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Multiple brake orbits on compact convex symmetric reversible hypersurfaces in $\mathbf{R}^{2n}$

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Abstract

In this paper, we prove that there exist at least $[\frac{n+1}{2}]+1$ geometrically distinct brake orbits on every C^2 compact convex symmetric hypersurface Σ in $\mathbf{R}^{2n}$ for $n \geqslant 2$ satisfying the reversible condition $N\Sigma = \Sigma$ with $N = \mathrm{diag}(-I_n, I_n)$. As a consequence, we show that there exist at least $[\frac{n+1}{2}]+1$ geometrically distinct brake orbits in every bounded convex symmetric domain in $\mathbf{R}^n$ with $n \geqslant 2$ which gives a positive answer to the Seifert conjecture of 1948 in the symmetric case for $n=3$. As an application, for $n=4$ and 5, we prove that if there are exactly n geometrically distinct closed characteristics on Σ, then all of them are symmetric brake orbits after suitable time translation.

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1. Introduction

Let $V \in C^2(\mathbf{R}^n, \mathbf{R})$ and $h > 0$ be such that $\Omega \equiv \{q \in \mathbf{R}^n \mid V(q) < h\}$ is nonempty, bounded, open and connected. Consider the following fixed energy problem of the second order autonomous Hamiltonian system

$$\ddot{q}(t) + V'\big(q(t)\big) = 0, \quad \text{for } q(t) \in \Omega, \tag{1.1}$$

$$\frac{1}{2}\big|\dot{q}(t)\big|^2 + V\big(q(t)\big) = h, \quad \forall t \in \mathbf{R}, \tag{1.2}$$

$$\dot{q}(0) = \dot{q}\left(\frac{\tau}{2}\right) = 0, \tag{1.3}$$

$$q\left(\frac{\tau}{2}+t\right) = q\left(\frac{\tau}{2}-t\right), \qquad q(t+\tau) = q(t), \quad \forall t \in \mathbf{R}. \tag{1.4}$$

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A solution (τ, q) of (1.1)–(1.4) is called a *brake orbit* in Ω. We call two brake orbits q_1 and $q_2 : \mathbf{R} \to \mathbf{R}^n$ *geometrically distinct* if $q_1(\mathbf{R}) \neq q_2(\mathbf{R})$.

We denote by $\mathcal{O}(\Omega)$ and $\tilde{\mathcal{O}}(\Omega)$ the sets of all brake orbits and geometrically distinct brake orbits in Ω respectively.

Let $J_k = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$ and $N_k = \begin{pmatrix} -I_k & 0 \\ 0 & I_k \end{pmatrix}$ with I_k being the identity in $\mathbf{R}^k$. If $k = n$ we will omit the subscript k for convenience, i.e., $J_n = J$ and $N_n = N$.

The symplectic group $\mathrm{Sp}(2k)$ for any $k \in \mathbf{N}$ is defined by

$$\mathrm{Sp}(2n) = \{M \in \mathcal{L}(\mathbf{R}^{2k}) \mid M^T J_k M = J_k\},$$

where M^T is the transpose of matrix M.

For any $\tau > 0$, the symplectic path in $\mathrm{Sp}(2k)$ starting from the identity I_{2k} is defined by

$$\mathcal{P}_\tau(2k) = \{\gamma \in C([0, \tau], \mathrm{Sp}(2k)) \mid \gamma(0) = I_{2k}\}.$$

Suppose that $H \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ satisfying

$$H(Nx) = H(x), \quad \forall x \in \mathbf{R}^{2n}. \tag{1.5}$$

We consider the following fixed energy problem

$$\dot{x}(t) = JH'(x(t)), \tag{1.6}$$
$$H(x(t)) = h, \tag{1.7}$$
$$x(-t) = Nx(t), \tag{1.8}$$
$$x(\tau + t) = x(t), \quad \forall t \in \mathbf{R}. \tag{1.9}$$

A solution (τ, x) of (1.6)–(1.9) is also called a *brake orbit* on $\Sigma := \{y \in \mathbf{R}^{2n} \mid H(y) = h\}$.

Remark 1.1. It is well known that via

$$H(p, q) = \frac{1}{2}|p|^2 + V(q), \tag{1.10}$$

$x = (p, q)$ and $p = \dot{q}$, the elements in $\mathcal{O}(\{V < h\})$ and the solutions of (1.6)–(1.9) are one-to-one correspondent.

In more general setting, let Σ be a C^2 compact hypersurface in $\mathbf{R}^{2n}$ bounding a compact set C with nonempty interior. Suppose Σ has non-vanishing Gaussian curvature and satisfies the reversible condition $N(\Sigma - x_0) = \Sigma - x_0 := \{x - x_0 \mid x \in \Sigma\}$ for some $x_0 \in C$. Without loss of generality, we may assume $x_0 = 0$. We denote the set of all such hypersurfaces in $\mathbf{R}^{2n}$ by $\mathcal{H}_b(2n)$. For $x \in \Sigma$, let $N_\Sigma(x)$ be the unit outward normal vector at $x \in \Sigma$. Note that by the reversible condition there holds $N_\Sigma(Nx) = NN_\Sigma(x)$. We consider the dynamics problem of finding $\tau > 0$ and an absolutely continuous curve $x : [0, \tau] \to \mathbf{R}^{2n}$ such that

$$\dot{x}(t) = JN_\Sigma(x(t)), \quad x(t) \in \Sigma, \tag{1.11}$$
$$x(-t) = Nx(t), \qquad x(\tau + t) = x(t), \quad \text{for all } t \in \mathbf{R}. \tag{1.12}$$

A solution (τ, x) of the problem (1.11)–(1.12) is a special closed characteristic on Σ, here we still call it a brake orbit on Σ.

We also call two brake orbits (τ_1, x_1) and (τ_2, x_2) *geometrically distinct* if $x_1(\mathbf{R}) \neq x_2(\mathbf{R})$, otherwise we say they are equivalent. Any two equivalent brake orbits are geometrically the same. We denote by $\mathcal{J}_b(\Sigma)$ the set of all brake orbits on Σ, by $[(\tau, x)]$ the equivalent class of $(\tau, x) \in \mathcal{J}_b(\Sigma)$ in this equivalent relation and by $\tilde{\mathcal{J}}_b(\Sigma)$ the set of $[(\tau, x)]$ for all $(\tau, x) \in \mathcal{J}_b(\Sigma)$. From now on, in the notation $[(\tau, x)]$ we always assume x has minimal period τ. We also denote by $\tilde{\mathcal{J}}(\Sigma)$ the set of all geometrically distinct closed characteristics on Σ.

Let (τ, x) be a solution of (1.6)–(1.9). We consider the boundary value problem of the linearized Hamiltonian system

$$\dot{y}(t) = JH''(x(t))y(t), \tag{1.13}$$
$$y(t + \tau) = y(t), \qquad y(-t) = Ny(t), \quad \forall t \in \mathbf{R}. \tag{1.14}$$

Denote by $\gamma_x(t)$ the fundamental solution of the system (1.13), i.e., $\gamma_x(t)$ is the solution of the following problem

$$\dot{\gamma_x}(t) = JH''\big(x(t)\big)\gamma_x(t), \tag{1.15}$$

$$\gamma_x(0) = I_{2n}. \tag{1.16}$$

We call $\gamma_x \in C([0, \tau/2], \mathrm{Sp}(2n))$ the *associated symplectic path* of (τ, x).

Let $B_1^n(0)$ denote the open unit ball $\mathbf{R}^n$ centered at the origin 0. In [20] of 1948, H. Seifert proved $\tilde{\mathcal{O}}(\Omega) \neq \emptyset$ provided $V' \neq 0$ on $\partial\Omega$, V is analytic and Ω is homeomorphic to $B_1^n(0)$. Then he proposed his famous conjecture: ${}^{\#}\tilde{\mathcal{O}}(\Omega) \geqslant n$ *under the same conditions*.

After 1948, many studies have been carried out for the brake orbit problem. S. Bolotin proved first in [4] (also see [5]) of 1978 the existence of brake orbits in general setting. K. Hayashi in [10], H. Gluck and W. Ziller in [8], and V. Benci in [2] in 1983–1984 proved ${}^{\#}\tilde{\mathcal{O}}(\Omega) \geqslant 1$ if V is C^1, $\bar{\Omega} = \{V \leqslant h\}$ is compact, and $V'(q) \neq 0$ for all $q \in \partial\Omega$. In 1987, P.H. Rabinowitz in [19] proved that if H satisfies (1.5), $\Sigma \equiv H^{-1}(h)$ is star-shaped, and $x \cdot H'(x) \neq 0$ for all $x \in \Sigma$, then ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geqslant 1$. In 1987, V. Benci and F. Giannoni gave a different proof of the existence of one brake orbit in [3].

In 1989, A. Szulkin in [21] proved that ${}^{\#}\tilde{\mathcal{J}}_b(H^{-1}(h)) \geqslant n$, if H satisfies conditions in [19] of Rabinowitz and the energy hypersurface $H^{-1}(h)$ is $\sqrt{2}$-pinched. E.W.C. van Groesen in [9] of 1985 and A. Ambrosetti, V. Benci, Y. Long in [1] of 1993 also proved ${}^{\#}\tilde{\mathcal{O}}(\Omega) \geqslant n$ under different pinching conditions.

Without pinching condition, in [17] Y. Long, C. Zhu and the first author of this paper proved the following result: *For $n \geqslant 2$, suppose H satisfies*

(H1) (*smoothness*) $H \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$,
(H2) (*reversibility*) $H(Ny) = H(y)$ *for all* $y \in \mathbf{R}^{2n}$,
(H3) (*convexity*) $H''(y)$ *is positive definite for all* $y \in \mathbf{R}^{2n} \setminus \{0\}$,
(H4) (*symmetry*) $H(-y) = H(y)$ *for all* $y \in \mathbf{R}^{2n}$.

Then for any given $h > \min\{H(y) \mid y \in \mathbf{R}^{2n}\}$ *and* $\Sigma = H^{-1}(h)$, *there holds*

$${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geqslant 2.$$

As a consequence they also proved that: *For $n \geqslant 2$, suppose $V(0) = 0$, $V(q) \geqslant 0$, $V(-q) = V(q)$ and $V''(q)$ is positive definite for all $q \in \mathbf{R}^n \setminus \{0\}$. Then for $\Omega \equiv \{q \in \mathbf{R}^n \mid V(q) < h\}$ with $h > 0$, there holds*

$${}^{\#}\tilde{\mathcal{O}}(\Omega) \geqslant 2.$$

Under the same condition of [17], in 2009 Liu and Zhang in [14] proved that ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geqslant [\frac{n}{2}] + 1$, also they proved ${}^{\#}\tilde{\mathcal{O}}(\Omega) \geqslant [\frac{n}{2}] + 1$ under the same condition of [17]. Moreover if all brake orbits on Σ are nondegenerate, Liu and Zhang in [14] proved that ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geqslant n + \mathfrak{A}(\Sigma)$, where $2\mathfrak{A}(\Sigma)$ is the number of geometrically distinct asymmetric brake orbits on Σ.

Definition 1.1. We denote

$$\mathcal{H}_b^c(2n) = \big\{\Sigma \in \mathcal{H}_b(2n) \bigm| \Sigma \text{ is strictly convex}\big\},$$
$$\mathcal{H}_b^{s,c}(2n) = \big\{\Sigma \in \mathcal{H}_b^c(2n) \bigm| -\Sigma = \Sigma\big\}.$$

Definition 1.2. For $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, a brake orbit (τ, x) on Σ is called symmetric if $x(\mathbf{R}) = -x(\mathbf{R})$. Similarly, for a C^2 convex symmetric bounded domain $\Omega \subset \mathbf{R}^n$, a brake orbit $(\tau, q) \in \mathcal{O}(\Omega)$ is called symmetric if $q(\mathbf{R}) = -q(\mathbf{R})$.

Note that a brake orbit $(\tau, x) \in \mathcal{J}_b(\Sigma)$ with minimal period τ is symmetric if $x(t + \tau/2) = -x(t)$ for $t \in \mathbf{R}$, a brake orbit $(\tau, q) \in \mathcal{O}(\Omega)$ with minimal period τ is symmetric if $q(t + \tau/2) = -q(t)$ for $t \in \mathbf{R}$.

In this paper, we denote by $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ the sets of positive integers, integers, rational numbers and real numbers respectively. We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathbf{R}^n$ or $\mathbf{R}^{2n}$, by $(\cdot, \cdot)$ the inner product of corresponding Hilbert space. For any $a \in \mathbf{R}$, we denote $E(a) = \inf\{k \in \mathbf{Z} \mid k \geqslant a\}$ and $[a] = \sup\{k \in \mathbf{Z} \mid k \leqslant a\}$.

The following are the main results of this paper.

Theorem 1.1. *For any* $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ *with* $n \geqslant 2$, *we have*

$$^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geqslant \left[\frac{n+1}{2}\right] + 1.$$

Corollary 1.1. *Suppose* $V(0) = 0$, $V(q) \geqslant 0$, $V(-q) = V(q)$ *and* $V''(q)$ *is positive definite for all* $q \in \mathbf{R}^n \setminus \{0\}$ *with* $n \geqslant 3$. *Then for any given* $h > 0$ *and* $\Omega \equiv \{q \in \mathbf{R}^n \mid V(q) < h\}$, *we have*

$$^{\#}\tilde{\mathcal{O}}(\Omega) \geqslant \left[\frac{n+1}{2}\right] + 1.$$

Remark 1.2. Note that for $n = 3$, Corollary 1.1 yields $^{\#}\tilde{\mathcal{O}}(\Omega) \geqslant 3$, which gives a positive answer to Seifert's conjecture in the convex symmetric case.

As a consequence of Theorem 1.1, we can prove

Theorem 1.2. *For* $n = 4, 5$ *and any* $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, *suppose*

$$^{\#}\tilde{\mathcal{J}}(\Sigma) = n.$$

Then all of them are symmetric brake orbits after suitable translation.

Example 1.1. A typical example of $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ is the ellipsoid $\mathcal{E}_n(r)$ defined as follows. Let $r = (r_1, \ldots, r_n)$ with $r_j > 0$ for $1 \leqslant j \leqslant n$. Define

$$\mathcal{E}_n(r) = \left\{ x = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbf{R}^{2n} \,\middle|\, \sum_{k=1}^{n} \frac{x_k^2 + y_k^2}{r_k^2} = 1 \right\}.$$

If $r_j / r_k \notin \mathbf{Q}$ whenever $j \neq k$, from [7] one can see that there are precisely n geometrically distinct symmetric brake orbits on $\mathcal{E}_n(r)$ and all of them are nondegenerate.

2. Index theories of (i_{L_j}, ν_{L_j}) and (i_ω, ν_ω)

Let $\mathcal{L}(\mathbf{R}^{2n})$ denote the set of $2n \times 2n$ real matrices and $\mathcal{L}_s(\mathbf{R}^{2n})$ denote its subset of symmetric ones. For any $F \in \mathcal{L}_s(\mathbf{R}^{2n})$, we denote by $m^*(F)$ the dimension of maximal positive definite subspace, negative definite subspace, and kernel of any F for $* = +, -, 0$ respectively.

In this section, we make some preparation for the proof of Theorem 3.1 below. We first briefly review the index function (i_ω, ν_ω) and (i_{L_j}, ν_{L_j}) for $j = 0, 1$, more details can be found in [11,12,14,16]. Following Theorem 2.3 of [23] we study the differences $i_{L_0}(\gamma) - i_{L_1}(\gamma)$ and $i_{L_0}(\gamma) + \nu_{L_0}(\gamma) - i_{L_1}(\gamma) - \nu_{L_1}(\gamma)$ for $\gamma \in \mathcal{P}_\tau(2n)$ by computing $\operatorname{sgn} M_\varepsilon(\gamma(\tau))$. We obtain some basic lemmas which will be used frequently in the proof of the main theorem of this paper.

For any $\omega \in \mathbf{U}$, the following codimension 1 hypersurface in $\mathrm{Sp}(2n)$ is defined by:

$$\mathrm{Sp}(2n)_\omega^0 = \{ M \in \mathrm{Sp}(2n) \mid \det(M - \omega I_{2n}) = 0 \}.$$

For any two continuous paths ξ and $\eta : [0, \tau] \to \mathrm{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, their joint path is defined by

$$\eta * \xi(t) = \begin{cases} \xi(2t) & \text{if } 0 \leqslant t \leqslant \frac{\tau}{2}, \\ \eta(2t - \tau) & \text{if } \frac{\tau}{2} \leqslant t \leqslant \tau. \end{cases}$$

Given any two $(2m_k \times 2m_k)$-matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ for $k = 1, 2$, as in [16], the $\diamond$-product of M_1 and M_2 is defined by the following $(2(m_1 + m_2) \times 2(m_1 + m_2))$-matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

A special path ξ_n is defined by

$$\xi_n(t) = \begin{pmatrix} 2-\frac{t}{\tau} & 0 \\ 0 & (2-\frac{t}{\tau})^{-1} \end{pmatrix}^{\diamond n}, \quad \forall t \in [0,\tau].$$

Definition 2.1. For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, define

$$\nu_\omega(M) = \dim_{\mathbf{C}} \ker(M - \omega I_{2n}).$$

For any $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega\big(\gamma(\tau)\big).$$

If $\gamma(\tau) \notin \mathrm{Sp}(2n)^0_\omega$, we define

$$i_\omega(\gamma) = \big[\mathrm{Sp}(2n)^0_\omega : \gamma * \xi_n\big], \tag{2.1}$$

where the right-hand side of (2.1) is the usual homotopy intersection number and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed endpoints. If $\gamma(\tau) \in \mathrm{Sp}(2n)^0_\omega$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf\big\{i_\omega(\beta) \bigm| \beta(\tau) \in U \text{ and } \beta(\tau) \notin \mathrm{Sp}(2n)^0_\omega\big\}.$$

Then $(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \dots, 2n\}$ is called the index function of γ at ω.

For any $M \in \mathrm{Sp}(2n)$ we define

$$\Omega(M) = \big\{P \in \mathrm{Sp}(2n) \bigm| \sigma(P) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and } \nu_\lambda(P) = \nu_\lambda(M),\ \forall \lambda \in \sigma(M) \cap \mathbf{U}\big\},$$

where we denote by $\sigma(P)$ the spectrum of P.

We denote by $\Omega^0(M)$ the path connected component of $\Omega(M)$ containing M, and call it the *homotopy component* of M in $\mathrm{Sp}(2n)$.

Definition 2.2. For any $M_1, M_2 \in \mathrm{Sp}(2n)$, we call $M_1 \approx M_2$ if $M_1 \in \Omega^0(M_2)$.

Remark 2.1. It is easy to check that $\approx$ is an equivalent relation. If $M_1 \approx M_2$, we have $M_1^k \approx M_2^k$ for any $k \in \mathbf{N}$ and $M_1 \diamond M_3 \approx M_2 \diamond M_4$ for $M_3 \approx M_4$. Also we have $PMP^{-1} \approx M$ for any $P, M \in \mathrm{Sp}(2n)$.

The following symplectic matrices were introduced as *basic normal forms* in [16]:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda = \pm 2,$$

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \pm 1,\ b = \pm 1, 0,$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in (0,\pi) \cup (\pi, 2\pi),$$

$$N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \quad \theta \in (0,\pi) \cup (\pi, 2\pi),$$

where $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbf{R}$ and $b_2 \neq b_3$.

For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, *splitting number* of M at ω is defined by

$$S^{\pm}_M(\omega) = \lim_{\epsilon \to 0^+} i_{\omega \exp(\pm\sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma)$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$.

Splitting numbers possesses the following properties.

Lemma 2.1. *(Cf. [15], Lemma 9.1.5 and List 9.1.12 of [16].) Splitting numbers $S_M^{\pm}(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau)=M$. For $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, $S_Q^{\pm}(\omega)=S_M^{\pm}(\omega)$ if $Q \approx M$. Moreover we have*

(1) $(S_M^+(\pm 1), S_M^-(\pm 1)) = (1,1)$ *for* $M = \pm N_1(1,b)$ *with* $b = 1$ *or* 0;
(2) $(S_M^+(\pm 1), S_M^-(\pm 1)) = (0,0)$ *for* $M = \pm N_1(1,b)$ *with* $b = -1$;
(3) $(S_M^+(e^{\sqrt{-1}\theta}), S_M^-(e^{\sqrt{-1}\theta})) = (0,1)$ *for* $M = R(\theta)$ *with* $\theta \in (0,\pi)\cup(\pi,2\pi)$;
(4) $(S_M^+(\omega), S_M^-(\omega)) = (0,0)$ *for* $\omega \in \mathbf{U}\setminus\mathbf{R}$ *and* $M = N_2(\omega,b)$ *is* ***trivial*** *i.e., for sufficiently small* $\alpha > 0$, $MR((t-1)\alpha)^{\diamond n}$ *possesses no eigenvalues on* $\mathbf{U}$ *for* $t \in [0,1)$;
(5) $(S_M^+(\omega), S_M^-(\omega)) = (1,1)$ *for* $\omega \in \mathbf{U}\setminus\mathbf{R}$ *and* $M = N_2(\omega,b)$ *is* ***non-trivial***;
(6) $(S_M^+(\omega), S_M^-(\omega)) = (0,0)$ *for any* $\omega \in \mathbf{U}$ *and* $M \in \mathrm{Sp}(2n)$ *with* $\sigma(M)\cap\mathbf{U} = \emptyset$;
(7) $S_{M_1\diamond M_2}^{\pm}(\omega) = S_{M_1}^{\pm}(\omega) + S_{M_2}^{\pm}(\omega)$, *for any* $M_j \in \mathrm{Sp}(2n_j)$ *with* $j = 1,2$ *and* $\omega \in \mathbf{U}$.

Let

$$F = \mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$$

possess the standard inner product. We define the symplectic structure of F by

$$\{v,w\} = (\mathcal{J}v, w), \quad \forall v,w \in F, \text{ where } \mathcal{J} = (-J)\oplus J = \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.$$

We denote by $\mathrm{Lag}(F)$ the set of Lagrangian subspaces of F, and equip it with the topology as a subspace of the Grassmannian of all $2n$-dimensional subspaces of F.

It is easy to check that, for any $M \in \mathrm{Sp}(2n)$ its graph

$$\mathrm{Gr}(M) \equiv \left\{ \begin{pmatrix} x \\ Mx \end{pmatrix} \,\middle|\, x \in \mathbf{R}^{2n} \right\}$$

is a Lagrangian subspace of F.

Let

$$V_1 = L_0 \times L_0 = \{0\} \times \mathbf{R}^n \times \{0\} \times \mathbf{R}^n \subset \mathbf{R}^{4n},$$
$$V_2 = L_1 \times L_1 = \mathbf{R}^n \times \{0\} \times \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{4n}.$$

By Proposition 6.1 of [18] and Lemma 2.8 and Definition 2.5 of [17], we give the following definition.

Definition 2.3. For any continuous path $\gamma \in \mathcal{P}_\tau(2n)$, we define the following Maslov-type indices:

$$i_{L_0}(\gamma) = \mu_F^{CLM}\big(V_1, \mathrm{Gr}(\gamma), [0,\tau]\big) - n,$$
$$i_{L_1}(\gamma) = \mu_F^{CLM}\big(V_2, \mathrm{Gr}(\gamma), [0,\tau]\big) - n,$$
$$\nu_{L_j}(\gamma) = \dim\big(\gamma(\tau)L_j \cap L_j\big), \quad j = 0,1,$$

where we denote by $i_F^{CLM}(V, W, [a,b])$ the Maslov index for Lagrangian subspace path pair (V,W) in F on $[a,b]$ defined by Cappell, Lee, and Miller in [6]. For any $M \in \mathrm{Sp}(2n)$ and $j = 0,1$, we also denote $\nu_{L_j}(M) = \dim(ML_j \cap L_j)$.

Definition 2.4. For two paths $\gamma_0, \gamma_1 \in \mathcal{P}_\tau(2n)$ and $j = 0,1$, we say that they are L_j-homotopic and denoted by $\gamma_0 \sim_{L_j} \gamma_1$, if there is a continuous map $\delta : [0,1] \to \mathcal{P}(2n)$ such that $\delta(0) = \gamma_0$ and $\delta(1) = \gamma_1$, and $\nu_{L_j}(\delta(s))$ is constant for $s \in [0,1]$.

Lemma 2.2. *(See [11].)*

(1) *If* $\gamma_0 \sim_{L_j} \gamma_1$, *there hold*

$$i_{L_j}(\gamma_0) = i_{L_j}(\gamma_1), \qquad \nu_{L_j}(\gamma_0) = \nu_{L_j}(\gamma_1).$$

(2) *If* $\gamma = \gamma_1 \diamond \gamma_2 \in \mathcal{P}(2n)$, *and correspondingly* $L_j = L'_j \oplus L''_j$, *then*

$$i_{L_j}(\gamma) = i_{L'_j}(\gamma_1) + i_{L''_j}(\gamma_2), \qquad \nu_{L_j}(\gamma) = \nu_{L'_j}(\gamma_1) + \nu_{L''_j}(\gamma_2).$$

(3) *If* $\gamma \in \mathcal{P}(2n)$ *is the fundamental solution of*

$$\dot{x}(t) = JB(t)x(t)$$

with symmetric matrix function $B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}$ *satisfying* $b_{22}(t) > 0$ *for any* $t \in R$, *then there holds*

$$i_{L_0}(\gamma) = \sum_{0<s<1} \nu_{L_0}(\gamma_s), \qquad \gamma_s(t) = \gamma(st).$$

(4) *If* $b_{11}(t) > 0$ *for any* $t \in \mathbf{R}$, *there holds*

$$i_{L_1}(\gamma) = \sum_{0<s<1} \nu_{L_1}(\gamma_s), \qquad \gamma_s(t) = \gamma(st).$$

Definition 2.5. For any $\gamma \in \mathcal{P}_\tau$ and $k \in \mathbf{N} \equiv \{1, 2, \ldots\}$, in this paper the k-time iteration γ^k of $\gamma \in \mathcal{P}_\tau(2n)$ in brake orbit boundary sense is defined by $\tilde{\gamma}|_{[0,k\tau]}$ with

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t - 2j\tau)(N\gamma(\tau)^{-1}N\gamma(\tau))^j, & t \in [2j\tau, (2j+1)\tau],\ j = 0, 1, 2, \ldots, \\ N\gamma(2j\tau + 2\tau - t)N(N\gamma(\tau)^{-1}N\gamma(\tau))^{j+1}, & t \in [(2j+1)\tau, (2j+2)\tau],\ j = 0, 1, 2, \ldots. \end{cases}$$

By [17] or Corollary 5.1 of [14] $\lim_{k\to\infty} \frac{i_{L_0}(\gamma^k)}{k}$ exists, as usual we define the mean i_{L_0} index of γ by $\hat{i}_{L_0}(\gamma) = \lim_{k\to\infty} \frac{i_{L_0}(\gamma^k)}{k}$.

For any $P \in \mathrm{Sp}(2n)$ and $\varepsilon \in \mathbf{R}$, we set

$$M_\varepsilon(P) = P^T \begin{pmatrix} \sin 2\varepsilon I_n & -\cos 2\varepsilon I_n \\ -\cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix} P + \begin{pmatrix} \sin 2\varepsilon I_n & \cos 2\varepsilon I_n \\ \cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix}.$$

Then we have the following

Theorem 2.1. *(See Theorem 2.3 of [23].) For* $\gamma \in \mathcal{P}_\tau(2k)$ *with* $\tau > 0$, *we have*

$$i_{L_0}(\gamma) - i_{L_1}(\gamma) = \frac{1}{2} \operatorname{sgn} M_\varepsilon\big(\gamma(\tau)\big),$$

where $\operatorname{sgn} M_\varepsilon(\gamma(\tau)) = m^+(M_\varepsilon(\gamma(\tau))) - m^-(M_\varepsilon(\gamma(\tau)))$ *is the signature of the symmetric matrix* $M_\varepsilon(\gamma(\tau))$ *and* $0 < \varepsilon \ll 1$. *We also have*

$$\big(i_{L_0}(\gamma) + \nu_{L_0}(\gamma)\big) - \big(i_{L_1}(\gamma) + \nu_{L_1}(\gamma)\big) = \frac{1}{2} \operatorname{sign} M_\varepsilon\big(\gamma(\tau)\big),$$

where $0 < -\varepsilon \ll 1$.

Remark 2.2. (See Remark 2.1 of [23].) For any $n_j \times n_j$ symplectic matrix P_j with $j = 1, 2$ and $n_j \in \mathbf{N}$, we have

$$M_\varepsilon(P_1 \diamond P_2) = M_\varepsilon(P_1) \diamond M_\varepsilon(P_2),$$
$$\operatorname{sgn} M_\varepsilon(P_1 \diamond P_2) = \operatorname{sgn} M_\varepsilon(P_1) + \operatorname{sgn} M_\varepsilon(P_2),$$

where $\varepsilon \in \mathbf{R}$.

In the following of this section we will give some lemmas which will be used frequently in the proof of our main theorem later.

Lemma 2.3. *For $k \in \mathbf{N}$ and any symplectic matrix $P = \begin{pmatrix} I_k & 0 \\ C & I_k \end{pmatrix}$, there holds $P \approx I_2^{\diamond p} \diamond N_1(1,1)^{\diamond q} \diamond N_1(1,-1)^{\diamond r}$ with p, q, r satisfying*

$$m^0(C) = p, \qquad m^-(C) = q, \qquad m^+(C) = r.$$

Proof. It is clear that

$$P \approx \begin{pmatrix} I_k & 0 \\ B & I_k \end{pmatrix},$$

where $B = \mathrm{diag}(0, -I_{m^-(C)}, I_{m^+(C)})$. Since $J_1 N_1(1, \pm 1)(J_1)^{-1} = \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}$, by Remark 2.1 we have $N_1(1, \pm 1) \approx \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}$. Then

$$P \approx I_2^{\diamond m^0(C)} \diamond N_1(1,1)^{\diamond m^-(C)} \diamond N_1(1,-1)^{\diamond m^+(C)}.$$

By Lemma 2.1 we have

$$S_P^+(1) = m^0(C) + m^-(C) = p + q. \tag{2.2}$$

By the definition of the relation $\approx$, we have

$$2p + q + r = \nu_1(P) = 2m^0(C) + m^+(C) + m^-(C). \tag{2.3}$$

Also we have

$$p + q + r = m^0(C) + m^+(C) + m^-(C) = k. \tag{2.4}$$

By (2.2)–(2.4) we have

$$m^0(C) = p, \qquad m^-(C) = q, \qquad m^+(C) = r.$$

The proof of Lemma 2.3 is complete. □

Definition 2.6. We call two symplectic matrices M_1 and M_2 in $\mathrm{Sp}(2k)$ *special homotopic* (or (L_0, L_1)-homotopic) and denote by $M_1 \sim M_2$, if there are $P_j \in \mathrm{Sp}(2k)$ with $P_j = \mathrm{diag}(Q_j, (Q_j^T)^{-1})$, where Q_j is a $k \times k$ invertible real matrix, and $\det(Q_j) > 0$ for $j = 1, 2$, such that

$$M_1 = P_1 M_2 P_2.$$

It is clear that $\sim$ is an equivalent relation.

Lemma 2.4. *For $M_1, M_2 \in \mathrm{Sp}(2k)$, if $M_1 \sim M_2$, then*

$$\operatorname{sgn} M_\varepsilon(M_1) = \operatorname{sgn} M_\varepsilon(M_2), \quad 0 \leqslant |\varepsilon| \ll 1, \tag{2.5}$$

$$N_k M_1^{-1} N_k M_1 \approx N_k M_2^{-1} N_k M_2. \tag{2.6}$$

Proof. By Definition 2.6, there are $P_j \in \mathrm{Sp}(2k)$ with $P_j = \mathrm{diag}(Q_j, (Q_j^T)^{-1})$, Q_j being $k \times k$ invertible real matrix, and $\det(Q_j) > 0$ such that

$$M_1 = P_1 M_2 P_2.$$

Since $\det(Q_j) > 0$ for $j = 1, 2$, we can joint Q_j to I_k by invertible matrix path. Hence we can joint $P_1 M_2 P_2$ to M_2 by symplectic path preserving the nullity ν_{L_0} and ν_{L_1}. By Lemma 2.2 of [23], (2.5) holds. Since $P_j N_k = N_k P_j$ for $j = 1, 2$. Direct computation shows that

$$N_k (P_1 M_2 P_2)^{-1} N_k (P_1 M_2 P_2) = P_2^{-1} N_k M_2^{-1} N_k M_2 P_2. \tag{2.7}$$

Thus (2.6) holds from Remark 2.1. The proof of Lemma 2.4 is complete. □

Lemma 2.5. *Let* $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2k)$*, where* A, B, C, D *are all* $k \times k$ *matrices. Then*

(i) $\frac{1}{2}\operatorname{sgn} M_\varepsilon(P) \leqslant k - \nu_{L_0}(P)$*, for* $0 < \varepsilon \ll 1$*. If* $B = 0$*, we have* $\frac{1}{2}\operatorname{sgn} M_\varepsilon(P) \leqslant 0$ *for* $0 < \varepsilon \ll 1$*.*
(ii) *Let* $m^+(A^T C) = q$*, we have*

$$\frac{1}{2}\operatorname{sgn} M_\varepsilon(P) \leqslant k - q, \quad 0 \leqslant |\varepsilon| \ll 1. \tag{2.8}$$

(iii) $\frac{1}{2}\operatorname{sgn} M_\varepsilon(P) \geqslant \dim\ker C - k$ *for* $0 < \varepsilon \ll 1$*. If* $C = 0$*, then* $\frac{1}{2}\operatorname{sgn} M_\varepsilon(P) \geqslant 0$ *for* $0 < \varepsilon \ll 1$*.*
(iv) *If both* B *and* C *are invertible, we have*

$$\operatorname{sgn} M_\varepsilon(P) = \operatorname{sgn} M_0(P), \quad 0 \leqslant |\varepsilon| \ll 1.$$

Proof. Since P is symplectic, so is for P^T. From $P^T J_k P = J_k$ and $P J_k P^T = J_k$ we get $A^T C, B^T D, AB^T, CD^T$ are all symmetric matrices and

$$AD^T - BC^T = I_k, \qquad A^T D - C^T B = I_k. \tag{2.9}$$

We denote $s = \sin 2\varepsilon$ and $c = \cos 2\varepsilon$. By definition of $M_\varepsilon(P)$, we have

$$\begin{aligned} M_\varepsilon(P) &= \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} sI_k & -cI_k \\ -cI_k & -sI_k \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} sI_k & cI_k \\ cI_k & -sI_k \end{pmatrix} \\ &= \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} sI_k & -2cI_k \\ 0 & -sI_k \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} sI_k & 2cI_k \\ 0 & -sI_k \end{pmatrix} \\ &= \begin{pmatrix} sA^TA - 2cA^TC - sC^TC + sI_k & * \\ sB^TA - 2cB^TC - sD^TC & sB^TB - 2cB^TD - sD^TD - sI_k \end{pmatrix} \\ &= \begin{pmatrix} sA^TA - 2cA^TC - sC^TC + sI_k & sA^TB - 2cC^TB - sC^TD \\ sB^TA - 2cB^TC - sD^TC & sB^TB - 2cB^TD - sD^TD - sI_k \end{pmatrix}, \end{aligned} \tag{2.10}$$

where in the second equality we have used that $P^T J_k P = J_k$, in the fourth equality we have used that $M_\varepsilon(P)$ is a symmetric matrix. So

$$M_0(P) = -2\begin{pmatrix} A^TC & C^TB \\ B^TC & B^TD \end{pmatrix} = -2\begin{pmatrix} C^T & 0 \\ 0 & B^T \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where we have used A^TC is symmetric. So if both B and C are invertible, $M_0(P)$ is invertible and symmetric, its signature is invariant under small perturbation, so (iv) holds.

If $\nu_{L_0}(P) = \dim\ker B > 0$, since $B^TD = D^TB$, for any $x \in \ker B \subseteq \mathbf{R}^k$, $x \neq 0$, and $0 < \varepsilon \ll 1$, we have

$$\begin{aligned} M_\varepsilon(P)\begin{pmatrix} 0 \\ x \end{pmatrix} \cdot \begin{pmatrix} 0 \\ x \end{pmatrix} &= \bigl(sB^TB - 2cD^TB - sD^TD - sI_k\bigr)x \cdot x \\ &= -s\bigl(D^TD + I_k\bigr)x \cdot x \\ &< 0. \end{aligned} \tag{2.11}$$

So $M_\varepsilon(P)$ is negative definite on $(0 \oplus \ker B) \subseteq \mathbf{R}^{2k}$. Hence $m^-(M_\varepsilon(p)) \geqslant \dim\ker B$ which yields that $\frac{1}{2}\operatorname{sgn} M_\varepsilon(P) \leqslant k - \dim\ker B = k - \nu_{L_0}(P)$, for $0 < \varepsilon \ll 1$. Thus (i) holds. Similarly we can prove (iii).

If $m^+(A^TC) = q > 0$, let A^TC be positive definite on $E \subseteq \mathbf{R}^k$, then for $0 \leqslant |s| \ll 1$, similar to (2.11) we have $M_\varepsilon(P)$ is negative on $E \oplus 0 \subseteq \mathbf{R}^{2k}$. Hence $m^-(M_\varepsilon(P)) \geqslant q$, which yields (2.8). □

Lemma 2.6. *(See [23].) For* $\gamma \in \mathcal{P}_\tau(2)$*,* $b > 0$*, and* $0 < \varepsilon \ll 1$ *small enough we have*

$$\begin{aligned} &\operatorname{sgn} M_{\pm\varepsilon}\bigl(R(\theta)\bigr) = 0, \quad \textit{for } \theta \in \mathbf{R}, \\ &\operatorname{sgn} M_\varepsilon(P) = 0, \quad \textit{if } P = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \textit{ or } \pm\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}, \end{aligned}$$

$$\operatorname{sgn} M_\varepsilon(P)=2, \quad \text{if } P=\pm\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix},$$

$$\operatorname{sgn} M_\varepsilon(P)=-2, \quad \text{if } P=\pm\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}.$$

3. Proofs of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2. The proof mainly depends on the method in [14] and the following

Theorem 3.1. *For any odd number $n\geqslant 3$, $\tau>0$ and $\gamma\in\mathcal{P}_\tau(2n)$, let $P=\gamma(\tau)$. If $i_{L_0}\geqslant 0$, $i_{L_1}\geqslant 0$, $i(\gamma)\geqslant n$, $\gamma^2(t)=\gamma(t-\tau)\gamma(\tau)$ for all $t\in[\tau,2\tau]$, and $P\sim(-I_2)\diamond Q$ with $Q\in\mathrm{Sp}(2n-2)$, then*

$$i_{L_1}(\gamma)+S^+_{P^2}(1)-\nu_{L_0}(\gamma)>\frac{1-n}{2}. \tag{3.1}$$

Proof. If the conclusion of Theorem 3.1 does not hold, then

$$i_{L_1}(\gamma)+S^+_{P^2}(1)-\nu_{L_0}(\gamma)\leqslant\frac{1-n}{2}. \tag{3.2}$$

In the following we shall obtain a contradiction from (3.2). Hence (3.1) holds and Theorem 3.1 is proved.

Since $n\geqslant 3$ and n is odd, in the following of the proof of Theorem 3.1 we write $n=2p+1$ for some $p\in\mathbf{N}$. We denote $Q=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$, where A,B,C,D are $(n-1)\times(n-1)$ matrices. Then since Q is a symplectic matrix we have

$$A^TC=C^TA,\qquad B^TD=D^TB,\qquad AB^T=BA^T,\qquad CD^T=DC^T, \tag{3.3}$$

$$AD^T-BC^T=I_{n-1},\qquad A^TD-C^TB=I_{n-1}, \tag{3.4}$$

$$\dim\ker B=\nu_{L_0}(\gamma)-1,\qquad \dim\ker C=\nu_{L_1}(\gamma)-1. \tag{3.5}$$

Since $\gamma^2(t)=\gamma(t-\tau)\gamma(\tau)$ for all $t\in[\tau,2\tau]$ we have γ^2 is also the twice iteration of γ in the periodic boundary value case, so by the Bott-type formula (cf. Theorem 9.2.1 of [16]) and the proof of Lemma 4.1 of [17] we have

$$\begin{aligned}
&i(\gamma^2)+2S^+_{P^2}(1)-\nu(\gamma^2)\\
&\quad=2i(\gamma)+2S^+_P(1)+\sum_{\theta\in(0,\pi)}S^+_P\big(e^{\sqrt{-1}\theta}\big)-\sum_{\theta\in(0,\pi)}S^-_P\big(e^{\sqrt{-1}\theta}\big)+\big(\nu(P)-S^-_P(1)\big)+\big(\nu_{-1}(P)-S^-_P(-1)\big)\\
&\quad\geqslant 2n+2S^+_P(1)-n\\
&\quad=n+2S^+_P(1)\\
&\quad\geqslant n,
\end{aligned} \tag{3.6}$$

where we have used the condition $i(\gamma)\geqslant n$ and $S^+_{P^2}(1)=S^+_P(1)+S^+_P(-1)$, $\nu(\gamma^2)=\nu(\gamma)+\nu_{-1}(\gamma)$. By Proposition C of [17] and Proposition 6.1 of [14] we have

$$i_{L_0}(\gamma)+i_{L_1}(\gamma)=i(\gamma^2)-n,\qquad \nu_{L_0}(\gamma)+\nu_{L_1}(\gamma)=\nu(\gamma^2). \tag{3.7}$$

So by (3.6) and (3.7) we have

$$\begin{aligned}
&\big(i_{L_1}(\gamma)+S^+_{P^2}(1)-\nu_{L_0}(\gamma)\big)+\big(i_{L_0}(\gamma)+S^+_{P^2}(1)-\nu_{L_1}(\gamma)\big)\\
&\quad=i(\gamma^2)+2S^+_{P^2}(1)-\nu(\gamma^2)-n\\
&\quad\geqslant n-n\\
&\quad=0.
\end{aligned} \tag{3.8}$$

By Theorem 2.1 and Lemma 2.6 we have

$$\begin{aligned}
&\big(i_{L_1}(\gamma)+S^+_{P^2}(1)-\nu_{L_0}(\gamma)\big)-\big(i_{L_0}(\gamma)+S^+_{P^2}(1)-\nu_{L_1}(\gamma)\big)\\
&\quad=i_{L_1}(\gamma)-i_{L_0}(\gamma)-\nu_{L_0}(\gamma)+\nu_{L_1}(\gamma)\\
&\quad=-\frac{1}{2}\operatorname{sgn}M_\varepsilon(Q)-\frac{1}{2}\operatorname{sgn}M_\varepsilon(-I_2)\\
&\quad=-\frac{1}{2}\operatorname{sgn}M_\varepsilon(Q)\\
&\quad\geqslant 1-n.
\end{aligned}\tag{3.9}$$

So by (3.8) and (3.9) we have

$$i_{L_1}(\gamma)+S^+_{P^2}(1)-\nu_{L_0}(\gamma)\geqslant\frac{1-n}{2}.\tag{3.10}$$

By (3.2), the inequality of (3.10) must be equality. Then both (3.6) and (3.9) are equality. So we have

$$i\big(\gamma^2\big)+2S^+_{P^2}(1)-\nu\big(\gamma^2\big)=n,\tag{3.11}$$

$$i_{L_1}(\gamma)+S^+_{P^2}(1)-\nu_{L_0}(\gamma)=\frac{1-n}{2},\tag{3.12}$$

$$i_{L_0}(\gamma)+\nu_{L_0}(\gamma)-i_{L_1}(\gamma)-\nu_{L_1}(\gamma)=n-1.\tag{3.13}$$

Thus by (3.6), (3.11), Theorem 1.8.10 of [16], and Lemma 2.1 we have

$$P\approx(-I_2)^{\diamond p_1}\diamond N_1(1,-1)^{\diamond p_2}\diamond N_1(-1,1)^{\diamond p_3}\diamond R(\theta_1)\diamond R(\theta_2)\diamond\cdots\diamond R(\theta_{p_4}),$$

where $p_j\geqslant 0$ for $j=1,2,3,4$, $p_1+p_2+p_3+p_4=n$ and $\theta_j\in(0,\pi)$ for $1\leqslant j\leqslant p_4$. Otherwise by (3.6) and Lemma 2.1 we have $i(\gamma^2)+2S^+_{P^2}(1)-\nu(\gamma^2)>n$ which contradicts to (3.11). So by Remark 2.1, we have

$$P^2\approx I_2^{\diamond p_1}\diamond N_1(1,-1)^{\diamond p_2}\diamond R(\theta_1)\diamond R(\theta_2)\diamond\cdots\diamond R(\theta_{p_3}),\tag{3.14}$$

where $p_i\geqslant 0$ for $1\leqslant i\leqslant 3$, $p_1+p_2+p_3=n$ and $\theta_j\in(0,2\pi)$ for $1\leqslant j\leqslant p_3$.

Note that, since $\gamma^2(t)=\gamma(t-\tau)\gamma(\tau)$, we have

$$\gamma^2(2\tau)=\gamma(\tau)^2=P^2.\tag{3.15}$$

By Definition 2.5 we have

$$\gamma^2(2\tau)=N\gamma(\tau)^{-1}N\gamma(\tau)=NP^{-1}NP.\tag{3.16}$$

So by (3.15) and (3.16) we have

$$P^2=NP^{-1}NP.\tag{3.17}$$

By (3.17), Lemma 2.4, and $P\sim(-I_2)\diamond Q$ we have

$$\begin{aligned}
P^2&=NP^{-1}NP\\
&\approx N\big((-I_2)\diamond Q\big)^{-1}N\big((-I_2)\diamond Q\big)\\
&=I_2\diamond\big(N_{n-1}Q^{-1}N_{n-1}Q\big).
\end{aligned}\tag{3.18}$$

So by (3.14), we have

$$p_1\geqslant 1.\tag{3.19}$$

Also by (3.18) and Lemma 2.5, we have

$$P^2\approx I_2\diamond\big(N_{n-1}Q'^{-1}N_{n-1}Q'\big),\quad\forall Q'\sim Q\text{ where }Q'\in\operatorname{Sp}(2n-2).\tag{3.20}$$

By (3.14) it is easy to check that

$$\operatorname{tr}\big(P^2\big)=2n-2p_3+2\sum_{j=1}^{p_3}\cos\theta_j.\tag{3.21}$$

By (3.11), (3.14) and Lemma 2.1 we have

$$n = i(\gamma^2) + 2S^+_{P^2}(1) - \nu(\gamma^2) = i(\gamma^2) - p_2 \geqslant i(\gamma^2) - n + 1.$$

So

$$i(\gamma^2) \leqslant 2n - 1. \tag{3.22}$$

By (3.7) we have

$$i(\gamma^2) = n + i_{L_0}(\gamma) + i_{L_1}(\gamma). \tag{3.23}$$

Since $i_{L_0}(\gamma) \geqslant 0$ and $i_{L_1}(\gamma) \geqslant 0$, we have $n \leqslant i(\gamma^2) \leqslant 2n - 1$. So we can divide the index $i(\gamma^2)$ into the following three cases.

Case I. $i(\gamma^2) = n$.

In this case, by (3.7), $i_{L_0}(\gamma) \geqslant 0$, and $i_{L_1}(\gamma) \geqslant 0$, we have

$$i_{L_0}(\gamma) = 0 = i_{L_1}(\gamma). \tag{3.24}$$

So by (3.13) we have

$$\nu_{L_0}(\gamma) - \nu_{L_1}(\gamma) = n - 1. \tag{3.25}$$

Since $\nu_{L_1}(\gamma) \geqslant 1$ and $\nu_{L_0}(\gamma) \leqslant n$, we have

$$\nu_{L_0}(\gamma) = n, \qquad \nu_{L_1}(\gamma) = 1. \tag{3.26}$$

By (3.7) we have

$$\nu(\gamma^2) = \nu(P^2) = n + 1. \tag{3.27}$$

By (3.12), (3.24) and (3.26) we have

$$S^+_{P^2}(1) = \frac{1-n}{2} + n = \frac{1+n}{2} = p + 1. \tag{3.28}$$

So by (3.14), (3.27), (3.28), and Lemma 2.1 we have

$$P^2 \approx I_2^{\diamond(p+1)} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_p), \tag{3.29}$$

where $\theta_j \in (0, 2\pi)$. By (3.5) and (3.26) we have $B = 0$. By (3.18), (3.3), and (3.4), we have

$$\begin{aligned} P^2 &= NP^{-1}NP \approx I_2 \diamond \left(N_{n-1}Q^{-1}N_{n-1}Q\right) \\ &= I_2 \diamond \begin{pmatrix} D^T & 0 \\ C^T & A^T \end{pmatrix} \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \\ &= I_2 \diamond \begin{pmatrix} D^T A & 0 \\ 2C^T A & AD^T \end{pmatrix} \\ &= I_2 \diamond \begin{pmatrix} I_{2p} & 0 \\ 2A^T C & I_{2p} \end{pmatrix}. \end{aligned}$$

Hence $\sigma(P^2) = \{1\}$ which contradicts to (3.29) since $p \geqslant 1$.

Case II. $i(\gamma^2) = n + 2k$, where $1 \leqslant k \leqslant p$.

In this case by (3.7) we have

$$i_{L_0}(\gamma) + i_{L_1}(\gamma) = 2k.$$

Since $i_{L_0}(\gamma) \geqslant 0$ and $i_{L_1}(\gamma) \geqslant 0$ we can write $i_{L_0}(\gamma) = k + r$ and $i_{L_1}(\gamma) = k - r$ for some integer $-k \leqslant r \leqslant k$. Then by (3.13) we have

$$n - 1 \geqslant \nu_{L_0}(\gamma) - \nu_{L_1}(\gamma) = n - 2r - 1. \tag{3.30}$$

Thus $r \geqslant 0$ and $0 \leqslant r \leqslant k$.

By Theorem 2.1 and (i) of Lemma 2.5 we have

$$2r = i_{L_0}(\gamma) - i_{L_1}(\gamma) = \frac{1}{2} M_\varepsilon(P) \leqslant n - \nu_{L_0}(P) \tag{3.31}$$

which yields that $\nu_{L_0}(\gamma) \leqslant n - 2r$. So by (3.30) and $\nu_{L_1}(\gamma) \geqslant 1$ we have

$$\nu_{L_0}(\gamma) = n - 2r, \qquad \nu_{L_1}(\gamma) = 1. \tag{3.32}$$

Then by (3.12) we have

$$S^+_{P^2}(1) = (n - 2r) + \frac{1 - n}{2} - (k - r) = \frac{1 + n}{2} - k - r = p + 1 - k - r. \tag{3.33}$$

Then by (3.14) and $\nu(P^2) = n - 2r + 1$ and Lemma 2.1 we have

$$P^2 \approx I_2^{\diamond (p+1-k-r)} \diamond N_1(1, -1)^{\diamond 2k} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_q), \tag{3.34}$$

where $q = n - (p + 1 - k - r) - 2k = p + r - k \geqslant 0$. Then we have the following three subcases (i)–(iii).

(i) $q = 0$.

The only possibility is $k = p$ and $r = 0$, in this case $P^2 \approx I_2 \diamond N_1(1, -1)^{\diamond 2p}$ and $B = 0$. By direct computation we have

$$N_1(1, -1)^{\diamond 2p} \approx N_{2p} Q^{-1} N_{2p} Q = \begin{pmatrix} I_{n-1} & 0 \\ 2A^T C & I_{n-1} \end{pmatrix}. \tag{3.35}$$

Then by Lemma 2.3 we have

$$m^+\left(A^T C\right) = 2p.$$

By (ii) of Lemma 2.5 we have

$$\frac{1}{2} \operatorname{sgn} M_\varepsilon(Q) \leqslant 2p - 2p = 0, \quad 0 < -\varepsilon \ll 1. \tag{3.36}$$

Thus by (3.36) and Theorem 2.1, for $0 < -\varepsilon \ll 1$ we have

$$\begin{aligned}
&\left(i_{L_0}(\gamma) + \nu_{L_0}(\gamma)\right) - \left(i_{L_1}(\gamma) + \nu_{L_1}(\gamma)\right) \\
&\quad = \frac{1}{2} \operatorname{sgn} M_\varepsilon(P) \\
&\quad = \frac{1}{2} \operatorname{sgn} M_\varepsilon(I_2) + \frac{1}{2} M_\varepsilon(Q) \\
&\quad = 0 + \frac{1}{2} M_\varepsilon(Q) \\
&\quad \leqslant 0
\end{aligned}$$

which contradicts (3.13).

(ii) $q > 0$ and $r = 0$.

In this case $\nu_{L_0}(\gamma) = n$ and $\nu_{L_1}(\gamma) = 1$, also we have $B = 0$. By the equality of (3.35) we have

$$\operatorname{tr}\left(P^2\right) = 2n$$

which contradicts to (3.21) with $p_3 = q > 0$.

(iii) $q > 0$ and $r > 0$.

In this case, by (3.33) we have $r < p$. (Otherwise, then $p = r = k$. From (3.19) there holds $S^+_{p2}(1) \geqslant 1$, so from (3.33) we have $1 \leqslant S^+_{p2}(1) = 1 - p \leqslant 0$ a contradiction.) Here it is easy to see rank $B = 2r$. Then there are two invertible $2p \times 2p$ matrices U and V with $\det U > 0$ and $\det V > 0$ such that

$$UBV = \begin{pmatrix} I_{2r} & 0 \\ 0 & 0 \end{pmatrix}.$$

So there holds

$$Q \sim \operatorname{diag}\bigl(U, (U^T)^{-1}\bigr) Q \operatorname{diag}\bigl((V^T)^{-1}, V\bigr) = \begin{pmatrix} A_1 & B_1 & I_{2r} & 0 \\ C_1 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & B_2 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix} := Q_1, \tag{3.37}$$

where for $j = 1, 2, 3$, A_j is a $2r \times 2r$ matrix, D_j is a $(2p-2r) \times (2p-2r)$ matrix for $j = 1, 2, 3$, B_j is a $2r \times (2p-2r)$ matrix, and C_j is $(2p-2r) \times 2r$ matrix. Since Q_1 is still a symplectic matrix, we have $Q_1^T J_{2p} Q_1 = J_{2p}$, then it is easy to check that

$$C_1 = 0, \qquad B_2 = 0. \tag{3.38}$$

So

$$Q_1 = \begin{pmatrix} A_1 & B_1 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & 0 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix}. \tag{3.39}$$

So for the case (iii) of Case II, we have the following Subcases 1–3.

Subcase 1. $A_3 = 0$.

In this case since Q_1 is symplectic, by direct computation we have

$$N_{2p} Q_1^{-1} N_{2p} Q_1 = \begin{pmatrix} I_{2r} & * & * & * \\ * & I_{2p-2r} & * & * \\ * & * & I_{2r} & * \\ * & * & * & I_{2p-2r} \end{pmatrix}.$$

Hence we have

$$\operatorname{tr}\bigl(N_{2p} Q_1^{-1} N_{2p} Q_1\bigr) = 4p.$$

Since $Q_1 \sim Q$, we have

$$P \sim (-I_2) \diamond Q_1. \tag{3.40}$$

Then by the proof of Lemma 2.4 we have

$$\begin{aligned} \operatorname{tr} P^2 &= \operatorname{tr}\bigl(NP^{-1}NP\bigr) \\ &= \operatorname{tr} N\bigl((-I_2) \diamond Q_1\bigr)^{-1} N\bigl((-I_2) \diamond Q_1\bigr) \\ &= \operatorname{tr} I_2 \diamond \bigl(N_{2p} Q_1^{-1} N_{2p} Q_1\bigr) \\ &= 4p + 2 = 2n. \end{aligned} \tag{3.41}$$

By (3.21) and $p_3 = q > 0$ we have

$$\operatorname{tr}\bigl(P^2\bigr) < 2n. \tag{3.42}$$

(3.41) and (3.42) yield a contradiction.

Subcase 2. A_3 is invertible.

By Q_1 is symplectic we have

$$\begin{pmatrix} A_1^T & 0 \\ B_1^T & D_1^T \end{pmatrix}\begin{pmatrix} A_2 & 0 \\ C_2 & D_2 \end{pmatrix} - \begin{pmatrix} A_3^T & C_3^T \\ B_3^T & D_3^T \end{pmatrix}\begin{pmatrix} I_{2r} & 0 \\ 0 & 0 \end{pmatrix} = I_{2p}. \tag{3.43}$$

Hence

$$D_1^T D_2 = I_{2p-2r}. \tag{3.44}$$

By direct computation we have

$$\begin{pmatrix} A_1 & B_1 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & 0 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix}\begin{pmatrix} I_{2r} & -A_3^{-1}B_3 & 0 & 0 \\ 0 & I_{2p-2r} & 0 & 0 \\ 0 & 0 & I_{2r} & 0 \\ 0 & 0 & B_3^T(A_3^T)^{-1} & I_{2p-2r} \end{pmatrix} = \begin{pmatrix} A_1 & \tilde{B}_1 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & 0 & A_2 & 0 \\ C_3 & \tilde{D}_3 & \tilde{C}_2 & D_2 \end{pmatrix}.$$

So by (3.44) we have

$$\begin{pmatrix} I_{2r} & -\tilde{B}_1 D_2^T & 0 & 0 \\ 0 & I_{2p-2r} & 0 & 0 \\ 0 & 0 & I_{2r} & 0 \\ 0 & 0 & D_2\tilde{B}_1^T & I_{2p-2r} \end{pmatrix}\begin{pmatrix} A_1 & \tilde{B}_1 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & 0 & A_2 & 0 \\ C_3 & \tilde{D}_3 & \tilde{C}_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 & I_{2r} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & 0 & A_2 & 0 \\ \tilde{C}_3 & \tilde{D}_3 & \hat{C}_2 & D_2 \end{pmatrix} := Q_2.$$

Then we have

$$Q_2 \sim Q_1 \sim Q. \tag{3.45}$$

Since Q_2 is a symplectic matrix, we have $Q_2^T J_{2p} Q_2 = J_{2p}$, then it is easy to check that

$$\tilde{C}_3 = 0, \qquad \hat{C}_2 = 0. \tag{3.46}$$

Hence we have

$$Q_2 = \begin{pmatrix} A_1 & I_{2r} \\ A_3 & A_2 \end{pmatrix} \diamond \begin{pmatrix} D_1 & 0 \\ \tilde{D}_3 & D_2 \end{pmatrix}. \tag{3.47}$$

Since

$$N_{2p-2r}\begin{pmatrix} D_1 & 0 \\ \tilde{D}_3 & D_2 \end{pmatrix}^{-1} N_{2p-2r}\begin{pmatrix} D_1 & 0 \\ \tilde{D}_3 & D_2 \end{pmatrix} = \begin{pmatrix} I_{2p-2r} & 0 \\ 2D_1^T\tilde{D}_3 & I_{2p-2r} \end{pmatrix}, \tag{3.48}$$

by (3.45), (3.20), and Lemma 2.4, there is a symplectic matrix W such that

$$P^2 \approx I_2 \diamond W \diamond \begin{pmatrix} I_{2p-2r} & 0 \\ 2D_1^T\tilde{D}_3 & I_{2p-2r} \end{pmatrix}. \tag{3.49}$$

Then by (3.14) and Lemma 2.3, $D_1^T\tilde{D}_3$ is semipositive and

$$1 + m^0(D_1^T\tilde{D}_3) \leqslant S^+_{P^2}(1).$$

So by (3.33) we have

$$m^0(D_1^T\tilde{D}_3) \leqslant p+1-k-r-1 = p-k-r = (2p-2r)-(p+k-r) \leqslant 2p-2r-1. \tag{3.50}$$

Since $D_1^T\tilde{D}_3$ is a semipositive $(2p-2r)\times(2p-2r)$ matrix, by (3.50) we have $m^+(D_1^T\tilde{D}_3) > 0$. Then by Theorem 2.1, (ii) of Lemma 2.5 and Lemma 2.6, for $0 < -\varepsilon \ll 1$ we have

$$\begin{aligned}
&\big(i_{L_0}(\gamma)+\nu_{L_0}(\gamma)\big)-\big(i_{L_1}(\gamma)+\nu_{L_1}(\gamma)\big)\\
&\quad=\frac{1}{2}\left(M_\varepsilon(-I_2)+M_\varepsilon\left(\begin{pmatrix} A_1 & I_{2r}\\ A_3 & A_2\end{pmatrix}\right)+M_\varepsilon\left(\begin{pmatrix} D_1 & 0\\ \tilde{D}_3 & D_2\end{pmatrix}\right)\right)\\
&\quad\leqslant\frac{1}{2}\big(0+4r+2(2p-2r-1)\big)\\
&\quad=2p-1\\
&\quad=n-2
\end{aligned}\tag{3.51}$$

which contradicts to (3.13).

Subcase 3. $A_3\neq 0$ and A_3 is not invertible.

In this case, suppose $\operatorname{rank}A_3=\lambda$, then $0<\lambda<2r$. There is an invertible $2r\times 2r$ matrix G with $\det G>0$ such that

$$GA_3G^{-1}=\begin{pmatrix}\Lambda & 0\\ 0 & 0\end{pmatrix},\tag{3.52}$$

where Λ is a $\lambda\times\lambda$ invertible matrix. Then we have

$$\begin{aligned}
&\begin{pmatrix}(G^T)^{-1} & 0 & 0 & 0\\ 0 & I_{2p-2r} & 0 & 0\\ 0 & 0 & G & 0\\ 0 & 0 & 0 & I_{2p-2r}\end{pmatrix}\begin{pmatrix}A_1 & B_1 & I_{2r} & 0\\ 0 & D_1 & 0 & 0\\ A_3 & B_3 & A_2 & 0\\ C_3 & D_3 & C_2 & D_2\end{pmatrix}\begin{pmatrix}(G)^{-1} & 0 & 0 & 0\\ 0 & I_{2p-2r} & 0 & 0\\ 0 & 0 & G^T & 0\\ 0 & 0 & 0 & I_{2p-2r}\end{pmatrix}\\
&\quad=\begin{pmatrix}\tilde{A}_1 & \tilde{B}_1 & I_{2r} & 0\\ 0 & D_1 & 0 & 0\\ GA_3G^{-1} & \tilde{B}_3 & \tilde{A}_2 & 0\\ \tilde{C}_3 & D_3 & \tilde{C}_2 & D_2\end{pmatrix}:=Q_3.
\end{aligned}\tag{3.53}$$

By (3.52) we can write Q_3 as the following block form

$$Q_3=\begin{pmatrix}U_1 & U_2 & F_1 & I_\lambda & 0 & 0\\ U_3 & U_4 & F_2 & 0 & I_{2r-\lambda} & 0\\ 0 & 0 & D_1 & 0 & 0 & 0\\ \Lambda & 0 & E_1 & W_1 & W_2 & 0\\ 0 & 0 & E_2 & W_3 & W_4 & 0\\ G_1 & G_2 & D_3 & K_1 & K_2 & D_2\end{pmatrix}.\tag{3.54}$$

Let $R_1=\begin{pmatrix}I_\lambda & 0 & 0\\ 0 & I_{2r-\lambda} & 0\\ -G_1\Lambda^{-1} & 0 & I_{2p-2r}\end{pmatrix}$ and $R_2=\begin{pmatrix}I_\lambda & 0 & -\Lambda^{-1}E_1\\ 0 & I_{2r-\lambda} & 0\\ 0 & 0 & I_{2p-2r}\end{pmatrix}$. By (3.54) we have

$$\operatorname{diag}\big((R_1^T)^{-1},R_1\big)Q_3\operatorname{diag}\big(R_2,(R_2^T)^{-1}\big)=\begin{pmatrix}U_1 & U_2 & \tilde{F}_1 & I_\lambda & 0 & 0\\ U_3 & U_4 & \tilde{F}_2 & 0 & I_{2r-\lambda} & 0\\ 0 & 0 & D_1 & 0 & 0 & 0\\ \Lambda & 0 & 0 & W_1 & W_2 & 0\\ 0 & 0 & E_2 & W_3 & W_4 & 0\\ 0 & G_2 & \tilde{D}_3 & \tilde{K}_1 & \tilde{K}_2 & D_2\end{pmatrix}:=Q_4.$$

Since Q_4 is a symplectic matrix we have

$$Q_4^TJQ_4=J.$$

Then by (3.55) and direct computation we have $U_2=0$, $U_3=0$, $W_2=0$, $W_3=0$, $\tilde{F}_1=0$, $\tilde{K}_1=0$, and U_1, U_4, W_1, W_4 are all symmetric matrices, and

$$U_4 W_4 = I_{2r-\lambda}, \tag{3.55}$$
$$D_1 D_2^T = I_{2p-2r}, \tag{3.56}$$
$$U_4 \tilde{E}_2 = G_2^T D_1, \tag{3.57}$$

So

$$Q_4 = \begin{pmatrix} U_1 & 0 & 0 & I_\lambda & 0 & 0 \\ 0 & U_4 & \tilde{F}_2 & 0 & I_{2r-\lambda} & 0 \\ 0 & 0 & D_1 & 0 & 0 & 0 \\ \Lambda & 0 & 0 & W_1 & 0 & 0 \\ 0 & 0 & \tilde{E}_2 & 0 & W_4 & 0 \\ 0 & G_2 & \tilde{D}_3 & 0 & K_2 & D_2 \end{pmatrix}. \tag{3.58}$$

By (3.55)–(3.57), we have both $\tilde{E}_2$ and G_2 are zero or nonzero. By Definition 2.3 we have $Q_4 \sim Q_3 \sim Q$. Then by (3.32), $\begin{pmatrix} \Lambda & 0 & 0 \\ 0 & 0 & \tilde{E}_2 \\ 0 & G_2 & \tilde{D}_3 \end{pmatrix}$ is invertible. So both $\tilde{E}_2$ and G_2 are nonzero.

Since Q_4 is symplectic, by (3.57) we have

$$\begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_4 & \tilde{F}_2 \\ 0 & 0 & D_1 \end{pmatrix}^T \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & 0 & \tilde{E}_2 \\ 0 & G_2 & \tilde{D}_3 \end{pmatrix} = \begin{pmatrix} U_1\Lambda & 0 & 0 \\ 0 & 0 & U_4\tilde{E}_2 \\ 0 & (U_4\tilde{E}_2)^T & D_1^T\tilde{D}_3 + \tilde{B}_2^T\tilde{E}_2 \end{pmatrix} \tag{3.59}$$

which is a symmetric matrix.

Denote $F = \begin{pmatrix} 0 & U_4\tilde{E}_2 \\ (U_4\tilde{E}_2)^T & D_1^T\tilde{D}_3+\tilde{B}_2^T\tilde{E}_2 \end{pmatrix}$. Since $U_4\tilde{E}_2$ is nonzero, in the following we prove that $m^+(F) \geqslant 1$.

Note that here $U_4\tilde{E}_2$ is a $(2r-\lambda)\times(2p-2r)$ matrix and $D_1^T\tilde{D}_3 + \tilde{B}_2^T\tilde{E}_2$ is a $(2p-2r)\times(2p-2r)$ matrix. Denote $U_4\tilde{E}_2 = (e_{ij})$ and $D_1^T\tilde{D}_3 + \tilde{B}_2^T\tilde{E}_2 = (d_{ij})$, where e_{ij} and d_{ij} are elements on the i-th row and j-th column of the corresponding matrix. Since $U_4\tilde{E}_2$ is nonzero, there exists an $e_{ij} \neq 0$ for some $1 \leqslant i \leqslant 2r-\lambda$ and $1 \leqslant j \leqslant 2p-2r$. Let $x = (0,\dots,0,e_{ij},0,\dots,0)^T \in \mathbf{R}^{2r-\lambda}$ whose i-th row is e_{ij} and other rows are all zero, and $y = (0,\dots,0,\rho,0,\dots,0)^T \in \mathbf{R}^{2p-2r}$ whose j-th row is ρ and other rows are all zero. Then we have

$$F\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2\rho e_{ij}^2 - \rho^2 d_{jj} > 0$$

for $\rho > 0$ small enough. Hence the dimension of positive definite space of F is at least 1, thus $m^+(F) \geqslant 1$. Then

$$m^+\left(\begin{pmatrix} U_1\Lambda & 0 & 0 \\ 0 & 0 & U_4\tilde{E}_2 \\ 0 & (U_4\tilde{E}_2)^T & D_1^T\tilde{D}_3 + \tilde{B}_2^T\tilde{E}_2 \end{pmatrix}\right) = m^+(\Lambda) + m^+(F) \geqslant 1. \tag{3.60}$$

Then by (3.59), (3.60) and (ii) of Lemma 2.5, we have

$$\frac{1}{2}\operatorname{sgn} M_\varepsilon(Q_4) \leqslant 2p-1 = n-2, \quad 0 < -\varepsilon \ll 1. \tag{3.61}$$

Since $Q \sim Q_4$, by (3.61) and Lemma 2.4 we have

$$\frac{1}{2}\operatorname{sgn} M_\varepsilon(Q) \leqslant 2p-1, \quad 0 < -\varepsilon \ll 1. \tag{3.62}$$

Then since $P \sim (-I_2) \diamond Q$, by Theorem 2.1, Remark 2.2 and Lemma 2.4 we have

$$\begin{aligned} &\big(i_{L_0}(\gamma) + \nu_{L_0}(\gamma)\big) - \big(i_{L_1}(\gamma) + \nu_{L_1}(\gamma)\big) \\ &\quad = \frac{1}{2} M_\varepsilon(P) \\ &\quad = \frac{1}{2}\operatorname{sgn} M_\varepsilon\big((-I_2)\diamond Q\big) \\ &\quad = \frac{1}{2}\operatorname{sgn} M_\varepsilon(-I_2) + \frac{1}{2}\operatorname{sgn} M_\varepsilon(Q) \end{aligned}$$

$$= 0 + \frac{1}{2} \operatorname{sgn} M_\varepsilon(Q)$$
$$\leqslant n - 2. \tag{3.63}$$

Thus (3.13) and (3.63) yields a contradiction. And in Case II we can always obtain a contradiction.

Case III. $i(\gamma^2) = n + 2k + 1$, where $0 \leqslant k \leqslant p - 1$.

In this case by (3.7) we have

$$i_{L_0}(\gamma) + i_{L_1}(\gamma) = 2k + 1. \tag{3.64}$$

Since $i_{L_0}(\gamma) \geqslant 0$ and $i_{L_1}(\gamma) \geqslant 0$ we can write $i_{L_0}(\gamma) = k + 1 + r$ and $i_{L_1}(\gamma) = k - r$ for some integer $-k \leqslant r \leqslant k$. Then by (3.13) we have

$$n - 1 \geqslant \nu_{L_0}(\gamma) - \nu_{L_1}(\gamma) = n - 2r - 2. \tag{3.65}$$

Thus $r \geqslant 0$ and $0 \leqslant r \leqslant k$.

By Theorem 2.1 and (i) of Lemma 2.5 we have

$$2r + 1 = i_{L_0}(\gamma) - i_{L_1}(\gamma) = \frac{1}{2} M_\varepsilon(P) \leqslant n - \nu_{L_0}(\gamma) \tag{3.66}$$

which yields $\nu_{L_0}(\gamma) \leqslant n - 2r - 1$. Then by (3.65) and $\nu_{L_1}(\gamma) \geqslant 1$ we have

$$\nu_{L_0}(\gamma) = n - 2r - 1, \qquad \nu_{L_1}(\gamma) = 1. \tag{3.67}$$

Then by (3.12) we have

$$S^+_{P^2}(1) = (n - 2r - 1) + \frac{1 - n}{2} - (k - r) = \frac{1 + n}{2} - k - r - 1 = p - k - r \geqslant 1. \tag{3.68}$$

Then by (3.14) and $\nu(P^2) = \nu_{L_0}(\gamma) + \nu_{L_1}(\gamma) = n - 2r$ and Lemma 2.1 we have

$$P^2 \approx I_2^{\diamond(p-k-r)} \diamond N_1(1, -1)^{\diamond(2k+1)} \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_q),$$

where $q = n - (p - k - r) - (2k + 1) = p + r - k \geqslant p - k \geqslant 1$.

Since in this case $\operatorname{rank} B = 2r + 1 \leqslant n - 2$, by the same argument of (iii) in Case II, we have

$$Q \sim Q_1 = \begin{pmatrix} A_1 & B_1 & I_{2r+1} & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & 0 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix}.$$

Then by the same argument of Subcases 1, 2, 3 of Case II, we can always obtain a contradiction in Case III. The proof of Theorem 3.1 is complete. □

Now we are ready to give a proof of Theorem 1.1. For $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, let $j_\Sigma : \Sigma \to [0, +\infty)$ be the gauge function of Σ defined by

$$j_\Sigma(0) = 0, \quad \text{and} \quad j_\Sigma(x) = \inf\left\{\lambda > 0 \;\middle|\; \frac{x}{\lambda} \in C\right\}, \quad \forall x \in \mathbf{R}^{2n} \setminus \{0\},$$

where C is the domain enclosed by Σ.

Define

$$H_\alpha(x) = \big(j_\Sigma(x)\big)^\alpha, \quad \alpha > 1, \qquad H_\Sigma(x) = H_2(x), \quad \forall x \in \mathbf{R}^{2n}. \tag{3.69}$$

Then $H_\Sigma \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^{1,1}(\mathbf{R}^{2n}, \mathbf{R})$.

We consider the following fixed energy problem

$$\dot{x}(t) = JH'_\Sigma\big(x(t)\big), \tag{3.70}$$

$$H_\Sigma\big(x(t)\big) = 1, \tag{3.71}$$

$$x(-t) = Nx(t), \tag{3.72}$$

$$x(\tau + t) = x(t), \quad \forall t \in \mathbf{R}. \tag{3.73}$$

Denote by $\mathcal{J}_b(\Sigma, 2)$ ($\mathcal{J}_b(\Sigma, \alpha)$ for $\alpha = 2$ in (3.69)) the set of all solutions (τ, x) of problem (3.70)–(3.73) and by $\tilde{\mathcal{J}}_b(\Sigma, 2)$ the set of all geometrically distinct solutions of (3.70)–(3.73). By Remark 1.2 of [14] or discussion in [17], elements in $\mathcal{J}_b(\Sigma)$ and $\mathcal{J}_b(\Sigma, 2)$ are one-to-one correspondent. So we have ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) = {}^{\#}\tilde{\mathcal{J}}_b(\Sigma, 2)$.

For readers' convenience in the following we list some known results which will be used in the proof of Theorem 1.1. In the following of this paper, we write $(i_{L_0}(\gamma, k), \nu_{L_0}(\gamma, k)) = (i_{L_0}(\gamma^k), \nu_{L_0}(\gamma^k))$ for any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ and $k \in \mathbf{N}$, where γ^k is defined by Definition 2.5. We have

Lemma 3.1. *(See Theorem 1.5 of [14] and Theorem 4.3 of [18].) Let $\gamma_j \in \mathcal{P}_{\tau_j}(2n)$ for $j = 1, \ldots, q$. Let $M_j = \gamma_j^2(2\tau_j) = N\gamma_j(\tau_j)^{-1}N\gamma_j(\tau_j)$, for $j = 1, \ldots, q$. Suppose*

$$\hat{i}_{L_0}(\gamma_j) > 0, \quad j = 1, \ldots, q.$$

Then there exist infinitely many $(R, m_1, m_2, \ldots, m_q) \in \mathbf{N}^{q+1}$ such that

(i) $\nu_{L_0}(\gamma_j, 2m_j \pm 1) = \nu_{L_0}(\gamma_j)$,
(ii) $i_{L_0}(\gamma_j, 2m_j - 1) + \nu_{L_0}(\gamma_j, 2m_j - 1) = R - (i_{L_1}(\gamma_j) + n + S^+_{M_j}(1) - \nu_{L_0}(\gamma_j))$,
(iii) $i_{L_0}(\gamma_j, 2m_j + 1) = R + i_{L_0}(\gamma_j)$.

and

(iv) $\nu(\gamma_j^2, 2m_j \pm 1) = \nu(\gamma_j^2)$,
(v) $i(\gamma_j^2, 2m_j - 1) + \nu(\gamma_j^2, 2m_j - 1) = 2R - (i(\gamma_j^2) + 2S^+_{M_j}(1) - \nu(\gamma_j^2))$,
(vi) $i(\gamma_j^2, 2m_j + 1) = 2R + i(\gamma_j^2)$,

where we have set $i(\gamma_j^2, n_j) = i(\gamma_j^{2n_j}, [0, 2n_j\tau_j])$, $\nu(\gamma_j^2, n_j) = \nu(\gamma_j^{2n_j}, [0, 2n_j\tau_j])$ for $n_j \in \mathbf{N}$.

Lemma 3.2. *(See Lemma 1.1 of [14].) Let $(\tau, x) \in \mathcal{J}_b(\Sigma, 2)$ be symmetric in the sense that $x(t + \frac{\tau}{2}) = -x(t)$ for all $t \in \mathbf{R}$ and γ be the associated symplectic path of (τ, x). Set $M = \gamma(\frac{\tau}{2})$. Then there is a continuous symplectic path*

$$\Psi(s) = P(s)MP(s)^{-1}, \quad s \in [0, 1],$$

such that

$$\Psi(0) = M, \qquad \Psi(1) = (-I_2) \diamond \tilde{M}, \qquad \tilde{M} \in \mathrm{Sp}(2n - 2),$$
$$\nu_1\big(\Psi(s)\big) = \nu_1(M), \qquad \nu_2\big(\Psi(s)\big) = \nu_2(M), \quad \forall s \in [0, 1],$$

where $P(s) = \begin{pmatrix} \psi(s)^{-1} & 0 \\ 0 & \psi(s)^T \end{pmatrix}$ and ψ is a continuous $n \times n$ matrix path with $\det \psi(s) > 0$ for all $s \in [0, 1]$.

For any $(\tau, x) \in \mathcal{J}_b(\Sigma, 2)$ and $m \in \mathbf{N}$, as in [14] we denote $i_{L_j}(x, m) = i_{L_j}(\gamma_x^m, [0, \frac{m\tau}{2}])$ and $\nu_{L_j}(x, m) = \nu_{L_j}(\gamma_x^m, [0, \frac{m\tau}{2}])$ for $j = 0, 1$ respectively. Also we denote $i(x, m) = i(\gamma_x^{2m}, [0, m\tau])$ and $\nu(x, m) = \nu(\gamma_x^{2m}, [0, m\tau])$. If $m = 1$, we denote $i(x) = i(x, 1)$ and $\nu(x) = \nu(x, 1)$. By Lemma 6.3 of [14] we have

Lemma 3.3. *Suppose ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) < +\infty$. Then there exist an integer $K \geqslant 0$ and an injection map $\phi : \mathbf{N} + K \mapsto \mathcal{J}_b(\Sigma, 2) \times \mathbf{N}$ such that*

(i) *For any* $k \in \mathbf{N} + K$, $[(\tau, x)] \in \mathcal{J}_b(\Sigma, 2)$ *and* $m \in \mathbf{N}$ *satisfying* $\phi(k) = ([(\tau, x)], m)$, *there holds*

$$i_{L_0}(x, m) \leqslant k - 1 \leqslant i_{L_0}(x, m) + \nu_{L_0}(x, m) - 1,$$

where x *has minimal period* τ.

(ii) *For any* $k_j \in \mathbf{N} + K$, $k_1 < k_2$, $(\tau_j, x_j) \in \mathcal{J}_b(\Sigma, 2)$ *satisfying* $\phi(k_j) = ([(\tau_j, x_j)], m_j)$ *with* $j = 1, 2$ *and* $[(\tau_1, x_1)] = [(\tau_2, x_2)]$, *there holds*

$$m_1 < m_2.$$

Lemma 3.4. *(See Lemma 7.2 of [14].) Let* $\gamma \in \mathcal{P}_\tau(2n)$ *be extended to* $[0, +\infty)$ *by* $\gamma(\tau + t) = \gamma(t)\gamma(\tau)$ *for all* $t > 0$. *Suppose* $\gamma(\tau) = M = P^{-1}(I_2 \diamond \tilde{M})P$ *with* $\tilde{M} \in \mathrm{Sp}(2n - 2)$ *and* $i(\gamma) \geqslant n$. *Then we have*

$$i(\gamma, 2) + 2S^+_{M^2}(1) - \nu(\gamma, 2) \geqslant n + 2.$$

Lemma 3.5. *(See Lemma 7.3 of [14].) For any* $(\tau, x) \in \mathcal{J}_b(\Sigma, 2)$ *and* $m \in \mathbf{N}$, *we have*

$$i_{L_0}(x, m + 1) - i_{L_0}(x, m) \geqslant 1,$$
$$i_{L_0}(x, m + 1) + \nu_{L_0}(x, m + 1) - 1 \geqslant i_{L_0}(x, m + 1) > i_{L_0}(x, m) + \nu_{L_0}(x, m) - 1.$$

Proof of Theorem 1.1. By Theorem 1.1 of [14] we have ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geqslant [\frac{n}{2}] + 1$ for $n \in \mathbf{N}$. So we only need to prove Theorem 1.1 for the case $n \geqslant 3$ and n is odd. The method of the proof is similar as that of [14].

It is suffices to consider the case ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) < +\infty$. Since $-\Sigma = \Sigma$, for $(\tau, x) \in \mathcal{J}_b(\Sigma, 2)$ we have

$$\begin{aligned} &H_\Sigma(x) = H_\Sigma(-x), \\ &H'_\Sigma(x) = -H'_\Sigma(-x), \\ &H''_\Sigma(x) = H''_\Sigma(-x). \end{aligned} \tag{3.74}$$

So $(\tau, -x) \in \mathcal{J}_b(\Sigma, 2)$. By (3.74) and the definition of γ_x we have that

$$\gamma_x = \gamma_{-x}.$$

So we have

$$\begin{aligned} &\big(i_{L_0}(x, m), \nu_{L_0}(x, m)\big) = \big(i_{L_0}(-x, m), \nu_{L_0}(-x, m)\big), \\ &\big(i_{L_1}(x, m), \nu_{L_1}(x, m)\big) = \big(i_{L_1}(-x, m), \nu_{L_1}(-x, m)\big), \quad \forall m \in \mathbf{N}. \end{aligned} \tag{3.75}$$

So we can write

$$\tilde{\mathcal{J}}_b(\Sigma, 2) = \big\{\big[(\tau_j, x_j)\big] \bigm| j = 1, \dots, p\big\} \cup \big\{\big[(\tau_k, x_k)\big], \big[(\tau_k, -x_k)\big] \bigm| k = p + 1, \dots, p + q\big\}, \tag{3.76}$$

with $x_j(\mathbf{R}) = -x_j(\mathbf{R})$ for $j = 1, \dots, p$ and $x_k(\mathbf{R}) \neq -x_k(\mathbf{R})$ for $k = p + 1, \dots, p + q$. Here we remind that (τ_j, x_j) has minimal period τ_j for $j = 1, \dots, p + q$ and $x_j(\frac{\tau_j}{2} + t) = -x_j(t)$, $t \in \mathbf{R}$ for $j = 1, \dots, p$.

By Lemma 3.3 we have an integer $K \geqslant 0$ and an injection map $\phi : \mathbf{N} + K \to \mathcal{J}_b(\Sigma, 2) \times \mathbf{N}$. By (3.75), (τ_k, x_k) and $(\tau_k, -x_k)$ have the same (i_{L_0}, ν_{L_0})-indices. So by Lemma 3.3, without loss of generality, we can further require that

$$\mathrm{Im}(\phi) \subseteq \big\{\big[(\tau_k, x_k)\big] \bigm| k = 1, 2, \dots, p + q\big\} \times \mathbf{N}. \tag{3.77}$$

By the strict convexity of H_Σ and (6.19) of [14], we have

$$\hat{i}_{L_0}(x_k) > 0, \quad k = 1, 2, \dots, p + q.$$

Applying Lemma 3.1 to the following associated symplectic paths

$$\gamma_1, \dots, \gamma_{p+q}, \gamma_{p+q+1}, \dots, \gamma_{p+2q}$$

of $(\tau_1, x_1), \dots, (\tau_{p+q}, x_{p+q}), (2\tau_{p+1}, x^2_{p+1}), \dots, (2\tau_{p+q}, x^2_{p+q})$ respectively, there exists a vector $(R, m_1, \dots, m_{p+2q}) \in \mathbf{N}^{p+2q+1}$ such that $R > K + n$ and

$$i_{L_0}(x_k, 2m_k + 1) = R + i_{L_0}(x_k), \tag{3.78}$$
$$i_{L_0}(x_k, 2m_k - 1) + \nu_{L_0}(x_k, 2m_k - 1) = R - \big(i_{L_1}(x_k) + n + S^+_{M_k}(1) - \nu_{L_0}(x_k)\big), \tag{3.79}$$

for $k = 1, \dots, p + q$, $M_k = \gamma_k^2(\tau_k)$, and

$$i_{L_0}(x_k, 4m_k + 2) = R + i_{L_0}(x_k, 2), \tag{3.80}$$
$$i_{L_0}(x_k, 4m_k - 2) + \nu_{L_0}(x_k, 4m_k - 2) = R - \big(i_{L_1}(x_k, 2) + n + S^+_{M_k}(1) - \nu_{L_0}(x_k, 2)\big), \tag{3.81}$$

for $k = p + q + 1, \dots, p + 2q$ and $M_k = \gamma_k^4(2\tau_k) = \gamma_k^2(\tau_k)^2$.

By Lemma 3.1, we also have

$$i(x_k, 2m_k + 1) = 2R + i(x_k), \tag{3.82}$$
$$i(x_k, 2m_k - 1) + \nu(x_k, 2m_k - 1) = 2R - \big(i(x_k) + 2S^+_{M_k}(1) - \nu(x_k)\big), \tag{3.83}$$

for $k = 1, \dots, p + q$, $M_k = \gamma_k^2(\tau_k)$, and

$$i(x_k, 4m_k + 2) = 2R + i(x_k, 2), \tag{3.84}$$
$$i(x_k, 4m_k - 2) + \nu(x_k, 4m_k - 2) = 2R - \big(i(x_k, 2) + 2S^+_{M_k}(1) - \nu(x_k, 2)\big), \tag{3.85}$$

for $k = p + q + 1, \dots, p + 2q$ and $M_k = \gamma_k^4(2\tau_k) = \gamma_k^2(\tau_k)^2$.

From (3.77), we can set

$$\phi\big(R - (s - 1)\big) = \big(\big[(\tau_{k(s)}, x_{k(s)})\big], m(s)\big), \quad \forall s \in S := \left\{1, 2, \dots, \left[\frac{n+1}{2}\right] + 1\right\},$$

where $k(s) \in \{1, 2, \dots, p + q\}$ and $m(s) \in \mathbf{N}$.

We continue our proof to study the symmetric and asymmetric orbits separately. Let

$$S_1 = \big\{s \in S \,\big|\, k(s) \leqslant p\big\}, \qquad S_2 = S \setminus S_1.$$

We shall prove that $^{\#}S_1 \leqslant p$ and $^{\#}S_2 \leqslant 2q$, together with the definitions of S_1 and S_2, these yield Theorem 1.1.

Claim 1. $^{\#}S_1 \leqslant p$.

Proof. By the definition of S_1, $([(\tau_{k(s)}, x_{k(s)})], m(s))$ is symmetric when $k(s) \leqslant p$. We further prove that $m(s) = 2m_{k(s)}$ for $s \in S_1$.

In fact, by the definition of ϕ and Lemma 3.3, for all $s = 1, 2, \dots, [\frac{n+1}{2}] + 1$ we have

$$\begin{aligned} i_{L_0}\big(x_{k(s)}, m(s)\big) &\leqslant \big(R - (s - 1)\big) - 1 = R - s \\ &\leqslant i_{L_0}\big(x_{k(s)}, m(s)\big) + \nu_{L_0}\big(x_{k(s)}, m(s)\big) - 1. \end{aligned} \tag{3.86}$$

By the strict convexity of H_Σ and Lemma 2.2, we have $i_{L_0}(x_{k(s)}) \geqslant 0$, so there holds

$$i_{L_0}\big(x_{k(s)}, m(s)\big) \leqslant R - s < R \leqslant R + i_{L_0}(x_{k(s)}) = i_{L_0}(x_{k(s)}, 2m_{k(s)} + 1), \tag{3.87}$$

for every $s = 1, 2, \dots, [\frac{n+1}{2}] + 1$, where we have used (3.78) in the last equality. Note that the proofs of (3.86) and (3.87) do not depend on the condition $s \in S_1$.

By Lemma 3.2, γ_{x_k} satisfies conditions of Theorem 3.1 with $\tau = \frac{\tau_k}{2}$. Note that by definition $i_{L_1}(x_k) = i_{L_1}(\gamma_{x_k})$ and $\nu_{L_0}(x_k) = \nu_{L_0}(\gamma_{x_k})$. So by Theorem 3.1 we have

$$i_{L_1}(x_k) + S^+_{M_k}(1) - \nu_{L_0}(x_k) > \frac{1 - n}{2}, \quad \forall k = 1, \dots, p. \tag{3.88}$$

Also for $1 \leqslant s \leqslant [\frac{n+1}{2}] + 1$, we have

$$-\frac{n+3}{2} = -\left(\left[\frac{n+1}{2}\right] + 1\right) \leqslant -s. \tag{3.89}$$

Hence by (3.86), (3.88) and (3.89), if $k(s) \leqslant p$ we have

$$\begin{aligned} & i_{L_0}(x_{k(s)}, 2m_{k(s)} - 1) + \nu_{L_0}(x_{k(s)}, 2m_{k(s)} - 1) - 1 \\ & \quad = R - \big(i_{L_1}(x_{k(s)}) + n + S^+_{M_{k(s)}}(1) - \nu_{L_0}(x_{k(s)})\big) - 1 \\ & \quad < R - \frac{1-n}{2} - 1 - n = R - \frac{n+3}{2} \leqslant R - s \\ & \quad \leqslant i_{L_0}\big(x_{k(s)}, m(s)\big) + \nu_{L_0}\big(x_{k(s)}, m(s)\big) - 1. \end{aligned} \tag{3.90}$$

Thus by (3.87) and (3.90) and Lemma 3.5 of [14] we have

$$2m_{k(s)} - 1 < m(s) < 2m_{k(s)} + 1. \tag{3.91}$$

Hence

$$m(s) = 2m_{k(s)}. \tag{3.92}$$

So we have

$$\phi(R - s + 1) = \big(\big[(\tau_{k(s)}, x_{k(s)})\big], 2m_{k(s)}\big), \quad \forall s \in S_1. \tag{3.93}$$

Then by the injectivity of ϕ, it induces another injection map

$$\phi_1 : S_1 \to \{1, \dots, p\}, \quad s \mapsto k(s). \tag{3.94}$$

Therefore $^{\#}S_1 \leqslant p$. Claim 1 is proved.

Claim 2. $^{\#}S_2 \leqslant 2q$.

Proof. By the formulas (3.82)–(3.85), and (59) of [13] (also Claim 4 on p. 352 of [16]), we have

$$m_k = 2m_{k+q} \quad \text{for } k = p+1, p+2, \dots, p+q. \tag{3.95}$$

We set $\mathcal{A}_k = i_{L_1}(x_k, 2) + S^+_{M_k}(1) - \nu_{L_0}(x_k, 2)$ and $\mathcal{B}_k = i_{L_0}(x_k, 2) + S^+_{M_k}(1) - \nu_{L_1}(x_k, 2)$, $p+1 \leqslant k \leqslant p+q$, where $M_k = \gamma_k(2\tau_k) = \gamma(\tau_k)^2$. By (3.7), we have

$$\mathcal{A}_k + \mathcal{B}_k = i(x_k, 2) + 2S^+_{M_k}(1) - \nu(x_k, 2) - n, \qquad p+1 \leqslant k \leqslant p+q. \tag{3.96}$$

By similar discussion of the proof of Lemma 3.2, for any $p+1 \leqslant k \leqslant p+q$ there exist $P_k \in \mathrm{Sp}(2n)$ and $\tilde{M}_k \in \mathrm{Sp}(2n-2)$ such that

$$\gamma(\tau_k) = P_k^{-1}(I_2 \diamond \tilde{M}_k)P_k.$$

Hence by Lemma 3.4 and (3.96), we have

$$\mathcal{A}_k + \mathcal{B}_k \geqslant n + 2 - n = 2. \tag{3.97}$$

By Theorem 2.1, there holds

$$|\mathcal{A}_k - \mathcal{B}_k| = \big|\big(i_{L_0}(x_k, 2) + \nu_{L_0}(x_k, 2)\big) - \big(i_{L_1}(x_k, 2) + \nu_{L_1}(x_k, 2)\big)\big| \leqslant n. \tag{3.98}$$

So by (3.97) and (3.98) we have

$$\mathcal{A}_k \geqslant \frac{1}{2}\big((\mathcal{A}_k + \mathcal{B}_k) - |\mathcal{A}_k - \mathcal{B}_k|\big) \geqslant \frac{2-n}{2}, \quad p+1 \leqslant k \leqslant p+q. \tag{3.99}$$

By (3.81), (3.86), (3.89), (3.95) and (3.99), for $p+1 \leqslant k(s) \leqslant p+q$ we have

$$\begin{aligned}
&i_{L_0}(x_{k(s)}, 2m_{k(s)} - 2) + \nu_{L_0}(x_{k(s)}, 2m_{k(s)} - 2) - 1 \\
&\quad = i_{L_0}(x_{k(s)}, 4m_{k(s)+q} - 2) + \nu_{L_0}(x_{k(s)}, 4m_{k(s)+q} - 2) - 1 \\
&\quad = R - \big(i_{L_1}(x_{k(s)}, 2) + n + S^+_{M_{k(s)}}(1) - \nu_{L_0}(x_{k(s)}, 2)\big) - 1 \\
&\quad = R - \mathcal{A}_{k(s)} - 1 - n \\
&\quad \leqslant R - \frac{2-n}{2} - 1 - n \\
&\quad = R - \left(2 + \frac{n}{2}\right) \\
&\quad < R - \frac{n+3}{2} \\
&\quad \leqslant R - s \\
&\quad \leqslant i_{L_0}\big(x_{k(s)}, m(s)\big) + \nu_{L_0}\big(x_{k(s)}, m(s)\big) - 1. \qquad (3.100)
\end{aligned}$$

Thus by (3.87), (3.100) and Lemma 3.5, we have

$$2m_{k(s)} - 2 < m(s) < 2m_{k(s)} + 1, \qquad p < k(s) \leqslant p + q.$$

So

$$m(s) \in \{2m_{k(s)} - 1, 2m_{k(s)}\}, \quad \text{for } p < k(s) \leqslant p + q.$$

Especially this yields that for any s_0 and $s \in S_2$, if $k(s) = k(s_0)$, then

$$m(s) \in \{2m_{k(s)} - 1, 2m_{k(s)}\} = \{2m_{k(s_0)} - 1, 2m_{k(s_0)}\}.$$

Thus by the injectivity of the map ϕ from Lemma 3.3, we have

$${}^{\#}\big\{s \in S_2 \bigm| k(s) = k(s_0)\big\} \leqslant 2$$

which yields Claim 2.

By Claim 1 and Claim 2, we have

$${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) = {}^{\#}\tilde{\mathcal{J}}_b(\Sigma, 2) = p + 2q \geqslant {}^{\#}S_1 + {}^{\#}S_2 = \left[\frac{n+1}{2}\right] + 1.$$

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. By [13], there are at least n closed characteristics on every C^2 compact convex central symmetric hypersurface Σ of $\mathbf{R}^{2n}$. Hence by Example 1.1 the assumption of Theorem 1.2 is reasonable. Here we prove the case $n = 5$, the proof of the case $n = 4$ is the same.

We call a closed characteristic x on Σ a *dual brake orbit* on Σ if $x(-t) = -Nx(t)$. Then by the similar proof of Lemma 3.1 of [22], a closed characteristic x on Σ can became a dual brake orbit after suitable time translation if and only if $x(\mathbf{R}) = -Nx(\mathbf{R})$. So by Lemma 3.1 of [22] again, if a closed characteristic x on Σ can both became brake orbits and dual brake orbits after suitable translation, then $x(\mathbf{R}) = Nx(\mathbf{R}) = -Nx(\mathbf{R})$. Thus $x(\mathbf{R}) = -x(\mathbf{R})$.

Since we also have $-N\Sigma = \Sigma$, $(-N)^2 = I_{2n}$ and $(-N)J = -J(-N)$, dually by the same proof of Theorem 1.1, there are at least $[(n+1)/2] + 1 = 4$ geometrically distinct dual brake orbits on Σ.

If there are exactly 5 closed characteristics on Σ. By Theorem 1.1, four closed characteristics of them must be brake orbits after suitable time translation, then the fifth, say y, must be brake orbits after suitable time translation, otherwise $Ny(-\cdot)$ will be the sixth geometrically distinct closed characteristic on Σ which yields a contradiction. Hence all closed characteristics on Σ must be brake orbits on Σ. By the same argument we can prove that all closed characteristics on Σ must be dual brake orbits on Σ. Then by the argument in the second paragraph of the proof of this theorem, all these five closed characteristics on Σ must be symmetric. Hence all of them must be symmetric brake orbits after suitable time translation. Thus we have proved the case $n = 5$ of Theorem 1.2 and the proof of Theorem 1.2 is complete. □

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