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Standing waves for linearly coupled Schrödinger equations with critical exponent ☆

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Abstract

We study the following linearly coupled Schrödinger equations:

$$\begin{cases} -\varepsilon^2\Delta u + a(x)u = u^p + \lambda v, & x \in \mathbb{R}^N, \\ -\varepsilon^2\Delta v + b(x)v = v^{2^*-1} + \lambda u, & x \in \mathbb{R}^N, \\ u, v > 0 \quad \text{in } \mathbb{R}^N, & u(x), v(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where $N \geqslant 3$, $2^* = \frac{2N}{N-2}$, $1 < p < 2^* - 1$, and $a(x), b(x)$ are positive continuous potentials which are both bounded away from 0. Under some assumptions on $a(x)$ and $\lambda > 0$, we obtain positive solutions of the coupled system for sufficiently small $\varepsilon > 0$, which have concentration phenomenon as $\varepsilon \to 0$. It is interesting that we do not need any further assumptions on $b(x)$.

1. Introduction

In this paper we study standing waves for the following system of time-dependent nonlinear Schrödinger equations

$$\begin{cases} -i\hbar\dfrac{\partial \psi_1}{\partial t} - \dfrac{\hbar^2}{2}\Delta\psi_1 + a(x)\psi_1 = |\psi_1|^{p-1}\psi_1 + \lambda\psi_2, & x \in \mathbb{R}^N,\ t > 0, \\ -i\hbar\dfrac{\partial \psi_2}{\partial t} - \dfrac{\hbar^2}{2}\Delta\psi_2 + b(x)\psi_2 = |\psi_2|^{2^*-2}\psi_2 + \lambda\psi_1, & x \in \mathbb{R}^N,\ t > 0, \\ \psi_j = \psi_j(x,t) \in \mathbb{C}, \quad j = 1, 2, \\ \psi_j(x,t) \to 0, \quad \text{as } |x| \to +\infty,\ t > 0,\ j = 1, 2, \end{cases} \tag{1.1}$$

where i denotes the imaginary unit, $\hbar$ is the Plank constant, $N \geqslant 3$, $1 < p < 2^* - 1$ and $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent.

Nonlinear Schrödinger equations (NLS) have been broadly investigated in many aspects. In particular, there has been an ever-growing interest in the study of standing wave solutions to NLS starting from the cerebrated works

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[8,14,28]. For system (1.1), a solution of the form $(\psi_1(x,t),\psi_2(x,t)) = (e^{-iEt/\hbar}u(x), e^{-iEt/\hbar}v(x))$ is called a standing wave. Then (ψ_1,ψ_2) is a solution of (1.1) if and only if (u,v) solves the following system

$$\begin{cases} -\dfrac{\hbar^2}{2}\Delta u + \big(a(x)-E\big)u = |u|^{p-1}u + \lambda v, & x\in\mathbb{R}^N,\\ -\dfrac{\hbar^2}{2}\Delta v + \big(b(x)-E\big)v = |v|^{2^*-2}v + \lambda u, & x\in\mathbb{R}^N,\\ u(x), v(x)\to 0 \quad \text{as } |x|\to\infty. \end{cases} \tag{1.2}$$

In this paper we are concerned with positive solutions for small $\hbar > 0$. For sufficiently small $\hbar > 0$, the standing waves are referred to as semiclassical states. Replacing $a(x)-E$, $b(x)-E$ by $a(x)$, $b(x)$ for convenience, we turn to consider the following system of NLS

$$\begin{cases} -\varepsilon^2\Delta u + a(x)u = u^p + \lambda v, & x\in\mathbb{R}^N,\\ -\varepsilon^2\Delta v + b(x)v = v^{2^*-1} + \lambda u, & x\in\mathbb{R}^N,\\ u,v>0 \quad \text{in } \mathbb{R}^N, \quad u(x),v(x)\to 0 \quad \text{as } |x|\to\infty, \end{cases} \tag{1.3}$$

where $\lambda > 0$, and $\varepsilon > 0$ is sufficiently small.

The mathematical interest in (1.3) relies on its criticality, due to the fact that 2^* is the critical exponent. Critical exponent elliptic problems create some difficulties in using variational methods due to the lack of compactness, and have received great interest since the cerebrated work by Brezis and Nirenberg [10]. Before saying more about the background for problems like (1.3), we would like to introduce our main result first. Let C_{p+1} be the sharp constant of the Sobolev embedding $H^1(\mathbb{R}^N)\hookrightarrow L^{p+1}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\,dx \geqslant C_{p+1}\Big(\int_{\mathbb{R}^N} |u|^{p+1}\,dx\Big)^{\frac{2}{p+1}},$$

and S the sharp constant of $D^{1,2}(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \geqslant S\Big(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\Big)^{\frac{2}{2^*}}. \tag{1.4}$$

Here, $D^{1,2}(\mathbb{R}^N) := \{u\in L^2(\mathbb{R}^N)\colon |\nabla u|\in L^2(\mathbb{R}^N)\}$ with the norm

$$\|u\|_{D^{1,2}} := \Big(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\Big)^{1/2}.$$

Define

$$\mu_0 := \left[\frac{2(p+1)}{N(p-1)} S^{\frac{N}{2}} C_{p+1}^{-\frac{p+1}{p-1}}\right]^{(\frac{p+1}{p-1}-\frac{N}{2})^{-1}}. \tag{1.5}$$

Assume that

(V_1) $a,b\in C(\mathbb{R}^N,\mathbb{R})$ and there exist some constants $a_0>0$, $b_0>0$ such that

$$\inf_{x\in\mathbb{R}^N} a(x) = a_0 \leqslant \mu_0, \qquad \inf_{x\in\mathbb{R}^N} b(x) = b_0.$$

(V_2) There exists a smooth open bounded domain $\Lambda\subset\mathbb{R}^N$ such that

$$\inf_{x\in\Lambda} a(x) \leqslant \mu_0 < a_1 := \inf_{x\in\partial\Lambda} a(x). \tag{1.6}$$

Then the main result of this paper is the following

Theorem 1.1. *Let $N \geqslant 3$ and assumptions (V_1)–(V_2) hold. Assume that*

$$0 < \lambda < \min\{\sqrt{a_0 b_0}, \sqrt{(a_1 - \mu_0) b_0}\}. \tag{1.7}$$

Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, problem (1.3) has a positive solution $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$, which satisfies

(i) *there exists a maximum point $\tilde{x}_\varepsilon$ of $\tilde{u}_\varepsilon + \tilde{v}_\varepsilon$ such that $\tilde{x}_\varepsilon \in \Lambda$;*

(ii) *for any such an $\tilde{x}_\varepsilon$, $(w_{1,\varepsilon}(x), w_{2,\varepsilon}(x)) := (\tilde{u}_\varepsilon(\varepsilon x + \tilde{x}_\varepsilon), \tilde{v}_\varepsilon(\varepsilon x + \tilde{x}_\varepsilon))$ converge (up to a subsequence) to a positive ground state solution $(w_1(x), w_2(x))$ of*

$$\begin{cases} -\Delta u + a(P_0)u = u^p + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + b(P_0)v = v^{2^*-1} + \lambda u, & x \in \mathbb{R}^N, \\ u, v > 0 \quad \text{in } \mathbb{R}^N, & u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.8}$$

where $\tilde{x}_\varepsilon \to P_0 \in \Lambda$ as $\varepsilon \to 0$;

(iii) *there exist $c, C > 0$ independent of $\varepsilon > 0$ such that*

$$(\tilde{u}_\varepsilon + \tilde{v}_\varepsilon)(x) \leqslant C \exp\left(-\frac{c}{\varepsilon}|x - \tilde{x}_\varepsilon|\right).$$

Remark 1.1. The constant μ_0, that is defined in (1.5) and appears in assumptions (V_1)–(V_2), plays a crucial role in Theorem 1.1. As we will see in Lemma 2.2, the assumptions $\inf_{x \in \mathbb{R}^N} a(x) \leqslant \mu_0$ and $\inf_{x \in \Lambda} a(x) \leqslant \mu_0$ are both necessary for Theorem 1.1.

Remark 1.2. It is interesting that we only assume $\inf_{x \in \mathbb{R}^N} b(x) > 0$ without any further assumptions on the potential $b(x)$. In contrast, there have been many papers working on other kinds of elliptic systems (see [21,23,24,26] for example), where further assumptions on $b(x)$ seem always to be needed in the literature.

Remark 1.3. It was pointed out in [16, Remark 1.4] that, by Pohozaev Identity, the limit problem (1.8) has no nontrivial solutions if $p = 2^* - 1$. That is why we assume $1 < p < 2^* - 1$ in this paper.

For the scalar case of (1.3)

$$-\varepsilon^2 \Delta u + a(x)u = u^p, \quad x \in \mathbb{R}^N, \ u > 0, \tag{1.9}$$

where $1 < p < \frac{N+2}{N-2}$, there are many works on the existence of solutions which concentrate and develop spike layers, peaks, around some points in $\mathbb{R}^N$ while vanishing elsewhere as $\varepsilon \to 0$. The related results can be seen in [11–13,17, 18,25,27] and the references therein. On the other hand, the following critical problem

$$-\varepsilon^2 \Delta u + a(x)u = f(u) + u^{2^*-1}, \quad x \in \mathbb{R}^N, \ u > 0, \tag{1.10}$$

has also been well studied, where $f(u)$ is a subcritical nonlinearity, see [3,29] for example. Remark that in [3], $a(x)$ is assumed to be locally Hölder continuous. Recently, under more general assumptions on both $a(x)$ and $f(u)$ than those in [3], Zhang and the authors [29] proved the same result as in [3].

Meanwhile, there are an increasing interest in studying coupled nonlinear Schrödinger equations in recent years, which is motivated by applications to nonlinear optics and Bose–Einstein condensation (cf. [1,2,19]). For example, the following coupled Schrödinger equations

$$\begin{cases} -\varepsilon^2 \Delta u + a(x)u = \mu_1 u^3 + \beta u v^2, & x \in \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + b(x)v = \mu_2 v^3 + \beta v u^2, & x \in \mathbb{R}^N, \\ u > 0, \quad v > 0 \quad \text{in } \mathbb{R}^N, \\ u(x), v(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.11}$$

in subcritical case $N \leqslant 3$ have been well studied by [21,23,24,26]. Remark that, further assumptions on $b(x)$ are needed in all these works.

Note that the coupling term of (1.11) is nonlinear. Recently, linearly coupled Schrödinger equations of the following type

$$\begin{cases} -\Delta u+\mu u=|u|^{p-1}u+\lambda v, & x\in\mathbb{R}^N,\\ -\Delta v+\nu v=|v|^{p-1}v+\lambda u, & x\in\mathbb{R}^N,\\ u,v\in H^1(\mathbb{R}^N), \end{cases} \tag{1.12}$$

have been well studied from the cerebrated works by Ambrosetti et al. [4–6]. Systems of this type arise in nonlinear optics (cf. [1]). In the case of $N\leqslant 3$, $\mu=\nu=1$, $p=3$ and $\lambda>0$ small enough, Ambrosetti, Colorado and Ruiz [5] proved that (1.12) has multi-bump solitons. When $|u|^{p-1}u$ and $|v|^{p-1}v$ are replaced by $f(x,u)=(1+c(x))|u|^{p-1}u$ and $g(x,v)=(1+d(x))|v|^{p-1}v$ respectively, system (1.12) has been studied by Ambrosetti [4] with dimension $N=1$ and Ambrosetti, Cerami and Ruiz [6] with dimension $N\geqslant 2$. When $|u|^{p-1}u$ and $|v|^{p-1}v$ are replaced by general subcritical nonlinearities $f(u)$ and $g(v)$ respectively, (1.12) was studied by the authors [15].

Note that all works mentioned above deal with subcritical case. Recently, the authors [16] studied the ground state solutions of the following system with critical exponent

$$\begin{cases} -\Delta u+\mu u=u^p+\lambda v, & x\in\mathbb{R}^N,\\ -\Delta v+\nu v=v^{2^*-1}+\lambda u, & x\in\mathbb{R}^N,\\ u,v>0 \quad \text{in } \mathbb{R}^N, & u,v\in H^1(\mathbb{R}^N), \end{cases} \tag{1.13}$$

where $\mu,\nu>0$ and $0<\lambda<\sqrt{\mu\nu}$. Note that system (1.13) appears as a limit problem after a suitable rescaling of (1.3), see Theorem 1.1(ii) for example. As far as the semiclassical states related to (1.13) are concerned, we are naturally led to study system (1.3), and this is the goal of this paper.

There are several useful approaches to study semiclassical states. One is the classical Lyapunov–Schmidt reduction method, which cannot be used here, since the uniqueness and nondegeneracy of the ground state solutions of (1.13) are not known. Another one is Byeon and Jeanjean's variational approach [11], which cannot be used here either, since we have no any further assumptions on $b(x)$. Here, to prove Theorem 1.1, we will mainly follow the variational penalization scheme introduced by Del Pino and Felmer [17]. However, we should point out that the compactness is the main difficulty since (1.3) is a critical problem. Therefore, the method of Del Pino and Felmer [17] cannot be used directly and some crucial modifications and new tricks are needed.

The paper is organized as follows. In Section 2, we give a general assumption (V_3) on $a(x)$ and $b(x)$ instead of (V_2), and then give a general result. Theorem 1.1 will be a direct corollary of this general result. Some comments about (V_2) and (V_3) are also given. We will prove the general result in Section 3.

We give some notations here. Throughout this paper, we denote the norm of $L^p(\mathbb{R}^N)$ by $|u|_p=(\int_{\mathbb{R}^N}|u|^p\,dx)^{\frac{1}{p}}$, and the norm of $H^1(\mathbb{R}^N)$ by $\|u\|=\sqrt{|\nabla u|_2^2+|u|_2^2}$. We denote positive constants (possibly different in different places) by $c, C, C_0, C_1, \ldots$, and $B(x,r):=\{y\in\mathbb{R}^N\colon |x-y|<r\}$.

2. A general result and some comments

Consider the following coefficient problem

$$\begin{cases} -\Delta u+a(P)u=u^p+\lambda v, & x\in\mathbb{R}^N,\\ -\Delta v+b(P)v=v^{2^*-1}+\lambda u, & x\in\mathbb{R}^N,\\ u,v>0 \quad \text{in } \mathbb{R}^N, & u,v\in H^1(\mathbb{R}^N). \end{cases} \tag{2.1}$$

Define $H:=H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ with the norm $\|(u,v)\|^2:=\|u\|^2+\|v\|^2$. It is well known that solutions of (2.1) correspond to the critical points of C^2 functional $L_{P,\lambda}: H\to\mathbb{R}$ given by

$$L_{P,\lambda}(u,v)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+a(P)u^2+|\nabla v|^2+b(P)v^2\right)dx - \frac{1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\,dx-\frac{1}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}\,dx-\lambda\int_{\mathbb{R}^N}uv\,dx. \tag{2.2}$$

Define the Nehari manifold

$$\mathcal{N}_{P,\lambda} := \{(u,v) \in H \setminus \{(0,0)\}: L'_{P,\lambda}(u,v)(u,v) = 0\}, \tag{2.3}$$

and

$$m_\lambda(P) := \inf_{(u,v)\in\mathcal{N}_{P,\lambda}} L_{P,\lambda}(u,v),$$

then $(u,v) \in \mathcal{N}_{P,\lambda}$ satisfying $L_{P,\lambda}(u,v) = m_\lambda(P)$ will be a *ground state solution* of (2.1) (see [16]). Let us introduce the following assumption

(V_3) There exists a smooth open bounded domain $\Lambda \subset \mathbb{R}^N$ such that

$$m_{0,\lambda} := \inf_{P\in\Lambda} m_\lambda(P) < \inf_{P\in\partial\Lambda} m_\lambda(P).$$

Define

$$\mathcal{M}_\lambda := \{P \in \Lambda: m_\lambda(P) = m_{0,\lambda}\}. \tag{2.4}$$

In the sequel, the subscript λ will be dropped when there is no possible misunderstanding. Now we can state a general result.

Theorem 2.1. *Let $N \geqslant 3$ and assumptions (V_1) hold. Assume that (V_3) holds for some $\lambda \in (0, \sqrt{a_0 b_0})$, then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, problem (1.3) has a positive solution $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$, which satisfies*

(i) *there exists a maximum point $\tilde{x}_\varepsilon$ of $\tilde{u}_\varepsilon + \tilde{v}_\varepsilon$ such that $\tilde{x}_\varepsilon \in \Lambda$ and*

$$\lim_{\varepsilon\to 0} \operatorname{dist}(\tilde{x}_\varepsilon, \mathcal{M}) = 0;$$

(ii) *for any such an $\tilde{x}_\varepsilon$, $(w_{1,\varepsilon}(x), w_{2,\varepsilon}(x)) = (\tilde{u}_\varepsilon(\varepsilon x + \tilde{x}_\varepsilon), \tilde{v}_\varepsilon(\varepsilon x + \tilde{x}_\varepsilon))$ converge (up to a subsequence) to a positive ground state solution $(w_1(x), w_2(x))$ of (2.1) with $P = P_0$, where $\tilde{x}_\varepsilon \to P_0 \in \mathcal{M}$ as $\varepsilon \to 0$, and it satisfies that $L_{P_0}(w_1, w_2) = m(P_0) = m_0$;*

(iii) *there exist $c, C > 0$ independent of $\varepsilon > 0$ such that*

$$(\tilde{u}_\varepsilon + \tilde{v}_\varepsilon)(x) \leqslant C \exp\left(-\frac{c}{\varepsilon}|x - \tilde{x}_\varepsilon|\right).$$

Remark 2.1. Assumption (V_3) is an abstract condition, since we cannot write down explicitly the function $m_\lambda(P)$. Such type of abstract assumptions for system (1.11) can be seen in [21,24]. As we will see in Lemma 2.1, for our problem (1.3), we can show that (V_2) implies (V_3). However, for (1.11), (V_2) cannot imply (V_3) obviously, and further assumptions on $b(x)$ are needed (see [21,24]).

Recall μ_0 in (1.5); the following results have been proved by the authors [16].

Lemma A. *(See [16, Lemma 2.4].) Let $0 < \lambda < \sqrt{a(P)b(P)}$.*

(1) *If $0 < a(P) \leqslant \mu_0$, then $m_\lambda(P) < \frac{1}{N} S^{N/2}$.*
(2) *If $a(P) > \mu_0$, then there exists $\lambda_P \in [\sqrt{(a(P)-\mu_0)b(P)}, \sqrt{a(P)b(P)})$, such that*
 (i) *if $0 < \lambda \leqslant \lambda_P$, then $m_\lambda(P) = \frac{1}{N} S^{N/2}$;*
 (ii) *if $\lambda_P < \lambda < \sqrt{a(P)b(P)}$, then $m_\lambda(P) < \frac{1}{N} S^{N/2}$.*

Lemma B. *(See [16, Lemma 2.6].) Let $0 < \lambda < \sqrt{a(P)b(P)}$. If $m_\lambda(P) < \frac{1}{N} S^{N/2}$, then problem (2.1) has a positive ground state $(u_0, v_0) \in C^2(\mathbb{R}^N, \mathbb{R})$ such that u_0, v_0 are both radially symmetric decreasing.*

Theorem A. *(See [16, Theorem 1.1].) Assume $N \geqslant 3$, $1 < p < 2^* - 1$ and $0 < \lambda < \sqrt{a(P)b(P)}$.*

(1) *If* $0 < a(P) \leqslant \mu_0$, *then problem* (2.1) *has a positive ground state* (u, v), *such that* $L_P(u, v) = m_\lambda(P)$ *and* $u, v \in C^2(\mathbb{R}^N, \mathbb{R})$ *are both radially symmetric decreasing.*

(2) *If* $a(P) > \mu_0$, *then*
  (i) *if* $\lambda < \lambda_P$, *then* $m_\lambda(P)$ *is not attained, that is, problem* (2.1) *has no ground state solutions.*
  (ii) *if* $\lambda > \lambda_P$, *then problem* (2.1) *has a positive ground state* (u, v), *such that* $L_P(u, v) = m_\lambda(P)$ *and* $u, v \in C^2(\mathbb{R}^N, \mathbb{R})$ *are both radially symmetric decreasing.*

Remark 2.2. The assumption $0 < \lambda < \sqrt{a(P)b(P)}$ is needed in Lemmas A and B and Theorem A to guarantee that the Nehari manifold $\mathcal{N}_{P,\lambda}$ is bounded away from 0, and $0 < \lambda < \sqrt{a_0 b_0}$ is assumed in Theorems 1.1 and 2.1 such that $\mathcal{N}_{P,\lambda}$ is bounded away from 0 for all $P \in \mathbb{R}^N$.

Lemma 2.1. *Assume* (V_2). *Then* (V_3) *holds for any* λ *satisfying* (1.7). *Therefore, Theorem* 1.1 *is a direct corollary of Theorem* 2.1.

Proof. Fix any λ that satisfies (1.7). By (V_2), there exists some $P_1 \in \Lambda$ such that $a(P_1) \leqslant \mu_0$, then we see from Lemma A that

$$\inf_{P \in \Lambda} m_\lambda(P) \leqslant m_\lambda(P_1) < \frac{1}{N} S^{N/2}.$$

On the other hand, for any $P \in \partial\Lambda$, we have

$$\lambda < \sqrt{(a_1 - \mu_0) b_0} \leqslant \sqrt{\big(a(P) - \mu_0\big) b(P)} \leqslant \lambda_P,$$

then we see from Lemma A that $m_\lambda(P) = \frac{1}{N} S^{N/2}$, that is,

$$\inf_{P \in \partial\Lambda} m_\lambda(P) = \frac{1}{N} S^{N/2}.$$

Therefore, (V_3) holds. □

Lemma 2.2. *Assumptions* $\inf_{x \in \mathbb{R}^N} a(x) \leqslant \mu_0$ *and* $\inf_{x \in \Lambda} a(x) \leqslant \mu_0$ *in* (V_1)–(V_2) *are both necessary for Theorem* 1.1.

Proof. Suppose that Theorem 1.1 holds. Assume by contradiction that $a_2 := \inf_{x \in \Lambda} a(x) > \mu_0$. Fix any λ that satisfies

$$0 < \lambda < \min\big\{\sqrt{a_0 b_0}, \sqrt{(a_1 - \mu_0) b_0}, \sqrt{(a_2 - \mu_0) b_0}\big\}.$$

Theorem 1.1(ii) says that there exists some $P_0 \in \Lambda$ such that (2.1) has a ground state solution when $P = P_0$. On the other hand, since

$$\lambda < \sqrt{(a_2 - \mu_0) b_0} \leqslant \sqrt{\big(a(P_0) - \mu_0\big) b(P_0)} \leqslant \lambda_{P_0},$$

Theorem A says that (2.1) has no ground state solutions when $P = P_0$, a contradiction. So $\inf_{x \in \Lambda} a(x) \leqslant \mu_0$, and this implies $\inf_{x \in \mathbb{R}^N} a(x) \leqslant \mu_0$. □

Lemma 2.3. *Let* $\{P_n\colon n \in \mathbb{N}\} \subset \mathbb{R}^N$ *such that* $P_n \to P_0 \in \mathbb{R}^N$ *as* $n \to \infty$, *and fix any* $\lambda \in (0, \sqrt{a(P_0) b(P_0)})$. *If* $m_\lambda(P_0) < \frac{1}{N} S^{N/2}$, *then*

$$\lim_{n \to \infty} m_\lambda(P_n) = m_\lambda(P_0).$$

Proof. Since $P_n \to P_0$, we may assume that

$$\lambda < \sqrt{a(P_n) b(P_n)} - \delta, \quad \forall n \in \mathbb{N}, \tag{2.5}$$

where $\delta > 0$ is a small constant. In this proof, the subscript λ will be dropped. Then it is standard to prove that

$$m(P_n) = \inf_{(u,v) \in \mathcal{N}_{P_n}} L_{P_n}(u, v) = \inf_{(u,v) \neq (0,0)} \max_{t > 0} L_{P_n}(tu, tv). \tag{2.6}$$

Since $m(P_0) < \frac{1}{N}S^{N/2}$, we see from Lemma B that problem (2.1) has a ground state $(u_0, v_0) \in C^2(\mathbb{R}^N, \mathbb{R})$ when $P = P_0$, and $L_{P_0}(u_0, v_0) = m(P_0)$. It is easy to see that there exists $t_n > 0$ such that $(t_n u_0, t_n v_0) \in \mathcal{N}_{P_n}$ and $t_n \to 1$ as $n \to \infty$. Then

$$\limsup_{n\to\infty} m(P_n) \leqslant \limsup_{n\to\infty} L_{P_n}(t_n u_0, t_n v_0) = L_{P_0}(u_0, v_0) = m(P_0), \tag{2.7}$$

and so we may assume that $m(P_n) < \frac{1}{N}S^{N/2}$ for any $n \in \mathbb{N}$. Combining this with (2.5) and Lemma B, we see that (2.1) has a positive ground state solution (U_n, V_n) when $P = P_n$, and $L_{P_n}(U_n, V_n) = m(P_n)$. Moreover, U_n, V_n are both radially symmetric decreasing. By (2.5) we have

$$\begin{aligned} m(P_n) &= L_{P_n}(U_n, V_n) - \frac{1}{p+1} L'_{P_n}(U_n, V_n)(U_n, V_n) \\ &\geqslant \frac{p-1}{2p+2} \int_{\mathbb{R}^N} \left(|\nabla U_n|^2 + a(P_n)U_n^2 + |\nabla V_n|^2 + b(P_n)V_n^2 - 2\lambda U_n V_n\right) \\ &\geqslant C\left(\|U_n\|^2 + \|V_n\|^2\right), \quad \forall n \in \mathbb{N}, \end{aligned}$$

where $C > 0$ is independent of n. Hence, (U_n, V_n) is uniformly bounded in H. Passing to a subsequence, we may assume that $(U_n, V_n) \rightharpoonup (U_0, V_0)$ weakly in H. Then by repeating the proof of Lemma B in [16] with minor modifications, we can prove that $(U_n, V_n) \to (U_0, V_0)$ strongly in H and (U_0, V_0) is a nontrivial critical point of L_{P_0}. By (2.7) we have

$$\limsup_{n\to\infty} m(P_n) \leqslant m(P_0) \leqslant L_{P_0}(U_0, V_0) = \lim_{n\to\infty} L_{P_n}(U_n, V_n) = \lim_{n\to\infty} m(P_n).$$

This completes the proof. □

In fact, assumption (V_3) is more general than (V_2), and we believe that there may be some other kinds of conditions on $a(x)$ and $b(x)$ such that (V_3) holds. For example, we can show the following result.

Proposition 2.1. *Suppose (V_1) holds. Assume that there exist a smooth open bounded domain $\Lambda \subset \mathbb{R}^N$ and some $P_1 \in \Lambda$ such that $\sup_{x\in\overline{\Lambda}} a(x) \leqslant \mu_0$, and either*

$$a(P_1) < \inf_{x\in\partial\Lambda} a(x), \qquad b(P_1) \leqslant \inf_{x\in\partial\Lambda} b(x),$$

or

$$a(P_1) \leqslant \inf_{x\in\partial\Lambda} a(x), \qquad b(P_1) < \inf_{x\in\partial\Lambda} b(x). \tag{2.8}$$

Then (V_3) holds for any $\lambda \in (0, \sqrt{a_0 b_0})$.

Proof. Without loss of generality, we assume that (2.8) holds. Fix any $\lambda \in (0, \sqrt{a_0 b_0})$. In this proof, the subscript λ will be dropped. Lemma A says that $m(P) < \frac{1}{N}S^{N/2}$ for any $P \in \overline{\Lambda}$, so we see from Lemma 2.3 that $m(P)$ is continuous with respect to $P \in \overline{\Lambda}$. Then there exists $P_2 \in \partial\Lambda$ such that $m(P_2) = \inf_{P\in\partial\Lambda} m(P)$. By Lemma B we know that (2.1) has a positive ground state solution (U, V) when $P = P_2$, and $L_{P_2}(U, V) = m(P_2)$. Noting that $a(P_1) \leqslant a(P_2)$ and $b(P_1) < b(P_2)$, it is easy to see that there exists $0 < t_0 < 1$ such that $(t_0 U, t_0 V) \in \mathcal{N}_{P_1}$. Therefore,

$$\begin{aligned} m(P_1) \leqslant L_{P_1}(t_0 U, t_0 V) &= \left(\frac{1}{2} - \frac{1}{p+1}\right) t_0^{p+1} |U|_{p+1}^{p+1} + \left(\frac{1}{2} - \frac{1}{2^*}\right) t_0^{2^*} |V|_{2^*}^{2^*} \\ &< \left(\frac{1}{2} - \frac{1}{p+1}\right) |U|_{p+1}^{p+1} + \left(\frac{1}{2} - \frac{1}{2^*}\right) |V|_{2^*}^{2^*} \\ &= L_{P_2}(U, V) = m(P_2), \end{aligned}$$

that is,

$$\inf_{P\in\Lambda} m(P) \leqslant m(P_1) < m(P_2) = \inf_{P\in\partial\Lambda} m(P),$$

that is, (V_3) holds. □

3. Proof of Theorem 2.1

In the rest of this paper, we only need to prove Theorem 2.1. From now on, we assume that (V_1) hold, and we fix any $\lambda \in (0, \sqrt{a_0 b_0})$ such that (V_3) holds.

Define $a_\varepsilon(x) := a(\varepsilon x)$, $b_\varepsilon(x) := b(\varepsilon x)$. To study (1.3), it suffices to consider the following system

$$\begin{cases} -\Delta u + a_\varepsilon u = u^p + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + b_\varepsilon v = v^{2^*-1} + \lambda u, & x \in \mathbb{R}^N, \\ u > 0, \quad v > 0, \quad x \in \mathbb{R}^N, \\ u(x), v(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{3.1}$$

Let $H^1_{a,\varepsilon}$ (resp. $H^1_{b,\varepsilon}$) be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{a,\varepsilon} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + a_\varepsilon u^2 \, dx \right)^{\frac{1}{2}} \quad \left(\text{resp. } \|u\|_{b,\varepsilon} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + b_\varepsilon u^2 \, dx \right)^{\frac{1}{2}} \right).$$

Define $H_\varepsilon := H^1_{a,\varepsilon} \times H^1_{b,\varepsilon}$ with a norm $\|(u,v)\|_\varepsilon = \sqrt{\|u\|^2_{a,\varepsilon} + \|v\|^2_{b,\varepsilon}}$.

Define $\Lambda_\varepsilon := \{x \in \mathbb{R}^N \colon \varepsilon x \in \Lambda\}$ and

$$\chi_{\Lambda_\varepsilon}(x) := \begin{cases} 1 & \text{if } x \in \Lambda_\varepsilon, \\ 0 & \text{if } x \notin \Lambda_\varepsilon. \end{cases}$$

Note that $\lambda < \sqrt{a_0 b_0}$ implies the existence of $\delta_0 > 0$ such that $a_0 - \delta_0 \lambda > 0$ and $b_0 - \lambda/\delta_0 > 0$. Define

$$\alpha := \left(\frac{1}{2} \min\left\{ 1, a_0 - \delta_0 \lambda, b_0 - \frac{\lambda}{\delta_0} \right\} \right)^{\frac{1}{p-1}}, \tag{3.2}$$

then $\alpha < 1$ and $\alpha^{2^*-2} < \alpha^{p-1}$. Define

$$f(s) := \begin{cases} |s|^{p-1} s & \text{if } |s| \leqslant \alpha, \\ \alpha^{p-1} s & \text{if } |s| > \alpha, \end{cases} \qquad g(s) := \begin{cases} |s|^{2^*-2} s & \text{if } |s| \leqslant \alpha, \\ \alpha^{2^*-2} s & \text{if } |s| > \alpha, \end{cases}$$

and

$$\begin{aligned} f_\varepsilon(x,s) &:= \chi_{\Lambda_\varepsilon}(x) |s|^{p-1} s + \big(1 - \chi_{\Lambda_\varepsilon}(x)\big) f(s); \\ g_\varepsilon(x,s) &:= \chi_{\Lambda_\varepsilon}(x) |s|^{2^*-2} s + \big(1 - \chi_{\Lambda_\varepsilon}(x)\big) g(s). \end{aligned} \tag{3.3}$$

Let $F_\varepsilon(x,s) := \int_0^s f_\varepsilon(x,t)\,dt$, $F(s) := \int_0^s f(t)\,dt$ and $G_\varepsilon(x,s) := \int_0^s g_\varepsilon(x,t)\,dt$, $G(s) := \int_0^s g(t)\,dt$. Then

$$\big|f_\varepsilon(x,s)\big| \leqslant |s|^p, \qquad \big|g_\varepsilon(x,s)\big| \leqslant |s|^{2^*-1}, \quad x \in \mathbb{R}^N, \tag{3.4}$$

$$(p+1) F_\varepsilon(x,s) = f_\varepsilon(x,s) s, \qquad 2^* G_\varepsilon(x,s) = g_\varepsilon(x,s) s, \quad x \in \Lambda_\varepsilon, \tag{3.5}$$

$$2F_\varepsilon(x,s) \leqslant f_\varepsilon(x,s) s \leqslant \alpha^{p-1} s^2, \qquad 2G_\varepsilon(x,s) \leqslant g_\varepsilon(x,s) s \leqslant \alpha^{p-1} s^2, \quad x \notin \Lambda_\varepsilon. \tag{3.6}$$

Define a functional $J_\varepsilon : H_\varepsilon \to \mathbb{R}$ by

$$J_\varepsilon(u,v) := \frac{1}{2}\|u\|^2_{a,\varepsilon} + \frac{1}{2}\|v\|^2_{b,\varepsilon} - \int_{\mathbb{R}^N} \big(F_\varepsilon\big(x,u^+\big) + G_\varepsilon\big(x,v^+\big) + \lambda uv \big)\, dx. \tag{3.7}$$

Here and in the following, $u^+(x) := \max\{u(x), 0\}$ and so is v^+. By (V_1) we see that $H_\varepsilon \hookrightarrow H$ and there exists some $C > 0$ independent of $\varepsilon > 0$ such that

$$\big\|(u,v)\big\| \leqslant C \big\|(u,v)\big\|_\varepsilon. \tag{3.8}$$

Then it is standard to show that J_ε is well defined and $J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$. Furthermore, any critical points of J_ε are weak solutions of the following system

$$\begin{cases} -\Delta u + a_\varepsilon u = f_\varepsilon\big(x, u^+\big) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + b_\varepsilon v = g_\varepsilon\big(x, v^+\big) + \lambda u, & x \in \mathbb{R}^N, \\ u(x), v(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{3.9}$$

For each small $\varepsilon > 0$, we will find a nontrivial solution of (3.9) by applying mountain-pass argument to J_ε. Then we shall prove that this solution is a positive solution of (3.1) for $\varepsilon > 0$ sufficiently small.

By (V_1), (3.2) and (3.6), there exists $\mathcal{C} > 0$ independent of $\varepsilon > 0$ such that

$$\|u\|_{a,\varepsilon}^2 + \|v\|_{b,\varepsilon}^2 - \int\limits_{\mathbb{R}^N \setminus \Lambda_\varepsilon} \big(f_\varepsilon(x,u)u + g_\varepsilon(x,v)v\big) - 2\lambda \int\limits_{\mathbb{R}^N} uv \geqslant 2\mathcal{C}\big\|(u,v)\big\|_\varepsilon^2. \tag{3.10}$$

Then by (3.6)–(3.8) we have

$$\begin{aligned} J_\varepsilon(u,v) &\geqslant \mathcal{C}\big\|(u,v)\big\|_\varepsilon^2 - \frac{1}{p+1}\int\limits_{\Lambda_\varepsilon} \big|u^+\big|^{p+1}\,dx - \frac{1}{2^*}\int\limits_{\Lambda_\varepsilon}\big|v^+\big|^{2^*}\,dx \\ &\geqslant \mathcal{C}\big\|(u,v)\big\|_\varepsilon^2 - C\|u\|^{p+1} - C\|v\|^{2^*} \\ &\geqslant \mathcal{C}\big\|(u,v)\big\|_\varepsilon^2 - C\big\|(u,v)\big\|_\varepsilon^{p+1} - C\big\|(u,v)\big\|_\varepsilon^{2^*}, \end{aligned} \tag{3.11}$$

where $C > 0$ is independent of $\varepsilon > 0$. Therefore, there exist small $r, \alpha_1 > 0$ independent of $\varepsilon > 0$, such that

$$\inf_{\|(u,v)\|_\varepsilon = r} J_\varepsilon(u,v) \geqslant \alpha_1 > 0, \quad \forall \varepsilon > 0. \tag{3.12}$$

Define

$$c_\varepsilon := \inf_{\gamma \in \Phi_\varepsilon} \sup_{t\in[0,1]} J_\varepsilon\big(\gamma(t)\big), \tag{3.13}$$

where $\Phi_\varepsilon = \{\gamma \in C([0,1], H_\varepsilon)\colon\ \gamma(0) = (0,0),\ J_\varepsilon(\gamma(1)) < 0\}$. Take $\psi_\varepsilon \in C_0^\infty(\Lambda_\varepsilon, \mathbb{R})$ such that $\psi_\varepsilon \geqslant 0$ and $\psi_\varepsilon \not\equiv 0$, then $J_\varepsilon(t\psi_\varepsilon, t\psi_\varepsilon) \to -\infty$ as $t \to +\infty$. So $\Phi_\varepsilon \neq \emptyset$ and c_ε is well defined. By (3.12) we get

$$c_\varepsilon \geqslant \alpha_1 > 0, \quad \forall \varepsilon > 0. \tag{3.14}$$

By Lemma A we have $m(P) \leqslant \frac{1}{N}S^{N/2}$ for all $P \in \mathbb{R}^N$, and by (V_3) there holds

$$m_0 < \frac{1}{N}S^{N/2}. \tag{3.15}$$

Then by repeating the proof of Lemma 2.3 we obtain $\mathcal{M} \neq \emptyset$.

Lemma 3.1. $\limsup_{\varepsilon\to 0} c_\varepsilon \leqslant m_0 < \frac{1}{N}S^{N/2}$.

Proof. Take any $P \in \mathcal{M}$, then $m(P) = m_0 < \frac{1}{N}S^{N/2}$. By Lemma B we know that (2.1) has a positive ground state (U, V) such that $L_P(U,V) = m_0$. Take $T > 1$ such that $L_P(TU, TV) \leqslant -1$. Noting that there exists $R > 0$ such that $B(P,R) := \{x\colon |x-P| < R\} \subset \Lambda$, we take $\phi \in C_0^1(\mathbb{R}^N, \mathbb{R})$ with $0 \leqslant \phi \leqslant 1$, $\phi(x) \equiv 1$ for $|x| \leqslant R/2$ and $\phi(x) \equiv 0$ for $|x| \geqslant R$. Define $\phi_\varepsilon(x) := \phi(\varepsilon x)$, then $\phi_\varepsilon(x - P/\varepsilon) \neq 0$ implies $x \in \Lambda_\varepsilon$. Combining this with (3.3) and the Lebesgue Dominated Convergence Theorem, one has that

$$\begin{aligned} &J_\varepsilon\big(t(\phi_\varepsilon U)(\cdot - P/\varepsilon), t(\phi_\varepsilon V)(\cdot - P/\varepsilon)\big) \\ &\quad = \frac{t^2}{2}\int\limits_{|x|\leqslant R/\varepsilon} \big(\big|\nabla(\phi_\varepsilon U)\big|^2 + a(\varepsilon x + P)\phi_\varepsilon^2 U^2 + \big|\nabla(\phi_\varepsilon V)\big|^2 + b(\varepsilon x + P)\phi_\varepsilon^2 V^2\big) \\ &\qquad - \frac{t^{p+1}}{p+1}\int\limits_{|x|\leqslant R/\varepsilon} |\phi_\varepsilon U|^{p+1} - \frac{t^{2^*}}{2^*}\int\limits_{|x|\leqslant R/\varepsilon} |\phi_\varepsilon V|^{2^*} - \lambda t^2 \int\limits_{|x|\leqslant R/\varepsilon} \phi_\varepsilon^2 UV \\ &\quad \to L_P(tU, tV), \quad \text{as } \varepsilon \to 0,\ \text{uniformly for } t \in [0,T]. \end{aligned}$$

So $J_\varepsilon(T(\phi_\varepsilon U)(\cdot - P/\varepsilon), T(\phi_\varepsilon V)(\cdot - P/\varepsilon)) < 0$ for $\varepsilon > 0$ sufficiently small, that is, $\gamma_\varepsilon(t) := (tT(\phi_\varepsilon U)(\cdot - P/\varepsilon), tT(\phi_\varepsilon V)(\cdot - P/\varepsilon)) \in \Phi_\varepsilon$, so

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leqslant \limsup_{\varepsilon\to 0} \max_{t\in[0,1]} J_\varepsilon\big(tT(\phi_\varepsilon U)(\cdot - P/\varepsilon), tT(\phi_\varepsilon V)(\cdot - P/\varepsilon)\big)$$
$$= \max_{t\in[0,1]} L_P(tTU, tTV) = L_P(U, V) = m_0.$$

This completes the proof. □

Lemma 3.2. *There exists small $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, J_ε has a nontrivial critical point $(u_\varepsilon, v_\varepsilon) \in H_\varepsilon$, which satisfies that $u_\varepsilon > 0$, $v_\varepsilon > 0$ and $J_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon$.*

Proof. By Lemma 3.1, there exists small $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, we have $c_\varepsilon < \frac{1}{N} S^{N/2}$. Fix any $\varepsilon \in (0, \varepsilon_1)$. By the Mountain Pass Theorem [7], there exists $(u_n, v_n) \in H_\varepsilon$ such that

$$\lim_{n\to\infty} J_\varepsilon(u_n, v_n) = c_\varepsilon, \qquad \lim_{n\to\infty} J'_\varepsilon(u_n, v_n) = 0.$$

By (3.4)–(3.6) and (3.10), it is easy to see that

$$c_\varepsilon + o\big(\|(u_n, v_n)\|_\varepsilon\big) + o(1) \geqslant J_\varepsilon(u_n, v_n) - \frac{1}{p+1} J'_\varepsilon(u_n, v_n)(u_n, v_n)$$
$$\geqslant \frac{p-1}{2(p+1)} \mathcal{C} \|(u_n, v_n)\|_\varepsilon^2, \tag{3.16}$$

which implies that (u_n, v_n) is uniformly bounded in H_ε. Up to a subsequence, we may assume that $(u_n, v_n) \rightharpoonup (u_\varepsilon, v_\varepsilon)$ weakly in H_ε. Then $J'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$.

Step 1. We prove that $u_n \to u_\varepsilon$ strongly in $H^1_{a,\varepsilon}$.

Take R_0 such that $\Lambda_\varepsilon \subset B(0, R_0/2)$. For any $R \geqslant R_0$, we take a cut-off function $\eta_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ such that $0 \leqslant \eta_R \leqslant 1$, $\eta_R \equiv 0$ on $B(0, R/2)$, $\eta_R \equiv 1$ on $\mathbb{R}^N \setminus B(0, R)$ and $|\nabla \eta_R| \leqslant 10/R$. Then $\eta_R \equiv 0$ on Λ_ε. By (V_1), (3.2) and (3.6), there exists $C > 0$ independent of n such that

$$\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + a_\varepsilon u_n^2 + |\nabla v_n|^2 + b_\varepsilon v_n^2\big)\eta_R\,dx - \int_{\mathbb{R}^N} \big(f_\varepsilon(x, u_n)u_n + g_\varepsilon(x, v_n)v_n + 2\lambda u_n v_n\big)\eta_R\,dx$$
$$\geqslant C \int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + a_\varepsilon u_n^2 + |\nabla v_n|^2 + b_\varepsilon v_n^2\big)\eta_R\,dx. \tag{3.17}$$

Then we deduce from $J'_\varepsilon(u_n, v_n)(\eta_R u_n, \eta_R v_n) = o(1)$ that

$$\int_{\mathbb{R}^N} \big(|\nabla u_n|^2 + a_\varepsilon u_n^2 + |\nabla v_n|^2 + b_\varepsilon v_n^2\big)\eta_R\,dx \leqslant C \int_{\mathbb{R}^N} \big(|\nabla u_n||\nabla \eta_R||u_n| + |\nabla v_n||\nabla \eta_R||v_n|\big)\,dx + o(1)$$
$$\leqslant \frac{C}{R} + o(1), \tag{3.18}$$

that is,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B(0,R)} \big(|\nabla u_n|^2 + a_\varepsilon u_n^2 + |\nabla v_n|^2 + b_\varepsilon v_n^2\big)\,dx \leqslant \frac{C}{R}. \tag{3.19}$$

On the other hand, we may assume that $(u_n, v_n) \to (u_\varepsilon, v_\varepsilon)$ strongly in $L^q_{loc}(\mathbb{R}^N) \times L^q_{loc}(\mathbb{R}^N)$, where $2 \leqslant q < 2^*$. So

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \big|a_\varepsilon u_n^2 - a_\varepsilon u_\varepsilon^2\big|\,dx \leqslant \limsup_{n\to\infty} \int_{B(0,R)} \big|a_\varepsilon u_n^2 - a_\varepsilon u_\varepsilon^2\big|\,dx + \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B(0,R)} \big|a_\varepsilon u_n^2\big| + \big|a_\varepsilon u_\varepsilon^2\big|\,dx$$
$$\leqslant \frac{C}{R} \quad \text{holds for any } R \geqslant R_0,$$

that is,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} a_\varepsilon u_n^2\,dx=\int_{\mathbb{R}^N} a_\varepsilon u_\varepsilon^2\,dx. \tag{3.20}$$

Similarly we can prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} b_\varepsilon v_n^2\,dx=\int_{\mathbb{R}^N} b_\varepsilon v_\varepsilon^2\,dx; \tag{3.21}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f_\varepsilon\big(x,u_n^+\big)u_n\,dx=\int_{\mathbb{R}^N} f_\varepsilon\big(x,u_\varepsilon^+\big)u_\varepsilon\,dx; \tag{3.22}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} g_\varepsilon\big(x,v_n^+\big)v_n\,dx=\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} g_\varepsilon\big(x,v_\varepsilon^+\big)v_\varepsilon\,dx; \tag{3.23}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F_\varepsilon\big(x,u_n^+\big)\,dx=\int_{\mathbb{R}^N} F_\varepsilon\big(x,u_\varepsilon^+\big)\,dx; \tag{3.24}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} G_\varepsilon\big(x,v_n^+\big)\,dx=\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} G_\varepsilon\big(x,v_\varepsilon^+\big)\,dx; \tag{3.25}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} u_n v_n\,dx=\int_{\mathbb{R}^N} u_\varepsilon v_\varepsilon\,dx. \tag{3.26}$$

Then by $J_\varepsilon'(u_n,v_n)(u_n,0)=o(1)$ we get that

$$\lim_{n\to\infty}\|u_n\|_{a,\varepsilon}^2=\lim_{n\to\infty}\int_{\mathbb{R}^N} f_\varepsilon\big(x,u_n^+\big)u_n+\lambda u_n v_n\,dx=\int_{\mathbb{R}^N} f_\varepsilon\big(x,u_\varepsilon^+\big)u_\varepsilon+\lambda u_\varepsilon v_\varepsilon\,dx=\|u_\varepsilon\|_{a,\varepsilon}^2,$$

that is, $u_n\to u_\varepsilon$ strongly in $H^1_{a,\varepsilon}$.

Step 2. We prove that $v_n\to v_\varepsilon$ strongly in $H^1_{b,\varepsilon}$.

Denote $\omega_n=v_n-v_\varepsilon$ and $A:=\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla\omega_n|^2\,dx$. Fix any $\delta>0$, then it is easy to prove the existence of $C_\delta>1$ such that

$$\big|\big|(a+b)^+\big|^{2^*}-\big|a^+\big|^{2^*}\big|\leqslant\delta|a|^{2^*}+C_\delta|b|^{2^*},\quad\forall a,b\in\mathbb{R}. \tag{3.27}$$

Here $a^+:=\max\{a,0\}$ and so is $(a+b)^+$. Note that (3.27) can be easily checked by considering three different cases separately: (1) $a\geqslant 0, a+b\geqslant 0$; (2) $a\geqslant 0, a+b<0$; (3) $a<0, a+b\geqslant 0$. Then

$$f_n^\delta:=\big(\big|\big|v_n^+\big|^{2^*}-\big|\omega_n^+\big|^{2^*}-\big|v_\varepsilon^+\big|^{2^*}\big|-\delta\big|\omega_n^+\big|^{2^*}\big)^+\leqslant(1+C_\delta)|v_\varepsilon|^{2^*}.$$

By the Lebesgue Dominated Convergence Theorem, we see that $\int_{\Lambda_\varepsilon} f_n^\delta\,dx\to 0$ as $n\to\infty$. Note that

$$\big|\big|v_n^+\big|^{2^*}-\big|\omega_n^+\big|^{2^*}-\big|v_\varepsilon^+\big|^{2^*}\big|\leqslant f_n^\delta+\delta\big|\omega_n^+\big|^{2^*},$$

so

$$\limsup_{n\to\infty}\int_{\Lambda_\varepsilon}\big|\big|v_n^+\big|^{2^*}-\big|\omega_n^+\big|^{2^*}-\big|v_\varepsilon^+\big|^{2^*}\big|\leqslant C\delta.$$

Since $\delta>0$ is arbitrary, we get that

$$\lim_{n\to\infty}\int_{\Lambda_\varepsilon}\big(\big|v_n^+\big|^{2^*}-\big|\omega_n^+\big|^{2^*}\big)=\int_{\Lambda_\varepsilon}\big|v_\varepsilon^+\big|^{2^*}.$$

Combining this with (3.20)–(3.26), we deduce from $J'_\varepsilon(u_n, v_n)(0, v_n) = o(1)$ that

$$\begin{aligned}\lim_{n\to\infty} \|v_n\|_{b,\varepsilon}^2 &= \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla\omega_n|^2\,dx + \|v_\varepsilon\|_{b,\varepsilon}^2 \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} g_\varepsilon\big(x, v_n^+\big)v_n + \lambda u_n v_n\,dx \\ &= \lim_{n\to\infty} \int_{\Lambda_\varepsilon} \big|\omega_n^+\big|^{2^*} + \int_{\mathbb{R}^N} g_\varepsilon\big(x, v_\varepsilon^+\big)v_\varepsilon + \lambda u_\varepsilon v_\varepsilon\,dx \\ &= \lim_{n\to\infty} \int_{\Lambda_\varepsilon} \big|\omega_n^+\big|^{2^*} + \|v_\varepsilon\|_{b,\varepsilon}^2,\end{aligned}$$

that is,

$$A = \lim_{n\to\infty} \int_{\Lambda_\varepsilon} \big|\omega_n^+\big|^{2^*} \leqslant \lim_{n\to\infty} \int_{\mathbb{R}^N} |\omega_n|^{2^*} \leqslant \lim_{n\to\infty} \Big(S^{-1} \int_{\mathbb{R}^N} |\nabla\omega_n|^2\Big)^{2^*/2} = S^{-2^*/2} A^{2^*/2}.$$

If $A > 0$, then $A \geqslant S^{N/2}$. Similarly as (3.16), we have

$$J_\varepsilon(u_\varepsilon, v_\varepsilon) = J_\varepsilon(u_\varepsilon, v_\varepsilon) - \frac{1}{p+1} J'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon) \geqslant \frac{p-1}{2(p+1)}\mathcal{C}\big\|(u_\varepsilon, v_\varepsilon)\big\|_\varepsilon^2 \geqslant 0. \tag{3.28}$$

Combining this with (3.20)–(3.26), we get that

$$\begin{aligned}c_\varepsilon &= \lim_{n\to\infty} J_\varepsilon(u_n, v_n) \\ &= \frac{1}{2}\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla\omega_n|^2 - \frac{1}{2^*}\lim_{n\to\infty} \int_{\Lambda_\varepsilon} \big|\omega_n^+\big|^{2^*} + J_\varepsilon(u_\varepsilon, v_\varepsilon) \\ &= \frac{1}{N} A + J_\varepsilon(u_\varepsilon, v_\varepsilon) \geqslant \frac{1}{N} S^{N/2},\end{aligned}$$

a contradiction. So $A = 0$, that is, $v_n \to v_\varepsilon$ strongly in $H_{b,\varepsilon}^1$.

Step 3. We prove that $(u_\varepsilon, v_\varepsilon)$ is a nontrivial critical point of J_ε, which satisfies $J_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon$ and $u_\varepsilon > 0$, $v_\varepsilon > 0$.

By Step 1 and Step 2, we have $J_\varepsilon(u_\varepsilon, v_\varepsilon) = \lim_{n\to\infty} J_\varepsilon(u_n, v_n) = c_\varepsilon > 0$, so $(u_\varepsilon, v_\varepsilon)$ is a nontrivial critical point of J_ε, and we may assume that $u_\varepsilon \not\equiv 0$. Then (3.9) implies that $v_\varepsilon \not\equiv 0$. Define $u^-(x) := \max\{-u(x), 0\}$, then we see from $J'_\varepsilon(u_\varepsilon, v_\varepsilon)(u_\varepsilon^-, v_\varepsilon^-) = 0$ that

$$\big\|u_\varepsilon^-\big\|_{a,\varepsilon}^2 + \big\|v_\varepsilon^-\big\|_{b,\varepsilon}^2 = -\lambda \int_{\mathbb{R}^N} u_\varepsilon v_\varepsilon^- + u_\varepsilon^- v_\varepsilon\,dx \leqslant 2\lambda \int_{\mathbb{R}^N} u_\varepsilon^- v_\varepsilon^-\,dx \leqslant \delta_0\lambda \int_{\mathbb{R}^N} \big|u_\varepsilon^-\big|^2\,dx + \lambda/\delta_0 \int_{\mathbb{R}^N} \big|v_\varepsilon^-\big|^2\,dx,$$

so $(u_\varepsilon^-, v_\varepsilon^-) = (0, 0)$, that is, $u_\varepsilon \geqslant 0$ and $v_\varepsilon \geqslant 0$. On the other hand, by a Brezis–Kato type argument [9] on $u_\varepsilon + v_\varepsilon$, we see that $u_\varepsilon + v_\varepsilon \in L^q(\mathbb{R}^N)$ for any $q \geqslant 2$, and so $u_\varepsilon, v_\varepsilon \in W_{loc}^{2,q}(\mathbb{R}^N)$ for any $q \geqslant 2$. Then by Sobolev embedding we get that $u_\varepsilon, v_\varepsilon \in C_{loc}^{1,\beta}(\mathbb{R}^N)$ for any $\beta \in (0, 1)$. Therefore, by the strong maximum principle, we have $u_\varepsilon > 0$ and $v_\varepsilon > 0$. $\square$

Lemma 3.3. *For any $\varepsilon \in (0, \varepsilon_1)$, there holds $\|u_\varepsilon + v_\varepsilon\|_{L^\infty(\Lambda_\varepsilon)} \geqslant \alpha > 0$.*

Proof. If $\|u_\varepsilon + v_\varepsilon\|_{L^\infty(\Lambda_\varepsilon)} < \alpha$ for some $\varepsilon \in (0, \varepsilon_1)$, then

$$f_\varepsilon(x, u_\varepsilon)u_\varepsilon \leqslant \alpha^{p-1}u_\varepsilon^2, \qquad g_\varepsilon(x, v_\varepsilon)v_\varepsilon \leqslant \alpha^{2^*-2}v_\varepsilon^2 \leqslant \alpha^{p-1}v_\varepsilon^2, \quad \forall x \in \mathbb{R}^N.$$

So

$$\|u_\varepsilon\|_{a,\varepsilon}^2 + \|v\|_{b,\varepsilon}^2 \leqslant \int_{\mathbb{R}^N} \left(\alpha^{p-1}u_\varepsilon^2 + \alpha^{p-1}v_\varepsilon^2 + 2\lambda u_\varepsilon v_\varepsilon\right) dx.$$

Combining this with (3.2), we deduce that $(u_\varepsilon, v_\varepsilon) = (0, 0)$, a contradiction. □

Lemma 3.4. *There exist $\varepsilon_3 \in (0, \varepsilon_1)$, $C_0 > 0$ and $\{y_\varepsilon \in \Lambda_\varepsilon\colon \varepsilon \in (0, \varepsilon_3)\}$ such that*

$$\int_{B(y_\varepsilon,1)} \left(u_\varepsilon^2 + v_\varepsilon^2\right) dx \geqslant C_0 > 0, \quad \forall \varepsilon \in (0, \varepsilon_3). \tag{3.29}$$

Proof. Assume by contradiction that there exists $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$\lim_{\varepsilon_n \to 0} \sup_{y\in\mathbb{R}^N} \int_{B(y,1/2)} \left(u_{\varepsilon_n}^2 + v_{\varepsilon_n}^2\right)\chi_{\Lambda_{\varepsilon_n}}\, dx = 0,$$

then Lions Lemma [22] says that $(u_{\varepsilon_n}\chi_{\Lambda_{\varepsilon_n}}, v_{\varepsilon_n}\chi_{\Lambda_{\varepsilon_n}}) \to (0, 0)$ strongly in $L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*)$, and so

$$\lim_{\varepsilon_n \to 0} \int_{\Lambda_{\varepsilon_n}} |u_{\varepsilon_n}|^{p+1}\, dx = 0.$$

We write ε for ε_n for convenience. Note that

$$\begin{aligned} S\left(\int_{\mathbb{R}^N} |v_\varepsilon|^{2^*}\, dx\right)^{2/2^*} &\leqslant \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2\, dx \\ &\leqslant \|u_\varepsilon\|_{a,\varepsilon}^2 + \|v_\varepsilon\|_{b,\varepsilon}^2 - \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left(f_\varepsilon(x, u_\varepsilon)u_\varepsilon + g_\varepsilon(x, v_\varepsilon)v_\varepsilon\right) - 2\lambda \int_{\mathbb{R}^N} u_\varepsilon v_\varepsilon \\ &= \int_{\Lambda_\varepsilon} |u_\varepsilon|^{p+1} + \int_{\Lambda_\varepsilon} |v_\varepsilon|^{2^*}. \end{aligned} \tag{3.30}$$

Letting $\varepsilon \to 0$ and denoting $A := \lim_{\varepsilon\to 0} \int_{\Lambda_\varepsilon} |v_\varepsilon|^{2^*}$, we get that $SA^{2/2^*} \leqslant A$, so $A = 0$ or $A \geqslant S^{N/2}$. Meanwhile, we see from Sobolev inequalities, (3.10) and (3.30) that

$$\left\|(u_\varepsilon, v_\varepsilon)\right\|_\varepsilon^2 \leqslant C\left\|(u_\varepsilon, v_\varepsilon)\right\|_\varepsilon^{p+1} + C\left\|(u_\varepsilon, v_\varepsilon)\right\|_\varepsilon^{2^*},$$

where $C > 0$ is independent of $\varepsilon > 0$. Since $(u_\varepsilon, v_\varepsilon) \neq (0, 0)$, so

$$\int_{\Lambda_\varepsilon} |u_\varepsilon|^{p+1} + \int_{\Lambda_\varepsilon} |v_\varepsilon|^{2^*} \geqslant C\left\|(u_\varepsilon, v_\varepsilon)\right\|_\varepsilon \geqslant C > 0, \quad \forall \varepsilon > 0.$$

Letting $\varepsilon \to 0$ we see that $A > 0$. So $A \geqslant S^{N/2}$, and by (3.5)–(3.6) we obtain

$$\lim_{\varepsilon\to 0} c_\varepsilon = \lim_{\varepsilon\to 0}\left(J_\varepsilon(u_\varepsilon, v_\varepsilon) - \frac{1}{2}J_\varepsilon'(u_\varepsilon, v_\varepsilon)(u_\varepsilon, v_\varepsilon)\right) \geqslant \lim_{\varepsilon\to 0} \frac{1}{N}\int_{\Lambda_\varepsilon} |v_\varepsilon|^{2^*} \geqslant \frac{1}{N}S^{N/2},$$

which contradicts with Lemma 3.1. Therefore there exists $C_0 > 0$ such that

$$\liminf_{\varepsilon\to 0} \sup_{y\in\mathbb{R}^N} \int_{B(y,1/2)} \left(u_\varepsilon^2 + v_\varepsilon^2\right)\chi_{\Lambda_\varepsilon}\, dx \geqslant 2C_0.$$

That is, there exist $\varepsilon_3 \in (0, \varepsilon_1)$ and $\{z_\varepsilon \in \mathbb{R}^N\colon\ \varepsilon \in (0, \varepsilon_3)\}$ such that

$$\int\limits_{B(z_\varepsilon,1/2)\cap\Lambda_\varepsilon} \left(u_\varepsilon^2 + v_\varepsilon^2\right) dx \geqslant C_0 > 0, \quad \forall \varepsilon \in (0, \varepsilon_3).$$

Taking $y_\varepsilon \in B(z_\varepsilon, 1/2) \cap \Lambda_\varepsilon$, we see that (3.29) holds. □

Lemma 3.5. *There hold* $\lim_{\varepsilon\to 0} \operatorname{dist}(\varepsilon y_\varepsilon, \mathcal{M}) = 0$ *and* $\lim_{\varepsilon\to 0} c_\varepsilon = m_0$*. Moreover, for any* $\varepsilon_n \to 0$*,* $(w_{1,\varepsilon_n}(x), w_{2,\varepsilon_n}(x)) := (u_{\varepsilon_n}(x + y_{\varepsilon_n}), v_{\varepsilon_n}(x + y_{\varepsilon_n}))$ *converge (up to a subsequence, in the sense of* $\|\cdot\|$ *in* H*) to a positive ground state solution* $(w_1(x), w_2(x))$ *of* (2.1) *with* $P = P_0$*, where* $\varepsilon_n y_{\varepsilon_n} \to P_0 \in \mathcal{M}$ *as* $\varepsilon_n \to 0$*, and it satisfies that* $L_{P_0}(w_1, w_2) = m(P_0) = m_0$*.*

Proof. By Lemma 3.1 and (3.28), we may assume that $\{(u_\varepsilon, v_\varepsilon)\colon\ \varepsilon \in (0, \varepsilon_3)\}$ is uniformly bounded in H_ε and so in H. Take any $\varepsilon_n \to 0$. Up to a subsequence, we may assume that $(w_{1,\varepsilon_n}, w_{2,\varepsilon_n}) \rightharpoonup (w_1, w_2)$ weakly in H. Then we see from Lemma 3.4 that

$$\int\limits_{B(0,1)} \left(w_1^2 + w_2^2\right) dx \geqslant C_0 > 0,$$

that is, $(w_1, w_2) \neq (0, 0)$ and $w_1, w_2 \geqslant 0$. For convenience, we write ε for ε_n. Since $y_\varepsilon \in \Lambda_\varepsilon$, up to a subsequence, we may assume that $\varepsilon y_\varepsilon \to P_0 \in \overline{\Lambda}$. Since Λ is smooth, up to a subsequence, we may assume that $\chi_{\Lambda_\varepsilon}(\cdot + y_\varepsilon)$ converges almost everywhere to χ, where $0 \leqslant \chi \leqslant 1$. In fact, χ is either the characteristic function of $\mathbb{R}^N$ or the characteristic function of a half space. Then (w_1, w_2) satisfies

$$\begin{cases} -\Delta u + a(P_0)u = \chi u^p + (1-\chi) f(u) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + b(P_0)v = \chi v^{2^*-1} + (1-\chi) g(v) + \lambda u, & x \in \mathbb{R}^N, \\ u, v \in H^1\left(\mathbb{R}^3\right), \quad u > 0, \quad v > 0. \end{cases} \tag{3.31}$$

Define the functional of (3.31) as

$$\begin{aligned} L(u, v) := {}& \frac{1}{2} \int\limits_{\mathbb{R}^N} \left(|\nabla u|^2 + a(P_0)u^2 + |\nabla v|^2 + b(P_0)v^2 - 2\lambda uv\right) dx \\ & - \int\limits_{\mathbb{R}^N} \left[\chi\left(\frac{|u|^{p+1}}{p+1} + \frac{|v|^{2^*}}{2^*}\right) + (1-\chi)\left(F(u) + G(v)\right)\right] dx. \end{aligned} \tag{3.32}$$

Noting that $F(u) \leqslant \frac{|u|^{p+1}}{p+1}$ and $G(v) \leqslant \frac{|v|^{2^*}}{2^*}$, we see from (2.6) that

$$L(w_1, w_2) = \max_{t>0} L(tw_1, tw_2) \geqslant \max_{t>0} L_{P_0}(tw_1, tw_2) \geqslant m(P_0) \geqslant m_0.$$

Then by (3.4)–(3.6) and Fatou Lemma we have

$$\begin{aligned} m_0 \leqslant m(P_0) \leqslant L(w_1, w_2) &= L(w_1, w_2) - \frac{1}{2} L'(w_1, w_2)(w_1, w_2) \\ &= \int\limits_{\mathbb{R}^N} \left[\left(\frac{1}{2} - \frac{1}{p+1}\right)\chi w_1^{p+1} + \left(\frac{1}{2} - \frac{1}{2^*}\right)\chi w_2^{2^*}\right] \\ &\quad + \int\limits_{\mathbb{R}^N} (1-\chi)\left(\frac{1}{2} f(w_1)w_1 - F(w_1) + \frac{1}{2} g(w_2)w_2 - G(w_2)\right) \\ &\leqslant \lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^N} \left[\frac{1}{2} f_\varepsilon(x + y_\varepsilon, w_{1,\varepsilon})w_{1,\varepsilon} - F_\varepsilon(x + y_\varepsilon, w_{1,\varepsilon})\right] \end{aligned}$$

$$
\begin{aligned}
&+\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left[\frac{1}{2}g_\varepsilon(x+y_\varepsilon,w_{2,\varepsilon})w_{2,\varepsilon}-G_\varepsilon(x+y_\varepsilon,w_{2,\varepsilon})\right]\\
&=\lim_{\varepsilon\to 0}\left[J_\varepsilon(u_\varepsilon,v_\varepsilon)-\frac{1}{2}J'_\varepsilon(u_\varepsilon,v_\varepsilon)(u_\varepsilon,v_\varepsilon)\right]=\lim_{\varepsilon\to 0}c_\varepsilon\\
&\leqslant m_0.
\end{aligned}
\tag{3.33}
$$

This means that all inequalities in (3.33) are identities. Firstly, $\lim_{\varepsilon\to 0}c_\varepsilon = m_0$. Secondly, we have $m(P_0) = m_0$, so $P_0 \in \mathcal{M}$, which implies $\lim_{\varepsilon\to 0}\operatorname{dist}(\varepsilon y_\varepsilon,\mathcal{M}) = 0$ and so $\Lambda_\varepsilon - y_\varepsilon \to \mathbb{R}^N$, that is, $\chi \equiv 1$. Then $L(w_1,w_2) = L_{P_0}(w_1,w_2) = m(P_0)$, that is, (w_1,w_2) is a ground state solution of (2.1) with $P = P_0$. By the strong maximum principle, $w_1, w_2 > 0$. Thirdly, by putting $\chi \equiv 1$ into (3.33), we have

$$
\lim_{\varepsilon\to 0}\int_{\Lambda_\varepsilon-y_\varepsilon} w_{1,\varepsilon}^{p+1}\,dx=\int_{\mathbb{R}^N} w_1^{p+1}\,dx,
$$

$$
\lim_{\varepsilon\to 0}\int_{\Lambda_\varepsilon-y_\varepsilon} w_{2,\varepsilon}^{2^*}\,dx=\int_{\mathbb{R}^N} w_2^{2^*}\,dx.
$$

By (V_1), (3.2) and (3.6), there exists small $\delta > 0$ independent of $\varepsilon > 0$ such that

$$
\begin{aligned}
A_\varepsilon :=&\int_{\mathbb{R}^N}\big(a_\varepsilon(x+y_\varepsilon)-\delta\big)w_{1,\varepsilon}^2+\big(b_\varepsilon(x+y_\varepsilon)-\delta\big)w_{2,\varepsilon}^2\,dx\\
&-\int_{x+y_\varepsilon\in\mathbb{R}^N\setminus\Lambda_\varepsilon}\big(f(w_{1,\varepsilon})w_{1,\varepsilon}+g(w_{2,\varepsilon})w_{2,\varepsilon}\big)-2\lambda\int_{\mathbb{R}^N} w_{1,\varepsilon}w_{2,\varepsilon}\geqslant 0.
\end{aligned}
\tag{3.34}
$$

Then by Fatou Lemma we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\big(|\nabla w_1|^2+\delta w_1^2+|\nabla w_2|^2+\delta w_2^2\big)+\int_{\mathbb{R}^N}\big[\big(a(P_0)-\delta\big)w_1^2+\big(b(P_0)-\delta\big)w_2^2-2\lambda w_1w_2\big]\,dx\\
&\quad=\int_{\mathbb{R}^N}\big(w_1^{p+1}+w_2^{2^*}\big)\,dx=\lim_{\varepsilon\to 0}\int_{\Lambda_\varepsilon-y_\varepsilon}\big(w_{1,\varepsilon}^{p+1}+w_{2,\varepsilon}^{2^*}\big)\,dx\\
&\quad=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\big(|\nabla w_{1,\varepsilon}|^2+\delta w_{1,\varepsilon}^2+|\nabla w_{2,\varepsilon}|^2+\delta w_{2,\varepsilon}^2\big)+\lim_{\varepsilon\to 0}A_\varepsilon\\
&\quad\geqslant\int_{\mathbb{R}^N}\big(|\nabla w_1|^2+\delta w_1^2+|\nabla w_2|^2+\delta w_2^2\big)+\int_{\mathbb{R}^N}\big[\big(a(P_0)-\delta\big)w_1^2+\big(b(P_0)-\delta\big)w_2^2-2\lambda w_1w_2\big]\,dx,
\end{aligned}
$$

which implies that $(w_{1,\varepsilon}, w_{2,\varepsilon}) \to (w_1, w_2)$ strongly in H. □

To continue our proof, we need the following lemma which is a special case of Lemma 8.17 in [20] for Δ.

Lemma 3.6. *(See [20, Lemma 8.17].) Let Ω be an open subset of $\mathbb{R}^N$. Suppose that $t > N$, $h \in L^{\frac{t}{2}}(\Omega)$ and $u \in H^1(\Omega)$ satisfies $-\Delta u(y) \leqslant h(y)$, $y \in \Omega$ in the weak sense. Then for any ball $B(y, 2r) \subset \Omega$,*

$$
\sup_{B(y,r)} u\leqslant C\big(\|u^+\|_{L^2(B(y,2r))}+\|h\|_{L^{t/2}(B(y,2r))}\big),
$$

where $C = C(N, t, r)$ is independent of y, and $u^+ = \max\{0, u\}$.

Lemma 3.7. *There exist $\varepsilon_4 \in (0, \varepsilon_3)$ and $C_2 > 0$ such that*

$$
\|u_\varepsilon+v_\varepsilon\|_{L^\infty(\mathbb{R}^N)}\leqslant C_2,\quad \forall 0<\varepsilon<\varepsilon_4.
$$

Proof. In this proof, we use the Moser iteration. Assume by contradiction that there exists $\varepsilon_n \to 0$ such that

$$\lim_{n\to\infty} \|u_{\varepsilon_n} + v_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^N)} = +\infty. \tag{3.35}$$

Recall $(w_{1,\varepsilon_n}, w_{2,\varepsilon_n})$ in Lemma 3.5, up to a subsequence, we may assume that $(w_{1,\varepsilon_n}, w_{2,\varepsilon_n}) \to (w_1, w_2)$ strongly in H. Denote $w_n = w_{1,\varepsilon_n} + w_{2,\varepsilon_n}$ and $w = w_1 + w_2$, then $w_n \to w$ strongly in $H^1(\mathbb{R}^N)$ and (3.35) implies that

$$\lim_{n\to\infty} \|w_n\|_{L^\infty(\mathbb{R}^N)} = +\infty. \tag{3.36}$$

Since $(u_{\varepsilon_n}, v_{\varepsilon_n})$ is a solution of (3.9), by (3.4) we see that

$$-\Delta w_n \leqslant w_n^p + w_n^{2^*-1} + \lambda w_n \leqslant 2w_n^{2^*-1} + \lambda_1 w_n, \quad x \in \mathbb{R}^N, \tag{3.37}$$

where $\lambda_1 = \lambda + 1 > 0$.

First, for any $s \geqslant 0$, we claim that

$$\sup_n |w_n|_{2(s+1)} \leqslant C_1(s) \quad \Longrightarrow \quad \sup_n |w_n|_{2^*(s+1)} \leqslant C_2(s), \tag{3.38}$$

where $C_i(s)$ $(i = 1, 2)$ are positive constants independent of n.

Choose $l > 0$ and set

$$\psi_{n,l} := \min\{w_n^s, l\}, \qquad \varphi_{n,l} = w_n \psi_{n,l}^2, \qquad \Omega_{n,l} = \{x \in \mathbb{R}^N\colon\ w_n^s \leqslant l\},$$
$$\chi_{\Omega_{n,l}} = \begin{cases} 1 & \text{if } x \in \Omega_{n,l}, \\ 0 & \text{if } u \notin \Omega_{n,l}. \end{cases}$$

Then

$$\nabla(w_n\psi_{n,l}) = (1 + s\chi_{\Omega_{n,l}})\psi_{n,l}\nabla w_n, \qquad \nabla\varphi_{n,l} = (1 + 2s\chi_{\Omega_{n,l}})\psi_{n,l}^2\nabla w_n,$$

and $\varphi_{n,l} \in H^1(\mathbb{R}^N)$. By (3.37) we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w_n|^2\psi_{n,l}^2 &\leqslant \int_{\mathbb{R}^N} \nabla w_n \cdot \nabla\varphi_{n,l} \leqslant \int_{\mathbb{R}^N} \left(\lambda_1 w_n + 2w_n^{2^*-1}\right) w_n \psi_{n,l}^2 \\
&\leqslant \lambda_1 \int_{\mathbb{R}^N} w_n^{2(s+1)} + 2\int_{\mathbb{R}^N} w_n^{2^*}\psi_{n,l}^2 \leqslant C + 2\int_{\mathbb{R}^N} w_n^{2^*}\psi_{n,l}^2.
\end{aligned}$$

While, by Sobolev embedding (1.4) we have

$$\begin{aligned}
\int_{\mathbb{R}^N} w_n^{2^*}\psi_{n,l}^2 &\leqslant \int_{\mathbb{R}^N} w^{2^*-2} w_n^2\psi_{n,l}^2 + \int_{\mathbb{R}^N} \left|w_n^{2^*-2} - w^{2^*-2}\right| w_n^2\psi_{n,l}^2 \\
&\leqslant k^{2^*-2}\int_{\{w\leqslant k\}} w_n^{2(s+1)} + \int_{\{w>k\}} w^{2^*-2}w_n^2\psi_{n,l}^2 + \left(\int_{\mathbb{R}^N} \left|w_n^{2^*-2} - w^{2^*-2}\right|^{N/2}\right)^{2/N} \left(\int_{\mathbb{R}^N} w_n^{2^*}\psi_{n,l}^{2^*}\right)^{2/2^*} \\
&\leqslant C(k,n)\int_{\mathbb{R}^N} \left|\nabla(w_n\psi_{n,l})\right|^2 + Ck^{2^*-2},
\end{aligned} \tag{3.39}$$

where

$$C(k,n) := S^{-1}\left(\int_{\mathbb{R}^N} \left|w_n^{2^*-2} - w^{2^*-2}\right|^{N/2}\right)^{2/N} + S^{-1}\left(\int_{\{w>k\}} w^{2^*}\right)^{2/N}. \tag{3.40}$$

Therefore,

$$\int_{\mathbb{R}^N} \left|\nabla(w_n\psi_{n,l})\right|^2 \leqslant (1+s)^2\int_{\mathbb{R}^N} |\nabla w_n|^2\psi_{n,l}^2 \leqslant 2(1+s)^2 C(k,n)\int_{\mathbb{R}^N} \left|\nabla(w_n\psi_{n,l})\right|^2 + Ck^{2^*-2} + C.$$

Since $w_n \to w$ in $H^1(\mathbb{R}^N)$, we have $w_n^{2^*-2} \to w^{2^*-2}$ in $L^{N/2}(\mathbb{R}^N)$. By (3.40), there exist $k_0 > 0$ and $n_0 > 0$ large enough, such that for any $n \geqslant n_0$ we have $2(1+s)^2 C(k_0, n) \leqslant \frac{1}{2}$, where k_0 is independent of $n \in \mathbb{N}$. This implies that

$$\int_{\Omega_{n,l}} |\nabla(w_n^{s+1})|^2 \leqslant \int_{\mathbb{R}^N} |\nabla(w_n \psi_{n,l})|^2 \leqslant 2Ck_0^{2^*-2} + 2C = C(s), \quad n \geqslant n_0.$$

Letting $l \to +\infty$, we get that $\int_{\mathbb{R}^N} |\nabla(w_n^{s+1})|^2 \leqslant C(s)$, $n \geqslant n_0$. By (1.4) again, we have that $\int_{\mathbb{R}^N} w_n^{2^*(s+1)} \leqslant S^{-2^*/2} C(s)^{2^*/2}$, $n \geqslant n_0$. On the other hand,

$$\int_{\mathbb{R}^N} w_n^{2^*} \psi_{n,l}^2 \leqslant k^{2^*-2} \int_{\{w_n \leqslant k\}} w_n^{2(s+1)} + \int_{\{w_n > k\}} w_n^{2^*-2} w_n^2 \psi_{n,l}^2 \leqslant \tilde{C}(k,n) \int_{\mathbb{R}^N} |\nabla(w_n \psi_{n,l})|^2 + Ck^{2^*-2}, \tag{3.41}$$

where

$$\tilde{C}(k,n) := S^{-1} \left(\int_{\{w_n > k\}} w_n^{2^*} \right)^{2/N}. \tag{3.42}$$

Since there exists $\tilde{k}_0 > 0$ large enough, such that $2(1+s)^2 \tilde{C}(\tilde{k}_0, n) \leqslant \frac{1}{2}$ for any $n \leqslant n_0$, then by repeating the arguments above, we have $\sup_{n \leqslant n_0} \int_{\mathbb{R}^N} w_n^{2^*(s+1)} \leqslant C$. This proves the claim (3.38).

Note that w_n is uniformly bounded in $H^1(\mathbb{R}^N)$ and so in $L^2(\mathbb{R}^N)$. Letting $s_1 = 0$ and using a bootstrap argument, we see from the claim that for any $q \geqslant 2$, there exists $C(q) > 0$ such that $\sup_n |w_n|_q \leqslant C(q)$. By (3.37) and Lemma 3.6 we see that $\{w_n\}_n$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$, a contradiction with (3.36). This completes the proof. □

Proof of Theorem 2.1. Define $(w_{1,\varepsilon}(x), w_{2,\varepsilon}(x)) := (u_\varepsilon(x+y_\varepsilon), v_\varepsilon(x+y_\varepsilon))$ and $w_\varepsilon := w_{1,\varepsilon} + w_{2,\varepsilon}$. Similarly as (3.37), by Lemma 3.7 and $N \geqslant 3$ we have

$$-\Delta w_\varepsilon \leqslant (1+\lambda) w_\varepsilon + 2 w_\varepsilon^{2^*-1} \leqslant C w_\varepsilon^{\frac{4}{N+1}}, \tag{3.43}$$

where $C > 0$ is independent of $\varepsilon > 0$.

Step 1. We prove that there exists $\varepsilon_5 \in (0, \varepsilon_4)$ such that

$$\lim_{R\to\infty} \int_{|x| \geqslant R} w_\varepsilon^2 \, dx = 0, \quad \text{uniformly for } 0 < \varepsilon < \varepsilon_5. \tag{3.44}$$

If not, then there exist $\varepsilon_n \to 0$, $R_n \to \infty$ and $\gamma > 0$ such that

$$\int_{|x| \geqslant R_n} w_{\varepsilon_n}^2 \, dx \geqslant \gamma > 0, \quad \forall n \in \mathbb{N}. \tag{3.45}$$

By Lemma 3.5, up to a subsequence, we may assume that $(w_{1,\varepsilon_n}, w_{2,\varepsilon_n}) \to (w_1, w_2)$ strongly in H. Denote $w = w_1 + w_2$, then $w_{\varepsilon_n} \to w$ strongly in $H^1(\mathbb{R}^N)$, and so

$$\lim_{n\to\infty} \int_{|x| \geqslant R_n} w_{\varepsilon_n}^2 \leqslant \lim_{n\to\infty} 2 \int_{\mathbb{R}^N} |w_{\varepsilon_n} - w|^2 + \lim_{n\to\infty} 2 \int_{|x| \geqslant R_n} w^2 = 0,$$

a contradiction with (3.45).

Note that $w_\varepsilon^{\frac{4}{N+1}} \in L^{\frac{N+1}{2}}(\mathbb{R}^N)$ and

$$\left| w_\varepsilon^{\frac{4}{N+1}} \right|_{L^{\frac{N+1}{2}}(B(y,2r))} = |w_\varepsilon|_{L^2(B(y,2r))}^{\frac{4}{N+1}},$$

then by (3.43)–(3.44) and Lemma 3.6, we deduce that

$$\lim_{|x|\to\infty} w_\varepsilon(x) = 0, \quad \text{uniformly for } 0 < \varepsilon < \varepsilon_5. \tag{3.46}$$

Step 2. We prove that there exists $\varepsilon_6 \in (0, \varepsilon_5)$ such that $(u_\varepsilon, v_\varepsilon)$ is a solution of the original problem (3.1) for all $0 < \varepsilon < \varepsilon_6$.

By (3.46), there exists $R > 0$ such that

$$w_\varepsilon(x) \leqslant \alpha/2, \quad \forall |x| \geqslant R, \ \forall 0 < \varepsilon < \varepsilon_5. \tag{3.47}$$

Assumption (V_3) implies that $\operatorname{dist}(\mathcal{M}, \mathbb{R}^N \setminus \Lambda) = 2\delta_1$ for some $\delta_1 > 0$. Note that $\varepsilon y_\varepsilon \in \Lambda$ and $\lim_{\varepsilon \to 0} \operatorname{dist}(\varepsilon y_\varepsilon, \mathcal{M}) = 0$, therefore we may assume that $\operatorname{dist}(\varepsilon y_\varepsilon, \mathbb{R}^N \setminus \Lambda) \geqslant \delta_1$ for all $0 < \varepsilon < \varepsilon_5$. Let $\varepsilon_6 := \min\{\varepsilon_5, \frac{\delta_1}{R}\}$ and fix any $\varepsilon \in (0, \varepsilon_6)$. Then $x \notin \Lambda_\varepsilon$ implies that $|x - y_\varepsilon| \geqslant \frac{\delta_1}{\varepsilon} \geqslant R$, so

$$u_\varepsilon(x) + v_\varepsilon(x) = w_\varepsilon(x - y_\varepsilon) \leqslant \alpha/2, \quad \forall x \notin \Lambda_\varepsilon, \tag{3.48}$$

that is, $f_\varepsilon(x, u_\varepsilon(x)) \equiv u_\varepsilon^p(x)$ and $g_\varepsilon(x, v_\varepsilon(x)) \equiv v_\varepsilon^{2^*-1}(x)$, and so $(u_\varepsilon, v_\varepsilon)$ is a solution of the original problem (3.1) for all $0 < \varepsilon < \varepsilon_6$.

Step 3. We prove that $w_\varepsilon(x) \leqslant C \exp(-c|x|)$, $\forall 0 < \varepsilon < \varepsilon_6$, where $c, C > 0$ are independent of $\varepsilon > 0$.

Recall δ_0 in (3.2); we denote $\tilde{w}_\varepsilon := w_{1,\varepsilon} + \delta_0 w_{2,\varepsilon}$. Then it is easy to see that

$$-\Delta \tilde{w}_\varepsilon + \delta_2 \tilde{w}_\varepsilon \leqslant -\Delta \tilde{w}_\varepsilon + (a_0 - \delta_0 \lambda) w_{1,\varepsilon} + \left(b_0 - \frac{\lambda}{\delta_0}\right) \delta_0 w_{2,\varepsilon} \leqslant w_{1,\varepsilon}^p + \delta_0 w_{2,\varepsilon}^{2^*-1},$$

where $\delta_2 > 0$ is a small constant independent of $\varepsilon > 0$. By (3.46) there exists $R_1 > 0$ such that

$$w_{1,\varepsilon}^p(x) + \delta_0 w_{2,\varepsilon}^{2^*-1}(x) \leqslant \frac{\delta_2}{2} \tilde{w}_\varepsilon(x), \quad \forall |x| \geqslant R_1, \ \forall 0 < \varepsilon < \varepsilon_6,$$

that is,

$$-\Delta \tilde{w}_\varepsilon + \frac{\delta_2}{2} \tilde{w}_\varepsilon \leqslant 0, \quad \forall |x| \geqslant R_1, \ \forall 0 < \varepsilon < \varepsilon_6.$$

Then by Lemma 3.7 and a comparison principle, there exist $c, C > 0$ independent of $\varepsilon > 0$ such that $\tilde{w}_\varepsilon(x) \leqslant C \exp(-c|x|)$, $\forall 0 < \varepsilon < \varepsilon_6$. Therefore,

$$w_\varepsilon(x) \leqslant C \exp\bigl(-c|x|\bigr), \quad \forall 0 < \varepsilon < \varepsilon_6. \tag{3.49}$$

Step 4. We complete the proof of Theorem 2.1.

By Lemma 3.3 and (3.48), there exists $x_\varepsilon \in \Lambda_\varepsilon$ such that

$$(u_\varepsilon + v_\varepsilon)(x_\varepsilon) = \max_{x \in \mathbb{R}^N} (u_\varepsilon + v_\varepsilon)(x) \geqslant \alpha, \quad \forall 0 < \varepsilon < \varepsilon_6. \tag{3.50}$$

Moreover, (3.47) implies $|x_\varepsilon - y_\varepsilon| < R$ for all $\varepsilon \in (0, \varepsilon_6)$.

Define $(\tilde{u}_\varepsilon(x), \tilde{v}_\varepsilon(x)) := (u_\varepsilon(x/\varepsilon), v_\varepsilon(x/\varepsilon))$ and $\tilde{x}_\varepsilon := \varepsilon x_\varepsilon \in \Lambda$. Then $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ is a positive solution of (1.3). Moreover, $\tilde{x}_\varepsilon$ is a maximum point of $\tilde{u}_\varepsilon + \tilde{v}_\varepsilon$, and

$$\lim_{\varepsilon \to 0} \operatorname{dist}(\tilde{x}_\varepsilon, \mathcal{M}) \leqslant \lim_{\varepsilon \to 0} \operatorname{dist}(\varepsilon y_\varepsilon, \mathcal{M}) + \lim_{\varepsilon \to 0} \varepsilon |y_\varepsilon - x_\varepsilon| = 0,$$

that is, Theorem 2.1(i) holds. By (3.49) and $|x_\varepsilon - y_\varepsilon| < R$ we have

$$\begin{aligned}
(\tilde{u}_\varepsilon + \tilde{v}_\varepsilon)(x) &= w_\varepsilon\left(\frac{x}{\varepsilon} - y_\varepsilon\right) \\
&\leqslant C \exp\left(-c\left|\frac{x}{\varepsilon} - y_\varepsilon\right|\right) \leqslant C \exp\left(-c\left|\frac{x}{\varepsilon} - x_\varepsilon\right|\right) \\
&= C \exp\left(-\frac{c}{\varepsilon}|x - \tilde{x}_\varepsilon|\right), \quad \forall 0 < \varepsilon < \varepsilon_6,
\end{aligned}$$

that is, Theorem 2.1(iii) holds. Finally, for any such $\tilde{x}_\varepsilon$ and $(\tilde{w}_{1,\varepsilon}(x), \tilde{w}_{2,\varepsilon}(x)) := (\tilde{u}_\varepsilon(\varepsilon x + \tilde{x}_\varepsilon), \tilde{v}_\varepsilon(\varepsilon x + \tilde{x}_\varepsilon))$, we have

$$\bigl(\tilde{w}_{1,\varepsilon}(x), \tilde{w}_{2,\varepsilon}(x)\bigr) = \bigl(u_\varepsilon(x + x_\varepsilon), v_\varepsilon(x + x_\varepsilon)\bigr) = \bigl(w_{1,\varepsilon}(x + x_\varepsilon - y_\varepsilon), w_{2,\varepsilon}(x + x_\varepsilon - y_\varepsilon)\bigr).$$

Combining this with $|x_\varepsilon - y_\varepsilon| < R$, it is easy to see that Theorem 2.1(ii) follows directly from Lemma 3.5. This completes the proof. □

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