\frac{|\nabla\rho|^p}{\rho^{p-\alpha}}\in L^1_{loc}(\Omega)$.

As a consequence of Theorem 3.1 (it suffices to take $\alpha=p+q$), we obtain the following Caccioppoli-type inequality for p-subharmonic functions, which is worth of mention:

Corollary 3.3 *(L^p-Caccioppoli-type inequality). Let $\rho \in L^1_{loc}(M)$ and $q > -1$. Assume that ρ is nonnegative on an open set $\Omega \subset M$, and $\rho \in W^{1,p}_{loc}(\Omega)$, $\rho^q|\nabla\rho|^p, \rho^{p+q} \in L^1_{loc}(\Omega)$. If $\Delta_p\rho \geqslant 0$ on Ω in weak sense, then we have*

$$\left(\frac{q+1}{p}\right)^p \int_\Omega \rho^q|\nabla\rho|^p|u|^p\,dv_g \leqslant \int_\Omega \rho^{p+q}|\nabla u|^p\,dv_g, \quad u \in C_0^\infty(\Omega). \tag{3.35}$$

Notice that for $q = 0$ and $p = 2$ the above theorem is a version of the classical Caccioppoli inequality on manifolds. See also [47] for a version of Caccioppoli inequality related to subharmonic functions on manifolds.

Now we present a possible generalization of Lemma 2.10, and some of its consequences, like the weighted Gagliardo–Nirenberg inequality and the uncertain principle on manifolds.

Lemma 3.4. *Let $h \in L^1_{loc}(\Omega)$ be a vector field and let $A_h \in L^1_{loc}(\Omega)$ be a nonnegative function such that*

i) $A_h \leqslant \operatorname{div} h$,
ii) $\frac{|h|^p}{A_h^{p-1}} \in L^1_{loc}(\Omega)$.

Then for every $u \in C^1_0(\Omega)$, $q \in \mathbb{R}$, $s > 0$ and $a > 1$ we have

$$\int_\Omega |u|^s|h|^q\,dv_g \leqslant p^{p/a}\left(\int_\Omega \frac{|h|^p}{A_h^{p-1}}|\nabla u|^p\,dv_g\right)^{1/a}\left(\int_\Omega \frac{|h|^{qa'}}{A_h^{a'-1}}|u|^{\frac{as-p}{a-1}}\,dv_g\right)^{1/a'}, \tag{3.36}$$

provided $|h|^q \in L^1_{loc}(\Omega)$.

In particular, setting $w := |h|A_h^{\frac{1-p}{p}}$, we have

1.
$$\left(\int_\Omega |u|^s|h|^q\,dv_g\right)^{1/s} \leqslant p^{q(p-1)/s}\left(\int_\Omega w^p|\nabla u|^p\,dv_g\right)^{b/p}\left(\int_\Omega w^{t\delta}|u|^\delta\,dv_g\right)^{(1-b)/\delta}, \tag{3.37}$$

where $t, \delta > 0$ and

$$\frac{1}{s} = \frac{b}{p} + \frac{1-b}{\delta}, \qquad \frac{1}{q} = \frac{1}{p'} + \frac{1}{t\delta}, \qquad b = \frac{t(p-1)}{1+t(p-1)}.$$

2.
$$\int_\Omega |u|^s\,dv_g \leqslant p^{p/a}\left(\int_\Omega w^p|\nabla u|^p\,dv_g\right)^{1/a}\left(\int_\Omega \frac{1}{A_h^{a'-1}}|u|^{\frac{as-p}{a-1}}\,dv_g\right)^{1/a'}, \tag{3.38}$$

where $s > 0$ and $a > 1$.

Proof. By Hölder inequality with exponent a we have

$$\int_\Omega |u|^s|h|^q\,dv_g = \int_\Omega |u|^{p/a}A_h^{1/a}|h|^qA_h^{-1/a}|u|^{s-p/a}\,dv_g$$
$$\leqslant \left(\int_\Omega |u|^pA_h\,dv_g\right)^{1/a}\left(\int_\Omega |h|^{qa'}A_h^{-a'/a}|u|^{\frac{as-p}{a-1}}\,dv_g\right)^{1/a'},$$

which by using (2.12), implies (3.36).

From (3.36) we get (3.37) by choosing $a = 1 + \frac{p'}{t\delta}$, and (3.38) by choosing $q = 0$. □

Specializing h and A_h we obtain from (3.37) and (3.38) a weighted Gagliardo–Nirenberg inequality and an uncertain principle respectively. In particular, choosing h and A_h as in (2.18), we have the following

Theorem 3.5. *Let $\rho \in W^{1,p}_{loc}(\Omega)$ be nonnegative. Assume that ρ is p-superharmonic function on $\Omega \subset M$ and satisfies the hypotheses of Theorem 2.1. Let $\delta > 0$ and $0 \leqslant b \leqslant 1$. Then for every $u \in C^1_0(\Omega)$, we have*

$$\left(\int_\Omega |u|^s \frac{|\nabla\rho|^{q(p-1)}}{\rho^{q(p-1)}}\, dv_g\right)^{1/s} \leqslant \left(\frac{p}{p-1}\right)^{q(p-1)/s} \left(\int_\Omega |\nabla u|^p\, dv_g\right)^{b/p} \left(\int_\Omega |u|^\delta\, dv_g\right)^{(1-b)/\delta}, \tag{3.39}$$

where

$$\frac{1}{s} = \frac{b}{p} + \frac{1-b}{\delta}, \qquad \frac{1}{q(p-1)} = \frac{1}{p} + \frac{1-b}{b\delta}.$$

In particular, if $\rho = d^\alpha$ for some $\alpha \neq 0$ with $|\nabla d| = 1$, then we have

$$\int_\Omega \frac{|u|^s}{d^{p-1}}\, dv_g \leqslant \left(\frac{p}{|\alpha|(p-1)}\right)^{(p-1)} \left(\int_\Omega |\nabla u|^p\, dv_g\right)^{1/p'} \left(\int_\Omega |u|^\delta\, dv_g\right)^{1/p}, \tag{3.40}$$

where $s = p - 1 + \frac{\delta}{p}$.

Notice that for $s = p = 2$ the inequality (3.40) is the weighted Gagliardo–Nirenberg inequality on manifold. Its counterpart in Euclidean setting is largely studied by many authors, see for instance [25]. Further examples of manifolds and functions ρ satisfying the hypotheses of the above theorem are given in Section 6.

Theorem 3.6. *Let $\rho \in W^{1,p}_{loc}(\Omega)$ be nonnegative. Assume that ρ is p-superharmonic function on $\Omega \subset M$ and satisfies the hypotheses of Theorem 2.1. Let $s > 0$ and $a > 1$. Then for every $u \in C^1_0(\Omega)$, we have*

$$\int_\Omega |u|^s\, dv_g \leqslant \left(\frac{p}{p-1}\right)^{p/a} \left(\int_\Omega |\nabla u|^p\, dv_g\right)^{1/a} \left(\int_\Omega |u|^{\frac{as-p}{a-1}} \frac{\rho^{p(a'-1)}}{|\nabla\rho|^{p(a'-1)}}\, dv_g\right)^{1/a'}. \tag{3.41}$$

In particular, if $\rho = d^\alpha$ for some $\alpha \neq 0$ with $|\nabla d| = 1$, then we have

$$\int_\Omega |u|^s\, dv_g \leqslant \left(\frac{p}{|\alpha|(p-1)}\right)^{p/a} \left(\int_\Omega |\nabla u|^p\, dv_g\right)^{1/a} \left(\int_\Omega |u|^{\frac{as-p}{a-1}} d^{p(a'-1)}\, dv_g\right)^{1/a'}. \tag{3.42}$$

Notice that if $a = s = p = 2$ the inequality (3.42) in the Euclidean setting coincides with the celebrated uncertain principle with $d = |x|$, the Euclidean norm.

Remark 3.7. Different choices of the vector field h and of the function A_h in Lemma 3.4 produce inequalities different than (3.39)–(3.42). For instance, one can define h and A_h as in (3.26), obtaining a version of (3.39)–(3.42) with further weights.

To end this section, we want to point out that it is possible to extend all the results of this paper considering vector fields of the type $\nabla_\mu u := \mu(\nabla u)$, where μ is a $(1,1)$-tensor (say C^1). In this case, replacing ∇ with ∇_μ, a Hardy-type inequality like (2.5) holds provided $\nabla^*_\mu(|\nabla_\mu u|^{p-2}\nabla_\mu u) \geqslant 0$, where ∇^*_μ stands for the adjoint of ∇_μ. We leave the details to the interested reader. Notice that the study of Hardy inequalities for the vector field ∇_μ was already studied in [20], when the support of the manifold is $\mathbb{R}^N$.

4. Remarks on the best constant

Theorems 2.1 and 3.1 affirm the validity of some Hardy inequalities with an explicit value of the constants involved. In many cases these constants, $(\frac{p-1}{p})^p$ and $(\frac{|p-1-\alpha|}{p})^p$, result to be sharp. For example in [20] the author proves the sharpness of the constant $(\frac{|p-1-\alpha|}{p})^p$ involved in the inequality of Theorem 3.1 in several cases. Moreover the question of the existence of functions that realize the best constant is analyzed in many papers (for the Euclidean case see for instance [10,21,41–43]). On the other hand, the knowledge of the best constants for the inequalities plays a crucial role in [2,14,50].

For the sake of simplicity, we shall focus our attention on the inequality (2.5). We denote by $c(\Omega)$ the best constant in (2.5), namely

$$c(\Omega) := \inf_{u\in D^{1,p}(\Omega),u\neq 0} \frac{\int_\Omega |\nabla u|^p\, dv_g}{\int_\Omega \frac{|u|^p}{\rho^p}|\nabla\rho|^p\, dv_g}. \tag{4.43}$$

Then, we have the following:

Theorem 4.1. *Under the same hypotheses of Theorem 2.1 we have:*

1) *If $\rho^{\frac{p-1}{p}} \in D^{1,p}(\Omega)$, then $c(\Omega) = (\frac{p-1}{p})^p$ and $\rho^{\frac{p-1}{p}}$ is a minimizer.*
2) *If $\rho^{\frac{p-1}{p}} \notin D^{1,p}(\Omega)$, $p \geqslant 2$ and $c(\Omega) = (\frac{p-1}{p})^p$, then the best constant $c(\Omega)$ is not achieved.*

Proof. 1) From (2.5) we have $c(\Omega) \geqslant (\frac{p-1}{p})^p$. Moreover, if $\rho^{\frac{p-1}{p}} \in D^{1,p}(\Omega)$, by computation

$$\begin{aligned}\int_\Omega \left|\nabla\rho^{\frac{p-1}{p}}\right|^p dv_g &= \int_\Omega \left(\frac{p-1}{p}\right)^p \left(\rho^{-1/p}\right)^p|\nabla\rho|^p\, dv_g \\ &= \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla\rho|^p}{\rho}\, dv_g \\ &= \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\rho^{\frac{p-1}{p}}|^p}{\rho^p}|\nabla\rho|^p\, dv_g.\end{aligned}$$

Thus, taking $u = \rho^{\frac{p-1}{p}}$, we obtain the infimum in (4.43).

2) Let $u \in C_0^\infty(\Omega)$. We define the functional I as

$$I(u) := \int_\Omega |\nabla u|^p\, dv_g - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{\rho^p}|\nabla\rho|^p\, dv_g.$$

We note that the functional I is nonnegative, since (2.5) holds, and the best constant will be achieved if and only if $I(u) = 0$ for some $u \in D^{1,p}(\Omega)$.

Let v be the new variable $v := \frac{u}{\rho^\gamma}$ with $\gamma := \frac{p-1}{p}$. By computation we have

$$\begin{aligned}|\nabla u|^2 &= \left|\nabla\left(v\rho^\gamma\right)\right|^2 \\ &= |\gamma|^2 v^2\rho^{2\gamma-2}|\nabla\rho|^2 + \rho^{2\gamma}|\nabla v|^2 + 2\gamma v\rho^{2\gamma-1}\nabla\rho\cdot\nabla v.\end{aligned} \tag{4.44}$$

(If ρ is not smooth enough, we can consider ψ_n as in the proof of Theorem 2.1 and after the computation take the limit as $n \to +\infty$.)

We remind that the inequality

$$(\xi-\eta)^s \geqslant \xi^s - s\eta\xi^{s-1} \tag{4.45}$$

holds for every $\xi, \eta, s \in \mathbb{R}$, with $\xi > 0$, $\xi > \eta$ and $s \geqslant 1$ (see [27]). Applying (4.45) and (4.44), with $s = p/2$, $\xi = |\gamma|^2 v^2 \rho^{2\gamma-2} |\nabla\rho|^2$ and $\eta = -2\gamma v \rho^{2\gamma-1} \nabla\rho \cdot \nabla v - \rho^{2\gamma} |\nabla v|^2$, we have

$$|\nabla u|^p \geqslant |\gamma|^p v^p \rho^{p\gamma-p} |\nabla\rho|^p + p|\gamma|^{p-2}\gamma |v|^{p-2} v |\nabla\rho|^{p-2} \nabla\rho \cdot \nabla v + \frac{p}{2} |\gamma|^{p-2} |v|^{p-2} \rho \nabla\rho^{p-2} |\nabla v|^2.$$

Then, taking into account that $v = \frac{u}{\rho^\gamma}$, we have

$$\begin{aligned} I(u) &= \int_\Omega |\nabla u|^p \, dv_g - |\gamma|^p \int_\Omega \frac{|u|^p}{\rho^p} |\nabla\rho|^p \, dv_g \\ &\geqslant \int_\Omega p|\gamma|^{p-2}\gamma |v|^{p-2} v |\nabla\rho|^{p-2} (\nabla\rho \cdot \nabla v) \, dv_g + \int_\Omega \frac{p}{2} |\gamma|^{p-2} |v|^{p-2} \rho |\nabla\rho|^{p-2} |\nabla v|^2 \, dv_g \\ &=: I_1(v) + I_2(v). \end{aligned} \tag{4.46}$$

Re-arranging the expression in $I_1(v)$ and integrating by parts we obtain

$$\begin{aligned} I_1(v) &= \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \nabla\big(|v|^p\big) \cdot |\nabla\rho|^{p-2} \nabla\rho \, dv_g \\ &= \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega |v|^p (-\Delta_p \rho) \, dv_g \geqslant 0, \end{aligned} \tag{4.47}$$

where we have used the hypothesis $-\Delta_p \rho \geqslant 0$. On the other hand we can rewrite $I_2(v)$ as

$$I_2(v) = \frac{2}{p} |\gamma|^{p-2} \int_\Omega \rho |\nabla\rho|^{p-2} \big|\nabla |v|^{p/2}\big|^2 \, dv_g. \tag{4.48}$$

Thus, we conclude that for every $u \in D^{1,p}(\Omega)$

$$I(u) \geqslant \frac{2}{p} |\gamma|^{p-2} \int_\Omega \rho |\nabla\rho|^{p-2} \big|\nabla |v|^{p/2}\big|^2 \, dv_g > 0,$$

and this inequality implies the nonexistence of minimizers in $D^{1,p}(\Omega)$. □

We end this section by showing a further result that arises from the fact that the best constant, in some cases, is not achieved. Indeed, if the best constant involved in an inequality is not achieved, it is natural to ask if a reminder term can be added. The next result shows that in the inequality (2.5) one can add a reminder term.

Theorem 4.2. *Let $p = 2$ and let ρ be as in Theorem 2.1. We define*

$$\Lambda_1 := \inf_{u \in C_0^1(\Omega)} \frac{\int_\Omega \rho |\nabla u|^2 \, dv_g}{\int_\Omega \rho |u|^2 \, dv_g}.$$

Assume that $\Lambda_1 > 0$. Then

$$\int_\Omega |\nabla u|^2 \, dv_g \geqslant \frac{1}{4} \int_\Omega \frac{|u|^2}{\rho^2} |\nabla\rho|^2 \, dv_g + \Lambda_1 \int_\Omega u^2 \, dv_g, \quad u \in C_0^1(\Omega). \tag{4.49}$$

Proof. We shall give a sketch of the proof since it is similar to the proof of Theorem 4.1. By using the same notation of the proof of Theorem 4.1, from (4.46) and (4.47), we deduce that

$$I(u) \geqslant \int_\Omega \rho |\nabla v|^2 \, dv_g \geqslant \Lambda_1 \int_\Omega \rho |v|^2 \, dv_g = \Lambda_1 \int_\Omega u^2 \, dv_g,$$

where we have used the fact that $-\Delta\rho \geqslant 0$, the definition of Λ_1 and $v = u/\rho^{1/2}$. This concludes the proof. □

An example of manifold where Theorem 4.2 applies is the following. Let ρ be a nonnegative superharmonic function on $\mathbb{R}^N$ and let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Then ρ belongs to the Muckenhoupt class A_1 and this implies that $\Lambda_1 > 0$ (indeed it suffices to combine Theorems 3.59 and 15.21 of [33]). In particular, with the choice $\rho := |x|^{2-N}$, $N > 2$ and $\Omega \subset \mathbb{R}^N$ a bounded open set, (4.49) reads as

$$\int_\Omega |\nabla u|^2\,dx \geqslant \frac{(N-2)^2}{4}\int_\Omega \frac{|u|^2}{|x|^2}\,dx + \Lambda_1 \int_\Omega u^2\,dx, \quad u \in C_0^1(\Omega),$$

which is the celebrated inequality proved in [11]. See also [19,27] for related results in Euclidean and subelliptic setting for $p > 1$ and for further references.

5. First order interpolation inequalities

In this section we shall study some inequalities of Hardy–Sobolev type. As already said above, interpolation inequalities as well as the knowledge of an estimate of the best constant have an important role in several areas of mathematical science. Thus we shall address some efforts to keep track of explicit values of the involved constants.

We shall assume that the Sobolev inequality

$$S(p)\left(\int_\Omega |u|^{p^*}\,dv_g\right)^{1/p^*} \leqslant \left(\int_\Omega |\nabla u|^p\,dv_g\right)^{1/p}, \quad u \in C_0^\infty(\Omega), \tag{S}$$

holds for some $p^* > 0$, and the Hardy inequality

$$H(\alpha,p)\int_\Omega \rho^\alpha \frac{|u|^p}{\rho^p}|\nabla\rho|^p\,dv_g \leqslant \int_\Omega \rho^\alpha |\nabla u|^p\,dv_g, \quad u \in C_0^\infty(\Omega), \tag{H_α}$$

holds for an exponent $\alpha \in \mathbb{R}$.

In some cases, the validity of (S) implies that (H_α) holds as well. Indeed, let $N > 2$ and let M be an N-dimensional complete and connected Riemannian manifold with infinite volume, if (S) holds with $p = 2$ and $p^* = 2N/(N-2)$ then M is hyperbolic (see [15]). In this case, from Theorem 2.6 we have that a Hardy inequality holds. Therefore, there exists a nonnegative nonconstant superharmonic function ρ ad hence (H_α) holds with $p = 2$ and $\alpha < 1$ (see Theorem 3.1 and Remark 3.2).

In order to state our main result of this section, we need the following preliminary theorem:

Theorem 5.1. *Assume that (S) holds on Ω. Let $\theta \in \mathbb{R}$ and $\rho \geqslant 0$ be a function such that (H_α) holds with $\alpha = p\theta$. Then there exists $C_2 > 0$ such that*

$$C_2\left(\int_\Omega \rho^{p^*\theta}|u|^{p^*}\,dv_g\right)^{1/p^*} \leqslant \left(\int_\Omega \rho^{p\theta}|\nabla u|^p\,dv_g\right)^{1/p}, \quad u \in C_0^\infty(\Omega). \tag{5.50}$$

Moreover

$$C_2 = S(p)\frac{H(p\theta,p)^{1/p}}{|\theta| + H(p\theta,p)^{1/p}}.$$

In particular, if $\rho \in L^1_{loc}(\Omega)$ is a nonnegative function satisfying the hypotheses of Theorem 3.1 with $\alpha = p\theta$ and (S) holds, then we obtain (5.50) with

$$C_2 = S(p)\frac{|p-1-p\theta|}{p|\theta| + |p-1-p\theta|}.$$

Proof. The case $\theta = 0$ corresponds to the Sobolev inequality. Let $\theta \neq 0$. Let $u \in C_0^\infty(\Omega)$ and define v as $v := \rho^\theta u$. By computation we have

$$\begin{aligned}
|\nabla v|^p &= \left|\nabla\left(\rho^\theta u\right)\right|^p = \left|\rho^\theta \nabla u + \theta \rho^{\theta-1} u \nabla\rho\right|^p \\
&\leqslant \left(\rho^\theta |\nabla u| + |\theta| \rho^{\theta-1} |u| |\nabla\rho|\right)^p \\
&= \left(\rho^\theta |\nabla u| + \frac{|\theta|}{H^{1/p}} H^{1/p} \rho^{\theta-1} |u| |\nabla\rho|\right)^p
\end{aligned} \tag{5.51}$$

where, for sake of brevity, $H = H(p\theta, p)$ and $S = S(p)$. By using the inequality

$$\left(a + \frac{1-\epsilon}{\epsilon} b\right)^p \leqslant \epsilon^{1-p} a^p + \frac{1-\epsilon}{\epsilon^p} b^p \quad (0 < \epsilon < 1,\ a, b > 0)$$

with $\epsilon := \frac{H^{1/p}}{H^{1/p}+|\theta|}$, $a := \rho^\theta |\nabla u|$ and $b := H^{1/p}\rho^{\theta-1}|u||\nabla\rho|$, we have

$$|\nabla v|^p \leqslant \epsilon^{1-p} \rho^{p\theta} |\nabla u|^p + \frac{1-\epsilon}{\epsilon^p} H \rho^{p\theta-p} |u|^p |\nabla\rho|^p. \tag{5.52}$$

Then, by (S) and using (H_α) with $\alpha = p\theta$, we obtain

$$\begin{aligned}
\left(\int_\Omega \rho^{p^*\theta} |u|^{p^*}\, dv_g\right)^{p/p^*} &= \left(\int_\Omega |v|^{p^*}\, dv_g\right)^{p/p^*} \leqslant S^{-p} \int_\Omega |\nabla v|^p \\
&\leqslant S^{-p}\left[\epsilon^{1-p} \int_\Omega \rho^{p\theta} |\nabla u|^p\, dv_g + \frac{1-\epsilon}{\epsilon^p} H \int_\Omega \rho^{p\theta} \frac{|u|^p}{\rho^p} |\nabla\rho|^p\, dv_g\right] \\
&\leqslant S^{-p}\left[\epsilon^{1-p} + \frac{1-\epsilon}{\epsilon^p}\right] \int_\Omega \rho^{p\theta} |\nabla u|^p\, dv_g = (S\epsilon)^{-p} \int_\Omega \rho^{p\theta} |\nabla u|^p\, dv_g,
\end{aligned} \tag{5.53}$$

which concludes the proof. □

Theorem 5.2. *Assume that (S) holds on Ω with $p^* > p$. Let $\theta \in \mathbb{R}$ and $\rho \geqslant 0$ be a function such that (H_α) holds with $\alpha = p\theta$. Let $r > 0$, $0 \leqslant a \leqslant 1$, γ, ϵ, σ and δ be real numbers satisfying the following relations*

$$\frac{1}{p} \geqslant \frac{1}{r} \geqslant \frac{1-a}{p} + \frac{a}{p^*}, \tag{5.54}$$

$$\gamma + \frac{p^*(r-p)}{r(p^*-p)} = (1-\theta)a + \delta(1-a) \tag{5.55}$$

and

$$\epsilon = \theta a + \sigma(1-a). \tag{5.56}$$

Then there exists $C_3 > 0$ such that

$$\begin{aligned}
&C_3\left(\int_\Omega \frac{|u|^r}{\rho^{\gamma r}} |\nabla\rho|^{(\gamma+\epsilon)r}\, dv_g\right)^{1/r} \leqslant \left(\int_\Omega \rho^{\theta p} |\nabla u|^p\, dv_g\right)^{a/p} \left(\int_\Omega |u|^p \frac{|\nabla\rho|^{(\delta+\sigma)p}}{\rho^{\delta p}}\, dv_g\right)^{(1-a)/p}, \\
&\quad u \in C_0^\infty(\Omega),
\end{aligned} \tag{5.57}$$

that is

$$C_3 \left| u \frac{|\nabla\rho|^{\gamma+\epsilon}}{\rho^\gamma} \right|_{L^r} \leqslant \left|\rho^\theta |\nabla u|\right|_{L^p}^a \left| u \frac{|\nabla\rho|^{\delta+\sigma}}{\rho^\delta} \right|_{L^p}^{1-a}, \tag{5.58}$$

provided $\frac{|\nabla\rho|^{p\sigma}}{\rho^{p\delta}} \in L^1_{loc}(\Omega)$.

Moreover

$$C_3 = C_2^{\frac{p^*(r-p)}{r(p^*-p)}} H(p\theta, p)^{\frac{a}{p} - \frac{p^*(r-p)}{pr(p^*-p)}}.$$

In particular, if $\rho \in L^1_{loc}(\Omega)$ is a nonnegative function satisfying the hypotheses of Theorem 3.1 with $\alpha = p\theta$ and (S) holds, then we obtain (5.57) with

$$C_3 = C_2^{\frac{p^*(r-p)}{r(p^*-p)}} \left(\frac{|p-1-p\theta|}{p}\right)^{a-\frac{p^*(r-p)}{r(p^*-p)}}.$$

Proof. From condition (5.54) it follows that $p^* \geqslant r \geqslant p$. We shall distinguish tree cases.

Case: $r = p^*$. From (5.54) necessarily we have $a = 1$ and hence from (5.55) and (5.56) $\epsilon = \theta = -\gamma$. The inequality to prove is actually the thesis of Theorem 5.1.

Case: $r = p$. If $a = 0$ there is nothing to prove. If $a = 1$, then the thesis is the inequality (H_α). Let $0 < a < 1$. By using (5.55) and (5.56) we have

$$\int_\Omega \frac{|u|^r}{\rho^{\gamma r}} |\nabla\rho|^{(\gamma+\epsilon)r}\, dv_g = \int_\Omega \frac{|u|^{ap}}{\rho^{(1-\theta)ap}} |\nabla\rho|^{ap} \frac{|u|^{(1-a)p}}{\rho^{(1-a)\delta p}} |\nabla\rho|^{(\delta+\sigma)p}\, dv_g.$$

Now the claim follows applying Hölder's inequality with exponent $1/a$ and then Hardy inequality (H_α).

Case: $p^* > r > p$. Let $q \in \mathbb{R}$ be a parameter that we shall fix later. Using Hölder's inequality with exponent $s > 1$ we obtain

$$\int_\Omega \frac{|u|^r}{\rho^{\gamma r}} |\nabla\rho|^{(\gamma+\epsilon)r}\, dv_g = \int_\Omega |u|^{r-q} \rho^{p^*\theta/s} \frac{|u|^q}{\rho^{\gamma r + p^*\theta/s}} |\nabla\rho|^{(\gamma+\epsilon)r}\, dv_g$$
$$\leqslant \left(\int_\Omega |u|^{(r-q)s} \rho^{p^*\theta}\, dv_g\right)^{1/s} \left(\int_\Omega \frac{|u|^{qs'}}{\rho^{(\gamma r+p^*\theta/s)s'}} |\nabla\rho|^{(\gamma+\epsilon)rs'}\, dv_g\right)^{1/s'}. \tag{5.59}$$

Now we apply Hölder's inequality with exponent $t > 1$ to the second term of (5.59) and obtain

$$\int_\Omega \frac{|u|^{qs'}}{\rho^{(\gamma r+p^*\theta/s)s'}} |\nabla\rho|^{(\gamma+\epsilon)rs'}\, dv_g$$
$$= \int_\Omega \frac{|u|^{qs'/t}}{\rho^{qs'/t}} |\nabla\rho|^{qs'/t} \rho^{p\theta/t} \frac{|u|^{qs'/t'} |\nabla\rho|^{(\gamma+\epsilon)rs'-qs'/t}}{\rho^{(\gamma r+p^*\theta/s)s'-qs'/t+p\theta/t}}\, dv_g$$
$$\leqslant \left(\int_\Omega \frac{|u|^{qs'}}{\rho^{qs'}} |\nabla\rho|^{qs'} \rho^{p\theta}\, dv_g\right)^{1/t} \left(\int_\Omega \frac{|u|^{qs'} |\nabla\rho|^{(\gamma+\epsilon)rs't'-qs't'/t}}{\rho^{(\gamma r+p^*\theta/s)s't'-qs't'/t+p\theta t'/t}}\, dv_g\right)^{1/t'}. \tag{5.60}$$

Now, requiring that the following conditions are satisfied

$$qs' = p, \qquad (r-q)s = p^*, \tag{5.61}$$

we get

$$s = \frac{p^*-p}{r-p} > 1, \tag{5.62}$$

since (5.54) holds. Using (5.61), by (5.59) and (5.60) we have

$$\int_\Omega \frac{|u|^r}{\rho^{\gamma r}} |\nabla\rho|^{(\gamma+\epsilon)r}\, dv_g$$
$$\leqslant \left(\int_\Omega |u|^{p^*} \rho^{p^*\theta}\, dv_g\right)^{1/s} \left(\int_\Omega \frac{|u|^p}{\rho^p} |\nabla\rho|^p \rho^{p\theta}\, dv_g\right)^{1/s't} \left(\int_\Omega \frac{|u|^p |\nabla\rho|^{(\gamma+\epsilon)rs't'-pt'/t}}{\rho^{(\gamma r+p^*\theta/s)s't'-pt'/t+p\theta t'/t}}\, dv_g\right)^{1/s't'}$$
$$\leqslant C_2^{-p^*/s} H(p\theta,p)^{-1/s't} \left(\int_\Omega \rho^{p\theta} |\nabla u|^p\, dv_g\right)^{p^*/ps+1/s't} \left(\int_\Omega \frac{|u|^p |\nabla\rho|^{(\gamma+\epsilon)rs't'-pt'/t}}{\rho^{(\gamma r+p^*\theta/s)s't'-pt'/t+p\theta t'/t}}\, dv_g\right)^{1/s't'}, \tag{5.63}$$

where, in the last inequality, we have used (5.50) and (H_α) with $\alpha = p\theta$. To conclude we have to choose $t > 1$ such that

$$(\gamma + \epsilon)rs't' - pt'/t = p(\delta + \sigma), \qquad \big(\gamma r + p^*\theta/s\big)s't' - pt'/t + p\theta t'/t = p\delta, \tag{5.64}$$

and

$$p^*/ps + 1/s't = ar/p, \qquad 1/s't' = (1-a)r/p. \tag{5.65}$$

First of all, note that in (5.60) we can make the choice $t = \frac{sp}{s'(asr-p^*)} > 1$, since (5.54) holds. Using the expressions of t and s, equalities (5.65) follow by simple computations. Moreover, from (5.65) we obtain that $s't' = \frac{p}{r(1-a)}$ and $\frac{t'}{t} = \frac{asr-p^*}{sr(1-a)}$. Using this two expressions and the conditions (5.55) and (5.56) we get also (5.64). This concludes the proof. □

Remark 5.3. The condition (5.56) takes into account the presence of the $|\nabla\rho|$ in the weights appearing in (5.57) and it is also a necessary condition. Indeed, to see the necessity of (5.56) we argue as follows. Assume that Theorem 5.2 were true. If (S) and (H_α) hold with a function ρ, then those inequalities still hold with the function $\lambda\rho$ for every $\lambda > 0$, and hence the conclusion of Theorem 5.2 holds replacing ρ with $\lambda\rho$. By homogeneous consideration one derives the necessity of (5.56).

Remark 5.4. Since the condition (5.56) is a requirement on the parameters ϵ and σ, if $|\nabla\rho| = 1$, these parameters do not appear in the inequality (5.57). Therefore, condition (5.56) is always fulfilled (i.e. choosing $\epsilon = a\theta$ and $\sigma = 0$). The next corollary deals with a generalization of this case.

Corollary 5.5. *Assume that (S) holds on Ω with $p^* > p$. Let $\theta \in \mathbb{R}$ and $\rho \geqslant 0$ be a function such that (H_α) holds with $\alpha = p\theta$ and $\rho = d^\beta$ with $\beta \in \mathbb{R}$ and $|\nabla d| = 1$. Let $r > 0$, $0 \leqslant a \leqslant 1$ and γ, δ be real numbers satisfying (5.54) and*

$$\gamma + \frac{p^*(r-p)}{r(p^*-p)} = (1-\beta\theta)a + \delta(1-a). \tag{5.66}$$

Then there exists $C_3' > 0$ such that

$$C_3'\bigg(\int_\Omega \frac{|u|^r}{d^{\gamma r}}\,dv_g\bigg)^{1/r} \leqslant \bigg(\int_\Omega d^{\beta\theta p}|\nabla u|^p\,dv_g\bigg)^{a/p}\bigg(\int_\Omega \frac{|u|^p}{d^{\delta p}}\,dv_g\bigg)^{(1-a)/p}, \quad u \in C_0^\infty(\Omega), \tag{5.67}$$

that is

$$C_3'\bigg|\frac{u}{\rho^\gamma}\bigg|_{L^r} \leqslant \big|\rho^{\beta\theta}|\nabla u|\big|_{L^p}^a\bigg|\frac{u}{\rho^\delta}\bigg|_{L^p}^{1-a}, \tag{5.68}$$

provided $\frac{1}{\rho^{p\delta}} \in L^1_{loc}(\Omega)$. In particular, $C_3' = C_3\beta^{\gamma+\beta\theta a-(1-a)\delta}$.

Remark 5.6. Notice that, if in the previous corollary we take $p^* = \frac{pN}{N-p}$, condition (5.66) becomes

$$\frac{1}{r} - \frac{\gamma}{N} = \frac{1}{p} + \frac{a}{N}(\beta\theta - 1) - \frac{\delta}{N}(1-a), \tag{5.69}$$

and (5.67) is a particular case on manifold of a result obtained by Caffarelli, Kohn and Nirenberg in [13] in Euclidean setting.

6. Some applications

In what follows we apply the results proved in the above sections to concrete cases, obtaining Hardy inequalities which in some cases are new. For sake of brevity we shall limit ourselves to show some applications of Theorems 2.1 and 3.1 by specializing the function ρ. With the same technique it is possible to obtain applications of the other theorems presented in the previous sections (Caccioppoli inequality, uncertain principle, Gagliardo–Nirenberg inequality, first order interpolation inequalities, and so on). We leave the details to the interested reader.

6.1. Hardy inequality involving the distance from the boundary

In order to prove a Hardy inequality involving the distance from the boundary, we need the following result, which is an immediate consequence of Theorems 2.1 and 3.1.

Theorem 6.1. *Let $\overline{M}$ be a compact Riemannian manifold with boundary of class C^∞, let φ_1 be the first eigenfunction related to the first eigenvalue of the p-Laplacian,*[4] *and $\alpha < p-1$. If $\varphi_1 > 0$, then the following inequality holds on M*

$$\left(\frac{p-1-\alpha}{p}\right)^p \int_M \varphi_1^\alpha \frac{|u|^p}{\varphi_1^p} |\nabla\varphi_1|^p \, dv_g \leqslant \int_M \varphi_1^\alpha |\nabla u|^p \, dv_g, \quad u \in C_0^\infty(M). \tag{6.71}$$

In particular we have

$$\left(\frac{p-1}{p}\right)^p \int_M \frac{|u|^p}{\varphi_1^p} |\nabla\varphi_1|^p \, dv_g \leqslant \int_M |\nabla u|^p \, dv_g, \quad u \in C_0^\infty(M). \tag{6.72}$$

The main theorem of this section is the following:

Theorem 6.2. *Let $\overline{M}$ be a compact Riemannian manifold with boundary of class C^∞, let φ_1 be the first eigenfunction related to the first eigenvalue of the p-Laplacian. Assume that $\varphi_1 \in C^1(\overline{M})$, $\varphi_1 > 0$ on M and $|\nabla\varphi_1| \neq 0$ on ∂M.*

Denoted by $d(x) := dist(x, \partial M)$, there exists a constant $c > 0$ such that

$$c \int_M \frac{|u|^p}{d^p} \, dv_g \leqslant \int_M |\nabla u|^p \, dv_g, \quad u \in C_0^\infty(M). \tag{6.73}$$

The proof of the above theorem relies on the following result, which is worth of mention:

Theorem 6.3. *Let $\overline{M}$ be a compact Riemannian manifold with boundary of class C^∞, let $\varphi_1 > 0$ be the first eigenfunction related to the first eigenvalue λ_1 of the p-Laplacian, and $0 < s < p-1$. Then the following inequality holds*

$$\lambda_1 \frac{(p-1-s)^{(p-1)}}{p^p} \int_M \varphi_1^s |u|^p \, dv_g \leqslant \int_M \varphi_1^s |\nabla u|^p \, dv_g, \quad u \in C_0^\infty(M). \tag{6.74}$$

Proof. Set $\phi := \varphi_1^{s/(p-1)}$. By computation we have

$$-\Delta_p \phi = \frac{|\nabla\phi|^p}{\phi}(p-1-s)\left(\frac{p-1}{s}\right) + \lambda_1 \left(\frac{s}{p-1}\right)^{p-1} \phi^{p-1}.$$

Choosing $h := -|\nabla\phi|^{p-2}\nabla\phi$ and $A_h := -\Delta_p\phi$, an application of Lemma 2.10 yields

$$\begin{aligned}
\lambda_1 \left(\frac{s}{p-1}\right)^{p-1} \int_M \phi^{p-1}|u|^p \, dv_g &\leqslant \int_M (-\Delta_p\phi)|u|^p \, dv_g \leqslant p^p \int_M \frac{|\nabla\phi|^{p(p-1)}}{(-\Delta_p\phi)^{p-1}} |\nabla u|^p \, dv_g \\
&\leqslant p^p \left(\frac{s}{p-1}\right)^{p-1} \left(\frac{1}{p-1-s}\right)^{p-1} \int_M \phi^{p-1} |\nabla u|^p \, dv_g.
\end{aligned}$$

This last chain of inequalities concludes the proof. □

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[4] This means that (λ_1, φ_1) is a solution of the problem

$$\begin{cases} -\Delta_p\varphi = \lambda|\varphi|^{p-2}\varphi & \text{on } M, \\ \varphi = 0 & \text{on } \partial M, \end{cases} \tag{6.70}$$

and $\lambda_1 := \min\{\lambda\colon (\lambda, \varphi) \text{ solves (6.70)}\}$.

Proof of Theorem 6.2. For a fixed number $\gamma > 0$, we denote by Ω^γ and Ω_γ respectively the sets $\Omega^\gamma := \{x\colon d(x) < \gamma\}$ and $\Omega_\gamma := \{x\colon d(x) \geqslant \gamma\}$.

Let φ_1 be such that $\|\varphi_1\|_\infty \leqslant 1$. By continuity argument, we have that there exist $\epsilon, b > 0$ such that

$$\left|\nabla\varphi_1(x)\right| \geqslant b > 0, \quad \text{for } x \in \Omega^\epsilon.$$

Since φ_1 is a Lipschitz continuous function, we obtain that there exists $L > 0$ such that

$$\varphi_1(x) \leqslant L d(x), \quad x \in M.$$

We set

$$l_\epsilon := \min_{\Omega_\epsilon} \varphi_1.$$

From (6.72) and (6.74) we get respectively

$$\int_M |\nabla u|^p \, dv_g \geqslant \left(\frac{p-1}{p}\right)^p \int_{\Omega^\epsilon} \frac{|u|^p}{\varphi_1^p} |\nabla\varphi_1|^p \, dv_g \geqslant \left(\frac{p-1}{p}\right)^p \frac{b^p}{L^p} \int_{\Omega^\epsilon} \frac{|u|^p}{d^p} \, dv_g,$$

$$\int_M |\nabla u|^p \, dv_g \geqslant \lambda_1 \frac{(p-1-s)^{(p-1)}}{p^p} \int_{\Omega_\epsilon} \varphi_1^s |u|^p \, dv_g \geqslant \lambda_1 \frac{(p-1-s)^{(p-1)}}{p^p} l_\epsilon^s \epsilon^p \int_{\Omega_\epsilon} \frac{|u|^p}{d^p} \, dv_g.$$

Choosing $2c := \min\{(\frac{p-1}{p})^p \frac{b^p}{L^p}, \lambda_1 \frac{(p-1-s)^{(p-1)}}{p^p} l_\epsilon^s \epsilon^p\}$ and summing up the above estimates we obtain the claim. $\square$

6.2. *Hardy inequality for p-hyperbolic manifold*

In this section we establish Hardy inequalities involving the Green function of the operator $-\Delta_p$. The case $p = 2$ is already proved in [40].

Examples of p-hyperbolic manifolds are the following. The Euclidean space $\mathbb{R}^N$ is p-hyperbolic for $N > p$. If M is a Cartan–Hadamard manifold (see Section 6.3) whose sectional curvature K_M is uniformly negative, that is $K_M \leqslant -a^2 < 0$, then M is p-hyperbolic for any $p > 1$ (see [34] and [36]).

We have the following.

Theorem 6.4. *Let (M, g) be a p-hyperbolic manifold, let G_x be the Green function for Δ_p with pole at x, and $\alpha \in \mathbb{R}$. Then the following inequality holds*

$$\left(\frac{|p-1-\alpha|}{p}\right)^p \int_{M\setminus\{x\}} G_x^\alpha \frac{|\nabla G_x|^p}{G_x^p} |u|^p \, dv_g \leqslant \int_{M\setminus\{x\}} G_x^\alpha |\nabla u|^p \, dv_g, \quad u \in C_0^\infty\big(M \setminus \{x\}\big). \tag{6.75}$$

In particular, we have

$$\left(\frac{p-1}{p}\right)^p \int_{M\setminus\{x\}} \frac{|\nabla G_x|^p}{G_x^p} |u|^p \, dv_g \leqslant \int_{M\setminus\{x\}} |\nabla u|^p \, dv_g, \quad u \in C_0^\infty\big(M \setminus \{x\}\big), \tag{6.76}$$

and, if $p < N$,

$$\left(\frac{p-1}{p}\right)^p \int_M \frac{|\nabla G_x|^p}{G_x^p} |u|^p \, dv_g \leqslant \int_M |\nabla u|^p \, dv_g, \quad u \in C_0^\infty(M). \tag{6.77}$$

Proof. We know that $G_x \in W^{1,p}_{loc}(M)$ is a nonnegative function on M. Moreover the hypotheses of Theorem 3.1 are fulfilled; in fact

i) $-\Delta_p G_x = 0$ in $M \setminus \{x\}$,
ii) $\frac{|\nabla G_x|^p}{G_x^{p-\alpha}}, G_x^\alpha \in L^1_{loc}(M \setminus \{x\})$.

Then, by Theorem 3.1, inequality (6.75) holds. In particular, taking $\alpha = 0$, we obtain the inequality (6.76). Moreover, if $p < N$, we are in the position to apply Corollary 2.3, because $\{x\}$ is a set of zero p-capacity (see Theorem 2.27 in [33]), and then we can use Proposition A.1; this proves that also (6.77) holds. □

6.3. Hardy inequality on Cartan–Hadamard manifold

In what follows (M, g) will denote a Cartan–Hadamard manifold, that is, a connected, simply connected, complete Riemannian manifold of dimension $N \geqslant 2$, of nonpositive sectional curvature (see [16,34,40] for further details). Examples of Cartan–Hadamard manifolds are the Euclidean space $\mathbb{R}^N$ with the usual metric (which has constant sectional curvature equal to zero), and the standard N-dimensional hyperbolic space $\mathbb{H}^N$ (which has constant sectional curvature equal to -1).

Let $o \in M$ be a fixed point and denote by r the distance function from o. We have the following.

Theorem 6.5. *Let (M, g) be a Cartan–Hadamard manifold and $\alpha \in \mathbb{R}$.*

If $(N-p)(p-1-\alpha) > 0$, we have

$$\left(\frac{(N-p)(p-1-\alpha)}{p(p-1)}\right)^p \int\limits_{M\setminus\{o\}} r^{\alpha\frac{p-N}{p-1}} \frac{|u|^p}{r^p}\, dv_g \leqslant \int\limits_{M\setminus\{o\}} r^{\alpha\frac{p-N}{p-1}} |\nabla u|^p\, dv_g, \quad u \in C_0^\infty\big(M \setminus \{o\}\big). \tag{6.78}$$

If $1 < p < N$, we have

$$\left(\frac{N-p}{p}\right)^p \int\limits_M \frac{|u|^p}{r^p}\, dv_g \leqslant \int\limits_M |\nabla u|^p\, dv_g, \quad u \in C_0^\infty(M). \tag{6.79}$$

If $1 < p \leqslant N$, setting $\Omega := r^{-1}([0, 1[)$, we have

$$\left(\frac{p-1}{p}\right)^p \int\limits_\Omega \frac{|u|^p}{|r \ln r|^p}\, dv_g \leqslant \int\limits_\Omega |\nabla u|^p\, dv_g, \quad u \in C_0^\infty(\Omega). \tag{6.80}$$

Let $1 < p \leqslant N$. If $p - 1 > \alpha$ we set $\Omega := r^{-1}(]0, 1[)$, else if $p - 1 < \alpha$ we set $\Omega := r^{-1}(]1, +\infty[)$. We have

$$\left(\frac{|p-1-\alpha|}{p}\right)^p \int\limits_\Omega |\ln r|^\alpha \frac{|u|^p}{|r \ln r|^p}\, dv_g \leqslant \int\limits_\Omega |\ln r|^\alpha |\nabla u|^p\, dv_g, \quad u \in C_0^\infty(\Omega). \tag{6.81}$$

Proof. In $M \setminus \{o\}$ we define $\rho = r^\beta$ with $\beta \in \mathbb{R}$ that will be chosen later. The function $\rho \in W^{1,p}_{loc}(M \setminus \{o\})$ is nonnegative on $M \setminus \{o\}$.

The function r satisfies the relations

$$|\nabla r| = 1, \qquad \Delta r \geqslant \frac{N-1}{r} \tag{6.82}$$

(see [16]).

By computation we obtain

$$\begin{aligned}
\Delta_p \rho &= \operatorname{div}\big(|\nabla\rho|^{p-2}\nabla\rho\big) \\
&= \operatorname{div}\big(|\beta|^{p-2} r^{(\beta-1)(p-2)} \beta r^{\beta-1} \nabla r\big) \\
&= |\beta|^{p-2}\beta \operatorname{div}\big(r^{(\beta-1)(p-1)}\nabla r\big) \\
&= |\beta|^{p-2}\beta\big[(\beta-1)(p-1) r^{(\beta-1)(p-1)-1} + r^{(\beta-1)(p-1)}\Delta r\big] \\
&= |\beta|^{p-2}\beta r^{(\beta-1)(p-1)-1}\big[(\beta-1)(p-1) + r\Delta r\big],
\end{aligned}$$

and hence

$$-(p-1-\alpha)\Delta_p\rho = -(p-1-\alpha)|\beta|^{p-2}\beta r^{(\beta-1)(p-1)-1}\big[(\beta-1)(p-1) + r\Delta r\big]. \tag{6.83}$$

Next choosing $\beta = \frac{p-N}{p-1}$, using (6.82), we have that $(\beta - 1)(p-1) + r\Delta r \geqslant 0$, and then $-(p-1-\alpha)\Delta_p \rho \geqslant 0$. That is the hypothesis i) of Theorem 3.1 is fulfilled. Since $\frac{|\nabla\rho|^p}{\rho^{p-\alpha}} = |\beta|^p \frac{1}{r^{p-\alpha}} \in L^1_{loc}(M \setminus \{o\})$ and $\rho^\alpha = r^{\beta\alpha} \in L^1_{loc}(M \setminus \{o\})$, we are in a position to apply Theorem 3.1, obtaining the inequality (6.78).

In particular, taking $\alpha = 0$ and using Corollary 2.3 we get (6.79). Indeed, since $p \leqslant N$, $\{o\}$ is a set of zero p-capacity (see Theorem 2.27 in [33]), and then we can use Proposition A.1.

Next we prove (6.81). To this end, by choosing $\rho := (\alpha - p + 1)\ln r$, we have that $\rho > 0$ in Ω (according to the different cases $\alpha > (<)\ p-1$). By computation we have

$$\begin{aligned}
-(p-1-\alpha)\Delta_p\rho &= (\alpha - p + 1)\operatorname{div}\bigl(|\nabla\rho|^{p-2}\nabla\rho\bigr) \\
&= (\alpha - p + 1)^p \operatorname{div}\bigl(r^{1-p}\nabla r\bigr) \\
&= (\alpha - p + 1)^p \left(\frac{1-p}{r^p} + \frac{r\Delta r}{r^p}\right) \\
&\geqslant (\alpha - p + 1)^p \left(\frac{N-p}{r^p}\right) \geqslant 0.
\end{aligned}$$

The claim follows applying Theorem 3.1.

We conclude the proof by proving (6.80). Choosing $\alpha = 0$ in (6.81), we have that inequality (6.80) holds for every $u \in C_0^\infty(\Omega \setminus \{o\})$. However in this case $\{o\}$ is a set of zero p-capacity and, applying Corollary 2.3, we complete the proof. □

Inequality (6.79) is present in [16] for $p = 2$. In [38] the authors prove (6.78) for $p = 2$ and for a special case of manifold M, namely, when M is the unit ball modeling the standard hyperbolic space $\mathbb{H}^N$. For this case the authors prove that the constant in (6.78) is sharp and they show that a remainder term can be added.

6.4. Hardy inequalities involving the distance from the soul of a manifold

Let (M, g) be a complete noncompact Riemannian manifold, of dimension $N \geqslant 2$, with nonnegative sectional curvatures. A result due to Cheeger and Gromoll asserts that there exists a compact embedded totally convex submanifold S with empty boundary, whose normal bundle is diffeomorphic to M (see [17]). The submanifold S, called "soul" of M, is not necessarily unique but every two souls of M are isometric. *"Totally convex"* means that any geodesic arc in M connecting two points in S (which may coincide) lies entirely in S. In particular, S is connected, totally geodesic in M, and has nonnegative sectional curvature. Moreover $0 \leqslant \dim S < \dim M$.

Denote by $r : M \setminus S \to \mathbb{R}$ the distance function to S. We have that r is smooth on $M \setminus S$ and $|\nabla r| = 1$ on $M \setminus S$. Now we suppose that radial sectional curvature K_r, that is sectional curvature of two-planes containing the direction ∇r, satisfies

$$0 \leqslant K_r \leqslant \frac{c_N(1-c_N)}{r^2}, \tag{6.84}$$

where $c_N = \frac{N-2}{N}$; then we have

$$\Delta r \geqslant \frac{c_N(N-s-1)}{r}, \tag{6.85}$$

where $s = \dim S$ (see [26]). We have the following:

Theorem 6.6. *Let (M, g) be a Riemannian manifold with nonnegative curvature. Suppose that (6.84) is fulfilled.*

Let $G := c_N(N-s-1) - p + 1$. If $G \cdot (p-1-\alpha) > 0$, we have

$$\left(\frac{G\cdot(p-1-\alpha)}{p(p-1)}\right)^p \int\limits_{M\setminus S} r^{-\alpha\frac{G}{p-1}} \frac{|u|^p}{r^p}\, dv_g \leqslant \int\limits_{M\setminus S} r^{-\alpha\frac{G}{p-1}} |\nabla u|^p\, dv_g, \quad u \in C_0^\infty(M \setminus S). \tag{6.86}$$

Moreover, if $G > 0$, we have

$$\left(\frac{G}{p}\right)^p \int_M \frac{|u|^p}{r^p}\, dv_g \leqslant \int_M |\nabla u|^p\, dv_g, \quad u \in C_0^\infty(M). \tag{6.87}$$

Proof. Let $\rho = r^\beta$ in $M \setminus S$ with $\beta = -\frac{G}{p-1}$. Arguing as in the proof of Theorem 6.5, using (6.83) and (6.85), we obtain $-(p-1-\alpha)\Delta_p \rho \geqslant 0$. Since $\rho \in W^{1,p}_{loc}(M \setminus S)$, $\frac{|\nabla \rho|^p}{\rho^{p-\alpha}} = |\beta|^p \frac{1}{r^{p-\alpha}} \in L^1_{loc}(M \setminus S)$ and $\rho^\alpha = r^{\beta\alpha} \in L^1_{loc}(M \setminus S)$, the hypotheses of Theorem 3.1 are fulfilled, and (6.86) follows.

In particular, if $\alpha = 0$, (6.86) becomes

$$\left(\frac{G}{p}\right)^p \int_{M\setminus S} \frac{|u|^p}{r^p}\, dv_g \leqslant \int_{M\setminus S} |\nabla u|^p\, dv_g, \quad u \in C_0^\infty(M \setminus S).$$

Now, the hypothesis $G > 0$ implies that $N - p > s$. In fact, by the fact that $G = c_N(N - s - 1) - p + 1 = \frac{N-2}{N}(N - s - 1) - p + 1 > 0$, by simple computations we get

$$\begin{aligned}(N-2)(N-s-1) - N(p+1) &= (N-2)(N-s) - Np \\ &= N(N-s) - 2(N-s) - Np \\ &= N(N-s-p) - 2(N-s) > 0,\end{aligned}$$

which implies $N - s - p > 2\frac{N-s}{N} > 0$. Then S is a set of zero p-capacity (see Theorem 2.27 in [33]), and we can use Proposition A.1 and Corollary 2.3 to obtain inequality (6.87). □

6.5. *Hardy–Poincaré inequality for the hyperbolic plane*

Let $\mathbb{C}_+ = \{z = x + iy:\ Im\, z = y > 0\}$ be the upper half-plane equipped with the Poincaré metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. This space is a Riemannian manifold modeling the *two-dimensional hyperbolic space*. In this case, the gradient ∇_H, the divergence div_H, the Laplacian Δ_H and the volume dv_g related to the metric are respectively the following

$$\begin{aligned}&\nabla_H u = y\nabla_E u,\\ &\mathrm{div}_H = y^2\,\mathrm{div}_E,\\ &\Delta_H u = y^2 \Delta_E u,\\ &dv_g = \frac{dx\,dy}{y^2},\end{aligned} \tag{6.88}$$

where we have denoted with ∇_E, div_E, Δ_E the related operator in the Euclidean setting, and $dx\,dy$ is the Lebesgue measure in $\mathbb{R}^2$.

By using Theorem 3.1 with $p = 2$, we deduce a Hardy inequality on the upper half-plane.

Theorem 6.7. *Let $\alpha \in \mathbb{R}$. For every $u \in C_0^\infty(\mathbb{C}_+)$ we have*

$$\frac{(1-\alpha)^2}{4} \int_{\mathbb{C}_+} y^\alpha |u|^2 \frac{dx\,dy}{y^2} \leqslant \int_{\mathbb{C}_+} y^\alpha |\nabla_H u|^2 \frac{dx\,dy}{y^2}, \quad u \in C_0^\infty(\mathbb{C}_+).$$

Proof. We consider the function $\rho(z) = y$, where $z = x + iy$. Clearly, ρ belongs to $W^{1,2}_{loc}(\mathbb{C}_+)$, and $\rho^\alpha = y^\alpha$ belongs to $L^1_{loc}(\mathbb{C}_+)$. Moreover, from (6.88), we have that

$$\frac{|\nabla_H \rho|^2}{\rho^{2-\alpha}} = \frac{y^2 |\nabla_E \rho|^2}{y^{2-\alpha}} = y^\alpha \in L^1_{loc}(\mathbb{C}_+),$$

and

$$\Delta_H \rho = y^2 \Delta_E \rho = 0.$$

Therefore, the hypotheses of Theorem 3.1 are satisfied and this concludes the proof. □

6.6. The Euclidean case

In this last section we show that our main results, Theorems 2.1 and 3.1, yield some well known sharp Hardy inequalities in the Euclidean space.

Since $\mathbb{R}^N$ is p-hyperbolic for $N > p$ and it is a Cartan–Hadamard manifold, Theorems 6.4 and 6.5 hold also on $\mathbb{R}^N$ with $G_x = |x|^{\frac{p-N}{p-1}}$ and $r = |x|$ respectively. However, the function $|x|^{\frac{p-N}{p-1}}$ is p-harmonic in $\mathbb{R}^N \setminus \{0\}$ also for $p > N$. Therefore we have the following:

Theorem 6.8. *Let $p \neq N$. For any $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ we have*

$$\left(\frac{|N-p| \cdot |p-1-\alpha|}{p(p-1)}\right)^p \int\limits_{\mathbb{R}^N \setminus \{0\}} |x|^{\alpha \frac{p-n}{p-1}} \frac{|u|^p}{|x|^p}\, dx \leqslant \int\limits_{\mathbb{R}^N \setminus \{0\}} |x|^{\alpha \frac{p-n}{p-1}} |\nabla u|^p\, dx.$$

In the half-space $\mathbb{R}^N_+$ there holds the following:

Theorem 6.9. *Let $\alpha \in \mathbb{R}$, let $N \geqslant 2$, let $\mathbb{R}^N_+ = \{(x_1, \dots, x_N) \in \mathbb{R}^N\colon x_1 > 0\}$, and let $\rho(x) := d(x, \partial\mathbb{R}^N_+)$ be the distance from the boundary of $\mathbb{R}^N_+$. Then we have*

$$\left(\frac{|p-1-\alpha|}{p}\right)^p \int\limits_{\mathbb{R}^N_+} \rho^\alpha \frac{|u|^p}{\rho^p}\, dx \leqslant \int\limits_{\mathbb{R}^N_+} \rho^\alpha |\nabla u|^p\, dx, \quad u \in C_0^\infty\big(\mathbb{R}^N_+\big). \tag{6.89}$$

Proof. The distance $\rho(x) = x_1 \in W^{1,p}_{loc}(\mathbb{R}^N_+)$ is nonnegative on $\mathbb{R}^N_+$ and it is easy to verify that the hypotheses of Theorem 3.1 are satisfied. Therefore the thesis follows. □

From Theorem 6.9, we can deduce, as a particular case, a well known Hardy inequality for the upper half-plane $\mathbb{C}_+ = \{z = x + iy\colon Im\, z = y > 0\}$ (see for instance [32] and references therein).

Corollary 6.10. *Let $\alpha \in \mathbb{R}$. Then, for every $u \in C_0^\infty(\mathbb{C}_+)$, we have the following:*

$$\left(\frac{|p-1-\alpha|}{p}\right)^p \int\limits_{\mathbb{C}_+} |u|^p \frac{dA(z)}{(Im\, z)^{p-\alpha}} \leqslant 2^{p/2} \int\limits_{\mathbb{C}_+} (Im\, z)^\alpha \left(\left|\partial u(z)\right|^2 + \left|\bar{\partial} u(z)\right|^2\right)^{p/2} dA(z), \tag{6.90}$$

where $dA(z) := \frac{dx\, dy}{\pi}$, and ∂, $\bar{\partial}$ are the Wirtinger operators, that is

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{6.91}$$

Actually a more general theorem for convex domains holds.

Theorem 6.11. *Let D be a proper convex open subset of $\mathbb{R}^N$, let $d := dist(\cdot, \partial D)$ be the distance from ∂D and let $\alpha < p - 1$. Then we have*

$$\left(\frac{p-1-\alpha}{p}\right)^p \int\limits_D d^\alpha \frac{|u|^p}{d^p}\, dx \leqslant \int\limits_D d^\alpha |\nabla u|^p\, dx, \quad u \in C_0^\infty(D),$$

and, in particular,

$$\left(\frac{p-1}{p}\right)^p \int\limits_D \frac{|u|^p}{d^p}\, dx \leqslant \int\limits_D |\nabla u|^p\, dx, \quad u \in C_0^\infty(D).$$

Proof. The thesis will follow by applying Theorems 2.1 and 3.1. To this end it suffices to prove that the function $d(x) := dist(x, \partial D)$ is p-superharmonic. Indeed, $D = \bigcap \Pi$, and $d(x) = \inf_{\Pi} dist(x, \partial \Pi)$ where the intersection and the infimum are taken over all the half-spaces Π containing D. Since $dist(x, \partial \Pi)$ is continuous and p-harmonic, we have that d is p-superharmonic (see [33]). This concludes the proof. □

Appendix A

Let us recall that p-capacity of a compact set K is defined as

$$cap_p(K, M) = \inf\left\{ \int_M |\nabla u|^p \, dv_g \colon u \in C_0^\infty(M),\ 0 \leqslant u \leqslant 1,\ u = 1 \text{ in a neighborhood of } K \right\}. \tag{A.1}$$

Proposition A.1. *Let M be a p-hyperbolic manifold of dimension N. Let $K \subset M$ be a compact set of zero p-capacity. Then*

$$D^{1,p}(M) \subset D^{1,p}(M \setminus K),$$

that is every function $u \in D^{1,p}(M)$ can be approximated by function $C_0^\infty(M \setminus K)$ in the norm $|\cdot|_{D^{1,p}}$.

Proof. Let $\varphi \in C_0^\infty(M)$. In order to prove the claim it is sufficient to prove that $\varphi \in D^{1,p}(M \setminus K)$. Since $cap_p(K, M) = 0$, there exists a sequence $(u_j)_{j \geqslant 1}$ such that, for any $j \geqslant 1$, $u_j \in C_0^\infty(M)$, $0 \leqslant u_j \leqslant 1$, $u_j = 1$ in a neighborhood of K and $u_j \to 0$ in $D^{1,p}(M)$. For every $j \geqslant 1$ the function $\varphi_j := (1 - u_j)\varphi$ belongs to $C_0^\infty(M \setminus K) \subset D^{1,p}(M \setminus K)$. We shall prove that $\varphi_j \to \varphi$ in $D^{1,p}(M \setminus K)$, that is

$$\int_{M \setminus K} |\nabla \varphi_j - \nabla \varphi|^p \, dv_g \to 0 \quad (\text{as } j \to +\infty). \tag{A.2}$$

In fact, we have

$$\begin{aligned}
\left(\int_{M \setminus K} |\nabla \varphi_j - \nabla \varphi|^p \, dv_g \right)^{1/p} &= \left(\int_{M \setminus K} \left|\nabla \varphi (1 - u_j) - \varphi \nabla u_j - \nabla \varphi\right|^p \, dv_g \right)^{1/p} \\
&\leqslant \left(\int_{M \setminus K} |\nabla \varphi|^p |u_j|^p \, dv_g \right)^{1/p} + \left(\int_{M \setminus K} |\varphi|^p |\nabla u_j|^p \, dv_g \right)^{1/p}.
\end{aligned} \tag{A.3}$$

The second term in (A.3) converges to 0 for $j \to +\infty$. Indeed, since $u_j \to 0$ in $D^{1,p}(M)$, we obtain

$$\begin{aligned}
\int_{M \setminus K} |\varphi|^p |\nabla u_j|^p \, dv_g &\leqslant \int_M |\varphi|^p |\nabla u_j|^p \, dv_g \\
&\leqslant |\varphi|_\infty^p \int_M |\nabla u_j|^p \, dv_g \to 0 \quad (\text{as } j \to +\infty).
\end{aligned}$$

It remains to prove that the first term in (A.3) converges to 0 as well. Let D be the support of φ; then we get

$$\begin{aligned}
\int_{M \setminus K} |\nabla \varphi|^p |u_j|^p \, dv_g &\leqslant \int_M |\nabla \varphi|^p |u_j|^p \, dv_g = \int_D |\nabla \varphi|^p |u_j|^p \, dv_g \\
&\leqslant |\nabla \varphi|_\infty^p \int_D |u_j|^p \, dv_g \leqslant |\nabla \varphi|_\infty^p C \int_M |\nabla u_j|^p \, dv_g \to 0 \quad (\text{as } j \to +\infty),
\end{aligned}$$

where, in the last inequality, we have used a characterization of the p-hyperbolic manifold (see Theorem 3 in [49]). □

References

[1] Adimurthi, A. Sekar, Role of the fundamental solution in Hardy–Sobolev-type inequalities, Proc. Roy. Soc. Edinburgh Sect. A 136 (2006) 1111–1130.
[2] L. Adriano, C. Xia, Hardy type inequalities on complete Riemannian manifolds, Monatsh. Math. 163 (2011) 115–129.
[3] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer-Verlag, Berlin, 1998.
[4] P. Baras, J. Goldstein, The heat equation with a singular potential, Trans. Amer. Math. Soc. 284 (1984) 121–139.
[5] G. Barbatis, S. Filippas, A. Tertikas, A unified approach to improved L^p Hardy inequalities with best constants, Trans. Amer. Math. Soc. 356 (2004) 2169–2196.
[6] M. van den Berg, P. Gilkey, A. Grigor'yan, K. Kirsten, Hardy inequality and heat semigroup estimates for Riemannian manifolds with singular data, Comm. Partial Differential Equations 37 (2012) 885–900.
[7] Y. Bozhkov, E. Mitidieri, Conformal Killing vector fields and Rellich type identities on Riemannian manifolds, I, in: Lect. Notes Semin. Interdiscip. Mat., vol. 7, 2008, pp. 65–80.
[8] Y. Bozhkov, E. Mitidieri, Conformal Killing vector fields and Rellich type identities on Riemannian manifolds, II, Mediterr. J. Math. 9 (2012) 1–20.
[9] H. Brezis, X. Cabré, Some simple nonlinear PDE's without solutions, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 8 (1) (1998) 223–262.
[10] H. Brezis, M. Marcus, Hardy's inequalities revisited, Ann. Sc. Norm. Super. Cl. Sci. (4) 25 (1997) 217–237.
[11] H. Brezis, J.L. Vazquez, Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complut. Madr. 10 (1997) 443–469.
[12] V. Burenkov, E.B. Davies, Spectral stability of the Neumann Laplacian, J. Differential Equations 186 (2002) 485–508.
[13] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights, Compos. Math. 53 (3) (1984) 259–275.
[14] M.P. do Carmo, C. Xia, Complete manifolds with non-negative Ricci curvature and the Caffarelli–Kohn–Nirenberg inequalities, Compos. Math. 140 (2004) 818–826.
[15] G. Carron, Inégalités isopérimétriques de Faber–Krahn et conséquences, in: Actes de la Table Ronde de Géométrie Différentielle, Luminy, 1992, in: Semin. Congr., vol. 1, 1996, pp. 205–232.
[16] G. Carron, Inégalités de Hardy sur les variétés Riemanniennes non-compactes, J. Math. Pures Appl. (9) 76 (1997) 883–891.
[17] J. Cheeger, D. Gromoll, On the structure of complete manifolds of nonnegative curvature, Ann. of Math. 96 (1972) 413–443.
[18] L. D'Ambrosio, Hardy inequalities related to Grushin type operators, Proc. Amer. Math. Soc. 132 (2004) 725–734.
[19] L. D'Ambrosio, Some Hardy inequalities on the Heisenberg group, Differ. Equ. 40 (2004) 552–564.
[20] L. D'Ambrosio, Hardy-type inequalities related to degenerate elliptic differential operators, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) IV (2005) 451–586.
[21] E.B. Davies, A. Hinz, Explicit constants for Rellich inequalities in L^p, Math. Z. 227 (1998) 511–523.
[22] E.B. Davies, M. Lianantonakis, Heat kernel and Hardy estimates for locally Euclidean manifolds with fractal boundaries, Geom. Funct. Anal. 3 (1993) 527–563.
[23] B. Devyver, M. Fraas, Y. Pinchover, Optimal Hardy-type inequality for second-order elliptic operator and applications, preprint, arXiv:1208.2342, 58 pp.
[24] B. Devyver, M. Fraas, Y. Pinchover, Optimal Hardy-type inequalities for elliptic operators, C. R. Acad. Sci. Paris 350 (2012) 475–479.
[25] J. Duoandikoetxea, L. Vega, Some weighted Gagliardo–Nirenberg inequalities and applications, Proc. Amer. Math. Soc. 135 (2007) 2795–2802.
[26] J. Escobar, A. Freire, The spectrum of the Laplacian of manifolds of positive curvature, Duke Math. J. 65 (1992) 1–21.
[27] F. Gazzola, H. Grunau, E. Mitidieri, Hardy inequalities with optimal constants and remainder terms, Trans. Amer. Math. Soc. 356 (2004) 2149–2168.
[28] A.A. Grigor'yan, On the existence of a Green function on a manifold, Russian Math. Surveys 38 (1) (1983) 190–191.
[29] A.A. Grigor'yan, On the existence of positive fundamental solutions of the Laplace equation on Riemannian manifold, Math. USSR Sb. 56 (2) (1987) 349–357.
[30] A.A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bull. Amer. Math. Soc. (N.S.) 36 (2) (1999) 135–249.
[31] A.A. Grigor'yan, Isoperimetric inequalities and capacities on Riemannian manifolds, Oper. Theory Adv. Appl. 109 (1999) 139–153.
[32] H. Hedenmalm, The Beurling operator for the hyperbolic plane, Ann. Acad. Sci. Fenn. Math. 37 (2012) 3–18.
[33] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Clarendon Press, Oxford, 1993.
[34] I. Holopainen, Asymptotic Dirichlet problem for the p-Laplacian on Cartan–Hadamard manifolds, Proc. Amer. Math. Soc. 130 (11) (2002) 3393–3400.
[35] I. Holopainen, Non linear potential theory and quasiregular mappings on Riemannian manifolds, Ann. Acad. Sci. Fenn. Ser. A 74 (1990).
[36] I. Holopainen, P. Pankka, p-Laplace operator, quasiregular mappings, and Picard-type theorems, in: Quasiconformal Mappings and Their Applications, Narosa, New Delhi, 2007, pp. 117–150.
[37] E. Jannelli, The role played by space dimension in elliptic critical problems, J. Differential Equations 156 (1999) 407–426.
[38] I. Kombe, M. Özaydin, Improved Hardy and Rellich inequalities on Riemannian manifolds, Trans. Amer. Math. Soc. 361 (12) (2009) 6191–6203.
[39] P. Li, L. Tam, Symmetric Green's functions on complete manifolds, Amer. J. Math. 109 (1987) 1129–1154.
[40] P. Li, J. Wang, Weighted Poincaré inequality and rigidity of complete manifolds, Ann. Sci. Éc. Norm. Supér. (4) 39 (6) (2006) 921–982.
[41] M. Marcus, V. Mizel, Y. Pinchover, On the best constant for Hardy's inequality in $\mathbb{R}^n$, Trans. Amer. Math. Soc. 350 (1998) 3237–3255.
[42] T. Matskewich, P. Sobolevskii, The best possible constant in generalized Hardy's inequality for convex domain in $\mathbb{R}^n$, Nonlinear Anal. 28 (1997) 1601–1610.

[43] T. Matskewich, P. Sobolevskii, The sharp constant in Hardy's inequality for complement of bounded domain, Nonlinear Anal. 33 (1998) 105–120.
[44] V. Miklyukov, M. Vuorinen, Hardy's inequality for $W_0^{1,p}$-functions on Riemannian manifolds, Proc. Amer. Math. Soc. 127 (9) (1999) 2745–2754.
[45] E. Mitidieri, A simple approach to Hardy inequalities, Mat. Zametki 67 (2000) 563–572.
[46] E. Mitidieri, S. Pohozaev, A-priori estimates and blow-up of solutions of nonlinear partial differential equations and inequalities, Proc. Steklov Inst. Math. 234 (2001) 1–362.
[47] S. Pigola, M. Rigoli, A. Setti, Vanishing and Finiteness Results in Geometric Analysis, Birkhäuser Verlag, Berlin, 2008.
[48] M. Troyanov, Parabolicity of manifolds, Siberian Adv. Math. 9 (1999) 125–150.
[49] M. Troyanov, Solving the p-Laplacian on manifolds, Proc. Amer. Math. Soc. 128 (2) (2000) 541–545.
[50] C. Xia, The Gagliardo–Nirenberg inequalities and manifolds of non-negative Ricci curvature, J. Funct. Anal. 224 (2005) 230–241.