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SciVerse ScienceDirect
Ann. I. H. Poincaré – AN 31 (2014) 591–614
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Constant Q-curvature metrics near the hyperbolic metric

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Received 22 May 2012; received in revised form 25 April 2013; accepted 30 April 2013

Available online 11 June 2013

Abstract

Let (M, g) be a Poincaré–Einstein manifold with a smooth defining function. In this note, we prove that there are infinitely many asymptotically hyperbolic metrics with constant Q-curvature in the conformal class of an asymptotically hyperbolic metric close enough to g. These metrics are parametrized by the elements in the kernel of the linearized operator of the prescribed constant Q-curvature equation. A similar analysis is applied to a class of fourth order equations arising in spectral theory.

1. Introduction

In this note we will discuss the prescribed constant Q-curvature problem for asymptotically hyperbolic manifolds. We obtain the existence of a family of constant Q-curvature metrics in a small neighborhood of any Poincaré–Einstein metric, parametrized by elements in the null space of the linearized operator L in (1.3). Much of the analysis follows from Mazzeo's microlocal analysis method for elliptic edge operators. Results in this setting have been proved for the scalar curvature equation, see [1].

For $n \geqslant 4$, a natural conformal invariant and the corresponding conformal covariant operator are the Q-curvature and the fourth order Paneitz operator. Let Ric_g and R_g be the Ricci curvature and the scalar curvature of (M, g). The *Q-curvature* and the *Paneitz operator* are defined as follows,

$$Q_g = \begin{cases} -\frac{1}{12}(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2), & n = 4, \\ -\frac{2}{(n-2)^2}|\mathrm{Ric}_g|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_g^2 - \frac{1}{2(n-1)}\Delta_g R_g, & n \geqslant 5, \end{cases}$$

$$P_g(\varphi) = \begin{cases} \Delta_g^2\varphi - \mathrm{div}(\frac{2}{3}R_g g - 2\,\mathrm{Ric}_g)\,d\varphi, & n = 4, \\ \Delta_g^2\varphi - \mathrm{div}_g(a_n R_g g - b_n \mathrm{Ric}_g)\nabla_g\varphi + \frac{n-4}{2}Q_g\varphi, & n \geqslant 5, \end{cases}$$

where $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $b_n = \frac{4}{n-2}$, $\mathrm{div}_g X = \nabla_i X^i$ for any smooth vector field X, and φ is any smooth function on M.

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0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2013.04.008

Let $\tilde{g} = \rho g$, with ρ a positive function on M, so that

$$\rho = \begin{cases} e^{2u}, & n = 4, \\ u^{\frac{4}{n-4}}, & n \geqslant 5. \end{cases}$$

The Q-curvature has the following transformation,

$$P_g u + 2Q_g = 2Q_{\tilde{g}} e^{4u}, \quad n = 4,$$
$$P_g u = \frac{n-4}{2} Q_{\tilde{g}} u^{\frac{n+4}{n-4}}, \quad n > 4.$$

Note that Paneitz operator satisfies the following conformal covariance property for $\varphi \in C^\infty(M)$,

$$P_{\tilde{g}} \varphi = e^{-4u} P_g \varphi, \quad n = 4,$$
$$P_{\tilde{g}}(\varphi) = u^{-\frac{n+4}{n-4}} P_g(u\varphi), \quad n > 4.$$

We want to find a function u so that the metric $\tilde{g}$ satisfies $Q_{\tilde{g}} = \tilde{Q}$ for a given function $\tilde{Q}$. For the prescribed Q-curvature problem on closed manifold M of dimension four there are many results, see [3,7,12,13]. In [24] a boundary value problem for this problem is solved. A flow approach is performed in [2], see also [4]. For $n \geqslant 5$, see [6,23,27].

There are some interesting results for complete non-compact manifolds. For Euclidean space $\mathbb{R}^n$, $n \geqslant 4$, see [17] and [26]. In [11], using shooting method, the authors proved that there are infinitely many complete metrics with constant Q-curvature in the conformal class of the Poincaré disk with dimension $n \geqslant 5$, which are radially symmetric ODE solutions to the initial value problem parametrized by distinct given initial data at the origin. It is not difficult to prove that similar results hold for $n = 4$. Mazzeo pointed out that there should be a more general result of this type. In this paper, we solve a perturbation problem in the setting of asymptotically hyperbolic metrics close to a Poincaré–Einstein metric. To give a precise statement we first need some definitions.

Definition 1.1. Let M be a smooth manifold of dimensional n, with smooth boundary ∂M of dimension $n-1$. Let g be a complete metric on $M = Int(\overline{M})$. We say that g is conformally compact if there exists a smooth function x on $\overline{M}$, with the property that $x > 0$ in M, and $x = 0$ on ∂M, so that the metric $h = x^2 g$ extends continuously to a Riemannian metric on $\overline{M}$. Here x is called a defining function of g. Moreover, if $h \in C^k$ or $C^{k,\alpha}$, for some positive integer k, we say that g is conformally compact of order C^k or $C^{k,\alpha}$.

Note that a direct calculation shows that the sectional curvatures of a conformally compact metric g approach $-|dx|_g$ near ∂M. We call (M, g) *asymptotically hyperbolic*, if $|dx|_h|_{\partial M} = 1$; and here g is called an *asymptotically hyperbolic metric*, for which the sectional curvatures approach -1 near ∂M.

Definition 1.2. Let (M, g) be an asymptotically hyperbolic manifold. If g is also Einstein, we call g a Poincaré–Einstein metric, and (M, g) a Poincaré–Einstein manifold.

Let (M^n, g) be an asymptotically hyperbolic manifold of dimension n, with x as its smooth defining function. Actually, we can choose x so that $|dx|_h = 1$ in a neighborhood of ∂M, see [8], and to simplify notation we always choose a defining function in this sense except in Section 4. We will mainly focus on the asymptotic behavior of the metric near ∂M, which is a local matter. Let y be local coordinates on ∂M. In a neighborhood of ∂M in $\overline{M}$, we introduce the local coordinates in the following way: $(x, y) \in [0, \varepsilon) \times \partial M$ represents the point moving from the point on ∂M with local coordinate y, along the geodesic which is the integral curve of $\nabla_h x$ for a length x in the metric h. In local coordinates (x, y),

$$h = x^2 g = dx^2 + \sum_{i,j=1}^{n-1} h_{ij}\, dy^i\, dy^j.$$

For convenience, let $\tilde{g} = \rho g$, with ρ a positive function on M, so that

$$\rho = \begin{cases} e^{2u}, & n = 4, \\ (1+u)^{\frac{4}{n-4}}, & n \geqslant 5. \end{cases}$$

For a given function $\tilde{Q}$, let the operator $\mathcal{E}$ be defined by

$$\mathcal{E}(u) = \begin{cases} P_g u + 2Q_g - 2\tilde{Q}e^{4u}, & \text{for } n = 4, \\ P_g(1+u) - \frac{n-4}{2}\tilde{Q}(1+u)^{\frac{n+4}{n-4}}, & \text{for } n \geqslant 5. \end{cases} \tag{1.1}$$

To solve the prescribed Q-curvature problem amounts to finding a solution to

$$\mathcal{E}(u) = 0. \tag{1.2}$$

For $\tilde{Q} = Q_g$, we define the linear operator $L = L_g$ of $\mathcal{E}$ as follows,

$$L(u) = \begin{cases} P_g u - 8Q_g u, & n = 4, \\ P_g u - \frac{n+4}{2}Q_g u, & n \geqslant 5. \end{cases} \tag{1.3}$$

Let (x, y) be the local coordinates of M near the boundary described above. Let $\mathcal{V}_e$ be the collection of the smooth vector fields on $\overline{M}$, which restricted to a neighborhood of ∂M are generated by $\{x\partial_x, x\partial_{y^1}, \dots, x\partial_{y^{n-1}}\}$ with smooth coefficients on $\overline{M}$.

Next we introduce the weighted spaces that we will be using. First, the weighted Sobolev spaces,

$$x^\delta H_e^m\big(M, \Omega^{\frac{1}{2}}\big) = \big\{u = x^\delta v\colon\ V_1 \dots V_j v \in L^2\big(M, \Omega^{\frac{1}{2}}\big),\ \forall j \leqslant m,\ V_i \in \mathcal{V}_e\big\},$$

where $m \in \mathbb{N}$, $\delta \in \mathbb{R}$, and $\Omega^{\frac{1}{2}} = \sqrt{dx\,dy}$ is the half-density. We also introduce the weighted Hölder space,

$$x^\delta \Lambda^{m,\alpha} = x^\delta \Lambda^{m,\alpha}\big(M, \Omega^{\frac{1}{2}}\big) = \big\{u = x^\delta v\sqrt{dx\,dy}\colon\ V_1 \dots V_j v \in \Lambda^{0,\alpha},\ \forall j \leqslant m,\ V_i \in \mathcal{V}_e\big\},$$

with $m \in \mathbb{N}$, $\delta \in \mathbb{R}$, and $0 < \alpha < 1$, where $\Lambda^{0,\alpha}(M)$ is the space of half-densities $u = v\sqrt{dx\,dy}$ such that

$$\|v\|_{\Lambda^{0,\alpha}(M)} = \sup|v| + \sup \frac{(x+\tilde{x})^\alpha |v(x,y) - v(\tilde{x},\tilde{y})|}{|x-\tilde{x}|^\alpha + |y-\tilde{y}|^\alpha} < \infty.$$

We will use the norm

$$\|u\|_{x^\delta \Lambda^{k,\alpha}(M)} = \sum_{m=0}^{k} \sum_{|\gamma|=m} \big\|\partial_e^\gamma v\big\|_{0,\alpha},$$

with $\partial_e \in \mathcal{V}_e$ and $u = x^\delta v$.

In this paper, we always assume $n \geqslant 4$ to be the dimension of M. With these definitions, we can now state our main result:

Theorem 1.3. *Let (M^n, g), $n \geqslant 4$, be a Poincaré–Einstein manifold with defining function x, and assume the metric $h = x^2 g$ is smooth up to the boundary. Let $L : x^\nu \Lambda^{4,\alpha}(M) \to x^\nu \Lambda^{0,\alpha}(M)$, where $0 < \nu < \frac{n-1}{2}$ and $0 < \alpha < 1$ be the linearized operator defined in (1.3). Then,*

(i) *The kernel of $L \subset x^\nu \Lambda^{4,\alpha}(M)$ is infinite dimensional, and L is surjective. Furthermore, given $\tilde{Q} \in \Lambda^{0,\alpha}(M, \sqrt{dx\,dy})$ with $\|\tilde{Q} - Q_g\|_{x^\nu \Lambda^{0,\alpha}}$ sufficiently small, for each v in the kernel of L there exists a unique solution u to the problem (1.2) with $\Pi_1(u) = v$, where $\Pi_1 : x^\nu \Lambda^{4,\alpha}(M) \to \ker(L)$ is the projection map (see Theorem 1.5).*

(ii) *Let u be a solution of (1.2) with $\tilde{Q} = Q_g$. Then u has an expansion near the boundary*

$$u(x, y) \sim \big(u_{00}(y)x^{\frac{n-1}{2}+i\beta} + u_{10}(y)x^{\frac{n-1}{2}-i\beta}\big) + o\big(x^{\frac{n-1}{2}}\big), \tag{1.4}$$

with $\beta = \frac{\sqrt{n^2+2n-9}}{2}$ and $i = \sqrt{-1}$, where u_{00} and u_{10} are generally distributions of negative order.

Moreover, suppose $v = \Pi_1 u$ *has an expansion of the form*

$$v(x,y) \sim \sum_{j=0}^{+\infty} \big(v_{0j}(y) x^{\frac{n-1}{2}+i\beta+j} + v_{1j}(y) x^{\frac{n-1}{2}-i\beta+j} + v_{2j}(y) x^{n+j} \big), \tag{1.5}$$

in the sense that

$$v(x,y) - \sum_{j=0}^{k} \big(v_{0j}(y) x^{\frac{n-1}{2}+i\beta+j} + v_{1j}(y) x^{\frac{n-1}{2}-i\beta+j} \big) = o\big(x^{\frac{n-1}{2}+k}\big),$$

for each $k \geqslant 0$, *where* $\beta = \frac{\sqrt{n^2+2n-9}}{2}$ *and the coefficient functions are smooth. Then* u *has an expansion of the same form with different smooth coefficients.*

For kernel elements having an expansion with smooth coefficients as in (1.5), one can prescribe the leading terms; see Remark 2.2.

Remark 1.1. Our proof uses in a crucial way that the Laplacian operator Δ has no embedded eigenvalues in its essential spectrum. In [21], Mazzeo showed that this holds for any asymptotically hyperbolic manifold (M, g) with smooth defining function x such that the compactified metric $h = x^2 g$ is smooth up to the boundary. The smoothness assumption comes up when he uses boundary regularity results of kernel elements and the unique continuation property on the boundary.

Remark 1.2. For examples of smooth Poincaré–Einstein manifolds (M^n, g), we have the Poincaré ball $(B_1^n(0), g_H)$, geometrically finite quotients of hyperbolic space $\mathbb{H}^n/\Gamma$ with infinite volume. For nontrivial examples, Graham and Lee [9] and Lee [16] proved that there are infinitely many Poincaré–Einstein metrics g near the hyperbolic metric and a class of known Poincaré–Einstein metrics g with prescribed data of $x^2 g|_{\partial M}$. Moreover, by the regularity result in [5], for g asymptotically hyperbolic of order C^2, with smooth defining function x and $x^2 g|_{\partial M}$ is a smooth metric on ∂M, for n even, or $n = 3$, up to a $C^{1,\alpha}$ diffeomorphism near the boundary, $h = x^2 g$ extends to boundary smoothly; while for n odd and $n > 3$, h has expansion with possible $\log(x)$-terms appearing at $x = 0$, which does not satisfy the regularity condition in [21].

Remark 1.3. The ODE result in [11] only gives existence of radially symmetric constant Q-curvature metrics in the conformal class of the hyperbolic metric, but allows the metric to be far away from the hyperbolic metric. As a perturbation result, our theorem gives the existence of solutions in the conformal class of metrics in a small neighborhood of the hyperbolic metric, more precisely, see Theorem 4.1.

Since this is a perturbation result, we first discuss the linear problem. We say that a bounded linear operator L is *essentially injective*, if the null space of L is at most finitely dimensional; and L is *essentially surjective* if L has closed range and with at most finitely dimensional cokernel. Using Mazzeo's approach in [18], we obtain the semi-Fredholm property for the linear operator (1.3):

Theorem 1.4. *Let* (M^n, g) *be an asymptotically hyperbolic manifold with defining function* x *and the metric* $h = x^2 g$ *smooth up to the boundary, then the linear operator* $L : x^\delta H_e^4(M) \to x^\delta L^2(M, \sqrt{dx\,dy})$ *as in (1.3), is essentially injective if* $\delta > \frac{n}{2}$ *and* $\delta \neq n + \frac{1}{2}$, *with infinite dimensional cokernel, and* L *is essentially surjective if* $\delta < \frac{n}{2}$ *and* $\delta \neq -\frac{1}{2}$, *with infinite dimensional kernel. Moreover, in both cases,* L *has closed range, and admits a generalized inverse* G *and orthogonal projectors* Π_1 *onto the nullspace and* Π_2 *onto orthogonal complement of the range of* L *which are edge operators, such that,*

$$GL = I - \Pi_1,$$
$$LG = I - \Pi_2.$$

The corresponding theorem for the weighted Hölder space is as follows.

Theorem 1.5. *Let (M^n, g) be an asymptotically hyperbolic manifold with defining function x and the metric $h = x^2 g$ smooth up to the boundary. Let $0 < \alpha < 1$. The linear operator $L : x^\nu \Lambda^{4,\alpha}(M) \to x^\nu \Lambda^{0,\alpha}(M)$ as in (1.3), is essentially injective if $\nu > \frac{n-1}{2}$ and $\nu \neq n$, with infinite dimensional cokernel; and L is essentially surjective if $\nu < \frac{n-1}{2}$ and $\nu \neq -1$, with infinite dimensional kernel. Moreover, in both cases, L has closed range. Also, $x^\nu \Lambda^{4,\alpha}(M)$ has the topological splitting of the following direct sum $x^\nu \Lambda^{4,\alpha}(M) = \Pi_1(x^\nu \Lambda^{4,\alpha}(M)) \oplus (I - \Pi_1)(x^\nu \Lambda^{4,\alpha}(M))$, which is the projection to the null space of L and its topological complement for the second case. Similarly as the theorem with weighted Sobolev spaces, there is a corresponding splitting of $x^\nu \Lambda^{0,\alpha}(M)$ for $\nu > \frac{n-1}{2}$.*

The paper is organized as follows. In Section 2, we study the linear elliptic edge operator L defined in (1.3), and obtain the semi-Fredholm property of the linear operator L. In Section 3, we obtain that if the linear operator L with respect to the initial asymptotically hyperbolic metric g is surjective in a suitable weighted Hölder space, there are infinitely many solutions to the prescribed Q-curvature problem with $\tilde{Q}$ a small perturbation of Q_g, and the solutions are parametrized by the elements in the kernel of L. Then we give the proof of Theorem 1.3. Using a special weighted Hölder space, in Section 4, we prove a perturbation result for the prescribed constant Q-curvature problem for a Poincaré–Einstein metric. In Section 5, we give a similar discussion to the prescribed U-curvature equations.

2. Semi-Fredholm properties of the linearized operator

In this section, we will discuss the local parametrix for L and the Fredholm property of L. An important feature is that the elliptic operator L under consideration here is degenerate near infinity. Here we review some of the material developed by Mazzeo and others in the theory of elliptic edge operators, see [19].

As in the introduction, let (M^n, g) be an asymptotically hyperbolic manifold of dimension n, with defining function x and the metric $h = x^2 g$ smooth up to the boundary. Let (x, y) be the local coordinates of M near the boundary, and $\mathcal{V}_e$ as defined in the introduction. The one forms dual to the vector fields which are elements in $\mathcal{V}_e$ are smooth one forms in M, restricted on the neighborhood of ∂M generated linearly by $\{\frac{dx}{x}, \frac{dy^1}{x}, \dots, \frac{dy^{n-1}}{x}\}$ with coefficients smooth up to ∂M. Generally, a left or right parametrix E of an elliptic operator L on M is a pseudo-differential operator with the property that

$$EL = \mathrm{Id} + R_1, \quad \text{or} \quad LE = \mathrm{Id} + R_2,$$

with R_1, R_2 compact operators.

The Schwartz kernel of an interior parametrix of the linear operator L is a distribution on $M \times M$, and for "interior" we mean that the parametrix has singularity near the boundary which will be explained in the following. Let (x, y) and $(\tilde{x}, \tilde{y})$ be local coordinates on each copy of M near the boundary. We know that the parametrix is smooth, except for the singularity along the diagonal $\Delta = \{x = \tilde{x},\ y = \tilde{y}\}$, as in the case of compact manifolds. Moreover, due to the degeneration of the edge operator L, as $x, \tilde{x} \to 0$, we also have the important additional singularity at the intersection of Δ and the corner, which is $S = \{x = \tilde{x} = 0,\ y = \tilde{y}\}$. To deal with the boundary singularity, we introduce a new manifold $M_0^2 = M \times_0 M$, by blowing-up $M \times M$ along S. Actually, if we use polar coordinates for $M \times M$ near the corner,

$$r = \left(x^2 + |y - \tilde{y}|^2 + \tilde{x}^2\right)^{1/2} \in \mathbb{R}^+,$$
$$\Theta = (x, y - \tilde{y}, \tilde{x})/r \in S^n_{++} = \left\{\Theta \in S^n,\ \Theta_0, \Theta_n \geqslant 0\right\},$$

we know that the level set of $r = R$ is a submanifold of dimensional $2n - 1$ for $R > 0$, while $S = \{r = 0\}$ is singular. More precisely, let M_0^2 be the lift of $M \times M$ such that it is the same as $M \times M$ away from S, but near the corner, it is represented by the lift of the polar coordinates, smoothly. Hence, $S_{11} = \{r = 0\}$ is a $(2n-1)$-dimensional submanifold of M_0^2. Let b be the natural projection map from M_0^2 to $M \times M$. For the convenience of calculation, as in [18], we introduce two systems of local coordinates on M_e^2, $(s, v, \tilde{x}, \tilde{y})$ and (x, y, t, w), where

$$s = x/\tilde{x}, \quad v = \frac{y - \tilde{y}}{\tilde{x}}; \qquad t = \tilde{x}/x, \quad w = \frac{\tilde{y} - y}{x}.$$

Changing variables in these two coordinates,

$$x\partial_x = s\partial_s = x\partial_x - w\partial_w - t\partial_t, \quad \text{and} \quad x\partial_y = s\partial_v = x\partial_y - \partial_w.$$

In the following without loss of generality we only need to consider $(s, v, \tilde{x}, \tilde{y})$. Viewing elements in $\mathcal{V}_e$ as first order differential operators, we denote $\mathrm{Diff}_e^*(M)$ the algebra generated by $\mathcal{V}_e$ with coefficients in the ring $C^\infty(\overline{M})$, and with the product given by composition of operators. We call an element in $\mathrm{Diff}_e^*(M)$ an *edge operator*. Let $\mathrm{Diff}_e^m(M)$ be the linear subspace of differential operators which are of m-th order. Then $L \in \mathrm{Diff}_e^m(M)$ has the form

$$L = \sum_{j+|\alpha|\leqslant m} a_{j,\alpha}(x, y)(x\partial_x)^j (x\partial_y)^\alpha, \tag{2.1}$$

with $a_{j,\alpha} \in C^\infty(\overline{M})$, in the coordinate chart (x, y). The symbol of L is

$$\sigma_e(L)(x, y; \xi, \eta) = \sum_{j+|\alpha|=m} a_{j,\alpha}(x, y)\xi^j \eta^\alpha.$$

L is elliptic if $\sigma_e(L)(x, y; \xi, \eta) \neq 0$, for $(\xi, \eta) \neq 0$. It is easy to check that Δ_g and the linear operator L in (1.3) are elliptic. L in (2.1) can be considered as a lift to M_e^2 as follows,

$$L = \sum_{j+|\alpha|\leqslant m} a_{j,\alpha}(x, y)(x\partial_x)^j (x\partial_y)^\alpha = \sum_{j+|\alpha|\leqslant m} a_{j,\alpha}(s\tilde{x}, \tilde{y} + \tilde{x}v)(s\partial_s)^j (s\partial_v)^\alpha.$$

Let $N(L)$ be the normal operator of L, so that

$$N(L) = \sum_{j+|\alpha|\leqslant m} a_{j,\alpha}(0, \tilde{y})(s\partial_s)^j (s\partial_v)^\alpha,$$

is the restriction to S_{11} of the lift of L to M_e^2. The normal operator is an important approximation of L near the boundary. For the linear operator L in (2.1),

$$L\phi = \sum_{j+|\alpha|\leqslant m} a_{j,\alpha}(0, y)(x\partial_x)^j (x\partial_y)^\alpha \phi + E\phi,$$

for any smooth function ϕ, with the error term

$$E\phi = x \sum_{j+|\alpha|\leqslant m} b_{j,\alpha}(x, y)(x\partial_x)^j (x\partial_y)^\alpha \phi,$$

for $x > 0$ small, with the coefficients $b_{j,\alpha}$ smooth up to the boundary.

Definition 2.1. The indicial family $I_\zeta(L)$ of $L \in \mathrm{Diff}_e^k(M)$ is defined to be the family of operators

$$L\big(x^\zeta \big(\log(x)\big)^p f(x, y)\big) = x^\zeta \big(\log(x)\big)^p I_\zeta(L) f(0, y) + O\big(x^\zeta \big(\log(x)\big)^{p-1}\big),$$

for $f \in C^\infty(M)$, $\zeta \in \mathbb{C}$, $p \in \mathbb{N}_0$.

There exists a unique dilation-invariant operator $I(L)$, which is called the indicial operator, such that

$$I(L)(y, s\partial_s)s^\zeta f(y) = s^\zeta I_\zeta(L) f(y).$$

In local coordinates near the boundary, $I(L) = \sum_{j\leqslant k} a_{j,0}(0, y)(s\partial_s)^j$.

Definition 2.2. If $L \in \mathrm{Diff}_e^*(M)$ is elliptic, we denote $spec_b(L)$ as the boundary spectrum of L, which is the set of $\zeta \in \mathbb{C}$, for which $I_\zeta(L) = 0$.

Let (M, g), x, and h be defined as above. Denote S_x as the level set of x (x is also denoted as y_0 for convenience), and the coordinates $(y_1, \dots, y_{n-1}) = y$. We now use this point of view to analyze our linearized operator (1.3).

In a neighborhood of ∂M, we have the following,

$$\mathrm{Ric}_g = \mathrm{Ric}_h + x^{-1}\big[(n-2)\,\mathrm{Hess}_h\, x + \Delta_h x h\big] - (n-1)x^{-2}|dx|_h^2 h, \tag{2.2}$$

and

$$R_g = -n(n-1)|dx|_h^2 + (2n-2)x(\Delta_h x) + x^2 R_h, \tag{2.3}$$

where $|dx|_h = 1$, and

$$(\mathrm{Hess}_h)_{ij}(x) = \nabla_i^h \nabla_j^h(x) = \partial_i\partial_j(x) - \Gamma_{ij}^s \partial_s(x) = -\Gamma_{ij}^0 = \frac{1}{2}\partial_x h_{ij} = B_{ij},$$

with B_{ij} the second fundamental form of S_x, for $i, j > 0$; while $(\mathrm{Hess}_h)_{ij}(x) = 0$ for $i = 0$ or $j = 0$. Also $\Delta_h x = \mathrm{tr}_h(\mathrm{Hess}_h) = H(h)$, with $H(h)$ the mean curvature of the level set of x in the metric h. Also Γ_{ij}^k is the Christoffel symbol with respect to h. Note that Δ_g in our paper is the trace of Hess_g, with negative eigenvalues:

$$\begin{aligned}\Delta_g u &= g^{ij}\big(\partial_i\partial_j - \Gamma_{ij}^k\partial_k\big)u && (2.4)\\ &= x^2\Delta_h u + (2-n)x(\nabla_h x, du) && (2.5)\\ &= (2-n)x\partial_x u + x^2\big(\partial_x^2 u + \Delta_y u + H(h)\partial_x u\big), && (2.6)\end{aligned}$$

where Δ_y is the Laplacian on the level set S_x of x, in the induced metric $h|_{S_x}$.

Near the boundary, the Q-curvature is

$$\begin{aligned}Q_g &= -\frac{2}{(n-2)^2}(n-1)^2 n + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}n^2(n-1)^2 + O(x)\\ &= \frac{n(n^2-4)}{8} + O(x),\end{aligned}$$

for $n \geqslant 5$, and $Q_g = 3 + O(x)$, for $n = 4$.

In the rest of this section we will consider the linear operator L in (1.3). Note that

$$\begin{aligned}L\phi &= \Delta_g^2\phi - \mathrm{div}_g(a_n R_g g - b_n \mathrm{Ric}_g)\nabla_g\phi - 4f\phi\\ &= \Delta_g^2\phi - a_n R_g\Delta_g\phi + b_n \mathrm{Ric}_{ij}^g\,\nabla_g^i\nabla_g^j\phi - a_n(\nabla_g R_g, \nabla_g\phi) + b_n\nabla_g^i\,\mathrm{Ric}_{ij}\,\nabla_g^j\phi - 4f\phi\\ &= \Delta_g^2\phi - a_n R_g\Delta_g\phi + b_n \mathrm{Ric}_{ij}^g\,\nabla_g^i\nabla_g^j\phi + \left(-a_n + \frac{b_n}{2}\right)(\nabla_g R_g, \nabla_g\phi) - 4f\phi\\ &= \Delta_g^2\phi - a_n R_g\Delta_g\phi + b_n \mathrm{Ric}_{ij}^g\,\nabla_g^i\nabla_g^j\phi + \frac{6-n}{2(n-1)}(\nabla_g R_g, \nabla_g\phi) - 4f\phi,\end{aligned}$$

with $f = Q_g$ for $n \geqslant 5$, and $f = 2Q_g$ for $n = 4$. For the third equality, we use the second Bianchi identity. Also,

$$\Delta_g\phi = x^2\Delta_h\phi - (n-2)x(\nabla_h x, d\phi)_h = x^2\Delta_h\phi - (n-2)x\partial_x\phi,$$

and

$$\begin{aligned}R_{ij}(g)\nabla_g^i\nabla_g^j\phi &\sim \big[-(n-1)x^2 h_{ij} + O\big(x^3\big)\big]x^{-4}\nabla^i\nabla_g^j\phi\\ &= -(n-1)\big(\Delta_g\phi + O(x)p(x, y, x\partial_x, x\partial_y)\phi\big),\end{aligned}$$

for some smooth function $p(\cdot)$. As a consequence,

$$\begin{aligned}L\phi &= \Delta_g^2\phi - a_n R_g\Delta_g\phi + b_n \mathrm{Ric}_{ij}^g\,\nabla_g^i\nabla_g^j\phi + \frac{6-n}{2(n-1)}(\nabla_g R_g, \nabla_g\phi) - 4f\phi\\ &= \Delta_g^2\phi - a_n\big(-n(n-1) + O(x)\big)\Delta_g\phi + b_n\big(-(n-1)\Delta_g\phi + O(x)p(x, y, x\partial_x, x\partial_y)\phi\big)\\ &\quad + \frac{6-n}{2(n-1)}\big(-(2n-2)x^2 H(h|_{S_x})\partial_x\phi + O\big(x^3\big)|\nabla_y\phi|\big) - \left(\frac{1}{2}n\big(n^2-4\big) + O(x)\right)\phi.\end{aligned}$$

By definition,

$$N(L) = \big[(s\partial_s)^2 - (n-1)s\partial_s + s^2\Delta_v - n\big]\left[(s\partial_s)^2 - (n-1)s\partial_s + s^2\Delta_v + \frac{n^2-4}{2}\right].$$

In addition,

$$I(L) = \big((s\partial_s)^2 - (n-1)s\partial_s - n\big)\left((s\partial_s)^2 - (n-1)s\partial_s + \frac{n^2-4}{2}\right).$$

Suppose $I(L)\phi = 0$, and write $\phi = s^\zeta$. Solving this equation, we get the indicial roots ζ, given by

$$spec_b(L) = \left\{n, -1, \frac{n-1}{2} - i\frac{\sqrt{n^2+2n-9}}{2}, \frac{n-1}{2} + i\frac{\sqrt{n^2+2n-9}}{2}\right\}.$$

Let Λ be the index set

$$\Lambda = \left\{\frac{1}{2} + \mathrm{Re}(\delta);\ \delta \in spec_b(L)\right\}. \tag{2.7}$$

The operator $N(L)$ acts on functions defined on $\mathbb{R}^+_s \times \mathbb{R}^{n-1}_v$ for each fixed $\tilde{y}$, with coordinates (s, v). For the linear operator L, $N(L)$ does not depend on $\tilde{y}$. We now take the Fourier transformation of $N(L)$ in the v direction,

$$\widehat{N(L)} = \sum_{j+|\alpha|\leqslant m} a_{i,\alpha}(s\partial_s)^j(is\eta)^\alpha.$$

We have the symmetry of dilation:

$$a_{j\alpha}(s\partial_s)^j(s\partial_y)^\alpha = a_{j\alpha}(ks\partial_{ks})^j(ks\partial_{ky})^\alpha,$$

for any $k \in \mathbb{R} - \{0\}$. Let $t = s|\eta|$, then

$$\widehat{N(L)}(s,\eta) = \sum_{j+|\alpha|\leqslant m} a_{i,\alpha}(0,\tilde{y})(t\partial_t)^j(it\hat{\eta})^\alpha,$$

which is denoted as $L_0(t, \hat{\eta})$, where $\hat{\eta} = \frac{\eta}{|\eta|}$. This is a family of totally characteristic operators on $\mathbb{R}^n_+$ and generally its coefficients depend on $\tilde{y}$. Note that we have fixed $\hat{\eta}$ in the formula, and there is no scaling freedom in this direction.

Let $\mathcal{H}^{m,\delta,l}$ be the weighted Sobolev space

$$\mathcal{H}^{m,\delta,l} = \big\{f\colon \phi(t)f \in t^\delta H^m_e\big(\mathbb{R}^+\big),\ \big(1-\phi(t)\big)f \in t^{-l}H^m\big(\mathbb{R}^+\big)\big\},$$

with $\phi \in C^\infty_0(\mathbb{R}^+)$, and $\phi(t) = 1$ in a neighborhood of $t = 0$. Note that

$$L_0 : t^\delta\mathcal{H}^{m,\delta,l} \to t^\delta\mathcal{H}^{m-4,\delta,l+4}$$

is bounded.

For the linear operator L,

$$\widehat{N(L)} = \big[(s\partial_s)^2 - (n-1)s\partial_s + s^2\big(-|\eta|^2\big) - n\big]\left[(s\partial_s)^2 - (n-1)s\partial_s + s^2\big(-|\eta|^2\big) + \frac{n^2-4}{2}\right],$$

hence

$$\begin{aligned} L_0(t,\hat{\eta}) &= \big[(t\partial_t)^2 - (n-1)t\partial_t - t^2 - n\big]\left[(t\partial_t)^2 - (n-1)t\partial_t - t^2 + \frac{n^2-4}{2}\right] \\ &= L_1 \circ L_2, \end{aligned}$$

with $s\partial_s = s|\eta|\partial_{s|\eta|} = t\partial_t$. Note that L_0 does not depend on $\tilde{y}$. We have made the formula into the simplest form.

Let us consider the relationship of Fredholm property among $N(L)$, $\widehat{N(L)}$ and L_0, in $t^\delta L^2$, for $\delta > \frac{n}{2}$. We know that the first two operators have the same properties of injectivity and surjectivity. Let

$$L_0\varphi(t) = 0,$$

by definition, it holds if and only if

$$\widehat{N(L)}\varphi\big(s|\eta|\big) = 0.$$

But then

$$\widehat{N(L)}\big(a(\eta)\varphi\big(s|\eta|\big)\big) = a(\eta)\widehat{N(L)}\varphi\big(s|\eta|\big) = 0,$$

for all $a(\eta)$ smooth, since the derivative is only in s direction, with fixed η. Then, using the inverse Fourier transformation,

$$N(L)\int_{\mathbb{R}^{n-1}} e^{2\pi i\langle y,\eta\rangle} a(\eta)\varphi\big(s|\eta|\big)\,d\eta = 0.$$

This means one-dimensional kernel of L_0 corresponds to the infinite dimensional kernel of $N(L)$, and this construction also gives the fact that the kernel of $N(L)$ is either trivial or of infinite dimension. But if $\widehat{N(L)}$ is injective, then L_0 is injective. Conversely, if L_0 is injective, then $\widehat{N(L)}$ is injective, and so is $N(L)$. We have a dual argument of the surjectivity for $\delta < \frac{n}{2}$. As in [18], L_0 is Fredholm when $\delta \notin \Lambda$, with the set Λ in (2.7), and $N(L)$ is semi-Fredholm with either infinite dimensional kernel or cokernel. Roughly speaking, L is a small perturbation of $N(L)$ near ∂M. When $N(L)$ is injective or surjective, L is essentially injective or essentially surjective, which will be Theorem 1.4 and Theorem 1.5.

To see the semi-Fredholm property of L, the strategy is to first study the Fredholm property of L_0 and $N(L)$, and finally obtain the semi-Fredholm property of L using Mazzeo's theorems which we list here as Theorem 2.4 and Corollary 2.5.

We consider the Fredholm property of L_0, L_1 and L_2 on weighted spaces. To this end, we introduce Bessel functions, which are solutions to the Bessel equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \big(x^2+\alpha^2\big)y = 0,$$

where α is a complex number.

The Bessel functions I_α and $I_{-\alpha}$ form a basis of linear space of solutions to the Bessel equation above, while $\{I_\alpha, K_\alpha\}$ is another basis. For $\mathrm{Re}(\alpha) > -\frac{1}{2}$, and $-\frac{\pi}{2} < arg(x) < \frac{\pi}{2}$, the integral representations of these solutions are as follows,

$$I_\alpha(x) = \frac{(\frac{x}{2})^\alpha}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{-xt}\big(1-t^2\big)^{\alpha-\frac{1}{2}}\,dt,$$

$$K_\alpha(x) = \frac{\pi}{2}\frac{I_\alpha(x)-I_{-\alpha}(x)}{\sin(\alpha\pi)} = \frac{\Gamma(\frac{1}{2})(\frac{x}{2})^\alpha}{\Gamma(\alpha+\frac{1}{2})}\int_{1}^{\infty} e^{-xt}\big(t^2-1\big)^{\alpha-\frac{1}{2}}\,dt,$$

with x a complex number (see pp. 172 and 77 in [25]). Note that I_α is bounded near $x = 0$, and it increases exponentially near $+\infty$, and

$$K_\alpha(x) \sim C(\varepsilon)x^{\mathrm{Re}(\alpha)}e^{-x+\varepsilon},$$

for any $\varepsilon > 0$, as $x \to +\infty$. Also $K_\alpha(x)$ is bounded for $\mathrm{Re}(\alpha) \geqslant 0$, near $x = 0$. The form $K_\alpha(x)$ is more useful near $x = \infty$, since it decays exponentially.

We want to solve the following ODE, by transforming it into the Bessel type equations as above:

$$L_1u = \big((t\partial_t)^2 - (n-1)t\partial_t - t^2 - n\big)u = 0.$$

Let $u = t^\beta\tilde{u}$, then we obtain that

$$t^\beta\big((t\partial_t)^2\tilde{u} + (2\beta+1-n)t\partial_t\tilde{u} + \big(-n-t^2+\beta^2-\beta(n-1)\big)\tilde{u}\big) = 0. \tag{2.8}$$

Then, letting $2\beta + 1 - n = 0$, Eq. (2.8) is just the form of the Bessel equation. In this case, $\beta = \frac{n-1}{2}$, and then the index $\alpha = \frac{n+1}{2}$.

Therefore,

$$u(t) = t^{\frac{n-1}{2}}\big(C_1I_{\frac{n+1}{2}}(t) + C_2K_{\frac{n+1}{2}}(t)\big).$$

In fact,

$$t^{\frac{n-1}{2}} I_{\frac{n+1}{2}}\big(t|\eta|\big) \sim t^n |\eta|^{\frac{n+1}{2}}, \qquad t^{\frac{n-1}{2}} K_{\frac{n+1}{2}}\big(t|\eta|\big) \sim t^{-1} |\eta|^{-\frac{n+1}{2}},$$

near $t = 0$. Moreover,

$$t^{\frac{n-1}{2}} I_{\frac{n+1}{2}}\big(t|\eta|\big) \sim t^{\frac{n}{2}-1} e^{t|\eta|}/\sqrt{2\pi|\eta|}, \qquad t^{\frac{n-1}{2}} K_{\frac{n+1}{2}}\big(t|\eta|\big) \sim t^{\frac{n}{2}-1} e^{-t|\eta|}\sqrt{\frac{\pi}{2|\eta|}},$$

as $t \to \infty$.

Similarly,

$$L_2 u = \left((t\partial_t)^2 - (n-1)t\partial_t - t^2 + \frac{n^2-4}{2}\right)u = 0.$$

Let $u(t) = t^\beta \tilde{u}(t)$, then

$$t^\beta\left((t\partial_t)^2\tilde{u} + (2\beta+1-n)t\partial_t\tilde{u} + \left(\frac{n^2-4}{2} - t^2 + \beta^2 - \beta(n-1)\right)\tilde{u}\right) = 0.$$

Set $2\beta + 1 - n = 0$, so that $\beta = \frac{n-1}{2}$, and then $\tilde{u}$ is a solution to the Bessel equation with $\alpha = \frac{i\sqrt{n^2+2n-9}}{2}$,

$$u(t) = t^{\frac{n-1}{2}}\big(C_1 I_{\frac{i\sqrt{n^2+2n-9}}{2}}(t) + C_2 K_{\frac{i\sqrt{n^2+2n-9}}{2}}(t)\big).$$

By the expansion of the series form of the Bessel functions, as in [15, p. 108], we have

$$t^{\frac{n-1}{2}} I_\alpha\big(t|\eta|\big) \sim t^{\frac{n-1}{2}+\alpha}|\eta|^\alpha/\big(2^\alpha\Gamma(1+\alpha)\big),$$

and

$$t^{\frac{n-1}{2}} I_{-\alpha}\big(t|\eta|\big) \sim t^{\frac{n-1}{2}-\alpha}|\eta|^\alpha/\big(2^\alpha\Gamma(1-\alpha)\big),$$

with $\alpha = \frac{i\sqrt{n^2+2n-9}}{2}$, near $t = 0$. Now it is easy to see that the linear combination

$$x^{\frac{n-1}{2}}\big(C_1 x^{i\frac{\sqrt{n^2+2n-9}}{2}} + C_2 x^{-i\frac{\sqrt{n^2+2n-9}}{2}}\big)$$

can never vanish to infinite order at $t = 0$ if either $C_1 \neq 0$ or $C_2 \neq 0$. Also,

$$t^{\frac{n-1}{2}} K_\alpha\big(t|\eta|\big) \sim t^{\frac{n-1}{2}}\frac{\pi}{2}\frac{I_\alpha(t|\eta|) - I_{-\alpha}(t|\eta|)}{\sin(\alpha\pi)},$$

with $\alpha = \frac{i\sqrt{n^2+2n-9}}{2}$, and $|\eta| \neq 0$, near $t = 0$.

Using the integral form as above, we have that $I_\alpha(t)$ grows exponentially, while K_α decays exponentially as $t \to +\infty$, for $\alpha = \frac{i\sqrt{n^2+2n-9}}{2}$.

Denote L_0^t to be the L^2 adjoint of L_0 in the measure dt, and

$$L_0^* = t^{2\delta} L_0 t^{-2\delta},$$

to be the adjoint of L_0 in $t^\delta L^2$ in the measure $t^{-2\delta}\,dt$. These are all elliptic operators, with boundary spectra:

$$spec_b\big(L_0^t\big) = \big\{-\zeta - 1:\ \zeta \in spec_b(L_0)\big\},$$
$$spec_b\big(L_0^*\big) = \big\{-\zeta + 2\delta - 1:\ \zeta \in spec_b(L_0)\big\}.$$

For example, for $L_1 = (t\partial_t)^2 - (n-1)(t\partial_t) - t^2 - n$,

$$\int L_1 uv\,dt = \int uL_1^t v\,dt.$$

Then

$$L_1^t = \big(-\partial_t(t\cdot)\big)^2 + (n-1)\big(\partial_t(t\cdot)\big) - t^2 - n,$$

with

$$\partial_t(t\cdot) = t\partial_t + 1,$$

and $p^t(\xi) = p(-(\xi+1))$, for the quadratic polynomial p. Also, for L_1^*, using the fact that

$$-\partial_t\big(tt^{-2\delta}\cdot\big) = -t^{-2\delta}(-2\delta+1+t\partial_t) = t^{-2\delta}(2\delta-1-t\partial_t),$$

and

$$\int L_1 uvt^{2\delta}\,dt = -\int ut^{-2\delta}L_1^t\big(t^{2\delta}v\big)t^{2\delta}\,dt,$$

we obtain the boundary spectra as listed above. For the fourth order differential equation, we have obtained four linearly independent solutions, and they generalize the solution space.

Let $\delta = \frac{n-1}{2} + \frac{1}{2} = \frac{n}{2}$, we have $L_1^* = L_1$, and $L_2^* = L_2$.

Definition 2.3. We say that an operator L has the unique continuation property on a boundary B if any solution of $Lu = 0$ vanishing to infinite order at B vanishes identically.

Hypothesis 1. For each $\tilde{y}$ and $\hat{\eta}$, both L_0 and its adjoint L_0^* (the dual of L_0 with respect to the space $t^{\mathrm{Re}\,\delta}L^2$ for any δ we need) have the unique continuation property at $\{t=0\}$.

We know from the discussion above that L_0 satisfies the unique continuation property. Under the continuation hypothesis, we have that for each element $(\tilde{y},\hat{\eta}) \in N_0$, L_0 is surjective on $x^\delta L^2$ or injective on $x^\delta L^2$ when δ is sufficiently negative or sufficiently large. For our case, we use $\delta = \frac{n}{2}$ in Hypothesis 1. Now let us define $\bar{\delta}$ to be the minimal value of δ so that L_0 is injective, and meanwhile $\underline{\delta}$ the maximal value so that L_0 is surjective dually. These values must lie in Λ. The following theorem and corollary tell us the relationship between semi-Fredholm properties of L and the Fredholm properties of L_0, for certain cases we need.

Theorem 2.4. *(See Theorem 6.1 in [18].) Suppose $L \in \mathrm{Diff}_e^m(M)$ is elliptic and satisfies the unique continuation hypothesis, and that $\mathrm{spec}_b(L)$ is discrete. Suppose also that $\delta \notin \Lambda$ is chosen so that either $\delta > \bar{\delta}$ or $\delta < \underline{\delta}$. Then $L : x^\delta H_e^{r+m}(M) \to x^\delta H_e^r(M)$ has closed range, and it is either essentially surjective, or essentially injective respectively. Therefore, it admits a generalized inverse G and orthogonal projectors Π_i onto the nullspace and orthogonal complement of the range of L which are edge operators, such that,*

$$GL = I - \Pi_1,$$
$$LG = I - \Pi_2.$$

Since the edge operators used in the proof of the weighted Sobolev spaces are bounded in the appropriate Hölder spaces, the corresponding result for Hölder spaces follows.

Corollary 2.5. *(See Corollary 6.4 in [18].) For L as in Theorem 2.4, $k \geqslant m$ a positive integer and $0 < \alpha < 1$ the mapping $L : x^\nu\Lambda^{k,\alpha} \to x^\nu\Lambda^{k-m,\alpha}$ is semi-Fredholm provided $\nu = \delta - \frac{1}{2}$ and $\delta \notin \Lambda$ is as in the previous theorem. If $\delta < \underline{\delta}$ or $\delta > \bar{\delta}$ so that L is essentially surjective or essentially injective, then topologically, we have the splitting,*

$$x^\nu\Lambda^{k,\alpha} = \Pi_1\big(x^\nu\Lambda^{k,\alpha}\big) \oplus (I-\Pi_1)\big(x^\nu\Lambda^{k,\alpha}\big),$$
$$x^\nu\Lambda^{k-m,\alpha} = \Pi_2\big(x^\nu\Lambda^{k-m,\alpha}\big) \oplus (I-\Pi_2)\big(x^\nu\Lambda^{k-m,\alpha}\big).$$

Let us compute $\bar{\delta}$ and $\underline{\delta}$ for L_0. First, for L_1, since $t^{\frac{n-1}{2}}I_{\frac{n+1}{2}}(t|\eta|)$ increases exponentially as t goes to ∞ (here $|\eta| \neq 0$), it does not lie in $t^\delta L^2$ for any $\delta > 0$; furthermore,

$$t^{\frac{n-1}{2}}K_{\frac{n+1}{2}}\big(t|\eta|\big) \in t^\delta L^2(\mathbb{R}_+),$$

for $\delta < -\frac{1}{2}$. Similarly, for L_2, $t^{\frac{n-1}{2}} I_{\frac{i\sqrt{n^2+2n-9}}{2}}(t|\eta|)$ grows exponentially when t goes to ∞ (with $|\eta| \neq 0$), and

$$t^{\frac{n-1}{2}} K_{\frac{i\sqrt{n^2+2n-9}}{2}}\big(t|\eta|\big) \in t^{\delta} L^2(\mathbb{R}_+),$$

for $\delta < \frac{n-1}{2} + \frac{1}{2} = \frac{n}{2}$. Therefore, L_1 and L_2 both have trivial kernel in the space $x^{\delta} L^2(M, \sqrt{dx\,dy})$ for $\delta > \frac{n}{2}$. But $\mathrm{Ker}(L_2)$ is nontrivial for $\delta < \frac{n}{2}$. Also the composition of two injective maps is still injective. Therefore, $\overline{\delta} = \frac{n}{2}$ for $L_0 = L_1 \circ L_2$. Since L_0 is self-adjoint in $t^{\frac{n}{2}} L^2(\mathbb{R}_+)$, we have that $\underline{\delta} = \frac{n}{2}$. Since it satisfies the conditions of Theorem 2.4 and Corollary 2.5, therefore Theorems 1.4 and 1.5 are proved.

To conclude this section, we want to see when L is injective or surjective in the special case of Poincaré–Einstein manifolds. For a Poincaré–Einstein manifold (M, g) with $g = x^{-2}h$, without loss of generality we assume $R_g = -n(n-1)$. Let us first consider it in the weighted Sobolev spaces. We have

$$L = (\Delta_g - n)\left(\Delta_g + \frac{(n+2)(n-2)}{2}\right) = \mathcal{T}_1 \circ \mathcal{T}_2.$$

We know that L is self-adjoint with respect to $x^{\frac{n}{2}} L^2(M, \sqrt{dx\,dy})$. Then to show that $L : x^{\delta} H_e^4(M) \to x^{\delta} L^2(M)$ is surjective for $0 \leqslant \delta < \frac{n}{2}$, since L has close range, we only need to $1 \leqslant \delta < \frac{n}{2}$, we only need to show that L is injective when $\delta > \frac{n}{2}$. For that, we only need to show that $\mathcal{T}_1$ and $\mathcal{T}_2$ are injective for $\delta > \frac{n}{2}$.

Lemma 2.6. *$\mathcal{T}_1, \mathcal{T}_2 : x^{\delta} H_e^{2+m}(M, \sqrt{dx\,dy}) \to x^{\delta} H_e^{m}(M, \sqrt{dx\,dy})$, are both injective for $\delta > \frac{n}{2}$, and all $m \geqslant 0$.*

Proof. By the regularity argument, we only need to discuss on the case $m = 0$. Also, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are injective in $L^2(M, g)$, which is $x^{\frac{n}{2}} L^2(M, \sqrt{dx\,dy})$, then we are done. The proof is as follows.

If $u \in x^{\delta} L^2(M, \sqrt{dx\,dy})$, for $\delta > \frac{n}{2}$, then $u \in L^2(M, g)$. Moreover, if also

$$\mathcal{T}_1 u = (\Delta_g - n)u = 0,$$

by Weyl's lemma, $u \in H^1(M, g)$. Now we multiply u on both sides of the equation, and integrate by parts, and then we have

$$-\int_M \big(|\nabla u|_g^2 + u^2\big)\, dV_g = 0.$$

Therefore, $u = 0$. Then we have that $\mathcal{T}_1$ is injective.

For the Poincaré ball (B, g_{-1}), we know that the Laplacian $-\Delta_g$ has pure continuous spectrum, consisting of $[\frac{(n-1)^2}{4}, \infty)$, with $\lambda_0 = \frac{(n-1)^2}{4}$.

For an asymptotically hyperbolic manifold (M, g) with its defining function x and the extending metric $h = x^2 g$ smooth up to the boundary, combining the boundary regularity result and the unique continuation result for $(-\Delta_g - \lambda)$, it was proved in [21] that if $\lambda > \frac{(n-1)^2}{4}$, $u \in L^2(M, g)$ and $(-\Delta_g - \lambda)u = 0$ then $u = 0$. That is to say, $-\Delta_g$ has essential spectrum $[\frac{(n-4)^2}{4}, +\infty)$, with no embedded eigenvalues. It is easy to check that when $n \geqslant 4$, $\frac{n^2-4}{2} > \frac{(n-1)^2}{4}$. Therefore, for $n \geqslant 4$, $\mathcal{T}_2$ is injective in $L^2(M, g) = x^{\frac{n}{2}} L^2(M, \sqrt{dx\,dy})$.

It follows that $\mathcal{T}_1$ and $\mathcal{T}_2$ are both injective when $\delta > \frac{n}{2}$. This proves the lemma. □

The lemma tells that L is injective for $\delta > \frac{n}{2}$ on the Poincaré–Einstein manifolds. Then dually L is surjective when $0 < \delta < \frac{n}{2}$.

The linear edge operators used above are all bounded linear operators in the weighted Hölder spaces, and can be used correspondingly in the weighted Hölder spaces. Then the corresponding statement for the weighted Hölder spaces is as follows. Let

$$L : x^{\nu} \Lambda^{4,\alpha}(M) \to x^{\nu} \Lambda^{0,\alpha}(M).$$

Here $0 < \alpha < 1$. Then L is injective when $\nu = \delta - \frac{1}{2} > \frac{n-1}{2}$, while L is surjective when $0 < \nu = \delta - \frac{1}{2} < \frac{n-1}{2}$ on the Poincaré–Einstein manifolds M.

Remark 2.1. Generally, on an asymptotically hyperbolic manifold (M, g) with a smooth defining function x and $h = x^2 g$ smooth, let $u \in \mathrm{Ker}(L)$ for L defined in (1.3) in the weighted Hölder spaces $x^\nu \Lambda^{4,\alpha}(M, \sqrt{dx\, dy})$, for $0 < \nu < \frac{n-1}{2}$ and $0 < \alpha < 1$. Then, $u \in x^\nu \Lambda^{m,\alpha}$ for all $m \in \mathbb{N}$, and u has the following weak expansion with coefficients which are generally distributions,

$$u(x, y) \sim \sum_{j=0}^{+\infty} \big(u_{0j}(y) x^{\frac{n-1}{2} + i\frac{\sqrt{n^2+2n-9}}{2} + j} + u_{1j}(y) x^{\frac{n-1}{2} - i\frac{\sqrt{n^2+2n-9}}{2} + j} + x^{n+j} u_{2j}(y)\big), \tag{2.9}$$

in the sense that

$$u(x, y) - \sum_{j=0}^{k} \big(u_{0j}(y) x^{\frac{n-1}{2} + i\frac{\sqrt{n^2+2n-9}}{2} + j} + u_{1j}(y) x^{\frac{n-1}{2} - i\frac{\sqrt{n^2+2n-9}}{2} + j}\big) = o\big(x^{\frac{n-1}{2} + k}\big),$$

for $k \geqslant 0$. If either u_{00} or u_{01} is smooth, then all the coefficients are smooth. The more precise regularity of the coefficients in a weighted Sobolev space setting can be found in Chapter 7 in [18].

Remark 2.2. On a Poincaré–Einstein manifold (M, g) with a smooth defining function x and $h = x^2 g$ smooth, for $0 < \nu < \frac{n-1}{2}$ and $0 < \alpha < 1$, since $\mathcal{T}_1$ is injective, an element u in the kernel of L is exactly an element in the kernel of $\mathcal{T}_2$. By Proposition 3.4 in [10], for any chosen $u_{00} \in C^\infty$ or $u_{10} \in C^\infty$, there exists a unique $u \in x^\nu \Lambda^{4,\alpha}(M, \sqrt{dx\, dy})$, for $0 < \nu < \frac{n-1}{2}$, in the kernel of L, so that u has the expansion (2.9) with smooth coefficients.

3. The nonlinear problem

Now let us return to the perturbation problem. It is more convenient to work in weighted Hölder spaces. Let (M, g) be an asymptotically hyperbolic manifold defined as in the introduction. Let $\tilde{g}$, u also be defined as in the introduction, and let the prescribed curvature $Q_{\tilde{g}} = f$. Define the operator $\mathcal{T} : x^\nu \Lambda^{4,\alpha}(M) \to x^\nu \Lambda^{0,\alpha}(M)$ as follows,

$$\mathcal{T}(u) = \begin{cases} 2fe^{4u} - 2Q_g - 8Q_g u, & n = 4, \\ \frac{n-4}{2}(1+u)^{\frac{n+4}{n-4}} f - \frac{n-4}{2} Q_g - \frac{n+4}{2} Q_g u, & n \geqslant 5. \end{cases}$$

We rewrite it in the form

$$\mathcal{T}(u) = \begin{cases} 2(e^{4u} - 1 - 4u)f + 2(f - Q_g) - 8(Q_g - f)u, & n = 4, \\ \frac{n-4}{2}((1+u)^{\frac{n+4}{n-4}} - 1 - \frac{n+4}{n-4}u)f + \frac{n-4}{2}(f - Q_g) + \frac{n+4}{2}(f - Q_g)u, & n \geqslant 5. \end{cases}$$

Let L be as in (1.3), then the prescribed Q-curvature equation is

$$Lu = \mathcal{T}(u). \tag{3.1}$$

Let $0 < \nu < \underline{\nu} = \frac{n-1}{2}$ and $0 < \alpha < 1$, so that L is essentially surjective. Moreover, in the following we assume that L is surjective. Then

$$L : V_1 = (I - \Pi_1)\big(x^\nu \Lambda^{4,\alpha}(M)\big) \to x^\nu \Lambda^{0,\alpha}(M)$$

is an isomorphism, using topological splitting of $x^\nu \Lambda^{4,\alpha}(M)$ in Theorem 1.5 and the open mapping theorem. That is,

$$C_1 \|u\|_{x^\nu \Lambda^{4,\alpha}(M)} \leqslant \|Lu\|_{x^\nu \Lambda^{0,\alpha}(M)} \leqslant C_2 \|u\|_{x^\nu \Lambda^{4,\alpha}(M)}, \tag{3.2}$$

for some constant $C_2 > C_1 > 0$, for all $u \in V_1$. We denote the inverse of L as

$$L^{-1} : x^\nu \Lambda^{0,\alpha}(M) \to V_1.$$

Let $f \in C^\alpha(M)$, and

$$(Q_g - f) \in x^\nu \Lambda^{0,\alpha},$$

with its small norm to be determined later. We want to use elements in kernel of L to parametrize the perturbation solutions to the nonlinear problem at 0. We will define a new map for each element in the kernel of L, and use it to construct a contraction map. For any fixed $u_1 \in \operatorname{Ker}(L)$, for any $u_2 \in V_1$, let $u = u_1 + u_2$, and

$$\mathcal{T}_{u_1}(u_2) = \mathcal{T}(u_1 + u_2).$$

Now $L^{-1} \circ \mathcal{T}_{u_1} : V_1 \to V_1$.

From now on, let u_1 be any fixed element in $B_\epsilon(0) \cap \operatorname{Ker}(L)$, and $u_2 \in B_\epsilon(0) \cap V_1$, with small $\epsilon \in (0,1)$ to be determined. Note that

$$\big\|\mathcal{T}_{u_1}(u_2)\big\|_{x^\nu \Lambda^{0,\alpha}} \leqslant \begin{cases} 2\|(e^{4u} - 1 - 4u)f\|_{x^\nu \Lambda^{0,\alpha}(M)} + 2\|(f - Q_g)\|_{x^\nu \Lambda^{0,\alpha}(M)} \\ \quad + 8\|(f - Q_g)u\|_{x^\nu \Lambda^{0,\alpha}(M)}, & n = 4, \\ \frac{n-4}{2}\|((1+u)^{\frac{n+4}{n-4}} - 1 - \frac{n+4}{n-4}u)f\|_{x^\nu \Lambda^{0,\alpha}(M)} \\ \quad + \frac{n-4}{2}\|(f - Q_g)\|_{x^\nu \Lambda^{0,\alpha}(M)} + \frac{n+4}{2}\|(f - Q_g)u\|_{x^\nu \Lambda^{0,\alpha}(M)}, & n \geqslant 5. \end{cases}$$

Then we have

$$\begin{aligned} \big\|\mathcal{T}_{u_1}(u_2)\big\|_{x^\nu \Lambda^{0,\alpha}} \leqslant{} & C(n)\big(\|f\|_{L^\infty}\big\|(u_1+u_2)^2\big\|_{x^\nu \Lambda^{0,\alpha}} + \big(1 + \|u_1 + u_2\|_{L^\infty}\big)\|f - Q_g\|_{x^\nu \Lambda^{0,\alpha}} \\ & + \big\|x^{-\nu}(u_1+u_2)\big\|_{L^\infty}\big\|(u_1+u_2)\big\|_{L^\infty}\big(\|f\|_{\Lambda^{0,\alpha}} + \|Q_g\|_{\Lambda^{0,\alpha}}\big) \\ & + \|f - Q_g\|_{L^\infty}\|u_1+u_2\|_{x^\nu \Lambda^{0,\alpha}}\big), \end{aligned}$$

where $C > 0$ is a constant depending only on n, the diameter of M with respect to $x^2 g$, and ν. By the definition of the weighted norm,

$$\|\phi\|_{L^\infty} \leqslant \|\phi\|_{\Lambda^{0,\alpha}}, \quad \text{and} \quad \|\phi\|_{L^\infty} \leqslant C_0\|\phi\|_{x^\nu \Lambda^{0,\alpha}}, \tag{3.3}$$

for a constant $C_0 > 0$ depending on the defining function and ν, for any $\phi \in x^\nu \Lambda^{0,\alpha}$. Therefore,

$$\big\|\mathcal{T}_{u_1}(u_2)\big\|_{x^\nu \Lambda^{0,\alpha}} \leqslant C_1\big(\big(\epsilon\big(\|f\|_{\Lambda^{0,\alpha}} + \|Q_g\|_{\Lambda^{0,\alpha}}\big) + \|f - Q_g\|_{L^\infty}\big)\|u_1 + u_2\|_{x^\nu \Lambda^{0,\alpha}} + (1+\epsilon)\|f - Q_g\|_{x^\nu \Lambda^{0,\alpha}}\big),$$

where C_1 depends on n, the defining function, the diameter of M with respect to $x^2 g$, and ν, so that

$$\big\|L^{-1} \circ \mathcal{T}_{u_1}(u_2)\big\|_{x^\nu \Lambda^{4,\alpha}} \tag{3.4}$$

$$\leqslant C_1\big\|L^{-1}\big\|\big(\big(\epsilon\big(\|f\|_{\Lambda^{0,\alpha}} + \|Q_g\|_{\Lambda^{0,\alpha}}\big) + \|f - Q_g\|_{L^\infty}\big)\|u_1 + u_2\|_{x^\nu \Lambda^{0,\alpha}} \tag{3.5}$$

$$+ (1+\epsilon)\|f - Q_g\|_{x^\nu \Lambda^{0,\alpha}}\big). \tag{3.6}$$

We now choose $\epsilon \in (0,1)$ small so that

$$16 C_1 \epsilon \big\|L^{-1}\big\|\|Q_g\|_{\Lambda^{0,\alpha}} < 1, \tag{3.7}$$

and let f satisfy that

$$\|f\|_{\Lambda^{0,\alpha}} \leqslant 2\|Q_g\|_{\Lambda^{0,\alpha}}, \quad \text{and} \quad \|f - Q_g\|_{x^\nu \Lambda^{0,\alpha}} \leqslant \min\left\{\frac{1}{4(1+\epsilon)C_1\|L^{-1}\|}\epsilon, \frac{\epsilon\|Q_g\|_{\Lambda^{0,\alpha}}}{C_0}\right\}. \tag{3.8}$$

Combining (3.3), we have

$$\big\|L^{-1} \circ \mathcal{T}_{u_1}(u_2)\big\|_{x^\nu \Lambda^{4,\alpha}} \leqslant \frac{3}{4}\epsilon.$$

Therefore, $L^{-1} \circ T_{u_1}$ maps $B_\epsilon(0) \cap V_1$ into $B_\epsilon(0) \cap V_1$.

For $u_3, u_4 \in V_1 \cap B_\epsilon(0)$,

$$\begin{aligned} & \big\|L^{-1} \circ \mathcal{T}_{u_1}(u_3) - L^{-1} \circ \mathcal{T}_{u_1}(u_4)\big\|_{x^\nu \Lambda^{4,\alpha}} \\ & \quad \leqslant \big\|L^{-1}\big\|\big\|\mathcal{T}_{u_1}(u_3) - \mathcal{T}_{u_1}(u_4)\big\|_{x^\nu \Lambda^{0,\alpha}} \\ & \quad = \begin{cases} \|L^{-1}\|\|2f(e^{4u_1}(e^{4u_3} - e^{4u_4}) - 4(u_3 - u_4)) - 8(Q_g - f)(u_3 - u_4)\|_{x^\nu \Lambda^{0,\alpha}}, & n = 4, \\ \|L^{-1}\|\|\frac{n-4}{2}((1+u_1+u_3)^{\frac{n+4}{n-4}} - (1+u_1+u_4)^{\frac{n+4}{n-4}} - \frac{n+4}{n-4}(u_3 - u_4))f \\ \quad + \frac{n+4}{2}(f - Q_g)(u_3 - u_4)\|_{x^\nu \Lambda^{0,\alpha}}, & n \geqslant 5. \end{cases} \end{aligned}$$

But

$$e^{4(u_1+u_3)} - e^{4(u_1+u_4)} - 4(u_3 - u_4) = 4(u_3 - u_4)w,$$

with

$$w = \left(\frac{e^{4(u_1+u_3)} - e^{4(u_1+u_4)}}{4(u_3 - u_4)} - 1 \right) = \left(\int_0^1 e^{4(u_1+u_4+t(u_3-u_4))}\,dt - 1 \right) \in x^\nu \Lambda^{0,\alpha} \cap B_{C\epsilon}(0),$$

with C which does not depend on u_3, u_4, or $\epsilon \in (0,1)$. We have similar results for $n \geqslant 5$. By the discussion above,

$$\begin{aligned} &\left\| L^{-1} \circ \mathcal{T}_{u_1}(u_3) - L^{-1} \circ \mathcal{T}_{u_1}(u_4) \right\|_{x^\nu \Lambda^{4,\alpha}} && (3.9)\\ &\quad \leqslant \left\| L^{-1} \right\| \widetilde{C_0} \left(\epsilon \|f\|_{\Lambda^{0,\alpha}} \|u_3 - u_4\|_{x^\nu \Lambda^{0,\alpha}} + \|Q_g - f\|_{x^\nu \Lambda^{0,\alpha}} \|u_3 - u_4\|_{x^\nu \Lambda^{0,\alpha}} \right) && (3.10)\\ &\quad = \left\| L^{-1} \right\| \widetilde{C_0} \left(\epsilon \|f\|_{\Lambda^{0,\alpha}} + \|Q_g - f\|_{x^\nu \Lambda^{0,\alpha}} \right) \|u_3 - u_4\|_{x^\nu \Lambda^{0,\alpha}}, \quad n \geqslant 4, && (3.11) \end{aligned}$$

where $\widetilde{C_0}$ depends only on the defining function, the diameter of M with respect to $x^2 g$, ν and n. Let ϵ be small so that

$$8\widetilde{C_0} \left\| L^{-1} \right\| \left(1 + \|Q_g\|_{\Lambda^{0,\alpha}} \right) \epsilon < 1, \tag{3.12}$$

and let

$$\|Q_g - f\|_{x^\nu \Lambda^{0,\alpha}} \leqslant \frac{1}{8\widetilde{C_0} \|L^{-1}\|}. \tag{3.13}$$

Then we have

$$\begin{aligned} \left\| L^{-1} \circ \mathcal{T}_{u_1}(u_3) - L^{-1} \circ \mathcal{T}_{u_1}(u_4) \right\|_{x^\nu \Lambda^{4,\alpha}} &\leqslant \frac{3}{8} \|u_3 - u_4\|_{x^\nu \Lambda^{0,\alpha}} \\ &\leqslant \frac{3}{8} \|u_3 - u_4\|_{x^\nu \Lambda^{4,\alpha}}. \end{aligned}$$

Note that $\|L^{-1}\|$ depends on the projection map Π_1 that we construct in Theorem 1.5. Therefore, if L is surjective for $\nu < \frac{n-1}{2}$, and also ϵ and f satisfy the above conditions, then for each $u_1 \in B_\epsilon(0) \cap \mathrm{Ker}(L)$,

$$L^{-1} \circ \mathcal{T}_{u_1} : V_1 \cap B_\epsilon(0) \to V_1 \cap B_\epsilon(0)$$

is a contraction map. This implies that there exists a unique $u_2 \in B_\epsilon(0) \cap V_1$, solving the equation

$$L(u_1 + u_2) = \mathcal{T}_{u_1}(u_2).$$

Note that the proof above holds for $h = x^2 g \in C^{4,\alpha}(\overline{M})$. Now we have proved the following theorem.

Theorem 3.1. *Let (M, g) be an asymptotically hyperbolic manifold of dimensional $n \geqslant 4$, with x the smooth defining function, and the metric $h = x^2 g \in C^{4,\alpha}(\overline{M})$. For $0 < \nu < \frac{n-1}{2}$ and $0 < \alpha < 1$, let*

$$L : x^\nu \Lambda^{4,\alpha}(M) \to x^\nu \Lambda^{0,\alpha}$$

be the linear operator defined in (1.3), which by Theorem 1.5 is essentially surjective. Assume that L is surjective. Then there exists a small constant $\epsilon_0 > 0$, depending on the diameter of M with respect to h, ν, n and also Π_1 and L, so that the following holds:

Let ϵ be any small real number satisfying $0 < \epsilon < \epsilon_0$, and let $f \in \Lambda^{0,\alpha}(M)$ satisfy

$$\|Q_g - f\|_{x^\nu \Lambda^{0,\alpha}} \leqslant \tilde{C}\epsilon,$$

for some positive constant $\tilde{C}$ depending on the diameter of M with respect to h, ν, n, also Π_1 and L.

Then for each $u_1 \in B_\epsilon(0) \cap \mathrm{Ker}(L)$, there exists a unique $u \in B_{2\epsilon}(0) \subseteq x^\nu \Lambda^{4,\alpha}(M)$, so that $Q_{\tilde{g}} = f$, where $\tilde{g} = (1+u)^{\frac{4}{n-4}} g$ for $n \geqslant 5$, and $\tilde{g} = e^{2u} g$ for $n = 4$, with $\Pi_1 u = u_1$.

By the discussion at the end of Section 2, for the cases in Theorem 1.3, L is surjective for $x^\nu \Lambda^{4,\alpha}(M)$, $0 < \nu < \frac{n-1}{2}$. This completes the proof of (i) of Theorem 1.3.

Since surjectivity is an open property, L is surjective for $x^\nu \Lambda^{4,\alpha}(M)$, $0 < \nu < \frac{n-1}{2}$, for smooth g that is close enough to these metrics. Theorem 3.1 holds for metrics in a small neighborhood of these metrics.

In the following, we will discuss about the boundary regularity of the solutions. For convenience, we assume that the defining function x and the metric $h = x^2 g$ are smooth up to the boundary. The discussion we use here is standard, see [20]. We will sketch the discussion. Composing the inverse G operator of L on both sides of (3.1),

$$u - \Pi_1 u = GLu = G\mathcal{T}(u), \tag{3.14}$$

with $u_1 = \Pi_1 u$ the projection of u to the null space of L.

For the regularity of u with respect to the derivative ∂_y, which is the derivative in some y direction, we introduce the following weighted space with $k \leqslant m$:

$$\begin{aligned} x^\nu \Lambda^{m,\alpha,k} = \big\{ & u \in x^\nu \Lambda^{m,\alpha}\big(M, \sqrt{dx\,dy}\big),\ \text{so that } (x\partial_x)^j (x\partial_y)^\beta \partial_y^\gamma u \in x^\nu \Lambda^{0,\alpha}, \\ & \text{for } j + |\beta| + |\gamma| \leqslant m,\ j \geqslant 0,\ \text{and } |\gamma| \leqslant k \big\}. \end{aligned}$$

An easy observation is that for $u \in x^\nu \Lambda^{m,\alpha}$ and $m \geqslant 1$, $\partial_y u = x\partial_y(x^{-1}u)$, so that

$$\partial_y u \in x^{\nu-1} \Lambda^{m-1,\alpha}. \tag{3.15}$$

Also for $u \in x^\nu \Lambda^{m,\alpha,k}$ and $1 \leqslant k \leqslant m$, $\partial_y u \in x^\nu \Lambda^{m-1,\alpha,k-1}$. In Proposition 2.9 in [20], it is proved that the inverse operator $G : x^\nu \Lambda^{m,\alpha,k} \to x^\nu \Lambda^{m+4,\alpha,k}$ is bounded for $m \geqslant 0$ and $0 \leqslant k \leqslant m$; also, $\Pi_1 : x^\nu \Lambda^{m+4,\alpha,k} \to x^\nu \Lambda^{m+4,\alpha,k}$ is bounded for $m \geqslant 0$ and $0 \leqslant k \leqslant m$.

Lemma 3.2. *Let $u \in x^\nu \Lambda^{4,\alpha}$ be a solution to (3.1) with $1 \leqslant \nu < \frac{n-1}{2}$ and $0 < \alpha < 1$. Assume that $(f - Q_g) \in x^\nu \Lambda^{m,\alpha,k}$, and $u_1 = \Pi_1 u \in x^\nu \Lambda^{m+4,\alpha,k}$, for $0 \leqslant k \leqslant m$. Then we have that $u \in x^\nu \Lambda^{m+4,\alpha,k}$.*

Proof. By assumption, x and the metric h are smooth up to the boundary, so that $Q_g \in C^\infty(\overline{M}) \subseteq \Lambda^{m,\alpha,k}$ for any $m \geqslant k$, and then we have $f \in \Lambda^{m,\alpha,k}$. For $m = 0$ the claim holds automatically. Now assume $m \geqslant 1$. Using (3.14) and boundedness of G for $k = 0$ we obtain that $u \in x^\nu \Lambda^{1+4,\alpha}$. Then we can substitute the regularity of u into the right-hand side of (3.14), to gain more regularity. Using this induction argument, we obtain $u \in x^\nu \Lambda^{m+4,\alpha} = x^\nu \Lambda^{m+4,\alpha,0}$. This proves the lemma for $k = 0$.

Define the function F on $\mathbb{R}$ as follows,

$$F(u) = \begin{cases} e^{4u} - 1 - 4u, & n = 4, \\ (1+u)^{\frac{n+4}{n-4}} - 1 - \frac{n+4}{n-4} u, & n \geqslant 5. \end{cases}$$

Noticing that for $u \in x^\nu \Lambda^{m,\alpha,k'}$ with $k' < k$, using (3.15) and the fact $\nu \geqslant 1$, we have that

$$u^2 f = xu\big(x^{-1}u\big) f \in x^\nu \Lambda^{m,\alpha,k'+1},$$

raising the third index by 1. This holds for the term $F(u)f$, since F is smooth on $\mathbb{R}$ and vanishes quadratically at 0. Similarly,

$$u(f - Q_g) = xu\big(x^{-1}(f - Q_g)\big) = x\big(x^{-1}u\big)(f - Q_g) \in x^\nu \Lambda^{m,\alpha,k'+1}.$$

By this fact, combining with Eq. (3.14), and also with boundedness of G, an induction argument as the case $k = 0$ proves the lemma. $\square$

Now we assume that $f = Q_g$. Generally, $u_1 = \Pi_1 u \in x^\nu \Lambda^{4,\alpha}$ does not have better regularity. In (3.14), the terms on the right-hand side behave better than $\Pi_1 u$, and u behaves like $\Pi_1 u$ near the boundary, and u only has the expansion (1.4) with the coefficients which are distributions of negative order, as discussed in Proposition 3.16 in [20]. If $1 \leqslant \nu < \frac{n-1}{2}$ and $u_1 = \Pi_1 u \in x^\nu \Lambda^{m,\alpha,k}$ for all $m \geqslant k \geqslant 0$, which as discussed in [18] is equivalent to say u_1 has a smooth expansion (2.9), then by Lemma 3.2, u has a smooth expansion as in (1.5). Also, for u_1 small enough, we already obtain the existence of u in Poincaré–Einstein manifolds. This completes the proof of Theorem 1.3.

Here we observe that the expansion of u gives us information on the asymptotic behavior of the curvature. For $n=4$, assume that g and $\tilde{g}$ are asymptotically hyperbolic metrics on M, with the transformation $\tilde{g}=e^{2u}g$, such that u has the expansion $u\sim x^{\frac{3}{2}+i\frac{\sqrt{15}}{2}}u_{00}(y)+x^{\frac{3}{2}-i\frac{\sqrt{15}}{2}}u_{10}(y)+o(x^{\frac{3}{2}})$. Let $(1+v)^2=e^{2u}$. Denote $\nu_0=\frac{3}{2}+i\frac{\sqrt{15}}{2}$, and $\nu_1=\bar{\nu_0}$. Then,

$$\begin{aligned}R_{\tilde{g}}&=(1+v)^{-3}(-6\Delta_g+R_g)(1+v)=e^{-3u}(-6\Delta_g+R_g)e^u\\&=-6e^{-2u}\big[-3x\partial_x u+(x\partial_x)^2u\big]+R_g-2R_gu+R_g\big(e^{-2u}-1+2u\big)\\&\quad+6x^2e^{-3u}\bigg(\Delta_y e^u+\frac{1}{2}\sum_{4\geqslant i,j\geqslant 2}h^{ij}\partial_x e^u\bigg).\end{aligned}$$

Therefore,

$$\begin{aligned}R_{\tilde{g}}-R_g&=-6e^{-2u}\big[-3x\partial_x u+(x\partial_x)^2u\big]-2R_gu+R_g\big(e^{-2u}-1+2u\big)\\&\quad+6x^2e^{-3u}\bigg(\Delta_y e^u+\frac{1}{2}\sum_{4\geqslant i,j\geqslant 2}h_{ij}\partial_x h_{ij}\partial_x e^u\bigg)\\&=-6\big(-3x\partial_x u+(x\partial_x)^2u\big)+6\big(1-e^{-2u}\big)\big(-3x\partial_x u+(x\partial_x)^2u\big)+24u\\&\quad-2u(12+R_g)+R_g\big(e^{-2u}-1+2u\big)+6x^2e^{-3u}\bigg[\Delta_y e^u+\frac{1}{2}\sum_{4\geqslant i,j\geqslant 2}h^{ij}\partial_x h_{ij}\partial_x e^u\bigg]\\&=-6\big(-3\nu_0x^{\nu_0}u_{00}(y)+\nu_0^2x^{\nu_0}u_{00}(y)-3\nu_1x^{\nu_1}u_{10}(y)+\nu_1^2x^{\nu_1}u_{10}(y)+O\big(x^{\frac{3}{2}+1}\big)\big)\\&\quad+24\big(x^{\nu_0}u_{00}(y)+x^{\nu_1}u_{10}(y)+O\big(x^{\frac{3}{2}+1}\big)\big)+O\big(x^{\frac{3}{2}+1}\big)\\&=-6\big(\big(\nu_0^2-3\nu_0-4\big)x^{\nu_0}u_{00}(y)+\big(\nu_1^2-3\nu_1-4\big)x^{\nu_1}u_{10}(y)\big)+O\big(x^{\frac{3}{2}+1}\big)\\&=120u+o\big(x^{\frac{3}{2}}\big).\end{aligned}$$

For asymptotically hyperbolic manifolds of higher dimension, with similar calculation, we obtain the formula

$$R_{\tilde{g}}-R_g=\frac{4(n-1)(n^2+2n-4)}{(n-4)}u+o\big(x^{\frac{n-1}{2}}\big).$$

4. Constant Q-curvature metrics for perturbed conformal structures

Let (M,g_0) be a Poincaré–Einstein manifold, with a defining function x and the metric $h_0=x^2g_0$ smooth up to the boundary. Let

$$\mathfrak{M}_\tau=\big\{h\text{: metrics on }\overline{M},\ \text{so that }h\in C^{4,\alpha}(\overline{M}),\ \text{with }\|h-h_0\|_{C^{4,\alpha}(M)}\leqslant\tau,\ \text{and }|dx|_h|_{\partial M}=1\big\},$$

for $\tau>0$ and $0<\alpha<1$. For $h\in\mathfrak{M}_\tau$, let $g=x^{-2}h$. We want to see when τ is small enough whether we can find a constant Q-curvature metric $\tilde{g}$ in the conformal class of g, with $Q_{\tilde{g}}=Q_{g_0}$. We use the same notation u, L_g and so on as above. Note that the choice of x with $|dx|_h=1$ in a neighborhood of the boundary in the preceding sections is only to make the notation simpler. In this section we only assume that $|dx|_h=1$ on ∂M, and this only introduces order terms in $E(L)$. Now let us state the main theorem in this section.

Theorem 4.1. *Let (M,g_0) be a Poincaré–Einstein manifold with defining function x, such that the metric $h_0=x^2g_0$ is smooth up to the boundary. There exists $\tau_0>0$, so that for $0<\tau\leqslant\tau_0$, and all $C^{4,\alpha}$ Riemannian metrics h on $\overline{M}$ with $\|h-h_0\|_{C^{4,\alpha}(M)}\leqslant\tau$, there exists a family of asymptotically hyperbolic metrics in the conformal class of $g=x^{-2}h$ with constant Q-curvature Q_{g_0}. Furthermore, these metrics are parametrized by elements in* $\operatorname{Ker}(L_{g_0})$.

For the proof of this theorem, we need the following lemma.

Lemma 4.2. *Let (M, g_0) be a Poincaré–Einstein manifold with defining function x and the metric $h_0 = x^2 g_0$ smooth up to the boundary, and let $\mathfrak{M}_\tau$ be as above, with $\tau > 0$. There exists $\tau_0 > 0$, so that for $0 < \tau \leqslant \tau_0$, and any metric $h \in \mathfrak{M}_\tau$, there always exist a family of asymptotically hyperbolic metrics in the conformal class of $g = x^{-2}h$ with constant Q-curvature Q_{g_0}, which are parametrized by elements in* $\mathrm{Ker}(L_{g_0})$.

Now let us first use the lemma to prove Theorem 4.1. For a metric h close enough to h_0 in $C^{4,\alpha}(\overline{M})$, let $h_1 = sh$ with s a smooth function on $\overline{M}$ so that $s = |dx|_h^2$ in a small neighborhood of ∂M, and $\|s-1\|_{C^{4,\alpha}(M)} \leqslant 10\|h-h_0\|_{C^{4,\alpha}(M)}$. Then $|dx|_{h_1}|_{\partial M} = 1$. Since $|dx|_{h_0} = 1$, if $\|h-h_0\|_{C^{4,\alpha}(M)}$ is small enough, then s is close enough to 1 and also h_1 is in the class $\mathfrak{M}_\tau$ for τ small. By Lemma 4.2, there are infinitely many asymptotically hyperbolic metrics in the conformal class of $x^{-2}h_1$ (which is also the conformal class of $g = x^{-2}h$) with constant Q-curvature Q_{g_0}, which are parametrized by elements in $\mathrm{Ker}(L_{g_0})$. This proves Theorem 4.1.

Proof of Lemma 4.2. It is easy to check that

$$x^\alpha \Lambda^{0,\alpha}(M, \sqrt{dx\,dy}) = \{u \in C^\alpha(\overline{M}),\ u|_{\partial M} = 0\}.$$

Let L_g and L_{g_0} be the linear operators (1.3) with respect to g and g_0. Recall that Ric_g and R_g satisfy (2.2) and (2.3). Also we know that

$$\left(|dx|_h^2 - 1\right) \in x^\alpha \Lambda^{0,\alpha}(M, \sqrt{dx\,dy}), \quad \text{and} \quad \left\| |dx|_h^2 - |dx|_{h_0}^2 \right\|_{x^\alpha \Lambda^{0,\alpha}} \leqslant C\tau,$$

for some constant C depending on the defining function and h_0. Also it is easy to see the following inequalities by direct calculation

$$\begin{aligned}
&\left\| \left(\Delta_g^2 - \Delta_{g_0}^2\right) u \right\|_{x^\alpha \Lambda^{0,\alpha}} \leqslant C\tau \|u\|_{x^\alpha \Lambda^{4,\alpha}},\\
&\left\| (R_g \Delta_g - R_{g_0} \Delta_{g_0}) u \right\|_{x^\alpha \Lambda^{0,\alpha}} \leqslant C\tau \|u\|_{x^\alpha \Lambda^{4,\alpha}},\\
&\left\| \left(\mathrm{Ric}_{ij}(g) \nabla_g^i \nabla_g^j - \mathrm{Ric}_{ij}(g_0) \nabla_{g_0}^i \nabla_{g_0}^j\right) u \right\|_{x^\alpha \Lambda^{0,\alpha}} \leqslant C\tau \|u\|_{x^\alpha \Lambda^{4,\alpha}},\\
&\left\| (\nabla_g R_g, \nabla_g u) - (\nabla_{g_0} R_{g_0}, \nabla_{g_0} u) \right\|_{x^\alpha \Lambda^{0,\alpha}} \leqslant C\tau \|u\|_{x^\alpha \Lambda^{4,\alpha}},\\
&\|Q_g - Q_{g_0}\|_{x^\alpha \Lambda^{0,\alpha}} \leqslant C\tau,
\end{aligned}$$

with C depending on the defining function x and the metric h_0. Moreover, from the discussion in Section 2, we know that L_{g_0} is surjective since g_0 is a smooth Poincaré–Einstein metric. Let

$$x^\alpha \Lambda^{4,\alpha}(M, \sqrt{dx\,dy}) = \mathrm{Ker}(L_{g_0}) \oplus V_1(g_0), \tag{4.1}$$

be the splitting as in Theorem 1.5. Restricted to $V_1(g_0)$, L_{g_0} is an isomorphism and satisfies (3.2). Let $L_{g_0}^{-1}$ be the inverse of the restriction map of $L_{g_0}|_{V_1(g_0)}$. By the above estimates, there exists $\tau_1 > 0$, so that for $0 < \tau \leqslant \tau_1$,

$$\|L_g - L_{g_0}\| \leqslant \frac{1}{8\|L_{g_0}^{-1}\|}. \tag{4.2}$$

We want to solve Eq. (3.1) with $f = Q_{g_0}$ and the starting metric g. We rewrite the equation as follows:

$$L_{g_0} u = F(u) \equiv (L_{g_0} - L_g) u + \mathcal{T}(u), \tag{4.3}$$

and it follows that,

$$u = L_{g_0}^{-1} \circ (L_{g_0} - L_g) u + L_{g_0}^{-1} \circ \mathcal{T}(u). \tag{4.4}$$

For any $u_1 \in \mathrm{Ker}(L_{g_0})$, let F_{u_1} be the function on V_1 so that $F_{u_1}(u) = F(u_1 + u)$ for $u \in V_1$, and also $\mathcal{T}_{u_1}(u) = \mathcal{T}(u_1 + u)$ as before.

As in Section 3, we want to show that for $\epsilon > 0$ small enough, $L_{g_0}^{-1} \circ F_{u_1}$ is a contraction map on $B_\epsilon(0) \cap V_1$ for $u_1 \in B_\epsilon(0) \cap \mathrm{Ker}(L_{g_0})$. To do this, in the following we will essentially follow the arguments of Section 3.

For any $\epsilon > 0$, let $u_1 \in \mathrm{Ker}(L_{g_0}) \cap B_\epsilon(0)$ and $u_2 \in B_\epsilon(0) \cap V_1$. Then we get (3.4), with $f = Q_{g_0}$ and the constant C_1 depends on the diameter of M with respect to $x^2 g_0$ instead. Now let

$$\epsilon_1 = \frac{1}{32C_1\|L_{g_0}^{-1}\|\|Q_{g_0}\|_{\Lambda^{0,\alpha}}},$$

and let $\epsilon \leqslant \epsilon_1$. Also, by the above estimates, there exists $\tau_2 = \tau_2(\epsilon) > 0$, so that for $0 < \tau \leqslant \tau_2$,

$$\|Q_g\|_{\Lambda^{0,\alpha}} \leqslant 2\|Q_{g_0}\|_{\Lambda^{0,\alpha}}, \quad \text{and} \quad \|Q_{g_0} - Q_g\|_{x^\alpha\Lambda^{0,\alpha}} \leqslant \min\left\{\frac{1}{4(1+\epsilon)C_1\|L_{g_0}^{-1}\|}\epsilon, \frac{\epsilon\|Q_{g_0}\|_{\Lambda^{0,\alpha}}}{C_0}\right\}, \tag{4.5}$$

for C_0 in (3.3). Combining (3.3), we have

$$\left\|L_{g_0}^{-1} \circ \mathcal{T}_{u_1}(u_2)\right\| \leqslant \frac{1}{2}\epsilon. \tag{4.6}$$

Now let $0 < \epsilon \leqslant \epsilon_1$ and $\tau \leqslant \min\{\tau_1, \tau_2(\epsilon)\}$. Using (4.2), (4.4) and (4.6), we have that

$$\left\|L_{g_0}^{-1} \circ F_{u_1}(u_2)\right\|_{x^\alpha\Lambda^{0,\alpha}} \leqslant \frac{3}{4}\epsilon. \tag{4.7}$$

Therefore, $L^{-1} \circ F_{u_1}$ maps $B_\epsilon \cap V_1$ into itself.

For $u_3, u_4 \in V_1 \cap B_\epsilon(0)$, using the same argument as in Section 3, we have similar inequality as (3.9),

$$\left\|L_{g_0}^{-1} \circ \mathcal{T}_{u_1}(u_3) - L_{g_0}^{-1} \circ \mathcal{T}_{u_1}(u_4)\right\|_{x^\alpha\Lambda^{4,\alpha}} \tag{4.8}$$

$$\leqslant \widetilde{C_0}\left\|L_{g_0}^{-1}\right\|\left(\epsilon\|Q_{g_0}\|_{\Lambda^{0,\alpha}} + \|Q_{g_0} - Q_g\|_{x^\alpha\Lambda^{0,\alpha}}\right)\|u_3 - u_4\|_{x^{0,\alpha}\Lambda^{0,\alpha}}, \tag{4.9}$$

with $\widetilde{C_0}$ depending on the defining function x, the diameter of M with respect to x^2g_0, α and n. Let

$$\epsilon_2 = \frac{1}{8\widetilde{C_0}\|L_{g_0}^{-1}\|(1 + \|Q_{g_0}\|_{\Lambda^{0,\alpha}})}.$$

There exists $\tau_3 > 0$, so that for $0 < \tau \leqslant \tau_3$,

$$\|Q_{g_0} - Q_g\|_{x^\alpha\Lambda^{0,\alpha}} \leqslant \frac{1}{8\widetilde{C_0}\|L_{g_0}^{-1}\|}. \tag{4.10}$$

Therefore we have that

$$\left\|L_{g_0}^{-1} \circ \mathcal{T}_{u_1}(u_3) - L_{g_0}^{-1} \circ \mathcal{T}_{u_1}(u_4)\right\|_{x^\alpha\Lambda^{4,\alpha}} \leqslant \frac{1}{4}\|u_3 - u_4\|_{x^\alpha\Lambda^{4,\alpha}}, \tag{4.11}$$

for $\epsilon \leqslant \epsilon_2$. Now let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, and $\tau = \min\{\tau_1, \tau_2(\epsilon), \tau_3\}$. Combing (4.2) and (4.11), we obtain that

$$\left\|L_{g_0}^{-1} \circ F_{u_1}(u_3) - L_{g_0}^{-1} \circ F_{u_1}(u_4)\right\|_{x^\alpha\Lambda^{4,\alpha}} \tag{4.12}$$

$$\leqslant \left\|L_{g_0}^{-1}\right\|\|L_{g_0} - L_g\|\|u_3 - u_4\|_{x^\alpha\Lambda^{4,\alpha}} + \frac{1}{4}\|u_3 - u_4\|_{x^\alpha\Lambda^{4,\alpha}} \tag{4.13}$$

$$\leqslant \left(\frac{1}{8} + \frac{1}{4}\right)\|u_3 - u_4\|_{x^\alpha\Lambda^{4,\alpha}} = \frac{3}{8}\|u_3 - u_4\|_{x^\alpha\Lambda^{4,\alpha}}. \tag{4.14}$$

Therefore, $L_{g_0}^{-1} \circ F_{u_1}$ is a contraction map on $B_\epsilon \cap V_1$. Then there exists a unique fixed point u_2. But then

$$L_{g_0}(u_1 + u_2) = L_{g_0}(u_2) = F_{u_1}(u_2) = F(u_1 + u_2). \tag{4.15}$$

So $u = u_1 + u_2$ is a solution to the constant Q-curvature equation with $u_1 = \Pi_1(u)$. We should also note that the dimension of $\mathrm{Ker}(L)$ is infinity. This completes the proof of the lemma. □

5. Critical metrics of regularized determinants

Let M be a fourth-dimensional asymptotically hyperbolic manifold, with complete metric g and its smooth defining function x, so that $h = x^2g$ is a smooth metric on $\overline{M}$. Consider the equation

$$U = U_g \equiv \gamma_1|W|^2 + \gamma_2 Q - \gamma_3\Delta R = C, \tag{5.1}$$

where $\gamma_1, \gamma_2, \gamma_3$ and C are some constants, W is the Weyl tensor, and Q, R the Q-curvature and the scalar curvature with respect to g. This equation arises as the Euler–Lagrange equation for the regularized determinants,

$$F_A[w] = \log\left(\frac{\det A_{\tilde{g}}}{\det A_g}\right),$$

of a conformally covariant operator $A = A_g$, under the conformal change of metrics $\tilde{g} = e^{2w} g$, see Chapter 6 in [14]. More precisely, under a conformal change of metric,

$$\tilde{U} e^{4w} = U + \left(\frac{1}{2}\gamma_2 + 6\gamma_3\right)\Delta^2 w + 6\gamma_3 \Delta|\nabla w|^2 - 12\gamma_3 \nabla^i\left[\left(\Delta w + |\nabla w|^2\right)\nabla_i w\right] \tag{5.2}$$

$$+ \gamma_2 R_{ij} \nabla_i \nabla_j w + \left(2\gamma_3 - \frac{1}{3}\gamma_2\right) R\Delta w + \left(2\gamma_3 + \frac{1}{6}\gamma_2\right)(\nabla R, \nabla w), \tag{5.3}$$

with $\tilde{U} = U_{\tilde{g}}$. Define $\alpha = \frac{\gamma_2}{12\gamma_3}$. The following are some examples that we are interested in.

Example 1. For the conformal Laplacian, $A = L$, we have that $(\gamma_1, \gamma_2, \gamma_3) = (1, -4, -\frac{2}{3})$, and $\alpha = \frac{1}{2}$.

Example 2. For the spin Laplacian, $A = D^2$, we have that $(\gamma_1, \gamma_2, \gamma_3) = (7, -88, -\frac{14}{3})$, and $\alpha = \frac{11}{7}$.

Example 3. For the Paneitz operator, $A = P$, we have that $(\gamma_1, \gamma_2, \gamma_3) = (-\frac{1}{4}, -14, \frac{8}{3})$, and $\alpha = \frac{-7}{16}$.

For convenience, dividing both sides of the function by $6\gamma_3$, we have the following equation,

$$\frac{\tilde{U}}{6\gamma_3} e^{4w} = (1+\alpha)\Delta^2 w + \Delta|\nabla w|^2 - 2\nabla^i\left[\left(\Delta w + |\nabla w|^2\right)\nabla_i w\right] + 2\alpha R_{ij} \nabla^i \nabla^j w \tag{5.4}$$

$$+ \left(\frac{1}{3} - \frac{2}{3}\alpha\right) R\Delta w + \left(\frac{1}{3} + \frac{1}{3}\alpha\right)(\nabla R, \nabla w) + \frac{U}{6\gamma_3}. \tag{5.5}$$

By the Bochner formula,

$$\Delta|\nabla w|^2 - 2\nabla^i(\Delta w \nabla_i w) = 2\,\mathrm{Ric}(\nabla w, \nabla w) + 2\left(\left|\nabla^2 w\right|_g^2 - (\Delta w)^2\right).$$

Moreover,

$$\nabla^i\left(|\nabla w|^2 \nabla_i w\right) = 2\nabla^i \nabla^j w \nabla_j w \nabla_i w + |\nabla w|^2 \Delta w,$$

therefore, the equation can be written in the following way,

$$\frac{\tilde{U}}{6\gamma_3} e^{4w} = (1+\alpha)\Delta^2 w + 2\,\mathrm{Ric}(\nabla w, \nabla w) + 2\left(\left|\nabla^2 w\right|_g^2 - (\Delta w)^2\right) \tag{5.6}$$

$$- 4\nabla_i \nabla_j w \nabla_j w \nabla_i w - 2|\nabla w|^2 \Delta w + 2\alpha R_{ij} \nabla^i \nabla^j w \tag{5.7}$$

$$+ \left(\frac{1}{3} - \frac{2}{3}\alpha\right) R\Delta w + \left(\frac{1}{3} + \frac{1}{3}\alpha\right)(\nabla R, \nabla w) + \frac{U}{6\gamma_3}. \tag{5.8}$$

We should point out that for $\alpha = -1$ and $\gamma_1 = 0$, the equation reduces to a second order differential equation, and in this case the U-curvature corresponds to the σ_2-curvature with respect to the Schouten tensor $A(g)$,

$$\begin{aligned}
\frac{1}{12\gamma_3} U(g) &= \frac{\gamma_1}{12\gamma_3} |W|_g^2 + \frac{\gamma_2}{12\gamma_3} Q_g - \frac{\Delta R_g}{12} \\
&= -\left(\frac{-1}{4}|\mathrm{Ric}_g|^2 + \frac{1}{12} R_g^2 - \frac{1}{12}\Delta_g R_g\right) - \frac{\Delta R_g}{12} \\
&= -\left(\frac{-1}{4}|\mathrm{Ric}_g|^2 + \frac{1}{12} R_g^2\right) = -2\sigma_2(A_g).
\end{aligned}$$

In this case, we have the equation

$$\begin{aligned}4\sigma_2(\tilde{g}) &= -2\operatorname{Ric}(\nabla_g w, \nabla_g w) - 2\big(\big|\nabla^2 w\big|_g^2 - (\Delta w)^2\big) + 4\nabla_{ij} w \nabla_i w \nabla_j w \\ &\quad + 2|\nabla w|^2 \Delta w + 2\operatorname{Ric}_{ij} \nabla^i \nabla^j w - R_g \Delta w + 4\sigma_2(A_g).\end{aligned}$$

The prescribed constant σ_2-curvature problem for asymptotically hyperbolic metrics is discussed in [22]. From now on, we assume that $\alpha \neq -1$.

The linearization of (5.6) is given by

$$Lw = (1+\alpha)\Delta^2 w + 2\alpha R_{ij} \nabla^i \nabla^j w + \left(\frac{1}{3} - \frac{2}{3}\alpha\right) R \Delta w + \left(\frac{1}{3} + \frac{1}{3}\alpha\right)(\nabla R, \nabla w) - \frac{2U}{3\gamma_3} w.$$

As $x \to 0$,

$$\begin{aligned}R_{ijkl}(g) &= x^{-2}\bigg[R_{ijkl}(h) - h_{ik}\left(x^{-1}\nabla_j^h \nabla_l x + \frac{1}{2}x^{-2}h_{jl}\right) - h_{jl}\left(-x^{-1}\nabla_i^h \nabla_k x + \frac{1}{2}x^{-2}h_{ik}\right) \\ &\quad + h_{il}\left(-x^{-1}\nabla_j^h \nabla_k x + \frac{1}{2}x^{-2}h_{jk}\right) + h_{jk}\left(-x^{-1}\nabla_i^h \nabla_l x + \frac{1}{2}x^{-2}h_{il}\right)\bigg] \\ &= x^{-4}\left[-\frac{1}{2}h_{ik}h_{jl} - \frac{1}{2}h_{jl}h_{ik} + h_{il}h_{jk} + \frac{1}{2}h_{jk}h_{il} + O(x)\right] \\ &= x^{-4}\big[-h_{ik}h_{jl} + h_{il}h_{jk} + O(x)\big],\end{aligned}$$

while

$$A(g) = \frac{1}{4-2}\left(\operatorname{Ric}(g) - \frac{1}{2(4-1)}R(g)g\right) = \frac{1}{2}\big(-3+2+O(x)\big)g = \left(-\frac{1}{2} + O(x)\right)g.$$

Therefore,

$$\begin{aligned}W_{ijkl}(g) &= R_{ijkl}(g) - g_{ik}A_{jl}(g) + g_{il}A_{jk}(g) + g_{jk}A_{il}(g) - g_{jl}A_{ik}(g) \\ &= x^{-4}\big(-h_{ik}h_{jl} + h_{il}h_{jk} + O(x)\big) + x^{-4}\bigg[-h_{ik}\left(-\frac{1}{2}h_{jl} + O(x)\right) \\ &\quad + h_{il}\left(-\frac{1}{2}h_{jk} + O(x)\right) + h_{jk}\left(-\frac{1}{2}h_{il} + O(x)\right) - h_{jl}\left(-\frac{1}{2}h_{ik} + O(x)\right)\bigg] \\ &= x^{-4}O(x).\end{aligned}$$

Moreover, using the fact $\Delta_h R = O(x)$ and $Q(g) = 3 + O(x)$, we have that $U(g) = 3\gamma_2 + O(x)$. We then obtain the main terms of Lw as follows:

$$\begin{aligned}Lw &= (1+\alpha)\Delta_g^2 w + \left(\frac{1}{3} - \frac{2}{3}\alpha\right) R_g \Delta_g w + 2\alpha \operatorname{Ric}_{ij}^g \nabla_g^i \nabla_g^j w + \frac{1}{3}(1+\alpha)(\nabla_g R_g, \nabla_g w) - \frac{2U}{3\gamma_3} w \\ &= (1+\alpha)\Delta_g^2 w + \left(\frac{1}{3} - \frac{2}{3}\alpha\right)\big(-12 + O(x)\big)\Delta_g w + 2\alpha\big(-3\Delta_g w + O(x)p(x, y, x\partial_x, x\partial_y)w\big) \\ &\quad + \frac{1}{3}(1+\alpha)\big(-(2\times 4 - 2)x^2 H(h|_{S_x})\partial_x w + O\big(x^3\big)|\nabla_y w|\big) - \big(8\times 3\alpha + O(x)\big)w \\ &= (1+\alpha)\Delta_g^2 w - 12\left(\frac{1}{3} - \frac{2}{3}\alpha\right)\Delta_g w - 6\alpha\Delta_g w - 24\alpha w + O(x)p(x, y, x\partial_x, x\partial_y)w \\ &= (1+\alpha)\Delta_g^2 w - (4 - 2\alpha)\Delta_g w - 24\alpha w + O(x)p(x, y, x\partial_x, x\partial_y)w \\ &= \big((1+\alpha)\Delta_g + 6\alpha\big)(\Delta_g - 4)w + O(x)p(x, y, x\partial_x, x\partial_y)w.\end{aligned}$$

Correspondingly,

$$\begin{aligned}
N(L)w &= (1+\alpha)\big((s\partial_s)^2+s^2\Delta_v-3s\partial_s\big)^2 w+(2\alpha-4)\big((s\partial_s)^2+s^2\Delta_v-3s\partial_s\big)w-24\alpha w,\\
L_0(t,\hat\eta)w &= (1+\alpha)\big((t\partial_t)^2+t^2-3t\partial_t\big)^2 w+(2\alpha-4)\big((t\partial_t)^2+t^2-3t\partial_t\big)w-24\alpha w\\
&= \big((1+\alpha)\big((t\partial_t)^2+t^2-3t\partial_t\big)+6\alpha\big)\big(\big((t\partial_t)^2+t^2-3t\partial_t\big)-4\big)w = L_3\circ L_1 w,\\
I(L)w &= (1+\alpha)\big((s\partial_s)^2-3s\partial_s\big)^2 w+(2\alpha-4)\big((s\partial_s)^2-3s\partial_s\big)w-24\alpha w\\
&= (1+\alpha)\big((s\partial_s)^2-3s\partial_s+6\alpha\big)\big((s\partial_s)^2-3s\partial_s-4\big)w.
\end{aligned}$$

Therefore, the indicial roots of L are as follows,

(i) For $\alpha=\frac{1}{2}$, $spec_b(L)=\{4,-1,1,2\}$.
(ii) For $\alpha=\frac{11}{7}$, $spec_b(L)=\{4,-1,\frac{3}{2}+i\frac{\sqrt{51}}{6},\frac{3}{2}-i\frac{\sqrt{51}}{6}\}$.
(iii) For $\alpha=\frac{-7}{16}$, $spec_b(L)=\{4,-1,\frac{3}{2}+\frac{\sqrt{249}}{6},\frac{3}{2}-\frac{\sqrt{249}}{6}\}$.

The solution of $L_1 w=0$ is exactly the same as discussed in Section 2. We solve $L_3 w=0$ by transferring it into the Bessel type equations discussed as above. Let $u(t)=t^\beta\tilde w(t)$, then

$$0=t^\beta\left((t\partial_t)^2\tilde w+(2\beta-3)t\partial_t\tilde w+\left(\beta^2-3\beta+\frac{6\alpha}{1+\alpha}-t^2\right)\tilde w\right).$$

Let $2\beta-3=0$, and then $\beta=\frac{3}{2}$. Consequently,

$$\left[(t\partial_t)^2-\left(t^2+\frac{9}{4}-\frac{6\alpha}{1+\alpha}\right)\right]\tilde w=0.$$

Let $\tilde\alpha^2=\frac{9}{4}-\frac{6\alpha}{1+\alpha}$, then the solution is

$$w=t^{\frac{3}{2}}\big(C_1 I_{\tilde\alpha}(t)+C_2 K_{\tilde\alpha}(t)\big). \tag{5.9}$$

Here $\tilde\alpha^2$ is $\frac{1}{4}$, $\frac{-17}{12}$, $\frac{83}{12}$, corresponding to the above three cases, with $\mathrm{Re}(\tilde\alpha)\geqslant 0$. For the case $\tilde\alpha^2=-\frac{17}{12}$, since $\tilde\alpha^2$ is negative, L_3 behaves the same as L_2 in Section 2, and it follows that Theorem 1.4 and Theorem 1.5 with $n=4$ hold for the linear operator L, using the same argument as in Section 2.

By the expansion of the series form of the Bessel functions, as in [15, p. 108], we have

$$t^{\frac{3}{2}}I_{\tilde\alpha}\big(t|\eta|\big)\sim t^{\frac{3}{2}+\tilde\alpha}|\eta|^{\tilde\alpha}/\big(2^{\tilde\alpha}\Gamma(1+\tilde\alpha)\big),$$

and

$$t^{\frac{3}{2}}I_{-\tilde\alpha}\big(t|\eta|\big)\sim t^{\frac{3}{2}-\tilde\alpha}|\eta|^{-\tilde\alpha}/\big(2^{-\tilde\alpha}\Gamma(1-\tilde\alpha)\big),$$

near $t=0$. Here we should note that the series expansion applies for all $\tilde\alpha\in\mathbb{C}$. Now it is easy to see that the linear combination

$$x^{\frac{3}{2}}\big(C_1x^{\tilde\alpha}+C_2x^{-\tilde\alpha}\big)$$

can never vanish to infinite order at $t=0$ if either $C_1\neq 0$ or $C_2\neq 0$. Also,

$$t^{\frac{3}{2}}K_{\tilde\alpha}\big(t|\eta|\big)\sim t^{\frac{3}{2}}\frac{\pi}{2}\frac{I_{\tilde\alpha}(t|\eta|)-I_{-\tilde\alpha}(t|\eta|)}{\sin(\tilde\alpha\pi)}\sim O\big(\big(t|\eta|\big)^{\frac{3}{2}-\tilde\alpha}\big),$$

near $t=0$, with $\tilde\alpha>0$ and $\tilde\alpha\neq 1,2,3,\ldots$.

Using the integral form as in Section 2, we have

$$t^{\frac{3}{2}}I_{\tilde\alpha}\big(t|\eta|\big)\quad\text{grows exponentially,}\qquad t^{\frac{3}{2}}K_{\tilde\alpha}\big(t|\eta|\big)\quad\text{decays exponentially}$$

near $t=+\infty$. Therefore, $t^{\frac{3}{2}}I_{\tilde\alpha}(t|\eta|)$ does not belong to $t^\delta L^2(\mathbb{R}^+)$ for any $\delta>0$, while

$$t^{\frac{3}{2}} K_{\tilde{\alpha}}\big(t|\eta|\big) \in t^{\delta} L^2(\mathbb{R}_+),$$

only for $\delta < \frac{3}{2} + \frac{1}{2} - \tilde{\alpha} = 2 - \tilde{\alpha}$. That is, L_3 is injective in $x^{\delta} L^2$ for $\delta > 2 - \tilde{\alpha}$.

Summarizing the above discussion, let us compute $\bar{\delta}$ and $\underline{\delta}$ for the linearized operator L

$$\bar{\delta} = \inf\{\delta\colon\ L_1 \text{ and } L_3 \text{ are injective in } t^{\delta} L^2\} = \sup\left\{-1 + \frac{1}{2}, 2 - \tilde{\alpha}\right\}, \quad \text{and dually,}$$

$$\underline{\delta} = \inf\left\{\left(\frac{3}{2} + \frac{1}{2}\right) \times 2 - \left(-1 + \frac{1}{2}\right), \left(\frac{3}{2} + \frac{1}{2}\right) \times 2 - (2 - \tilde{\alpha})\right\} = \inf\left\{\frac{9}{2}, 2 + \tilde{\alpha}\right\}.$$

For the case $\alpha = \frac{1}{2}$, $\bar{\delta} = \frac{3}{2}$, and $\underline{\delta} = \frac{5}{2}$ (surjectivity). For the case $\alpha = -\frac{7}{16}$, $\bar{\delta} = -1 + \frac{1}{2} = -\frac{1}{2}$, and $\underline{\delta} = \frac{9}{2}$. Then we can use Theorem 1.4 and Theorem 1.5, to obtain the semi-Fredholm property for these linear operators.

For the Poincaré–Einstein manifold (M, g), we have that the U-curvatures defined above are all constant on M. We want to consider the corresponding problem of finding constant U-curvature metrics in the same conformal class. Now $Lw = ((1+\alpha)\Delta_g + 6\alpha)(\Delta_g - 4)w$. Define the operator $\mathcal{T} : x^{\nu} \Lambda^{4,\alpha}(M) \to x^{\nu} \Lambda^{0,\alpha}(M)$ as follows,

$$\begin{aligned}\mathcal{T}(w) = {}& \left(\frac{\tilde{U}}{6\gamma_3} e^{4w} - \frac{U}{6\gamma_3} - \frac{2}{3\gamma_3} U w\right) - 2\,\mathrm{Ric}(\nabla w, \nabla w) \\ & - 2\big(\big|\nabla^2 w\big|_g^2 - (\Delta w)^2\big) + 4\nabla_j \nabla_i w \nabla^j w \nabla^i w + 2|\nabla w|^2 \Delta w.\end{aligned}$$

We rewrite it in the form

$$\begin{aligned}\mathcal{T}(w) = {}& \frac{\tilde{U}}{6\gamma_3}\big(e^{4w} - 1 - 4w\big) + (\tilde{U} - U)\left(\frac{1}{6\gamma_3} + \frac{2}{3\gamma_3} w\right) - 2\,\mathrm{Ric}(\nabla w, \nabla w) \\ & - 2\big(\big|\nabla^2 w\big|_g^2 - (\Delta w)^2\big) + 4\nabla_j \nabla_i w \nabla^j w \nabla^i w + 2|\nabla w|^2 \Delta w.\end{aligned}$$

In this formula, comparing with the nonlinear term defined for Q-curvature equation, a few square terms of w and its derivatives of order up to 2 are involved, which are small terms in the argument of the perturbation problem. Now, the nonlinear equation becomes

$$L_g w = \mathcal{T}(w).$$

To solve this, we argue as we did in Section 3 and Section 4. The only difference is the choice of weighted Hölder spaces. Note that the index of the weight for the Hölder space is $\frac{1}{2}$ less than the index of the weight of the corresponding Sobolev spaces.

5.1. Summary

Perturbation results for the curvatures defined in (5.1) can be proved along the same lines as the Q-curvature. For instance, assume (M, g) is a Poincaré–Einstein manifold. For the case $\alpha = -\frac{7}{16}$, by maximum principle, $((1+\alpha)\Delta_g + 6\alpha)$ and $(\Delta_g - 4)$ are both injective on $L^2(M, g)$. Then similar to the discussion for the Q-curvature equation, there are infinitely many solutions $u \in x^{\nu} \Lambda^{4,\beta}(M, \sqrt{dx\,dy})$ for $0 < \beta < 1$ to this equation parametrized by the projection $\Pi_1 u$ to the kernel of the linearized operator L, for $\nu \in (0, \frac{3}{2})$. Moreover, if $\tilde{U} = U$, then w has the weak expansion $w(x, y) \sim w_{00}(y) x^4 + o(x^4)$, and also w has a smooth expansion if $1 \leqslant \nu < \frac{3}{2}$ and $\Pi_1 w$ has a smooth expansion. For the case $\alpha = \frac{11}{7}$, it is the same as the Q-curvature problem, and the only difference is that here we use $i\sqrt{51}$ in the indicial roots and in the formula of expansion to replace $i\sqrt{15}$. For the case $\alpha = \frac{1}{2}$, $((1+\alpha)\Delta_g + 6\alpha)$ is essentially injective on $x^{\nu} \Lambda^{4,\beta}(M, \sqrt{dx\,dy})$ for $\nu > 1$ and $\nu \neq 2$, while it is essentially surjective on $x^{\nu} \Lambda^{4,\beta}(M, \sqrt{dx\,dy})$ for $\nu < 2$, also $\nu \neq 1$ and $0 < \beta < 1$. Since $(\frac{3}{2}\Delta_g + 3)$ may have finite dimensional kernel, we do not have perturbation result for ν in this interval. But note that, using the same argument as in Lemma 2.6 in weighted Hölder spaces, for $\nu > 2$, the operator

$$\left(\frac{3}{2}\Delta + 3\right) : x^{\nu} \Lambda^{2+m,\beta} \to x^{\nu} \Lambda^{m,\beta},$$

is injective, for $0 < \beta < 1$ and $m \geqslant 0$. Then dually the operator $(\frac{3}{2}\Delta + 3)$ is surjective for $\nu \in (0, 1)$. Also we know that the operator $(\Delta_g - 4)$ is surjective in the weighted Hölder space with $0 < \nu < \frac{3}{2}$, then the linearized operator

$$L : x^\nu \Lambda^{4+m,\beta} \to x^\nu \Lambda^{m,\beta},$$

with $m \geqslant 0$ is surjective for $0 < \nu < 1$ and $0 < \beta < 1$. Therefore, for the case $\alpha = \frac{1}{2}$, the existence result as in (i) in Theorem 1.3 holds for $0 < \nu < 1$. For the boundary expansion when $\tilde{U} = U$, since all the indicial roots are integers in this case, there may be $\log(x)$ terms in the expansion. Also, since $\nu < 1$, the smooth expansion result does not hold.

Acknowledgements

It is my pleasure to express my sincere gratitude to my advisor Professor Matthew Gursky for introducing this question, many helpful discussions and great patience in the course of this work. I am also indebted to Professor Rafe Mazzeo for inviting me to Stanford University to explain his theory of edge operators, his many comments on preliminary versions of the manuscript and his continued interest in this work. Thanks are also due to Professor C. Robin Graham for useful discussions and introducing me his paper [10]. I would also like to thank Professor Huicheng Yin in Nanjing University, who introduced the PDE area to me and taught me the theory of elliptic operators, for his continuous interest in this problem and constant support. I would like to thank the support of CSC program of Chinese government in my study. At last, I would like to thank my friends Ye Li and Yueh-Ju Lin for their encouragement and constant support.

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