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Ann. I. H. Poincaré – AN 31 (2014) 639–654

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

On linear instability of solitary waves for the nonlinear Dirac equation

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Received 5 September 2012; received in revised form 29 April 2013; accepted 3 June 2013

Available online 11 June 2013

Abstract

We consider the nonlinear Dirac equation, also known as the Soler model:

$$i\partial_t\psi = -i\boldsymbol{\alpha}\cdot\nabla\psi + m\beta\psi - \left(\psi^*\beta\psi\right)^k\beta\psi, \quad m>0, \qquad \psi(x,t)\in\mathbb{C}^N, \quad x\in\mathbb{R}^n,\ k\in\mathbb{N}.$$

We study the point spectrum of linearizations at small amplitude solitary waves in the limit $\omega\to m$, proving that if $k>2/n$, then one positive and one negative eigenvalue are present in the spectrum of the linearizations at these solitary waves with ω sufficiently close to m, so that these solitary waves are linearly unstable. The approach is based on applying the Rayleigh–Schrödinger perturbation theory to the nonrelativistic limit of the equation. The results are in formal agreement with the Vakhitov–Kolokolov stability criterion.

Résumé

Nous considérons l'équation de Dirac non linéaire, aussi connue comme modèle de Soler. Nous étudions le spectre ponctuel des linéarisations autour d'ondes solitaires de petite amplitude dans la limite $\omega\to m$, et montrons que si $k>2/n$ une valeur propre positive et une négative sont présentes dans le spectre des linéarisations autour de ces ondes solitaires lorsque ω est suffisamment proche de m, ce qui entraîne que ces ondes solitaires sont linéairement instables. L'approche est basée sur l'application de la théorie des perturbations de Rayleigh–Schrödinger à la limite non relativiste de l'équation. Les résultats sont en accord formel avec le critère de stabilité de Vakhitov–Kolokolov.
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1. Introduction

A natural simplification of the Dirac–Maxwell system [23] is the nonlinear Dirac equation, such as the massive Thirring model [41] with vector–vector self-interaction and the Soler model [36] with scalar–scalar self-interaction

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1 Research supported by an NSERC Discovery Grant.

0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2013.06.001

(known in dimension $n = 1$ as the massive Gross–Neveu model [21,27]). These models with self-interaction of local type have been receiving a lot of attention in particle physics (see e.g. [34]), as well as in the theory of Bose–Einstein condensates [26,29].

There is an enormous body of research devoted to the nonlinear Dirac equation, which we cannot cover comprehensively here. The existence of standing waves in the nonlinear Dirac equation was studied in [36,17,28,20]. The question of stability of solitary waves is of utmost importance: perturbations ensure that we only ever encounter stable configurations. Recent attempts at asymptotic stability of solitary waves in the nonlinear Dirac equation [8,9,32,5,16] rely on the fundamental question of *spectral stability*:

Consider the Ansatz $\psi(x,t) = (\phi_\omega(x) + \rho(x,t))e^{-i\omega t}$, with $\phi_\omega(x)e^{-i\omega t}$ a solitary wave solution. Let $\partial_t\rho = A_\omega\rho$ be the linearized equation for ρ. Does A_ω have eigenvalues in the right half-plane?

Definition 1.1. If $\sigma(A_\omega) \subset i\mathbb{R}$, we say that the solitary wave $\phi_\omega e^{-i\omega t}$ is spectrally stable. Otherwise, we say that the solitary wave is linearly unstable.

Remark 1.2. Let us note that the "spectral stability" $\sigma(A_\omega) \subset i\mathbb{R}$ does not guarantee stability. One of the possibilities which may still lead to instability is the presence of eigenvalues of higher algebraic multiplicity. This occurs e.g. at $\lambda = 0$ in the case $dQ(\omega)/d\omega = 0$ (where $Q(\omega)$ is the charge of ϕ_ω – see below) at a particular value of ω, leading to instability of the corresponding solitary wave; see [13]. In [15], to reflect this situation in the context of the nonlinear Schrödinger equation, linear instability is defined as either the presence of eigenvalues with positive real part or the presence of particular eigenvalues with higher algebraic multiplicity.

There is a very clear picture of the spectral stability for nonlinear Schrödinger and Klein–Gordon equations [43,35,44,37] and general results for abstract Hamiltonian systems with $\mathbf{U}(1)$ symmetry [24], which stimulated many attempts at spectral stability in the nonlinear Dirac context. We mention the numerical simulations [1] and the analysis of the energy minimization under charge-preserving dilations and similar transformations [7,2,39,11]. In spite of this, the question of spectral stability of solitary waves of nonlinear Dirac equation is still completely open. Numerical results [3] show that in the 1D Soler model (cubic nonlinearity) all solitary waves are spectrally stable. We also mention the related numerical results in [14,10].

According to [43], if $\phi_\omega(x)e^{-i\omega t}$ is a family of solitary wave solutions to the nonlinear Schrödinger equation

$$i\dot{u} = -\frac{1}{2m}\Delta u - f\left(|u|^2\right)u, \quad u(x,t) \in \mathbb{C},\ x \in \mathbb{R}^n,\ n \geqslant 1,$$

where f is smooth and real-valued, and if ϕ_ω have no nodes (such solitary waves are called *ground states*), then the linearization at the solitary wave corresponding to a particular value of ω has a positive eigenvalue if and only if at this value of ω one has $dQ(\omega)/d\omega > 0$, where $Q(\omega) = \|\phi_\omega\|^2_{L^2}$ is the charge (or *mass*) of the solitary wave. The opposite condition,

$$\frac{d}{d\omega}Q(\omega) < 0, \tag{1.1}$$

is called the Vakhitov–Kolokolov stability criterion; it ensures the absence of eigenvalues with positive real part. In the case of the nonlinear Dirac equation, the condition (1.1) gives a less definite answer about the spectral stability. All we know is that at the value of ω where $\partial_\omega Q(\omega)$ vanishes, with $Q(\omega)$ being the charge of the solitary wave $\phi_\omega e^{-i\omega t}$, two eigenvalues of the linearized equation collide at $\lambda = 0$, but we do not know where these eigenvalues are located when $\partial_\omega Q(\omega) \neq 0$ (see e.g. [12]).

Yet, it is natural to expect that the condition (1.1) remains meaningful in the nonrelativistic limit, as it was suggested in [11]. While it is a common practice to obtain solitary wave solutions for relativistic equations as bifurcations from the solitary waves to the equation corresponding to the nonrelativistic limit (see e.g. the review [19]), we show that the spectrum of the linearization at a solitary wave could also be learned from the nonrelativistic limit. More precisely, we develop the idea that the family of real eigenvalues of the linearization at a solitary wave of the nonlinear Dirac equation bifurcating from $\lambda = 0$ is a deformed family of eigenvalues of the linearization of the corresponding nonlinear Schrödinger equation. As a result, we prove that if the Vakhitov–Kolokolov stability criterion [43] guarantees linear instability for the NLS, then the same conclusion also holds for solitary waves with $\omega \lesssim m$ in the nonlinear Dirac equation. Let us mention that our results only apply to the solitary wave solutions which we obtain from the solitary waves of the nonlinear Schrödinger equation in the nonrelativistic limit of the nonlinear Dirac.

The model and the main results are described in Section 2. The necessary constructions in the context of the nonlinear Schrödinger equation are presented in Section 3. The existence and asymptotics of solitary waves of the nonlinear Dirac equation is covered in Section 4. The main result (Theorem 2.3) follows from Lemma 4.2 (existence of solitary wave solutions and their asymptotics) and Proposition 5.2 (presence of a positive eigenvalue in the spectrum of the linearized operator), which we prove using the Rayleigh–Schrödinger perturbation theory.

2. Main result

We consider the nonlinear Dirac equation

$$i\partial_t\psi = -i\boldsymbol{\alpha}\cdot\nabla\psi + m\beta\psi - f\big(\psi^*\beta\psi\big)\beta\psi, \qquad \psi(x,t)\in\mathbb{C}^N, \quad x\in\mathbb{R}^n, \tag{2.1}$$

with $m>0$ and ψ^* being the Hermitian conjugate of ψ. We assume that the nonlinearity $f(s)$ is smooth and real-valued, and that

$$f(0)=0. \tag{2.2}$$

Above, $\boldsymbol{\alpha}\cdot\nabla = \sum_{j=1}^n \alpha_j \frac{\partial}{\partial x_j}$, and the Hermitian matrices α_j and β are chosen so that

$$(-i\boldsymbol{\alpha}\cdot\nabla+\beta m)^2 = \big(-\Delta+m^2\big)I_N,$$

where I_N is the $N\times N$ unit matrix. That is, α_j and β are to satisfy

$$\alpha_j\alpha_k+\alpha_k\alpha_j = 2\delta_{jk}I_N, \qquad \beta^2 = I_N; \qquad \alpha_j\beta+\beta\alpha_j = 0. \tag{2.3}$$

The generalized massive Gross–Neveu model (the scalar–scalar case with $k>0$ in the terminology of [11]) corresponds to the nonlinearity $f(s)=|s|^k$.

According to the Dirac–Pauli theorem (cf. [18,42,30] and [40, Lemma 2.25]), the particular choice of the matrices α_j and β does not matter:

Lemma 2.1 *(Dirac–Pauli theorem). Let $n\in\mathbb{N}$. For any sets of Dirac matrices α_j, β and $\tilde\alpha_j$, $\tilde\beta$ of the same dimension N, with $1\leqslant j\leqslant n$, there is a unitary matrix S such that*

$$\tilde\alpha_j = S^{-1}\alpha_j S, \quad 1\leqslant j\leqslant n, \qquad \tilde\beta = S^{-1}\beta S \tag{2.4}$$

if n is odd, and such that

$$\tilde\alpha_j = \sigma S^{-1}\tilde\alpha_j S, \quad 1\leqslant j\leqslant n, \qquad \tilde\beta = \sigma S^{-1}\beta S, \qquad \sigma=\pm 1, \tag{2.5}$$

if n is even.

For more details, see [4].

Lemma 2.1 allows one, by a simple change of variable, to transform the nonlinear Dirac equation, changing the set of Dirac matrices. Thus, when studying the spectral stability, we can choose the Dirac matrices at our convenience. We use the standard Pauli matrices,

$$\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{2.6}$$

to make the following choice:

$$n=1: \quad N=2, \qquad \alpha = -\sigma_2; \tag{2.7}$$

$$n=2: \quad N=2, \qquad \alpha_j = \sigma_j, \quad 1\leqslant j\leqslant 2; \tag{2.8}$$

$$n=3: \quad N=4, \qquad \alpha_j = \begin{pmatrix}0&\sigma_j\\ \sigma_j&0\end{pmatrix}, \quad 1\leqslant j\leqslant 3; \tag{2.9}$$

and in all these cases we choose

$$\beta = \begin{pmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{pmatrix}. \tag{2.10}$$

Remark 2.2. If n is even, Lemma 2.1 may only allow us to transform Eq. (2.1) with a particular set of the Dirac matrices to the set of the Dirac matrices as in (2.7), (2.8), (2.9), and (2.10), but with the opposite signs (this corresponds to $\sigma = -1$ in (2.5)). In this case, the signs of α_j are flipped by taking the spatial reflections, while β being opposite to (2.10) corresponds to considering the nonrelativistic limit $\omega \to -m$, with appropriate changes to Theorem 2.3.

For a large class of nonlinearities $f(s)$, there are solitary wave solutions of the form

$$\psi(x,t) = \phi_\omega(x)e^{-i\omega t}, \quad \phi_\omega \in H^1(\mathbb{R}, \mathbb{C}^N), \qquad |\omega| < m. \tag{2.11}$$

In dimension $n = 1$, one can take

$$\phi_\omega(x) = \begin{bmatrix} v(x,\omega) \\ u(x,\omega) \end{bmatrix}, \tag{2.12}$$

with $v(x,\omega)$ positive and even and $u(x,\omega)$ real-valued and odd; under these conditions, the solitary wave $\phi_\omega(x)$ is unique (see Section 4.1 for details).

For $n = 2$ and $n = 3$, respectively,

$$\phi_\omega(x) = \begin{bmatrix} v(r,\omega) \\ ie^{i\phi}u(r,\omega) \end{bmatrix}, \qquad \phi_\omega(x) = \begin{bmatrix} v(r,\omega)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iu(r,\omega)\begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} \end{bmatrix}, \tag{2.13}$$

where $v(r,\omega)$ and $u(r,\omega)$ are real-valued, radially symmetric functions; (r,ϕ) are standard polar coordinates in $\mathbb{R}^2$, and (r,θ,ϕ) are standard spherical coordinates in $\mathbb{R}^3$. The existence of solitary waves of this form is proved, for example, in [17]. The particular solutions we consider here, however, are those constructed for ω close to m, as outlined in Section 4.2, from the nonrelativistic limit $\omega \to m$.

Due to the $\mathbf{U}(1)$-invariance, for solutions to (2.1) the value of the charge functional

$$Q(\psi) = \int_{\mathbb{R}^n} \left|\psi(x,t)\right|^2 dx$$

is formally conserved. For brevity, we also denote by $Q(\omega)$ the charge of the solitary wave $\phi_\omega(x)e^{-i\omega t}$:

$$Q(\omega) = \int_{\mathbb{R}^n} \left|\phi_\omega(x)\right|^2 dx. \tag{2.14}$$

We are interested in the spectrum of linearization of the nonlinear Dirac equation (2.1) at a solitary wave solution (2.11).

Theorem 2.3. *Let $n \leqslant 3$. Assume that $f(s) = s^k$, where $k \in \mathbb{N}$ satisfies $k > 2/n$ (and $k < 2$ for $n = 3$). Then there is $\omega_1 < m$ such that the solitary wave solutions $\phi_\omega e^{-i\omega t}$ to (2.1) described above, are linearly unstable for $\omega \in (\omega_1, m)$. More precisely, let A_ω be the linearization of the nonlinear Dirac equation at a solitary wave $\phi_\omega(x)e^{-i\omega t}$. Then for $\omega \in (\omega_1, m)$ there are eigenvalues*

$$\pm\lambda_\omega \in \sigma_p(A_\omega), \qquad \lambda_\omega > 0, \qquad \lambda_\omega = O(m-\omega).$$

See Fig. 1.

Remark 2.4. The existence of an eigenvalue with positive real part in the linearization at a particular solitary wave generally implies the dynamic, or *nonlinear*, instability of this wave. We expect that this could be proved following the argument of [22] given in the context of the nonlinear Schrödinger equation.

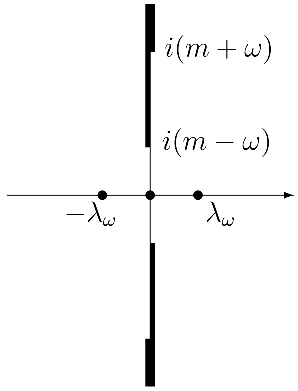

Fig. 1. Main result: The point spectrum of the linearization of the nonlinear Dirac equation in $\mathbb{R}^n$, $n \leqslant 3$, with $f(s) = s^k$, $k > \frac{2}{n}$, at a solitary wave with $\omega \lesssim m$ contains two nonzero real eigenvalues, $\pm\lambda_\omega$, with $\lambda_\omega = O(m-\omega)$. See Theorem 2.3. Also plotted on this picture is the essential spectrum, with the edges at $\lambda = \pm i(m-\omega)$ and with the embedded threshold points (branch points of the dispersion relation) at $\lambda = \pm i(m+\omega)$.

Remark 2.5. Theorem 2.3 extends easily to nonlinearities $f \in C^2(\mathbb{R})$ of the form

$$f(s) = as^k + O\left(s^{k+1}\right), \quad a > 0.$$

Remark 2.6. In Theorem 2.3, the value $\omega_1 < m$ could be taken to be the smallest point such that there is a C^1 family of solitary waves $\omega \mapsto \phi_\omega$ for $\omega \in (\omega_1, m)$ and moreover $\partial_\omega Q(\omega)$ does not vanish on (ω_1, m). Indeed, by [12], the positive and negative eigenvalues remain trapped on the real axis, not being able to collide at $\lambda = 0$ for $\omega \in (\omega_1, m)$ as long as $\partial_\omega Q(\omega)$ does not vanish on this interval. These eigenvalues cannot leave into the complex plane, either, since they are simple, while the spectrum of the operator is symmetric with respect to the real and imaginary axes.

Remark 2.7. If the family of solitary waves $\omega \mapsto \phi_\omega$ is defined for $\omega \in (\omega_0, m)$ with $\omega_0 < \omega_1$ and $\partial_\omega Q(\omega)$ vanishes at ω_1, we do not know what happens for $\omega \lesssim \omega_1$. If $\partial_\omega Q(\omega)$ changes the sign at ω_1, then, generically, either the pair of real eigenvalues, having collided at $\lambda = 0$ when $\omega = \omega_1$, turn into a pair of purely imaginary eigenvalues (linear instability disappears), or instead two purely imaginary eigenvalues, having met at $\lambda = 0$, turn into the second pair of real eigenvalues (linear instability persists). If $\partial_\omega Q(\omega)$ vanishes at ω_1 but does not change the sign, then generically the eigenvalues touch and separate again, remaining on $\mathbb{R}_+$ and $\mathbb{R}_-$. More details are in [12].

Remark 2.8. Theorem 2.3 is in formal agreement with the Vakhitov–Kolokolov stability criterion [43], since for $\omega \lesssim m$ one has $Q'(\omega) > 0$ for $k > \frac{2}{n}$. Let us mention that the sign of the stability criterion, $Q'(\omega) < 0$, differs from [43] because of their writing the solitary waves in the form $\varphi(x)e^{+i\omega t}$.

Remark 2.9. It has been shown that in the 1D case with $k = 1$, the small amplitude solitary waves are spectrally stable [4].

Remark 2.10. We expect that in the 1D case with $k = 2$ ("quintic nonlinearity") the small solitary wave solutions of the nonlinear Dirac equation in 1D are spectrally stable. For the corresponding nonlinear Schrödinger equation (quintic nonlinearity in 1D), the charge is constant, thus the zero eigenvalue of a linearized operator is always of higher algebraic multiplicity. For the Dirac equation, this degeneracy is "resolved": using the expression for the charge $Q(\omega)$ from [11, Section 2A], one can see that the charge is now a decaying function for $\omega \lesssim m$ (with nonzero limit as $\omega \to m$), suggesting that there are two purely imaginary eigenvalues $\pm\lambda_\omega$ in the spectrum of A_ω, with $\lambda_\omega = o(m-\omega)$, but no eigenvalues with nonzero real part.

Remark 2.11. Let us notice that in the 3D case for the cubic nonlinearity $f(s) = s$ (this is the original Soler model from [36]), based on the numerical evidence from [36,1], the charge $Q(\omega)$ has a local minimum at $\omega_1 \approx 0.936m$, suggesting that the solitary waves with $\omega_1 < \omega < 1$ are linearly unstable, but then at $\omega = \omega_1$ the real eigenvalues collide at $\lambda = 0$, and there are no nonzero real eigenvalues in the spectrum for $\omega \lesssim \omega_1$. Incidentally, this agrees with the "dilation-stability" results of [39] (one studies whether the energy is minimized or not under the charge-preserving dilation transformations).

Remark 2.12. We cannot rule out the possibility that the eigenvalues with nonzero real part could bifurcate directly from the imaginary axis into the complex plane. Such a mechanism is absent for the nonlinear Schrödinger equation linearized at a ground state, for which the point eigenvalues always remain on the real or imaginary axes. At present, though, we do not have examples of such bifurcations in the context of nonlinear Dirac equation.

Remark 2.13. For $n = 3$, we only consider the case $k = 1$, and we do not consider dimensions $n > 3$. This is because of the fact that the equation $-\Delta u + u = |u|^{2k} u$ in $\mathbb{R}^n$ has nontrivial solutions in $H^1(\mathbb{R}^n)$ if and only if $0 < k < 2/(n-2)$, as follows from the virial identities; see [31] and [6, Example 1]. This is why our method does not allow us to construct solitary wave solutions to the nonlinear Dirac equation (2.1) in $\mathbb{R}^n$, $n \geqslant 3$, with $k \geqslant 2/(n-2)$.

Remark 2.14. We consider here only integer powers k (and only dimensions $n = 1$, 2, and 3), being physically the most important cases. Mathematically, this merely avoids some minor technical complications associated with a non-smooth nonlinearity, and the instability argument can be extended to handle the corresponding equation with $f(s) = |s|^k$, in any dimension $n \geqslant 1$, under the condition $k > 2/n$. (The restriction $k < 2/(n-2)$ is needed so that there are nontrivial solitary waves for the NLS; see the previous remark.)

3. Nonlinear Schrödinger and its solitary waves

We are going to use the fact that the nonrelativistic limit of the nonlinear Dirac equation yields the nonlinear Schrödinger equation,

$$i\partial_t \psi = -\frac{1}{2m}\Delta\psi - |\psi|^{2k}\psi, \qquad \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}^n,\ k > 0,\ n \in \mathbb{N}. \tag{3.1}$$

3.1. Solitary waves

The properties of solitary wave solutions

$$\psi(x,t) = \phi_\omega(x) e^{-i\omega t}, \quad \phi_\omega \in H^1(\mathbb{R}^n),$$

with the amplitude $\phi_\omega(x)$ satisfying the stationary equation

$$-\frac{1}{2m}\Delta\phi(x) - |\phi|^{2k}\phi = \omega\phi, \quad x \in \mathbb{R}^n, \tag{3.2}$$

are well-known [38,6]. For any $k > 0$ when $n \leqslant 2$ and for $0 < k < 2/(n-2)$ when $n \geqslant 3$, for each $\omega \in (-\infty, 0)$, there is a unique positive, radially symmetric solution

$$\phi_\omega(x) = \phi_\omega(|x|) > 0,$$

which decays exponentially. This family of solitary waves, known as the *ground states*, is generated by rescaling a single amplitude function:

$$\phi_\omega(x) = |\omega|^{\frac{1}{2k}} F\left(\sqrt{2m|\omega|}x\right), \quad \omega < 0, \tag{3.3}$$

where $F(x) = F(|x|) > 0$ solves

$$-\Delta F - F^{2k+1} = -F, \quad x \in \mathbb{R}^n. \tag{3.4}$$

In one space dimension ($n = 1$), for $k > 0$, $F(x)$ is given by the explicit formula

$$F(x) = \left(\frac{k+1}{\cosh^2 kx}\right)^{\frac{1}{2k}}.$$

3.2. Linearization at a solitary wave

To derive the linearization of the nonlinear Schrödinger equation (3.1) at a solitary wave $\psi(x,t)=\phi_\omega(x)e^{-i\omega t}$, we use the Ansatz

$$\psi(x,t)=\big(\phi_\omega(x)+\rho(x,t)\big)e^{-i\omega t},\quad \rho(x,t)\in\mathbb{C},\ x\in\mathbb{R}^n,$$

and arrive at the linearized equation

$$\partial_t\boldsymbol{\rho}=\mathbf{j}\mathbf{l}(\omega)\boldsymbol{\rho},\qquad \boldsymbol{\rho}(x,t)=\begin{bmatrix}\operatorname{Re}\rho(x,t)\\ \operatorname{Im}\rho(x,t)\end{bmatrix}, \tag{3.5}$$

where

$$\mathbf{j}=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\qquad \mathbf{l}=\begin{bmatrix}\mathrm{l}_-&0\\0&\mathrm{l}_+\end{bmatrix}, \tag{3.6}$$

with $\mathrm{l}_\pm$ self-adjoint Schrödinger operators

$$\mathrm{l}_-(\omega)=-\frac{1}{2m}\Delta-|\phi_\omega|^{2k}-\omega,\qquad \mathrm{l}_+(\omega)=\mathrm{l}_-(\omega)-2k|\phi_\omega|^{2k}. \tag{3.7}$$

Since the solitary wave amplitudes $\phi_\omega(x)=\phi_\omega(|x|)$ we take here are radially symmetric, we may consider the operators

$$\mathbf{l}_{rad},\ \mathrm{l}_{\pm,rad}:=\mathbf{l},\ \mathrm{l}_\pm \text{ restricted to radially symmetric functions.} \tag{3.8}$$

The linear stability theory of NLS ground states is well understood, and can be summarized, in terms of their charge (2.14), as follows:

Lemma 3.1 *(Vakhitov–Kolokolov stability criterion). (See [43].) For the linearization (3.5) at a ground state solitary wave $\phi_\omega(x)e^{-i\omega t}$, there are real nonzero eigenvalues $\pm\lambda\in\sigma_d(\mathbf{j}\mathbf{l})$, $\lambda>0$, if and only if $\frac{d}{d\omega}Q(\omega)>0$ at this value of ω. If so, then $\pm\lambda\in\sigma_d(\mathbf{j}\mathbf{l}_{rad})$ are simple eigenvalues, and moreover* $\ker\mathrm{l}_{+,rad}=\{0\}$.

Using (3.3), we compute:

$$Q(\omega)=\int_{\mathbb{R}^n}\big|\phi_\omega(x)\big|^2\,dx=|\omega|^{\frac{1}{k}}\int_{\mathbb{R}^n}F^2\big(\sqrt{2m|\omega|}x\big)\,dx=C|\omega|^{\frac{1}{k}-\frac{n}{2}},\quad \omega<0, \tag{3.9}$$

where $C=\int_{\mathbb{R}^n}F^2(\sqrt{2m}y)\,dy>0$. We see from (3.9) that for $\omega<0$ one has $Q'(\omega)<0$ for $k<2/n$, $Q'(\omega)=0$ for $k=2/n$, and $Q'(\omega)>0$ for $k>2/n$. Thus:

Lemma 3.2. *Let $n\in\mathbb{N}$. If $k>n/2$ (and $k<2/(n-2)$ if $n\geqslant 3$) then $\sigma_p(\mathbf{j}\mathbf{l}_{rad})\ni\{\pm\lambda\}$, for some $\lambda>0$, and in particular the NLS ground states are linearly unstable.*

4. Nonlinear Dirac and its solitary waves

Solitary waves are solutions to (2.1) of the form

$$\psi(x,t)=\phi_\omega(x)e^{-i\omega t};\quad \phi_\omega\in H^1\big(\mathbb{R},\mathbb{C}^N\big),\ \omega\in\mathbb{R}$$

and as such the amplitude $\phi_\omega(x)$ must satisfy

$$\omega\phi_\omega=-i\boldsymbol{\alpha}\cdot\nabla\psi+m\beta\psi-f\big(\psi^*\beta\psi\big)\beta\psi,\quad x\in\mathbb{R}^n. \tag{4.1}$$

4.1. Solitary waves in one dimension

We first give a simple demonstration of the existence and uniqueness of solitary waves in one dimension, following the article [3], and allowing for more general nonlinearities $f(s)$.

Lemma 4.1. *Let $f(0)=0$. Denote $g(s)=m-f(s)$, and let $G(s)$ be the antiderivative of $g(s)$ such that $G(0)=0$. Assume that there is $\omega_0<m$ such that for given $\omega\in(\omega_0,m)$ there exists $\Gamma_\omega>0$ such that*

$$\omega\Gamma_\omega=G(\Gamma_\omega),\qquad \omega\ne g(\Gamma_\omega),\quad \text{and}\quad \omega s<G(s)\quad \text{for } s\in(0,\Gamma_\omega). \tag{4.2}$$

Then there is a solitary wave solution $\psi(x,t)=\phi_\omega(x)e^{-i\omega t}$ to (2.1), where

$$\phi_\omega(x)=\begin{bmatrix} v(x,\omega)\\ u(x,\omega)\end{bmatrix}, \tag{4.3}$$

with both v and u real-valued, belonging to $H^1(\mathbb{R})$ as functions of x, v being even and u odd.

More precisely, for $x\in\mathbb{R}$ and $\omega\in(\omega_0,m)$, let us define $\mathscr{X}(x,\omega)$ and $\mathscr{Y}(x,\omega)$ by

$$\mathscr{X}=v^2-u^2,\qquad \mathscr{Y}=vu. \tag{4.4}$$

Then $\mathscr{X}(x,\omega)$ is the unique positive symmetric solution to

$$\partial_x^2\mathscr{X}=-\partial_{\mathscr{X}}\big(-2G(\mathscr{X})^2+2\omega^2\mathscr{X}^2\big),\qquad \lim_{x\to\pm\infty}\mathscr{X}(x,\omega)=0, \tag{4.5}$$

and $\mathscr{Y}(x,\omega)=-\frac{1}{4\omega}\partial_x\mathscr{X}(x,\omega)$. This solution satisfies $\mathscr{X}(0,\omega)=\Gamma_\omega$.

Proof. Substituting $\phi_\omega(x)e^{-i\omega t}$, with ϕ_ω from (4.3), into (4.1), we obtain:

$$\begin{cases}\omega v=\partial_x u+g\big(|v|^2-|u|^2\big)v,\\ \omega u=-\partial_x v-g\big(|v|^2-|u|^2\big)u.\end{cases} \tag{4.6}$$

Since we assume that both v and u are real-valued, we may rewrite (4.6) as the following Hamiltonian system:

$$\begin{cases}\partial_x u=\omega v-g\big(v^2-u^2\big)v=\partial_v h(v,u),\\ -\partial_x v=\omega u+g\big(v^2-u^2\big)u=\partial_u h(v,u),\end{cases} \tag{4.7}$$

where the Hamiltonian $h(v,u)$ is given by

$$h(v,u)=\frac{\omega}{2}\big(v^2+u^2\big)-\frac{1}{2}G\big(v^2-u^2\big). \tag{4.8}$$

The solitary wave with a particular $\omega\in(\omega_0,m)$ corresponds to a trajectory of this Hamiltonian system such that

$$\lim_{x\to\pm\infty}v(x,\omega)=\lim_{x\to\pm\infty}u(x,\omega)=0,$$

hence $\lim_{x\to\pm\infty}\mathscr{X}=0$. Since $G(s)$ satisfies $G(0)=0$, we conclude that $h(v(x),u(x))\equiv 0$, which leads to

$$\omega\big(v^2+u^2\big)=G\big(v^2-u^2\big). \tag{4.9}$$

We conclude from (4.9) that solitary waves may only correspond to $|\omega|<m$, $\omega\ne 0$.

The functions $\mathscr{X}(x,\omega)$ and $\mathscr{Y}(x,\omega)$ introduced in (4.4) are to solve

$$\begin{cases}\partial_x\mathscr{X}=-4\omega\mathscr{Y},\\[4pt] \partial_x\mathscr{Y}=-\big(v^2+u^2\big)g(\mathscr{X})+\omega\mathscr{X}=-\dfrac{1}{\omega}G(\mathscr{X})g(\mathscr{X})+\omega\mathscr{X},\end{cases} \tag{4.10}$$

and to have the asymptotic behavior $\lim_{|x|\to\infty}\mathscr{X}(x)=0$, $\lim_{|x|\to\infty}\mathscr{Y}(x)=0$. In the second equation in (4.10), we used the relation (4.9). The system (4.10) can be written as the following equation on $\mathscr{X}$:

$$\partial_x^2\mathscr{X}=-\partial_{\mathscr{X}}\big(-2G(\mathscr{X})^2+2\omega^2\mathscr{X}^2\big)=4\big(G(\mathscr{X})g(\mathscr{X})-\omega^2\mathscr{X}\big). \tag{4.11}$$

This equation describes a particle in the potential $-2G(s)^2 + 2\omega^2 s^2$. The condition (4.2) is needed so that $s = \Gamma_\omega$ is the turning point for the zero energy trajectory in this potential. The existence of a positive solution $\mathscr{X}(x,\omega)$ follows. This solution is unique up to a translation, and it will be made symmetric in x by requiring $\mathscr{X}(0,\omega) = \Gamma_\omega$. $\square$

4.2. Solitary waves in the nonrelativistic limit

In dimensions $n = 1, 2$ and 3, we consider solitary wave amplitudes $\phi_\omega(x)$ of the forms given in (2.12) and (2.13). Substituting these into the nonlinear Dirac equation (2.1), a straightforward calculation results in the system

$$\begin{cases} \omega v = \partial_r u + \dfrac{n-1}{r} u + mv - f\big(v^2 - u^2\big)v, \\ \omega u = -\partial_r v - mu + f\big(v^2 - u^2\big)u \end{cases} \tag{4.12}$$

for the pair of real-valued functions $v = v(r,\omega)$, $u = u(r,\omega)$. Notice that Eq. (4.12) includes the 1-dimensional case (4.6) if we interpret $r = x$.

Recalling that $f(s) = s^k$, we arrive at

$$(\omega - m)v = \partial_r u + \frac{n-1}{r} u - fv, \qquad (\omega + m)u = -\partial_r v + fu,$$

with

$$f := \big(v^2 - u^2\big)^k.$$

To consider the nonrelativistic limit, we set

$$m^2 - \omega^2 = \epsilon^2, \quad 0 < \epsilon \ll m,$$

and rescale $v(r,\omega)$ and $u(r,\omega)$ as follows:

$$v(r,\omega) = \epsilon^{\frac{1}{k}} V(\epsilon r, \epsilon), \qquad u(r,\omega) = \epsilon^{1+\frac{1}{k}} U(\epsilon r, \epsilon).$$

Then V, U should satisfy

$$\begin{cases} (\omega - m)\epsilon^{\frac{1}{k}} V = \epsilon^{2+\frac{1}{k}} \left(\partial_R U + \dfrac{n-1}{R} U\right) - \epsilon^{1/k} f V, \\ (\omega + m)\epsilon^{1+\frac{1}{k}} U = -\epsilon^{1+\frac{1}{k}} \partial_R V + \epsilon^{1+\frac{1}{k}} f U, \end{cases}$$

where $R = \epsilon r$ denotes the "rescaled variable". Using $\omega = m - \frac{1}{2m}\epsilon^2 + O(\epsilon^4)$, and taking into account that

$$f = \big(\epsilon^{\frac{2}{k}} V^2 - \epsilon^{2+\frac{2}{k}} U^2\big)^k = \epsilon^2 V^{2k} + \epsilon^4 O\big(U^{2k} + V^{2k}\big),$$

we rewrite the system as

$$\begin{cases} \left(-\dfrac{1}{2m} + O\big(\epsilon^2\big)\right) V = \partial_R U + \dfrac{n-1}{R} U - V^{2k+1} + \epsilon^2 O\big(\big(U^{2k} + V^{2k}\big)|V|\big), \\ \big(2m + O\big(\epsilon^2\big)\big) U = -\partial_R V + \epsilon^2 V^{2k} U + \epsilon^4 O\big(\big(U^{2k} + V^{2k}\big)|U|\big). \end{cases} \tag{4.13}$$

The rescaled system (4.13) has an obvious limit as $\epsilon \to 0$. Formally (for now) setting

$$\hat{V}(r) = \lim_{\epsilon \to 0} V(r,\epsilon), \qquad \hat{U}(r) = \lim_{\epsilon \to 0} U(r,\epsilon),$$

we arrive at

$$-\frac{1}{2m}\hat{V} = \partial_R \hat{U} + \frac{n-1}{R}\hat{U} - \hat{V}^{2k+1}, \qquad 2m\hat{U} = -\partial_R \hat{V}. \tag{4.14}$$

Substituting the second equation into the first one yields

$$-\frac{1}{2m}\left(\partial_R^2 + \frac{n-1}{R}\partial_R\right)\hat{V} - \hat{V}^{2k+1} = -\frac{1}{2m}\hat{V}, \qquad \hat{U} = -\frac{1}{2m}\partial_R \hat{V}.$$

This equation for $\hat{V}(r)$ is precisely Eq. (3.2) for NLS solitary wave amplitudes ϕ_ω with $\omega = -\frac{1}{2m}$. Thus we let $\hat{V}(r)$ be the (unique) NLS ground state:

$$\hat{V}(r) := (2m)^{-\frac{1}{2k}} F(r), \qquad \hat{U}(r) := -(2m)^{-\frac{1}{2k}-1} F'(r), \tag{4.15}$$

with $F(r)$ the unique positive spherically symmetric solution to (3.4). We can use this nonrelativistic limit to construct nonlinear Dirac solitary waves for $\epsilon^2 = m^2 - \omega^2 \ll m^2$:

Lemma 4.2. *There is $\omega_0 < m$ such that for $\omega = \sqrt{m^2 - \epsilon^2} \in (\omega_0, m)$, there are solutions of (4.12) of the form*

$$v(r,\omega) = \epsilon^{\frac{1}{k}}\big[\hat{V}(\epsilon r) + \tilde{V}(\epsilon r)\big], \qquad u(r,\omega) = \epsilon^{1+\frac{1}{k}}\big[\hat{U}(\epsilon r) + \tilde{U}(\epsilon r)\big],$$
$$\|\tilde{V}\|_{H^2} + \|\tilde{U}\|_{H^2} = O(\epsilon^2).$$

Remark 4.3. In the one-dimensional case, since the solitary waves are unique (up to symmetries), it follows that these asymptotics describe *every* solitary wave for ω close to m, or equivalently every small amplitude solitary wave.

Proof. The argument parallels that of [25], where the (more general) nonlinearity $f(s) = |s|^\theta$, $0 < \theta < 2$ is considered for $n = 3$. Writing

$$V(R,\epsilon) = \hat{V}(R) + \tilde{V}(R,\epsilon), \qquad U(R,\epsilon) = \hat{U}(R) + \tilde{U}(R,\epsilon),$$

and subtracting equations (4.13) and (4.14), we arrive at

$$\begin{aligned}
-\frac{1}{2m}\tilde{V} + O(\epsilon^2)V &= \left(\partial_R + \frac{n-1}{R}\right)\tilde{U} - (2k+1)\hat{V}^{2k}\tilde{V} \\
&\quad + O\big(|\hat{V}|^{2k+1}\tilde{V}^2 + |\tilde{V}|^{2k+1}\big) + \epsilon^2 O\big(\big(U^{2k} + V^{2k}\big)|V|\big), \\
2m\tilde{U} + O(\epsilon^2)U &= -\partial_R\tilde{V} + \epsilon^2 V^{2k}U + \epsilon^4 O\big(\big(U^{2k} + V^{2k}\big)|U|\big),
\end{aligned}$$

which, setting

$$\Xi(R,\epsilon) := \begin{bmatrix} \tilde{V}(R,\epsilon) \\ \tilde{U}(R,\epsilon) \end{bmatrix},$$

we may rewrite as

$$\mathbf{H}\Xi = O_{H^1}(\epsilon^2) + O\big(\epsilon^2|\Xi| + |\Xi|^2 + |\Xi|^{2k+1}\big),$$

where

$$\mathbf{H} := \begin{bmatrix} -\frac{1}{2m} + (2k+1)\hat{V}^{2k} & -(\partial_R + \frac{n-1}{R}) \\ \partial_R & 2m \end{bmatrix}. \tag{4.16}$$

Since

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \ker \mathbf{H} \iff \xi \in \ker \mathrm{l}_+, \quad \eta = -\frac{1}{2m}\partial_R \xi,$$

and $\ker \mathrm{l}_{+,rad} = \{0\}$ (cf. definitions (3.7), (3.8)), we see that $\ker \mathbf{H} = \{0\}$. It then follows from the fact that $\mathrm{l}_{+,rad}^{-1}$ is bounded from $L^2_r(\mathbb{R}^n, \mathbb{C})$ to $H^2(\mathbb{R}^n, \mathbb{C})$, that $\mathbf{H}^{-1}$ is bounded from $H^1_r(\mathbb{R}^n, \mathbb{C}^2)$ to $H^2(\mathbb{R}^n, \mathbb{C}^2)$ (here L^2_r, H^1_r are the corresponding subspaces of spherically symmetric functions). Hence

$$\Xi = \mathbf{H}^{-1}\big\{O_{H^1}(\epsilon^2) + O\big(\epsilon^2|\Xi| + |\Xi|^2 + |\Xi|^{2k+1}\big)\big\}, \tag{4.17}$$

and since $\|\Xi\|_{L^\infty} \leqslant C\|\Xi\|_{H^2}$ (recall that $n \leqslant 3$), we arrive easily at

$$\begin{aligned}
\big\|\text{R.H.S. (4.17)}\big\|_{H^2} &\leqslant C\big\|O_{H^1}(\epsilon^2) + O\big(\epsilon^2|\Xi| + |\Xi|^2 + |\Xi|^{2k+1}\big)\big\|_{H^1} \\
&\leqslant C\big\{\epsilon^2 + \epsilon\|\Xi\|_{H^2} + \|\Xi\|^2_{H^2} + \|\Xi\|^{2k+1}_{H^2}\big\},
\end{aligned}$$

and so we see that for small enough ϵ, the map on the r.h.s. of (4.17) maps the ball of radius ϵ in H^2 into itself. A similar estimate shows that this map is a contraction, and hence has a unique fixed point Ξ in this ball. Finally, we see from (4.17) that $\|\Xi\|_{H^2} = O(\epsilon^2)$. □

5. Linear instability of small amplitude solitary waves

Our first observation here is that on spinor fields of the form

$$\psi(x,t) = \begin{bmatrix} \Psi_1(x,t) \\ \Psi_2(x,t) \end{bmatrix}, \qquad \psi(x,t) = \begin{bmatrix} \Psi_1(r,t) \\ i e^{i\phi}\Psi_2(r,t) \end{bmatrix},$$

$$\psi(x,t) = \begin{bmatrix} \Psi_1(r,t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i\Psi_2(r,t)\begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix} \end{bmatrix}$$

in dimensions $n=1$, $n=2$, and $n=3$ respectively, the nonlinear Dirac equation (2.1) reduces to the system

$$\begin{cases} i\partial_t\Psi_1 = \left(\partial_r + \dfrac{n-1}{r}\right)\Psi_2 + m\Psi_1 - f\left(|\Psi_1|^2 - |\Psi_2|^2\right)\Psi_1, \\ i\partial_t\Psi_2 = -\partial_r\Psi_1 - m\Psi_2 + f\left(|\Psi_1|^2 - |\Psi_2|^2\right)\Psi_2 \end{cases} \tag{5.1}$$

(with the convention $r = x$ for $n = 1$). The solitary waves considered in (2.12) and (2.13) lie in this class of fields, corresponding to

$$\Psi_1(r,t) = v(r,\omega)e^{-i\omega t}, \qquad \Psi_2(r,t) = u(r,\omega)e^{-i\omega t}. \tag{5.2}$$

To prove the instability of these solitary waves, it suffices to show that they are unstable as solutions of (5.1).

5.1. Linearization at a solitary wave

To derive the linearization of system (5.1) at a solitary wave (5.2) we consider solutions in the form of the Ansatz

$$\begin{bmatrix} \Psi_1(r,t) \\ \Psi_2(r,t) \end{bmatrix} = \left(\begin{bmatrix} v(r,\omega) \\ u(r,\omega) \end{bmatrix} + \begin{bmatrix} \rho_1(r,t) \\ \rho_2(r,t) \end{bmatrix}\right)e^{-i\omega t}, \tag{5.3}$$

where

$$\rho(r,t) := \begin{bmatrix} \rho_1(r,t) \\ \rho_2(r,t) \end{bmatrix} \in \mathbb{C}^2.$$

Inserting this Ansatz into system (5.1) with $f(s) = s^k$ and recalling that $u(r,\omega)$ and $v(r,\omega)$ are real-valued, we find that the linearized system for ρ is

$$\begin{aligned} i\partial_t\rho = {}& \begin{bmatrix} m - \omega - (v^2-u^2)^k & \partial_r + \frac{n-1}{r} \\ -\partial_r & -m - \omega + (v^2-u^2)^k \end{bmatrix}\rho \\ & - 2k\left(v^2-u^2\right)^{k-1}\begin{bmatrix} v^2 & -uv \\ -uv & u^2 \end{bmatrix}\operatorname{Re}\rho. \end{aligned} \tag{5.4}$$

We note that the above equation is $\mathbb{R}$-linear but not $\mathbb{C}$-linear, due to the presence of the term with $\operatorname{Re}\rho = \begin{bmatrix} \operatorname{Re}\rho_1 \\ \operatorname{Re}\rho_2 \end{bmatrix}$. We rewrite Eq. (5.4) in terms of $\operatorname{Re}\rho \in \mathbb{R}^2$ and $\operatorname{Im}\rho \in \mathbb{R}^2$:

$$\partial_t\begin{bmatrix} \operatorname{Re}\rho \\ \operatorname{Im}\rho \end{bmatrix} = \mathbf{J}\mathbf{L}(\omega)\begin{bmatrix} \operatorname{Re}\rho \\ \operatorname{Im}\rho \end{bmatrix} \tag{5.5}$$

where $\mathbf{J}$ corresponds to $1/i$:

$$\mathbf{J} = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix},$$

and the 4×4 matrix operator $\mathbf{L}(\omega)$ is defined by

$$\mathbf{L}(\omega) = \begin{bmatrix} \mathrm{L}_+(\omega) & 0 \\ 0 & \mathrm{L}_-(\omega) \end{bmatrix},$$

where, writing

$$f := f\big(\phi_\omega^* \beta \phi_\omega\big) = \big(v^2(r,\omega) - u^2(r,\omega)\big)^k, \qquad f' := f'\big(\phi_\omega^* \beta \phi_\omega\big),$$

we have

$$\mathrm{L}_-(\omega) = \begin{bmatrix} m - \omega - f & \partial_r + \frac{n-1}{r} \\ -\partial_r & -m - \omega + f \end{bmatrix} \tag{5.6}$$

and

$$\mathrm{L}_+(\omega) = \mathrm{L}_-(\omega) - 2f' \begin{bmatrix} v^2 & -uv \\ -uv & u^2 \end{bmatrix}. \tag{5.7}$$

Let us remind the reader that $v = v(r,\omega)$ and $u = u(r,\omega)$ in (5.6)–(5.7) both depend on ω.

Then Eq. (5.5) which describes the linearization of the reduced system (5.1) at the solitary wave $\phi_\omega e^{-i\omega t}$, takes the form

$$\partial_t \begin{bmatrix} \operatorname{Re}\rho \\ \operatorname{Im}\rho \end{bmatrix} = \mathbf{J}\mathbf{L}(\omega) \begin{bmatrix} \operatorname{Re}\rho \\ \operatorname{Im}\rho \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{L}_-(\omega) \\ -\mathrm{L}_+(\omega) & 0 \end{bmatrix} \begin{bmatrix} \operatorname{Re}\rho \\ \operatorname{Im}\rho \end{bmatrix}.$$

For the sake of completeness we record here the essential spectrum of the linearized operator:

Lemma 5.1. $\sigma_{\mathrm{ess}}(\mathbf{J}\mathbf{L}(\omega)) = i\mathbb{R} \backslash i(|\omega| - m, m - |\omega|)$.

Proof. The proof follows from noticing that, due to the exponential spatial decay of $v(r,\omega)$, $u(r,\omega)$ and the Weyl theorem on the essential spectrum [33, Theorem XIII.14, Corollary 2], which leads to

$$\sigma_{\mathrm{ess}}\big(\mathbf{J}\mathbf{L}(\omega)\big) = \sigma_{\mathrm{ess}}\big(\mathbf{J}(\mathbf{D}_m - \omega)\big),$$

where

$$\mathbf{D}_m = \begin{bmatrix} D_m & 0 \\ 0 & D_m \end{bmatrix}, \qquad D_m = i\sigma_2 \partial_r + \sigma_3 m.$$

At the same time, since $D_m^2 = -\Delta + m^2$, $\sigma_{\mathrm{ess}}(D_m) = \mathbb{R} \backslash (-m, m)$, while $\mathbf{J}$ commutes with $\mathbf{D}_m$ and $\sigma(\mathbf{J}) = \{\pm i\}$, one concludes that

$$\sigma_{\mathrm{ess}}\big(\mathbf{J}(\mathbf{D}_m - \omega)\big) = \sigma_{\mathrm{ess}}\big(i(D_m - \omega)\big) \cup \sigma_{\mathrm{ess}}\big(-i(D_m - \omega)\big) = i\mathbb{R} \backslash i\big(|\omega| - m, m - |\omega|\big). \qquad \square$$

5.2. *Unstable eigenvalue of* $\mathbf{J}\mathbf{L}$ *for* $\omega \lesssim m$

Proposition 5.2. *Let $k \in \mathbb{N}$ satisfy $k > 2/n$. There is $\omega_1 < m$ (which depends on n and k) such that for $\omega \in (\omega_1, m)$ there are two families of eigenvalues*

$$\pm\lambda_\omega \in \sigma_p\big(\mathbf{J}\mathbf{L}(\omega)\big), \quad \textit{with } \lambda_\omega > 0, \qquad \lambda_\omega = O(m - \omega).$$

Proof. The relation $\left[\begin{smallmatrix} 0 & \mathrm{L}_- \\ -\mathrm{L}_+ & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} \varphi \\ \vartheta \end{smallmatrix}\right] = \lambda \left[\begin{smallmatrix} \varphi \\ \vartheta \end{smallmatrix}\right]$, with $\varphi, \vartheta \in \mathbb{C}^2$, can be written explicitly as follows:

$$\begin{bmatrix} -\lambda & 0 & m - \omega - f & \partial_r + \frac{n-1}{r} \\ 0 & -\lambda & -\partial_r & f - m - \omega \\ \omega - m + f + 2f'v^2 & -\partial_r - 2f'vu & -\lambda & 0 \\ \partial_r - 2f'vu & m + \omega - f + 2f'u^2 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vartheta_1 \\ \vartheta_2 \end{bmatrix} = 0. \tag{5.8}$$

We divide the first and the third rows by $\epsilon^2 = m^2 - \omega^2$, the second and the fourth rows by ϵ, and substitute $R = \epsilon r$, $\varphi_2 = \epsilon\Phi_2$, $\vartheta_2 = \epsilon\Theta_2$, to get

$$\begin{bmatrix} -\frac{\lambda}{\epsilon^2} & 0 & \frac{m-\omega-f}{\epsilon^2} & \partial_R + \frac{n-1}{R} \\ 0 & -\lambda & -\partial_R & -m-\omega \\ \frac{\omega-m+f+2f'v^2}{\epsilon^2} & -(\partial_R + \frac{n-1}{R}) - \frac{2f'vu}{\epsilon} & -\frac{\lambda}{\epsilon^2} & 0 \\ \partial_R - \frac{1}{\epsilon}2f'vu & m+\omega-f+2f'u^2 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \Phi_2 \\ \vartheta_1 \\ \Theta_2 \end{bmatrix} = 0. \tag{5.9}$$

Anticipating the $\epsilon \to 0$ limit, formally set $\Lambda = \lim_{\epsilon\to 0}\frac{\lambda}{\epsilon^2}$, and introduce the matrices

$$\mathbf{A}_\Lambda = \begin{bmatrix} -\Lambda & 0 & \frac{1}{2m} - \hat{V}^{2k}(R) & \partial_R + \frac{n-1}{R} \\ 0 & 0 & -\partial_R & -2m \\ -\frac{1}{2m} + (2k+1)\hat{V}^{2k}(R) & -(\partial_R + \frac{n-1}{R}) & -\Lambda & 0 \\ \partial_R & 2m & 0 & 0 \end{bmatrix}, \tag{5.10}$$

$$\mathbf{K}_1 = \mathrm{diag}[1,0,1,0], \qquad \mathbf{K}_2 = \mathrm{diag}[0,1,0,1], \tag{5.11}$$

where $\hat{V}(R)$, the NLS ground state, was introduced in (4.15). We write (5.9) in the form

$$\mathbf{A}_\Lambda\eta = \left(\frac{\lambda}{\epsilon^2} - \Lambda\right)\mathbf{K}_1\eta + \lambda\mathbf{K}_2\eta + W\eta, \qquad \eta = \begin{bmatrix} \varphi_1 \\ \Phi_2 \\ \vartheta_1 \\ \Theta_2 \end{bmatrix} \in \mathbb{C}^4, \tag{5.12}$$

where $W(R,\epsilon)$ is a zero order differential operator with L^∞ coefficients.

Lemma 5.3. $\|W(\cdot,\epsilon)\|_{L^\infty(\mathbb{R}^+,\mathbb{C}^4\to\mathbb{C}^4)} \leqslant O(\epsilon^2)$.

Proof. By Lemma 4.2 (and the Sobolev inequality), one has

$$v(r,\omega) = \epsilon^{\frac{1}{k}}V(\epsilon r) = \epsilon^{\frac{1}{k}}\big(\hat{V}(\epsilon r) + \tilde{V}(\epsilon r,\epsilon)\big),$$
$$u(r,\omega) = \epsilon^{1+\frac{1}{k}}U(\epsilon r) = \epsilon^{1+\frac{1}{k}}\big(\hat{U}(\epsilon r) + \tilde{U}(\epsilon r,\epsilon)\big),$$
$$\|\tilde{V}\|_{L^\infty} + \|\tilde{U}\|_{L^\infty} = O\big(\epsilon^2\big).$$

Then

$$f\big(v^2-u^2\big) = \epsilon^2\big(V^2-U^2\big)^k,$$
$$f'\big(v^2-u^2\big) = k\epsilon^{2-\frac{2}{k}}\big(V^2-\epsilon^2U^2\big)^{k-1},$$
$$-\frac{1}{2m} + \frac{m-\omega-f-2f'v^2}{\epsilon^2} = O\big(\epsilon^2\big) - \big(V^2-\epsilon^2U^2\big)^k - 2kV^2\big(V^2-\epsilon^2U^2\big)^{k-1}$$
$$= -(1+2k)\hat{V}^{2k} + O_{L^\infty}\big(\epsilon^2\big),$$
$$\frac{f'vu}{\epsilon} = k\epsilon^2UV\big(V^2-\epsilon U^2\big)^{k-1} = O_{L^\infty}\big(\epsilon^2\big),$$
$$m-\omega+f = O\big(\epsilon^2\big) + \epsilon^2\big(V^2-\epsilon^2U^2\big)^k = O_{L^\infty}\big(\epsilon^2\big),$$
$$f'u^2 = k\epsilon^4U^2\big(V^2-\epsilon U^2\big)^{k-1} = O_{L^\infty}\big(\epsilon^4\big),$$

and

$$\frac{1}{2m} + \frac{\omega-m+f}{\epsilon^2} = +O\big(\epsilon^2\big) + \big(V^2-\epsilon^2U^2\big)^k = \hat{V}^{2k} + O_{L^\infty}\big(\epsilon^2\big),$$

and the lemma follows directly from this list of estimates. □

Lemma 5.4. $\dim\ker \mathbf{A}_\Lambda = \dim\ker(\mathrm{j}\mathfrak{l}_{rad} - \Lambda)$, *where*

$$\mathrm{j}\mathfrak{l}_{rad} = \begin{bmatrix} 0 & \mathfrak{l}_{+,rad} \\ -\mathfrak{l}_{-,rad} & 0 \end{bmatrix},$$

and where, we recall,

$$\mathfrak{l}_{-,rad} = -\frac{1}{2m}\left(\partial_R + \frac{n-1}{R}\right)\partial_R + \frac{1}{2m} - \hat{V}(R)^{2k},$$
$$\mathfrak{l}_{+,rad} = -\frac{1}{2m}\left(\partial_R + \frac{n-1}{R}\right)\partial_R + \frac{1}{2m} - (2k+1)\hat{V}(R)^{2k}.$$

Moreover, if $k > n/2$ ($k = 1$ if $n = 3$), there is $\Lambda > 0$ such that $\pm\Lambda \in \sigma_d(\mathrm{j}\mathrm{H})$ are simple eigenvalues. Here H is the operator defined in (4.16).

Proof. An easy computation shows that $\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{bmatrix} \in \ker \mathbf{A}_\Lambda$ if and only if

$$2m\Phi_2 = -\partial_R\Phi_1, \qquad 2m\Phi_4 = -\partial_R\Phi_3, \qquad (\mathrm{j}\mathfrak{l}_{rad} - \Lambda)\begin{bmatrix} \Phi_3 \\ -\Phi_1 \end{bmatrix} = 0,$$

and the first statement of the lemma follows from this observation. The second statement then follows from Lemma 3.2. □

So we may assume that there is $\Lambda > 0$ such that $\pm\Lambda \in \sigma_d(\mathbf{A}_\Lambda)$, with eigenfunctions

$$\ker \mathbf{A}_{\pm\Lambda} \ni \Phi_{\pm\Lambda} = \begin{bmatrix} \pm\Phi_1 \\ \mp\frac{1}{2m}\partial_R\Phi_1 \\ \Phi_3 \\ -\frac{1}{2m}\partial_R\Phi_3 \end{bmatrix}, \qquad \mathfrak{l}_{+,rad}\Phi_1 = -\Lambda\Phi_3, \qquad \mathfrak{l}_{-,rad}\Phi_3 = \Lambda\Phi_1.$$

We will use the Rayleigh–Schrödinger perturbation theory to show that there are eigenvalues $\pm\lambda \in \sigma_d(\mathbf{J}\mathbf{L})$ with $\lambda = \epsilon^2\Lambda + o(\epsilon^2)$.

Writing

$$\mathbf{A}_\Lambda = \mathbf{J}\mathrm{L}_0 - \Lambda\mathbf{K}_1, \qquad \mathrm{L}_0 = \begin{bmatrix} \mathrm{L}_{0,+} & 0 \\ 0 & \mathrm{L}_{0,-} \end{bmatrix},$$

with

$$\mathrm{L}_{0,+} = \begin{bmatrix} \frac{1}{2m} - (2k+1)\hat{V}^{2k} & \partial_R + \frac{n-1}{R} \\ -\partial_R & -2m \end{bmatrix}, \qquad \mathrm{L}_{0,-} = \begin{bmatrix} \frac{1}{2m} - \hat{V}^{2k} & \partial_R + \frac{n-1}{R} \\ -\partial_R & -2m \end{bmatrix}$$

self-adjoint, we see that

$$\mathbf{A}_\Lambda^* = -\mathrm{L}_0\mathbf{J} - \Lambda\mathbf{K}_1 = \mathrm{F}\mathbf{A}_\Lambda\mathrm{F}, \qquad \mathrm{F} := \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}.$$

Hence $\ker \mathbf{A}_\Lambda^*$ is spanned by $\Phi_\Lambda^* := \mathrm{F}\Phi_\Lambda$. Let $\mathbf{P}_\Lambda$ denote the orthogonal projection onto Φ_Λ^*.

Seeking η and λ in the form

$$\eta = \Phi_\Lambda + \zeta, \quad \zeta \perp \Phi_\Lambda, \qquad \lambda = \epsilon^2(\Lambda + \mu),$$

then (5.12) becomes

$$\mathbf{A}_\Lambda\zeta = \mu\mathbf{K}_1(\Phi_\Lambda + \zeta) + \epsilon^2(\Lambda + \mu)\mathbf{K}_2(\Phi_\Lambda + \zeta) + W(\Phi_\Lambda + \zeta). \tag{5.13}$$

Applying $\mathbf{P}_\Lambda$ and $1 - \mathbf{P}_\Lambda$ to (5.13), one has:

$$0 = \mu\langle\Phi_\Lambda^*, \mathbf{K}_1(\Phi_\Lambda + \zeta)\rangle + \epsilon^2(\Lambda + \mu)\langle\Phi_\Lambda^*, \mathbf{K}_2(\Phi_\Lambda + \zeta)\rangle + \langle\Phi_\Lambda^*, W(\Phi + \zeta)\eta\rangle, \tag{5.14}$$

$$\mathbf{A}_\Lambda\zeta = (1 - \mathbf{P}_\Lambda)\big(\mu\mathbf{K}_1 + \epsilon^2(\Lambda + \mu)\mathbf{K}_2 + W\big)(\Phi_\Lambda + \zeta). \tag{5.15}$$

Lemma 5.5.

$$\langle \Phi_\Lambda^*, \mathbf{K}_1 \Phi_\Lambda \rangle \neq 0.$$

Proof. Note that

$$\langle \Phi_\Lambda^*, \mathbf{K}_1 \Phi_\Lambda \rangle = \langle \mathrm{F}\Phi_\Lambda, \mathbf{K}_1 \Phi_\Lambda \rangle = 2\,\mathrm{Re}\langle \Phi_3, \Phi_1 \rangle.$$

Now the fact that $\Phi_\Lambda \in \ker \mathbf{A}_\Lambda$ means in particular that $L_- \Phi_3 = \Lambda \Phi_1$. Hence, since L_- is self-adjoint,

$$\mathbb{R} \ni \langle \Phi_3, L_- \Phi_3 \rangle = \Lambda \langle \Phi_3, \Phi_1 \rangle,$$

and so $\mathrm{Re}\langle \Phi_3, \Phi_1 \rangle = 0$ only if $\langle \Phi_3, L_- \Phi_3 \rangle = 0$. As is well known, since $L_- \hat{V} = 0$ and $\hat{V}(r) > 0$, we have $L_- \geqslant 0$. Thus $\mathrm{Re}\langle \Phi_3, \Phi_1 \rangle = 0$ only if $L_- \Phi_3 = 0$. This, in turn, would imply that either $\Lambda = 0$ or $\Phi_\Lambda = 0$, both of which are false. This finishes the proof. □

Denote by $L_r^2(\mathbb{R}^n, \mathbb{C}^4) \subset L^2(\mathbb{R}^n, \mathbb{C}^4)$ the subspace of spherically symmetric functions. Now using Lemma 5.5 and the existence of the bounded inverse $\mathbf{A}_\Lambda^{-1} : \mathrm{Range}(1 - \mathbf{P}_\Lambda) \to \Phi_\Lambda^\perp$, Eqs. (5.14), (5.15) can be written as

$$\mu = M(\mu, \zeta), \qquad \zeta = Z(\mu, \zeta),$$

with functions $M : \mathbb{R} \times L_r^2 \to \mathbb{R}$, $Z : \mathbb{R} \times L_r^2 \to L_r^2$ given by

$$M(\mu, \zeta) = -\frac{1}{\langle \Phi_\Lambda^*, \mathbf{K}_1 \Phi_\Lambda \rangle} \big[\mu \langle \Phi_\Lambda^*, \mathbf{K}_1 \zeta \rangle + \langle \Phi_\Lambda^*, \epsilon^2 (\Lambda + \mu) \mathbf{K}_2 (\Phi_\Lambda + \zeta) + W \rangle \big], \tag{5.16}$$

$$Z(\mu, \zeta) = \mathbf{A}_\Lambda^{-1} (1 - \mathbf{P}_\Lambda) \big(\mu \mathbf{K}_1 + \epsilon^2 (\Lambda + \mu) \mathbf{K}_2 + W \big) (\Phi_\Lambda + \zeta). \tag{5.17}$$

Pick $\Gamma \geqslant 1$ such that

$$\Gamma \geqslant 2 \big\| \mathbf{A}_\Lambda^{-1} (1 - \mathbf{P}_\Lambda) \mathbf{K}_1 \Phi_\Lambda \big\|_{L^2}. \tag{5.18}$$

Lemma 5.6. *Consider $\mathbb{R} \times L_r^2$ endowed with the metric*

$$\big\| (\mu, \zeta) \big\|_\Gamma = \Gamma |\mu| + \| \zeta \|_{L^2}.$$

There is $\omega_1 \in (\omega_0, m)$ such that for $\omega \in (\omega_1, m)$ the map

$$M \times Z : \mathbb{R} \times L_r^2 \to \mathbb{R} \times L_r^2, \quad (\mu, \zeta) \mapsto \big(M(\mu, \zeta), Z(\mu, \zeta) \big), \tag{5.19}$$

restricted onto the set

$$\mathcal{B}_\epsilon = \big\{ (\mu, \zeta) \in \mathbb{R} \times L^2\big(\mathbb{R}, \mathbb{C}^4\big);\ \big\| (\mu, \zeta) \big\|_\Gamma \leqslant \epsilon \big\} \subset \mathbb{R} \times L_r^2$$

is an endomorphism and a contraction with respect to $\| \cdot \|_\Gamma$.

Proof. Assuming $\Gamma |\mu| + \| \zeta \|_{L^2} \leqslant \epsilon < 1$, and using Lemma 5.3, we have the estimates

$$\big| M(\mu, \zeta) \big| \leqslant C \big(|\mu| \| \zeta \|_{L^2} + \epsilon^2 \big(1 + |\mu| \big) \big(1 + \| \zeta \|_{L^2} \big) + \| W \|_{L^\infty} \big) \leqslant C \epsilon^2,$$

$$\big\| Z(\mu, \zeta) \big\|_{L^2} \leqslant \frac{1}{2} \Gamma |\mu| + C \big(|\mu| \| \zeta \|_{L^2} + \epsilon^2 \big(1 + |\mu| \big) \big(1 + \| \zeta \|_{L^2} \big) + \| W \|_{L^\infty} \big(1 + \| \zeta \|_{L^2} \big) \big) \leqslant \frac{1}{2} \epsilon + C \epsilon^2,$$

which show that for all sufficiently small ϵ, $M \times Z$ maps $\mathcal{B}_\epsilon$ into itself. Similar estimates show that $(M \times Z)|_{\mathcal{B}_\epsilon}$ is a contraction in the metric $\| \cdot \|_\Gamma$. □

According to Lemma 5.6, by the contraction mapping theorem, the map (5.19) has a unique fixed point $(\mu_0(\omega), \zeta_0(\omega)) \in \mathcal{B}_\epsilon \subset \mathbb{R} \times L_r^2$ (as long as $\omega \in (\omega_1, m)$). Thus, we have $\pm \epsilon^2 (\Lambda + \mu_0(\omega)) \in \sigma_p(\mathbf{JL}(\omega))$, $\omega \in (\omega_1, m)$, with $\Gamma |\mu_0(\omega)| \leqslant \epsilon$, finishing the proof of the proposition. □

By Remark 2.6, Proposition 5.2 finishes the proof of Theorem 2.3.

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