Singular integrals and a problem on mixing flows ☆

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Abstract

We prove a result related to Bressan's mixing problem. We establish an inequality for the change of Bianchini semi-norms of characteristic functions under the flow generated by a divergence free time dependent vector field. The approach leads to a bilinear singular integral operator for which we prove bounds on Hardy spaces. We include additional observations about the approach and a discrete toy version of Bressan's problem.
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1. Introduction

1.1. Mixing flows

We consider subsets A of $\mathbb{T}^d \equiv \mathbb{R}^d / \mathbb{Z}^d$. For $0 < r < 1/4$, $x \in \mathbb{R}^d$ let $B_r(x)$ denote the ball of radius r centered at x, with respect to the usual geodesic distance on $\mathbb{T}^d$. A measurable set $E \subset \mathbb{T}^d$ is *mixed at scale r,* with mixing constant $\kappa \in (0, 1/2)$, if

$$\kappa \leq \frac{|E \cap B_r(x)|}{|B_r(x)|} \leq 1 - \kappa, \quad \forall x \in \mathbb{T}^d. \tag{1}$$

Let v be a time-dependent, a priori smooth vector field, defined on $\mathbb{T}^d \times [0, T]$ with values in the tangent bundle of the torus. The vector field can be considered a vector field $(x, t) \mapsto v(x, t)$ on $\mathbb{R}^d$ which is periodic in x, i.e.

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$$v(x + k, t) = v(x, t) \text{ for all } (x, t) \in \mathbb{R}^d \times \mathbb{R}, \ k \in \mathbb{Z}^d.$$

We assume that

$$\operatorname{div}_x v(x, t) = 0$$

and let Φ be the flow generated by v. I.e. Φ satisfies

$$\frac{\partial}{\partial t} \Phi(x, t) = v(\Phi(x, t), t),$$

$$\Phi(x, 0) = x.$$

For every t the map $x \mapsto \Phi(x, t)$ is a volume preserving diffeomorphism on $\mathbb{R}^d$ satisfying

$$\Phi(x + k, t) - k = \Phi(x, t), \quad x \in \mathbb{R}^d, k \in \mathbb{Z}^d.$$

In what follows we shall also use the notation $\Phi_t(x) = \Phi(x, t)$. We are interested in mixing flows which transport an unmixed set Ω at time $t = 0$ to a set $\Phi_T(\Omega)$ mixed at scale ε at time $t = T$.

1.2. Bressan's problem

Split $\mathbb{T}^d$ as $\Omega_L \cup \Omega_R$ with $\Omega_R = \Omega_L^{\complement}$ where

$$\Omega_L = \{x : 0 \le x_1 < \frac{1}{2}\}, \quad \Omega_R = \{x : \frac{1}{2} \le x_1 < 1\}. \tag{2}$$

Let $0 < \varepsilon < 1/4$. Consider a periodic flow Φ_t generated by a smooth time dependent divergence free vector field, and *assume that at time $t = T$ the flow mixes Ω_L at scale ε*; i.e. the set $E = \Phi_T(\Omega_L)$ satisfies (1) with $r = \varepsilon$. Bressan [5] asks (setting $\kappa = 1/3$) whether there is a universal constant $c_d > 0$ such that

$$\int_0^T \int_{[0,1)^d} |D_x v(x, t)| \, dx \, dt \ge c_d \log(1/\varepsilon). \tag{3}$$

As noted in [5] it suffices to consider the case $T = 1$, by replacing $v(x, t)$ with $T v(x, t/T)$. In [4], Bressan formulated a more general conjecture for mildly compressible flows.

Bressan's conjecture is still open at the time of this writing. Therefore it is of interest to ask for corresponding lower bounds if the $L^1(\mathbb{T}^d)$ norm is replaced by a larger norm. That is, under the assumption that the flow generated by v mixes the set at scale ε with mixing constant γ, do we have a universal lower bound of the form

$$\int_0^T \|D_x v(\cdot, t)\|_{\mathcal{Y}} \, dt \ge c_{\mathcal{Y}}(\kappa) \log(1/\varepsilon) \tag{4}$$

for suitable function spaces $\mathcal{Y} \subset L^1(\mathbb{T}^d)$ or even $\mathcal{Y} \subset M(\mathbb{T}^d)$ with $M(\mathbb{T}^d)$ the space of bounded Borel measures on $\mathbb{T}^d$? Crippa and De Lellis [7] showed this for $\mathcal{Y} = L^p(\mathbb{T}^d)$, $1 < p < \infty$ and also for the space $\mathcal{Y}$ consisting of functions for which the Hardy–Littlewood maximal function $M_{HL} f$ belongs to $L^1(\mathbb{T}^d)$, i.e. for $\mathcal{Y} = L \log L(\mathbb{T}^d)$. We shall discuss two ways to improve $\mathcal{Y}$ to a local Hardy space. In §7 we consider a discrete toy problem on $\mathbb{T}^2$ for which we prove an analogue of the L^1 conjecture, although this toy model does not yield significant information for the general Bressan problem. It should be noted that the lower bound $\log(1/\varepsilon)$ is sharp and cannot even be improved by working with L^p spaces, see the recent results by Yao and Zlatoš [21] and by Alberti, Crippa and Mazzucato [1].

1.3. An approach to Bressan's problem via a Bianchini semi-norm

We denote by

$$\fint_{B_r(x)} f(y) dy = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) dy$$

the average of f over the ball $B_r(x)$. For $\varepsilon < 1/8$ define the truncated Bianchini semi-norm by

$$\|f\|_{\mathcal{B}(\varepsilon)} := \int\limits_{\varepsilon}^{1/4} \int\limits_{\mathbb{T}^d} \Big| f(x) - \fint\limits_{B_r(x)} f(y) dy \Big| dx \, \frac{dr}{r}$$

and let the *Bianchini space* consist of all $L^1(\mathbb{T}^d)$ functions for which

$$\|f\|_{\mathcal{B}} := \sup_{\varepsilon < 1/4} \|f\|_{\mathcal{B}(\varepsilon)} < \infty.$$

This space was proposed by Bianchini in [2] as a measure for mixing in a one-dimensional shuffling problem. There it was denoted $\dot{B}^{0,1,1}$ in reference to Besov although this space does not actually belong to the usual scale of Besov spaces. The connection with mixing is given by the following

Observation: If E is mixed at scale $\varepsilon > 0$, with mixing constant κ, then

$$\Big| \mathbb{1}_E(x) - \fint\limits_{B_r(x)} \mathbb{1}_E(y) \, dy \Big| \geq \kappa \text{ a.e.} \quad \forall r > \varepsilon.$$

Hence integrating in r and x one gets

$$\|\mathbb{1}_E\|_{\mathcal{B}} \geq \|\mathbb{1}_E\|_{\mathcal{B}(\varepsilon/2)} \gtrsim \kappa \log(1/\varepsilon).$$

Also by straightforward computation $\|\mathbb{1}_{\Omega_L}\|_{\mathcal{B}} \lesssim 1$ for Ω_L as in (2). Our main result is an inequality for the change of the Bianchini norm of a characteristic function under the flow, which does not itself refer to mixing. In this result $h^1(\mathbb{T}^d)$ denotes the local Hardy space ([10]); note that for $p > 1$, we have the embeddings $L^p(\mathbb{T}^d) \subsetneq L \log L(\mathbb{T}^d) \subsetneq h^1(\mathbb{T}^d) \subsetneq L^1(\mathbb{T}^d)$.

Theorem 1.1. *Let v, ϕ be as above. Then the inequality*

$$\|\mathbb{1}_{\phi_T(A)}\|_{\mathcal{B}} \leq \|\mathbb{1}_A\|_{\mathcal{B}} + C_d \int\limits_0^T \|Dv(\cdot, t)\|_{h^1(\mathbb{T}^d)} \, dt$$

holds for any measurable subset $A \subset \mathbb{T}^d$, with C_d a universal constant.

Theorem 1.1 gives an alternative approach to the results by Crippa and De Lellis. By the above discussion the following implication on the mixing problem is immediate.

Corollary 1.2. *Let $0 < \varepsilon < 1/4$ and let the vector field v satisfy the assumptions in the Bressan problem stated in §1.2. Then inequality (4) holds with $\mathcal{Y} = h^1(\mathbb{T}^d)$.*

A weaker form of Theorem 1.1, with $Dv(\cdot, t) \in L^p$, $p > 1$, was cited in [17, eq. (1.5)] with reference to the current project, and served as initial motivation for the harmonic analysis results of that paper. Flavien Léger [13] independently found a related approach to mixing which leads to a limiting version of the singular integral forms in (9) below. Instead of the change of the Bianchini norm of characteristic functions he considers the change of the square of a logarithmic L^2-Sobolev norm of an arbitrary passive scalar advected under a divergence free vector field. For more comments about this see §5.2 below.

This paper

A computation reducing the problem to an inequality for bilinear singular integral operators is given in §2.1. In §2.2 we recall the connection with Christ–Journé operators. In §2.3 we describe the natural decomposition of our singular integral form and state the two main Propositions 2.4 and 2.5 which lead to $h^1 \to L^1$ boundedness. These propositions are proved in §3 and §4. In §5 we make additional remarks about the approach by Crippa and De Lellis and the results by Léger. In §6 we prove a result concerning the (non)-feasability of the singular integral estimate for Bressan's L^1 conjecture and formulate a related discrete problem. Finally, in §7 we include some positive results on a toy model for the L^1 version of Bressan's conjecture.

2. The reduction to singular integrals

2.1. The main computation

Given $A \subset \mathbb{T}^d$ we define

$$f_A(x) = \mathbb{1}_A(x) - \mathbb{1}_{A^{\complement}}(x).$$

Since constants (and thus $\mathbb{1}_A + \mathbb{1}_{A^{\complement}}$) have semi-norm equal to 0 in $\mathcal{B}(\varepsilon)$ we have $\|\mathbb{1}_A\|_{\mathcal{B}} = \|\mathbb{1}_{A^{\complement}}\|_{\mathcal{B}}$ and thus

$$\|\mathbb{1}_A\|_{\mathcal{B}(\varepsilon)} = \frac{1}{2}\|f_A\|_{\mathcal{B}(\varepsilon)}. \tag{5}$$

For a periodic time independent vector field b and functions f, g on $\mathbb{T}^d$ we define

$$\mathfrak{S}_\varepsilon^{per}[f, g, b] = \iint\limits_{\substack{(x,y)\in\mathbb{T}^d\times\mathbb{T}^d \\ \varepsilon\le|x-y|\le1/4}} \frac{\langle x - y, b(x) - b(y)\rangle}{|x - y|^{d+2}} g(y) f(x)\, dy\, dx. \tag{6}$$

Proposition 2.1. *Let v, ϕ be as in the introduction. Then*

$$\left\|\mathbb{1}_{\Phi_T(A)}\right\|_{\mathcal{B}_\varepsilon} - \left\|\mathbb{1}_A\right\|_{\mathcal{B}_\varepsilon} = \frac{1}{2V_d}\int_0^T \mathfrak{S}_\varepsilon^{per}\left[f_{\Phi_t(A)}, f_{\Phi_t(A)}, v(\cdot, t)\right] dt$$

where V_d denotes the volume of the unit ball in $\mathbb{R}^d$.

Proof. We compute using the incompressibility of the flow,

$$\|f_A \circ \Phi_T^{-1}\|_{\mathcal{B}(\varepsilon)} - \|f_A\|_{\mathcal{B}(\varepsilon)}$$

$$= \int_\varepsilon^{1/4} \Big[\int_{\mathbb{T}^d} \Big| f_A(\Phi_T^{-1}(x)) - \fint_{B_r(x)} f_A(\Phi_T^{-1}(y))dy\Big| dx$$

$$- \int \Big| f_A(x) - \fint_{B_r(x)} f_A(y)dy\Big| dx\Big] \frac{dr}{r}$$

$$= \int_\varepsilon^{1/4} \Big[\int_{\mathbb{T}^d} \Big| f_A(z) - \fint_{\Phi_T^{-1}B_r(\Phi_T(z))} f_A(w)dw\Big| dz$$

$$- \int \Big| f_A(x) - \fint_{B_r(x)} f_A(y)dy\Big| dx\Big] \frac{dr}{r}.$$

Now $f_A(x) = 1$ for $x \in A$ and $f_A(x) = -1$ for $x \in A^{\complement}$. Thus from the above, as we use that $-|E| \le \int_E f_A(y)dy \le |E|$ for all measurable sets E, we obtain

$$\|f_A \circ \Phi_T^{-1}\|_{\mathcal{B}(\varepsilon)} - \|f_A\|_{\mathcal{B}(\varepsilon)} =$$

$$\int\limits_\varepsilon^{1/4} \Big\{ \int\limits_A \Big[f_A(x) - \fint\limits_{\Phi_T^{-1} B_r(\Phi_T(x))} f_A(y) dy \Big] dx$$

$$- \int\limits_A \Big[f_A(x) - \fint\limits_{B_r(x)} f_A(y) dy \Big] dx$$

$$+ \int\limits_{A^\complement} \Big[\fint\limits_{\Phi_T^{-1} B_r(\Phi_T(x))} f_A(y) dy - f_A(x) \Big] dx$$

$$- \int\limits_{A^\complement} \Big[\fint\limits_{B_r(x)} f_A(y) dy - f_A(x) \Big] dx \Big\} \frac{dr}{r}$$

and this implies

$$\|f_A \circ \Phi_T^{-1}\|_{\mathcal{B}(\varepsilon)} - \|f_A\|_{\mathcal{B}(\varepsilon)}$$

$$= \int\limits_\varepsilon^{1/4} \int f_A(x) \Big[\fint\limits_{B_r(x)} f_A(y) dy - \fint\limits_{\Phi_T^{-1} B_r(\Phi_T(x))} f_A(y) dy \Big] dx \, \frac{dr}{r}$$

$$= -\int f_A(x) \int\limits_0^T \frac{d}{dt} \Big[\int\limits_\varepsilon^{1/4} \fint\limits_{\Phi_t^{-1} B_r(\Phi_t(x))} f_A(y) dy \, \frac{dr}{r} \Big] dt \, dx \tag{7}$$

Now let V_d denote the measure of the unit ball in $\mathbb{R}^d$. Then

$$\int\limits_\varepsilon^{1/4} \fint\limits_{\Phi_t^{-1} B_r(\Phi_t(x))} f_A(y) dy \, \frac{dr}{r}$$

$$= V_d^{-1} \int\limits_\varepsilon^{1/4} r^{-d-1} \int\limits_{\{y : |\Phi_t(x) - \Phi_t(y)| \le r\}} f_A(y) dy \, dr$$

$$= \int H_\varepsilon(\Phi_t(x) - \Phi_t(y)) f_A(y) \, dy$$

where

$$H_\varepsilon(u) = \begin{cases} d^{-1} V_d^{-1} (\varepsilon^{-d} - (1/4)^{-d}) & \text{if } |u| \le \varepsilon \\ d^{-1} V_d^{-1} (|u|^{-d} - (1/4)^{-d}) & \text{if } \varepsilon < |u| \le 1/4 \\ 0 & \text{if } |u| > 1/4 \end{cases}$$

H_ε is a Lipschitz function, and has a bounded gradient given by

$$\nabla H_\varepsilon(u) = -V_d^{-1} \frac{u}{|u|^{d+2}} \mathbb{1}_{\mathcal{A}(\varepsilon, 1/4)}(u)$$

where $\mathcal{A}(\varepsilon, 1/4)(u) = \{u \in \mathbb{R}^d : \varepsilon \le |u| \le 1/4\}$. Thus

$$\frac{d}{dt}\Big[\int\limits_{\varepsilon}^{1/4}\int\limits_{\Phi_t^{-1}B_r(\Phi_t(x))} f_A(y)dy\,\frac{dr}{r}\Big]$$

$$= \int \Big\langle \tfrac{d}{dt}(\Phi_t(x) - \Phi_t(y)), \nabla H_\varepsilon(\Phi_t(x) - \Phi_t(y))\Big\rangle f_A(y)\,dy$$

$$= - \int\limits_{\varepsilon \le |\Phi_t(x)-\Phi_t(y)| \le \frac{1}{4}} f_A(y)\frac{\langle v(\Phi_t(x),t) - v(\Phi_t(y),t), \Phi_t(x) - \Phi_t(y)\rangle}{V_d|\Phi_t(x) - \Phi_t(y)|^{d+2}}\,dy.$$

Using this in (7) and changing variables we obtain

$$\|f_{\Phi_T(A)}\|_{\mathcal{B}(\varepsilon)} - \|f_A\|_{\mathcal{B}(\varepsilon)}$$

$$= \int\limits_{0}^{T}\iint\limits_{\varepsilon \le |x-y| \le \frac{1}{4}} f_{\Phi_t(A)}(x) f_{\Phi_t(A)}(y)\frac{\langle v(x,t) - v(y,t), x - y\rangle}{V_d|x - y|^{d+2}}\,dy\,dx\,dt$$

which gives the assertion. □

In order to complete the proof of Theorem 1.1 it suffices to prove, for divergence free vector fields b, the inequality

$$\big|\mathfrak{S}_\varepsilon^{per}[\mathbb{1}_A, \mathbb{1}_B, b]\big| \lesssim \|Db\|_{h^1(\mathbb{T})} \tag{8}$$

for measurable subsets $A, B \subset \mathbb{T}^d$ and apply Proposition 2.1. Without loss of generality (after localization) one can assume that the diameters of A and B are small. We can then transfer the problem to $\mathbb{R}^d$ and look at the analogous singular integral form on $\mathbb{R}^d$, defined by

$$\mathfrak{S}_{\varepsilon,R}[f,g,b] = \iint\limits_{\varepsilon \le |x-y| \le R} \frac{\langle x - y, b(x) - b(y)\rangle}{|x - y|^{d+2}}g(y)f(x)\,dy\,dx. \tag{9}$$

Now (8) follows from

Theorem 2.2. *(i) For $\varepsilon < R$,*

$$\big|\mathfrak{S}_{\varepsilon,R}[f,g,b]\big| \le C_d\|Db\|_{H^1(\mathbb{R}^d)}\|g\|_\infty\|f\|_\infty \tag{10}$$

with C_d independent of ε, R.
(ii) If in addition $R < 1$ the Hardy space H^1 may be replaced in (10) with the local Hardy space h^1.

Remark. An examination of the proof of Theorem 2.2 also shows that for $f, g \in L^\infty$, $Db \in H^1$,

$$\lim\limits_{\substack{\varepsilon \to 0 \\ R \to \infty}} \mathfrak{S}_{\varepsilon,R}[f,g,b] = \mathfrak{S}[f,g,b] \tag{11}$$

where $\mathfrak{S}$ is a singular integral form satisfying

$$\big|\mathfrak{S}[f,g,b]\big| \le C_d\|Db\|_{H^1}\|g\|_\infty\|f\|_\infty. \tag{12}$$

2.2. Connection with Christ–Journé operators

There is a close relation with the operators considered by Christ and Journé [6], and in more generality by three of the authors [17]. The result of Proposition 2.1 is cited in [17] and served as a motivation for the harmonic analysis results of that paper.

For $\beta \in L^1_{loc}$ we can define for almost every pair $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$

$$m_{x,y}[\beta] = \int\limits_0^1 \beta(sx + (1-s)y)\,ds, \tag{13}$$

the mean of β over the line segment connecting the points x and y. Given a Calderón–Zygmund convolution kernel K in $\mathbb{R}^d$, $d \geq 2$, and $a \in L^\infty(\mathbb{R}^d)$ (or $L^q(\mathbb{R}^d)$) the so called d-commutator of first order is defined by

$$\mathcal{C}_K[f, \beta](x) = \int_{\mathbb{R}^d} K(x - y) m_{x,y}[\beta] f(y) \, dy. \tag{14}$$

For a divergence free vector field b set

$$\beta_{ij} = \frac{\partial b_i}{\partial x_j} \tag{15}$$

so that

$$b_i(x) - b_i(y) = \sum_{j=1}^d (x_j - y_j) m_{x,y}[\beta_{ij}].$$

By the assumption $\mathrm{div}(b) = 0$ we have $\beta_{dd}(x) = -\sum_{i=1}^{d-1} \beta_{ii}(x)$; hence

$$\frac{\langle x - y, b(x) - b(y) \rangle}{|x - y|^{d+2}} = \sum_{i=1}^{d-1} K_i(x - y) m_{x,y}[\beta_{ii}] + \sum_{\substack{1 \leq i,j \leq d \\ i \neq j}} K_{ij}(x) m_{x,y}[\beta_{ij}] \tag{16}$$

where

$$K_i(x) = \frac{(x_i - y_i)^2 - (x_d - y_d)^2}{|x - y|^{d+2}} \tag{17a}$$

and

$$K_{ij}(x) = \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^{d+2}}. \tag{17b}$$

Consequently,

$$\mathfrak{S}[f, g, b] = \sum_{i=1}^{d-1} \int \mathcal{C}_{K_i}[g, \beta_{ii}](x) f(x) \, dx + \sum_{i \neq j} \int \mathcal{C}_{K_{ij}}[g, \beta_{ij}](x) f(x) \, dx. \tag{18}$$

This identity turns our problem into a problem on d-commutators. Note that $K_i(x)$ and $K_{ij}(x)$ above are of the form $\Omega(x/|x|)|x|^{-d}$ where $\Omega \in C^\infty(S^{d-1})$ is even with $\int_{S^{d-1}} \Omega(\theta) d\theta = 0$. From (18) and the results in [17] one obtains

$$|\mathfrak{S}[f, g, b]| \leq C(p_1, p_2, p_3) \|f\|_{p_1} \|g\|_{p_2} \|Db\|_{p_3} \tag{19}$$

for $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$, with $1 < p_i \leq \infty$. S. Hofmann suggested in personal communication that this result might also follow from (the isotropic version) of his off-diagonal $T(1)$ theorem in [11]. These results do not seem to give enough information in the case $p_3 = 1$ which is relevant for the focus of this paper. The weak type $(1, 1)$ result in [16] can be modified to see that for $g \in L^\infty$ and $\beta \in L^1$ we have $\mathcal{C}_K[g, \beta] \in L^{1,\infty}$ and this can be used to prove a bound for compactly supported b with $Db \in L \log L$; however there does not seem to be an $H^1 \to L^1$ result for d-commutators which can be used to establish Theorem 2.2. Our approach will be more direct; we rely on some regularizations for the kernels, and use the original $T(1)$ theorem by David and Journé for one of the terms and Littlewood–Paley estimates for the others. The atomic decomposition will be used for the Hardy space estimates.

2.3. Further reductions

We now begin with the proof of Theorem 2.2 and first make an easy observation about single scale contributions. Using

$$b(x) - b(y) = \int_0^1 Db(sx + (1 - s)y) ds \, (x - y)$$

we observe, using a straightforward application of Hölder's inequality, that for each $R > 0$

$$\iint\limits_{R \le |x-y| \le 2R} \frac{|\langle x-y, b(x)-b(y)\rangle|}{|x-y|^{d+2}} |g(y)||h(x)|dy\,dx \lesssim \|Db\|_{p_1}\|g\|_{p_2}\|h\|_{p_3}, \tag{20}$$

for $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$, $1 \le p_1, p_2, p_3 \le \infty$.

Let χ be a *radial* C^∞ function supported in $\{x : 1/2 < |x| < 2\}$ such that $\sum_{k \in \mathbb{Z}} \chi(2^k x) = 1$ for $x \ne 0$. Define

$$\chi_k(x) = \chi(2^k x)$$

and set

$$S_k[g, b](x) = \int \chi_k(x-y) \frac{\langle x-y, b(x)-b(y)\rangle}{|x-y|^{d+2}} g(y)dy. \tag{21}$$

Using (20) it is easy to see that Theorem 2.2 follows from

Theorem 2.3.

$$\left| \sum_{k \in \mathcal{Z}} \int h(x) S_k[g, b](x) dx \right| \le C \|Db\|_{H^1} \|g\|_\infty \|h\|_\infty \tag{22}$$

where the summation over k is over a finite set $\mathcal{Z}$ of integers and the constant C does not depend on the cardinality of this set.

We need further decompositions. Let ϕ be a C^∞ function with support in $\{x : |x| \le 1/2\}$ such that

$$\int \phi(x)dx = 1 \tag{23a}$$

and

$$\int \phi(x)x_i dx = 0, \quad i = 1, \dots, d. \tag{23b}$$

Define

$$\phi_k(x) = 2^{kd}\phi(2^k x)$$
$$\psi_l(x) = \phi_l(x) - \phi_{l-1}(x)$$

For every k we have, in the sense of distributions,

$$\phi_k + \sum_{n=1}^\infty \psi_{k+n} = \delta; \tag{24}$$

here δ is the Dirac measure. Note that $\int \psi_l(x)\pi(x)dx = 0$ for all affine linear functions π.

Theorem 2.3 follows immediately from the second parts of the following two propositions. All constants will be independent of the cardinality of $\mathcal{Z}$.

Proposition 2.4. *(i) For $1 < p < \infty$,*

$$\left\| \sum_{k \in \mathcal{Z}} S_k[g, \phi_k * b] \right\|_p \le C(p_1, p_2)\|g\|_\infty\|Db\|_p.$$

(ii)

$$\left\| \sum_{k \in \mathcal{Z}} S_k[g, \phi_k * b] \right\|_1 \le C\|g\|_\infty\|Db\|_{H^1}.$$

Proposition 2.5. *(i) Let $1 < p_1, p_2, q < \infty$ and $1/p_1 + 1/p_2 = 1/q$. Then for $n = 1, 2, 3, \dots$*

$$\left\| \sum_{k \in \mathcal{Z}} S_k[g, \psi_{k+n} * b] \right\|_q \le C(p_1, p_2) 2^{-n} \|g\|_{p_2} \|Db\|_{p_1}.$$

(ii)

$$\left\| \sum_{k \in \mathcal{Z}} S_k[g, \psi_{k+n} * b] \right\|_1 \le Cn2^{-n} \|g\|_\infty \|Db\|_{H^1}.$$

The proofs of the two propositions will be given in §3 and §4.

Remark 2.6. Our proofs will show that if the index set $\mathcal{Z}$ is a subset of $\mathbb{Z}_+$ then the Hardy space H^1 in Propositions 2.4 and 2.5 can be replaced by the local Hardy space h^1 (cf. Remarks 2.8 below).

Remark 2.7. The condition $\mathrm{div}(b) = 0$ is crucial for Proposition 2.4 but not needed for Proposition 2.5. There are also $L^{p_1} \times L^{p_2} \to L^q$ estimates for other exponents with $p_1^{-1} + p_2^{-1} = q^{-1}$ in Proposition 2.4 but they will not be relevant for Theorem 1.1.

Remarks 2.8. *On Hardy spaces and atomic decompositions.* The proof of the Hardy space inequalities will rely on the atomic decomposition (see e.g. [19] for an exposition and historical references). Let $1 < r \le \infty$. We say that a is an r-atom associated with a cube Q if a is supported in Q, if $\|a\|_{L^r(Q)} \le |Q|^{-1+1/r}$ and if $\int a(x)dx = 0$. Note that $\|a\|_1 \le 1$ for atoms. The atomic characterization of H^1 states that any $f \in H^1$ can be decomposed as $f = \sum_Q \lambda_Q a_Q$ with convergence in L^1, where a_Q are r-atoms and $\sum_Q |\lambda_Q| < \infty$. The norm $\|f\|_{H^1}$ is equivalent to $\inf \sum_Q |\lambda_Q|$ where the infimum is taken over all such decompositions of f. We shall assume $r < \infty$. An operator T maps $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ if and only we have $\|Ta\|_1 \lesssim C$ for all r-atoms; the infimum over such C is equivalent to the $H^1 \to L^1$ operator norm of T. We refer to [3], [15] for the reason why it is preferable to work with r-atoms for $r < \infty$.

For compact manifolds the appropriate Hardy space is the local Hardy space h^1, introduced by Goldberg [10], which can be identified with the Triebel–Lizorkin space $F_{p,q}^0$ for $p = 1$ and $q = 2$, [20]. Functions in h^1 can be localized, i.e. if $f \in h^1$ and if $\chi \in C_0^\infty$ then $\chi f \in h^1$. More generally, classical pseudo-differential operators of order 0 are bounded on h^1 (see [10]). Finally an operator T maps h^1 to L^1 if we have $\|Ta\|_1 \lesssim C$ for all r-atoms associated to cubes with diameter $\le c_0$ and if in addition $\|Tb\|_1 \lesssim 1$ for all L^r functions b with $\|b\|_r \le 1$, which are supported on sets of bounded diameter.

3. Proof of Proposition 2.4

We shall use the $T1$ theorem of David and Journé [8]. For each term $S_k[g, \phi_k * b]$ we use the identity (16) with $\phi_k * b$ in place of b, and with $\phi_k * \beta_{ij}$ in place of β_{ij}. This reduces matters to the estimate of a singular integral operator $T \equiv T_{[g]}$ which acts on functions h, and is, for fixed $g \in L^\infty$, defined by

$$Th(x) = \sum_{k \in \mathcal{Z}} \int \chi_k(x-y)\kappa(x-y)g(y) \int_0^1 \phi_k * h(sx + (1-s)y)ds\, dy. \tag{25}$$

Here κ is smooth away from the origin, homogeneous of degree $-d$, with mean value 0 over S^{d-1}; in particular it can be any of the kernels in (17a), (17b). Proposition 2.4 follows from the inequalities

$$\|Th\|_p \lesssim \|g\|_\infty \|h\|_p, \tag{26}$$

$$\|Th\|_1 \lesssim \|g\|_\infty \|h\|_{H^1}. \tag{27}$$

We now have to verify the hypothesis of the David–Journé theorem [8]. Let K be the Schwartz kernel of T, i.e. we have

$$Th(x) = \int K(x, z)h(z)dz$$

for $h \in L^1 + L^\infty$; by our assumption on the index set $\mathcal{Z}$ we see $K(x, \cdot)$ is bounded and compactly supported (although $\mathcal{Z}$ and these assumptions are not supposed to quantitatively enter in our estimates). We need to check that K and its derivatives satisfy standard bounds for singular kernels, which are controlled by the L^∞ norm of g; *i.e.*

$$|K(x, z)| \lesssim \|g\|_\infty |x - z|^{-d} \tag{28}$$

and

$$|\nabla_x K(x, z)| + |\nabla_z K(x, z)| \lesssim \|g\|_\infty |x - z|^{-d-1}. \tag{29}$$

Secondly, T needs to satisfy the *weak boundedness property*. Let $\mathcal{N}$ be the class of C^1 functions supported in $\{x : |x| \le 1\}$ such that $\|u\|_\infty + \|\nabla u\|_\infty \le 1$. For $u \in \mathcal{N}$ define the translated and dilated versions u_R^w, $R > 0$, $w \in \mathbb{R}^d$, by $u_R^w(x) = u(R^{-1}(x - w))$. Then we need to verify for all $u, \tilde{u} \in \mathcal{N}$

$$\sup_{w \in \mathbb{R}^d} \sup_{R > 0} R^{-d} \left| \langle T u_R^w, \tilde{u}_R^w \rangle \right| \lesssim \|g\|_\infty. \tag{30}$$

Finally, we need the crucial BMO-conditions

$$\|T1\|_{BMO} + \|T^*1\|_{BMO} \lesssim \|g\|_\infty. \tag{31}$$

We begin by checking (28) and (29). We have $K(x, z) = \sum_k K_k(x, z)$ where

$$K_k(x, z) = \int \chi_k(x - y) \kappa(x - y) g(y) \int_0^1 \phi_k(sx + (1 - s)y - z) ds \, dy.$$

Observe that

$$K_k(x, z) = 0 \text{ for } |x - z| \ge C2^{-k}.$$

It is immediate from the definition that

$$|K_k(x, z)| \lesssim 2^{kd} \|g\|_\infty$$

and

$$|\nabla_x K_k(x, z)| + |\nabla_z K_k(x, z)| \lesssim 2^{k(d+1)} \|g\|_\infty.$$

Fix x, z and sum over k with $2^k \lesssim |x - z|^{-1}$, and (28) and (29) follow.

Next, we check the weak boundedness property (30). Let T_k denote the operator with Schwartz kernel K_k. We estimate $\langle T_k u_R^w, \tilde{u}_R^w \rangle$ and distinguish the cases $2^k R \le 1$ and $2^k R \ge 1$.

Write

$$\langle T_k u_R^w, \tilde{u}_R^w \rangle = \iint K_k(x, z) u_R^w(z) \tilde{u}_R^w(x) \, dz \, dx$$

$$= \iint \int \chi_k(x - y) \kappa(x - y) g(y) \int_0^1 \phi_k(sx + (1 - s)y - z) ds \, dy$$

$$\times u_R^w(z) \tilde{u}_R^w(x) \, dx \, dz$$

and since we have the conditions $|x - w| \lesssim R$, $|z - w| \lesssim R$, $|y - x| \lesssim 2^{-k}$ for the domains of integration, a straightforward estimation yields

$$\left| \langle T_k u_R^w, \tilde{u}_R^w \rangle \right| \lesssim 2^{kd} R^{2d} \|g\|_\infty \text{ if } R \le 2^{-k}.$$

For $R \ge 2^{-k}$ we use that the integrals of κ over spheres centered at the origin are zero. Since χ_k is radial we also have

$$\int \chi_k(x) \kappa(x) \, dx = 0, \tag{32}$$

for all $k \in \mathbb{Z}$. We may write (after performing a change of variable)

$$\iint K_k(x,z)u_R^w(z)\tilde{u}_R^w(x)\,dx\,dz$$

$$= \int g(y)\int_0^1\int\int \phi_k((1-s)y-z)[\cdots]dz\,ds\,dy \tag{33}$$

where

$$[\cdots] = \int u_R^w(z+sx)\tilde{u}_R^w(x)\chi_k(x-y)\kappa(x-y)\,dx$$

$$= \int \left(u_R^w(z+sx)\tilde{u}_R^w(x) - u_R^w(z+sy)\tilde{u}_R^w(y)\right)\chi_k(x-y)\kappa(x-y)\,dx$$

$$= O(2^{-k}R^{-1}).$$

Here we have of course used the cancellation property (32). Using the last estimate in (33) we see that

$$\left|\langle T_k u_R^w, \tilde{u}_R^w\rangle\right| \lesssim (2^k R)^{-1}\|g\|_\infty \int\limits_{|y-w|\le CR}\int_0^1\int |\phi_k((1-s)y-z)|\,dz\,ds\,dy$$

$$\lesssim (2^k R)^{-1}R^d\|g\|_\infty \text{ if } R \ge 2^{-k}.$$

Summing in k over $2^{-k} \le R$ yields (30).

Finally we need to verify the BMO bounds for $T1$ and T^*1. First,

$$T_k 1(x) = \int K_k(x,z)dz$$

$$= \int \chi_k(x-y)\kappa(x-y)g(y)\int_0^1\int \phi_k(sx+(1-s)y-z)dz\,ds\,dy$$

$$= (\chi_k\kappa)*g(x).$$

In view of the assumptions on κ the operator $g \mapsto \sum_k(\chi_k\kappa)*g = \kappa*g$ is a standard Calderón–Zygmund convolution operator and thus bounded from L^∞ to BMO. Thus we get

$$\|T1\|_{BMO} \lesssim \|g\|_\infty.$$

Next,

$$T_k^*1(z) = \int K_k(x,z)dx$$

$$= \int\int_0^1\int \chi_k(x-y)\kappa(x-y)g(y)\phi_k(sx+(1-s)y-z)dx\,ds\,dy$$

$$= \int g(y)\int\int s^{-d}\chi_k(s^{-1}(w-y))\kappa(s^{-1}(w-y))\phi_k(w-z)\,dw\,ds\,dy$$

where for fixed y,s we changed variables $w = sx+(1-s)y$.

Hence setting $\kappa_{k,s}(x) = \chi_k(s^{-1}x)s^{-d}\kappa(s^{-1}x)$, we have

$$T^*1 = \sum_{k\in\mathcal{Z}}\phi_k * \int_0^1 \kappa_{k,s}ds * g.$$

For fixed s we use the cancellation of $\chi_k\kappa$ to get an estimate for the Fourier transform of $\kappa_{k,s}$,

$$|\widehat{\kappa_{k,s}}(\xi)| \le C_N s2^{-k}|\xi|(1+s2^{-k}|\xi|)^{-N}.$$

It follows that $\sup_{\xi,s} \sum_k |\widehat{\kappa_{k,s}}(\xi)| \le C$ and since $\widehat{\phi_k} = O(1)$ we see that the Fourier transform of $\sum_k \phi_k * \kappa_{k,s}$ is bounded, independently of s. Integrating over $s \in [0, 1]$ we see that

$$\Big\| \sum_{k \in \mathcal{Z}} \phi_k * \int_0^1 \kappa_{k,s} ds * f \Big\|_2 \lesssim \|f\|_2.$$

It is also clear that the convolution kernel satisfies standard size and differentiability estimates in Calderón–Zygmund theory and consequently we get $L^\infty \to BMO$ boundedness. It follows that

$$\|T^* 1\|_{BMO} \lesssim \|g\|_\infty$$

and (31) is proved. This completes the proof of the L^p estimates (26).

The Hardy space estimate (27) follows from the corresponding estimates on atoms which are standard [19]. For completeness we include the argument. Let a be a 2-atom associated with a cube Q centered at y_Q and let Q^* be the triple cube. Then

$$\int_{Q^*} |Ta(x)| \, dx \le |Q^*|^{1/2} \|Ta\|_2 \lesssim |Q^*|^{1/2} \|g\|_\infty \|a\|_2 \lesssim \|g\|_\infty.$$

Since $\int a(y) dy = 0$ we get

$$\int_{\mathbb{R}^d \setminus Q^*} |Ta(x)| \, dx = \int_{\mathbb{R}^d \setminus Q^*} \int (K(x,y) - K(x, y_Q)) a(y) \, dy \, dx \lesssim \|g\|_\infty$$

given the size and derivative assumptions in (28) and (29) and $\|a\|_1 \le 1$. This finishes the proof of Proposition 2.4. $\square$

4. Proof of Proposition 2.5

This will be straightforward from standard estimates for singular convolution operators. Let

$$\mathcal{K}_{i,k}(x) = \chi_k(x) \frac{2^{-k} x_i}{|x|^{d+2}}.$$

We observe the commutator relation

$$S_k[g, h](x) = 2^k \sum_{i=1}^d \left(\mathcal{K}_{i,k} * g(x) h_i(x) - \mathcal{K}_{i,k} * [gh_i](x) \right) \tag{34}$$

which we use with the choice $h_i = \psi_{k+n} * b_i$. Notice that $\mathcal{K}_{i,k}$ is an odd kernel and therefore

$$\int \mathcal{K}_{i,k} * g(x) h_i(x) f(x) \, dx = -\int \mathcal{K}_{i,k} * [fh_i](x) g(x) \, dx \tag{35}$$

Hence, in order to prove part (i) of Proposition 2.5 it suffices to show

$$\Big| \sum_{k \in \mathcal{Z}} 2^k \int \mathcal{K}_{i,k} * g(x) \, \psi_{k+n} * b_i(x) \, f(x) dx \Big| \lesssim 2^{-n} \|f\|_{p_1} \|g\|_{p_2} \|\nabla b_i\|_{p_3},$$

$$\text{with } p_1^{-1} + p_2^{-1} + p_3^{-1} = 1 \text{ and } 1 < p_1, p_2, p_3 < \infty. \tag{36}$$

Moreover, to prove part (ii) it suffices to show

$$\Big| \sum_{k \in \mathcal{Z}} 2^k \int \mathcal{K}_{i,k} * g(x) \, \psi_{k+n} * b_i(x) \, f(x) dx \Big| \lesssim n 2^{-n} \|f\|_\infty \|g\|_\infty \|\nabla b_i\|_{H^1}. \tag{37}$$

We first simplify by rewriting the left hand sides as an expression which acts on ∇b_i. Let ϕ be as in (23a), (23b) and define for $j = 1, \ldots d$

$$\Psi^{[j]}(x) = \int\limits_{-\infty}^{x_j} 2^j \phi(2x_1, \ldots, 2x_{j-1}, 2s, x_{j+1}, \ldots, x_d)\, ds$$

$$- \int\limits_{-\infty}^{x_j} 2^{j-1} \phi(2x_1, \ldots, 2x_{j-1}, s, x_{j+1}, \ldots, x_d)\, ds$$

Since ϕ is supported in $[-1/2, 1/2]$ it is then easy to check using (23a) that $\Psi^{[j]}$ is also supported in $[-1/2, 1/2]$; moreover from (23b) and integration by parts we get

$$\int \Psi^{[j]}(x)\, dx = 0.$$

Now let $\Psi_l^{[j]}(x) = 2^{ld}\Psi^{[j]}(2^l x)$, and we verify that

$$\psi_l = 2^{-l} \sum_{j=1}^{d} \frac{\partial \Psi_l^{[j]}}{\partial x_j}.$$

Thus by integration by parts

$$\psi_{k+n} * b_i = 2^{-k-n} \sum_{j=1}^{d} \Psi_{k+n}^{[j]} * \frac{\partial b_i}{\partial x_j}.$$

Let Ψ be any smooth function supported in $[-1/2, 1/2]^d$ such that $\int \Psi(x)\, dx = 0$, and $\Psi_l = 2^{ld}\Psi(2^l \cdot)$. The above considerations imply that in order to establish (36), (37) it suffices to prove

$$\left| \sum_{k \in \mathcal{Z}} \int \mathcal{K}_{i,k} * g(x)\, \Psi_{k+n} * h(x)\, f(x)\, dx \right| \lesssim \|f\|_{p_1} \|g\|_{p_2} \|h\|_{p_3}, \tag{38}$$

with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, and $1 < p_1, p_2, p_3 < \infty$, and

$$\left| \sum_{k \in \mathcal{Z}} \int \mathcal{K}_{i,k} * g(x)\, \Psi_{k+n} * h(x)\, f(x)\, dx \right| \lesssim n \|f\|_\infty \|g\|_\infty \|h\|_{H^1}. \tag{39}$$

Proof of (38). We apply Hölder's inequality several times and dominate the left hand side of (38) by

$$\|f\|_{p_1} \left\| \sum_{k \in \mathcal{Z}} (\mathcal{K}_{i,k} * g)(\Psi_{k+n} * h) \right\|_{p_1'}$$

$$\leq \|f\|_{p_1} \left\| \left(\sum_k |\mathcal{K}_{i,k} * g|^2 \right)^{1/2} \left(\sum_k |\Psi_{k+n} * h|^2 \right)^{1/2} \right\|_{p_1'}$$

$$\leq \|f\|_{p_1} \left\| \left(\sum_k |\mathcal{K}_{i,k} * g|^2 \right)^{1/2} \right\|_{p_2} \left\| \left(\sum_k |\Psi_{k+n} * h|^2 \right)^{1/2} \right\|_{p_3} \tag{40}$$

where we have used $1/p_1' = 1/p_2 + 1/p_3$.

For any bounded sequence $\gamma = \{\gamma_k\}$ with $\|\gamma\|_\infty \leq 1$, $\sum_k \gamma_k \mathcal{K}_{i,k}$ defines a standard Calderón–Zygmund convolution kernel in $\mathbb{R}^d$ with bounds uniformly in γ. In particular we may randomly choose $\gamma = \pm 1$ and by the standard averaging argument using Khinchine's inequality for Rademacher functions (see e.g. [18, ch. II.5]) (or alternatively, arguments for vector-valued Calderón–Zygmund operators, cf. [18, Appendix D]) we get the inequality

$$\left\| \left(\sum_k |\mathcal{K}_{i,k} * g|^2 \right)^{1/2} \right\|_{p_2} \leq C(p_2) \|g\|_{p_2}, \tag{41}$$

for $1 < p_2 < \infty$. Similarly, we also have the Littlewood–Paley inequality (cf. [18, ch. II.5.])

$$\left\| \left(\sum_{l \in \mathbb{Z}} |\Psi_l * h|^2 \right)^{1/2} \right\|_{p_3} \leq \widetilde{C}(p_3) \|h\|_{p_3}, \tag{42}$$

for $1 < p_3 < \infty$. Now (38) follows by using (41) and (42) in (40). $\quad \square$

Proof of (39)**.** Let $r \in (1, \infty)$. It suffices to prove (39) for $h = a$ with a an r-atom associated to a cube Q. Let y_Q be the center of Q and Q^* be the double cube with same center. Let Q^{**} be the expanded cube with tenfold sidelength. Let L be such that the side length of Q is between 2^{-L} and 2^{-L+1}. We need to prove that

$$\left\| \sum_{k \in \mathbb{Z}} (\mathcal{K}_{i,k} * g)(\Psi_{k+n} * a) \right\|_1 \lesssim n \|g\|_\infty. \tag{43}$$

We split the sum in k in three parts, according to whether $k \geq L$, $L - n \leq k \leq L$ or $k \leq L - n$.

First let $k > L$. The support properties of a, Ψ_{k+n} and $\mathcal{K}_{k,i}$ show that $\Psi_{k+n} * a$ is supported in Q^* and that $\mathcal{K}_{k,i} * [g \mathbb{1}_{\mathbb{R}^d \setminus Q^{**}}](x) = 0$ for $x \in Q^*$. Hence

$$\Psi_{k+n} * a(x) \, \mathcal{K}_{i,k} * g(x) = \Psi_{k+n} * a(x) \, \mathcal{K}_{k,i} * [g \mathbb{1}_{Q^{**}}](x)$$

in this case. We choose $p_2, p_3 \in (1, \infty)$ such that $1/p_2 + 1/p_3 + 1/r = 1$, and $p_3 \leq r$; for example $p_2 = p_3 = r = 3$. Now use the already proven estimate (38) together with Hölder's inequality to get

$$\left\| \sum_{\substack{k \in \mathbb{Z} \\ k > L}} (\mathcal{K}_{i,k} * g)(\Psi_{k+n} * a) \right\|_1 \lesssim |Q^*|^{1/r} \left\| \sum_{\substack{k \in \mathbb{Z} \\ k \geq L}} (\mathcal{K}_{i,k} * [g \mathbb{1}_{Q^{**}}])(\Psi_{k+n} * a) \right\|_{r'}$$

$$\lesssim |Q^*|^{1/r} \|g \mathbb{1}_{Q^{**}}\|_{p_2} \|a\|_{p_3} \lesssim |Q^*|^{1/r} \|g\|_\infty |Q^{**}|^{1/p_2} |Q|^{1/p_3 - 1/r} \|a\|_r \lesssim \|g\|_\infty$$

since $\|a\|_r \leq |Q|^{-1+1/r}$.

Next for the case $L - n \leq k \leq L$ we use the straightforward bound

$$\|(\mathcal{K}_{i,k} * g)(\Psi_{k+n} * a)\|_1 \leq \|\mathcal{K}_{i,k} * g\|_\infty \|\Psi_{k+n} * a\|_1 \lesssim \|g\|_\infty \|a\|_1 \lesssim \|g\|_\infty$$

and then obtain

$$\left\| \sum_{\substack{k \in \mathbb{Z} \\ L-n \leq k \leq L}} (\mathcal{K}_{i,k} * g)(\Psi_{k+n} * a) \right\|_1 \lesssim n \|g\|_\infty.$$

Finally, if $k < L - n$ we use $\int a(x) dx = 0$ to get

$$\Psi_{k+n} * a = \int \left(\Psi_{k+n}(x - y) - \Psi_{k+n}(x - y_Q) \right) a(y) dy$$

and thus $\|\Psi_{k+n} * a\|_1 \lesssim 2^{k+n-L} \|a\|_1$. Hence

$$\left\| \sum_{\substack{k \in \mathbb{Z} \\ k < L-n}} (\mathcal{K}_{i,k} * g)(\Psi_{k+n} * a) \right\|_1 \leq \sum_{k < L-n} \|\mathcal{K}_{i,k} * g\|_\infty \|\Psi_{k+n} * a\|_1$$

$$\lesssim \|g\|_\infty \sum_{k \leq L-n} 2^{k+n-L} \|a\|_1 \lesssim \|g\|_\infty.$$

We combine the three cases and obtain (43). This completes the proof of Proposition 2.5. $\quad \square$

5. Additional remarks

5.1. On the result by Crippa and de Lellis

Corollary 1.2 can also be proved by a modification of the approach by Crippa and de Lellis. The elegant argument outlined in [9, §8] reduces matters to an estimate for vector fields $x \mapsto b(x)$, namely

$$\frac{|b(x) - b(y)|}{|x - y|} \leq \mathfrak{M}b(x) + \mathfrak{M}b(y) \tag{44}$$

where $\mathfrak{M}$ is a maximal operator to be determined, with

$$\|\mathfrak{M}b\|_{L^1} \lesssim \|\nabla b\|_{h_1}. \tag{45}$$

Assume that $|x - y| \le 10^{-2}$. Now let $\phi \in C_c^\infty$ supported on $\{y : |y| \le 1/4\}$ such that $\int \phi(y) \, dy = 1$, and $\int y_i \phi(y) dy = 0$ for $i = 1, \ldots, d$. Let $\phi_k(x) = 2^{kd}\phi(2^k x)$, and $\psi_k = \phi_k - \phi_{k-1}$ so that for any $\ell > 0$,

$$b = \phi_\ell * b + \sum_{k=\ell+1}^{\infty} \psi_k * b.$$

Now assume $2^{-\ell-1} \le |x - y| \le 2^{-\ell}$.

$$\frac{|\phi_\ell * b(x) - \phi_\ell * b(y)|}{|x - y|} = \left| \left\langle \frac{x - y}{|x - y|}, \int_0^1 \phi_\ell * \nabla b((1 - s)x + sy) \right\rangle ds \right|$$

$$\le \mathcal{M}_0(\nabla b)(x) + \mathcal{M}_0(\nabla b)(y)$$

where

$$\mathcal{M}_0 g(x) = \sup_{\ell > 4} \sup_{|h| \le 2^{-\ell}} |\phi_\ell * g(x + h)|.$$

By standard Hardy space theory,

$$\|\mathcal{M}_0 g\|_{L^1} \lesssim \|g\|_{h_1}$$

(which will be applied here to $g = \partial b_i / \partial x_j$).

Secondly, for $k \ge \ell$,

$$\frac{|\psi_k * b(x) - \psi_k * b(y)|}{|x - y|} \le 2^{\ell+2} \sup_k (|\psi_k * b(x)| + |\psi_k * b(y)|)$$

$$\le M_1 b(x) + M_1 b(y)$$

with

$$M_1 b(x) = \sup_{k > 0} 2^k |\psi_k * b(x)|.$$

Now, by the cancellation property of ψ, $\int \psi(y) l(y) dy = 0$ for all affine linear functions l, we have

$$\|M_1 b\|_1 \le \left\| \left(\sum_{k=1}^{\infty} 2^{2k} |\psi_k * b|^2 \right)^{1/2} \right\|_1 \lesssim \|\nabla b\|_{h^1};$$

in fact by definition of M_1 we have the better estimate in terms of the Triebel–Lizorkin $F_{1,\infty}^0$-norm of ∇b (cf. [20]). We have now proved (44) with $\mathfrak{M}b = \mathcal{M}_0(\nabla b) + M_1(b)$ and $\mathfrak{M}$ satisfies (45).

5.2. On Léger's result for transport equations

In a recent preprint Léger [13] considers solutions $\theta(t, x)$ of the initial value problem

$$\partial_t \theta + \operatorname{div}(v\theta) = 0$$

$$\theta(0, \cdot) = \theta_0$$

on $\mathbb{R}^d$; here v is a given divergence-free time-dependent vector field v on $[0, \infty) \times \mathbb{R}^d$. See also [14], [12] for related versions of the mixing problem. Léger introduces the functional

$$\mathcal{V}(f) = \int |\widehat{f}(\xi)|^2 \log |\xi| d\xi$$

which in physical space is computed to

$$c_1(d)\Big(\frac{1}{2} \iint\limits_{|x-y|\leq 1} \frac{|f(x)-f(y)|^2}{|x-y|^d} dx\, dy - \iint\limits_{|x-y|\geq 1} \frac{f(x)f(y)}{|x-y|^d}\Big) + c_2(d)\|f\|_{L^2}^2$$

for suitable constants $c_i(d)$. He then shows that

$$\partial_t \mathcal{V}(\theta(t,\cdot)) = c_d\, \mathfrak{S}[\theta(t,\cdot), \theta(t,\cdot), v(t,\cdot)] \tag{46}$$

with $\mathfrak{S}$ as in (9), (11). This is closely related to the computation in Proposition 2.1. Note that Léger's reduction to an estimate for $\mathfrak{S}$ works for arbitrary initial data θ_0 while Proposition 2.1 is limited to indicator functions of sets. Léger uses the results in [17] (cf. §2.2 above) to dominate, for $\theta(t,\cdot) \in L^\infty \cap L^{p'}$, the right hand side of (46) by $\|\theta(t,\cdot)\|_\infty \|\theta(t\cdot)\|_{p'} \|Dv(t,\cdot)\|_p$. Our estimate (12) yields the endpoint bound

$$|\partial_t \mathcal{V}(\theta(t,\cdot))| \leq C_d \|\theta(t,\cdot)\|_\infty^2 \|Dv(t,\cdot)\|_{H^1}. \tag{47}$$

This inequality can be used to extend other results in [13]. For example one obtains the inequality

$$\mathcal{V}(\theta(t,\cdot)) - \mathcal{V}(\theta_0) \lesssim \|\theta_0\|_\infty^2 \int_0^t \|Dv(s,\cdot)\|_{H^1} ds.$$

6. Failure of a singular integral estimate

Deviating slightly from our previous notation in (2) we now let $\Omega_L = (-1,0) \times (-1,1)$, $\Omega_R = (0,1) \times (-1,1)$. For a resolution of Bressan's problem on $\mathbb{T}^2$ it would be relevant if the inequality

$$\Big| \iint \frac{\langle x-y, b(x)-b(y)\rangle}{|x-y|^4} \chi_A(x)\chi_B(y)\, dx\, dy \Big| \leq C(A,B)\|Db\|_1 \tag{48}$$

held for subsets $A \subset \Omega_L$, $B \subset \Omega_R$ and divergence free vector fields b, with a constant independent of A and B. In particular we could consider regularized versions of

$$b(x) = \begin{cases} (0,1) \text{ for } x_1 < 0, \\ (0,-1) \text{ for } x_1 > 0. \end{cases}$$

Notice that

$$Db(x) = \begin{pmatrix} 0 & 0 \\ -2\delta(x_1) & 0 \end{pmatrix}$$

where δ is the Dirac measure in one dimension, and thus $\operatorname{div}(b) = 0$. For this choice of b the expression (48) becomes $|\mathcal{I}(A,B)|$ with

$$\mathcal{I}(A,B) = \iint\limits_{(x,y)\in A\times B} K_{|x_1-y_1|}(x_2-y_2)dx\, dy \tag{49a}$$

where

$$K_r(s) = \frac{s}{(r^2+s^2)^2} = -\frac{1}{2}\frac{d}{ds}\frac{1}{r^2+s^2}. \tag{49b}$$

We show that $\mathcal{I}(A,B)$ is *not* bounded independently of $A \subset \Omega_L$, $B \subset \Omega_R$. One gets a precise upper and lower bound in terms of some separation condition on A and B.

Proposition 6.1. *Let*

$$\mathcal{U}(\varepsilon) = \sup\big\{|\mathcal{I}(A,B)| : \, dist(A,B) \geq \varepsilon, \, A \subset \Omega_L, \, B \subset \Omega_R\big\}.$$

Then for $0 < \epsilon < 1/2$ we have

$$\mathcal{U}(\varepsilon) \approx \log(1/\varepsilon).$$

6.1. Upper bounds

Suppose g satisfies

$$\sup_s (1 + |s|)^{\delta+1}|g(s)| < \infty, \tag{50}$$

for some $\delta > 0$. Note that $K_r(s) = r^{-2}r^{-1}g(s/r)$ if we take $g(s) = s(1 + s^2)^{-2}$, and thus the following estimate gives the upper bound in the proposition.

Lemma 6.2. *Suppose $A \subset \Omega_L$, $B \subset \Omega_R$ and $dist(A, B) > \varepsilon$. Then, with g as in* (50),

$$\iint_{\Omega_L \times \Omega_R} |x_1 - y_1|^{-3}|g(\tfrac{x_2 - y_2}{|x_1 - y_1|})|\chi_B(y)\chi_A(x)dx\,dy \lesssim \log(1/\varepsilon).$$

Proof. Observe that for $x \in \Omega_L$, $y \in \Omega_R$ we have $|x_1 - y_1| = |x_1| + |y_1|$.

We consider separately the regions with (i) $|x_2 - y_2| \le |x_1 - y_1|$ (which for $x \in A$, $y \in B$ implies $|x_1 - y_1| \ge \varepsilon/2$) and (ii) $2^{m-1}|x_1 - y_1| \le |x_2 - y_2| < 2^m|x_1 - y_1|$ for some $m \ge 1$ (which for $x \in A$, $y \in B$ implies $|x_1 - y_1| \ge 2^{-m-2}\varepsilon$).

First,

$$\iint_{\substack{(x,y)\in\Omega_L \times \Omega_R \\ |x_2-y_2|\le|x_1-y_1|}} |x_1 - y_1|^{-3}|g(\tfrac{x_2 - y_2}{|x_1 - y_1|})|\chi_B(y)\chi_A(x)dx\,dy$$

$$\lesssim \iint_{\substack{(x_1,y_1)\in[-1,0]\times[0,1] \\ |x_1-y_1|\ge\varepsilon/2}} |x_1 - y_1|^{-2} \iint_{[-1,1]^2} \tfrac{1}{|x_1-y_1|}|g(\tfrac{x_2-y_2}{|x_1-y_1|})|dx_2dy_2\,dx_1dy_1$$

$$\lesssim \|g\|_{L^1(\mathbb{R})} \iint_{\varepsilon/2<|x_1|+|y_1|\le2} \frac{1}{(|x_1| + |y_1|)^2}dx_1dy_1 \lesssim \log(1/\varepsilon)$$

Next, when $|x_2 - y_2| \approx 2^m|x_1 - y_1|$ we have $|g(\tfrac{x_2-y_2}{|x_1-y_1|})| \lesssim 2^{-m(1+\delta)}$ and thus

$$\iint_{\substack{(x,y)\in\Omega_L \times \Omega_R \\ |x_2-y_2|\approx2^m|x_1-y_1|}} |x_1 - y_1|^{-3}|g(\tfrac{x_2 - y_2}{|x_1 - y_1|})|\chi_B(y)\chi_A(x)dx\,dy$$

$$\lesssim 2^{-m\delta} \iint_{\substack{(x_1,y_1)\in[-1,0]\times[0,1] \\ |x_1-y_1|\ge2^{-m-2}\varepsilon}} |x_1 - y_1|^{-2}dx_1dy_1$$

$$\lesssim 2^{-m\delta} \iint_{2^{-m-2}\varepsilon\le|x_1|+|y_1|\le2} \frac{1}{(|x_1| + |y_1|)^2}dx_1dy_1 \lesssim 2^{-m\delta}\log(2^m/\varepsilon).$$

Now sum in m to finish the proof. $\square$

6.2. Lower bounds

We now take $g(s) = \frac{s}{(1+s^2)^2}$ and construct a specific pair A, B for which $dist(A, B) \ge \varepsilon$ and $|\mathcal{I}(A, B)| \gtrsim \log(1/\varepsilon)$. It suffices to take $\varepsilon = 2^{-LM}$ for some integer L (and M be a sufficiently large fixed integer, $M > 10$).

Define

$$I_k^L = [-2^{-kM}, -2^{-kM-1}],$$
$$I_k^R = [2^{-kM-1}, 2^{-kM}],$$

$$J_{k,n}^L = [(Mn + 2)2^{-kM}, (Mn + 3)2^{-kM}],$$
$$J_{k,n}^R = [Mn2^{-kM}, (Mn + 1)2^{-kM}]$$

and

$$A = \bigcup_{1 \le k \le L-1} \bigcup_{0 \le n \le \frac{2^{kM}}{M+1}} I_k^L \times J_{k,n}^L,$$

$$B = \bigcup_{1 \le k \le L-1} \bigcup_{0 \le n \le \frac{2^{kM}}{M+1}} I_k^R \times J_{k,n}^R.$$

Observe that B is a vertical translation of the horizontal reflection of A and that both sets consist of columns of squares at $L - 1$ many different scales. Clearly $A \in \Omega_L$, $B \in \Omega_R$ and $dist(A, B) \ge 2^{-LM}$.

Let

$$\mathcal{I}(k_L, k_R, n_L, n_R) = \int_{x \in I_{k_L}^L \times J_{k_L,n_L}^L} \int_{y \in I_{k_R}^R \times J_{k_R,n_R}^L} K_{|x_1-y_1|}(x_2 - y_2) \, dy \, dx$$

and split $\mathcal{I}(A, B) = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$ where

$$\mathcal{E}_1 = \sum_{1 \le k \le L-1} \sum_{0 \le n \le \frac{2^{kM}}{M+1}} \mathcal{I}(k, k, n, n)$$

$$\mathcal{E}_2 = \sum_{1 \le k \le L-1} \sum_{\substack{0 \le n_L, n_R \le \frac{2^{kM}}{M+1} \\ n_L \ne n_R}} \mathcal{I}(k, k, n_L, n_R)$$

$$\mathcal{E}_3 = \sum_{\substack{1 \le k_L, k_R \le L-1 \\ k_L \ne k_R}} \sum_{0 \le n_L, n_R \le \frac{2^{kM}}{M+1}} \mathcal{I}(k_L, k_R, n_L, n_R).$$

We prove a lower bound for $\mathcal{E}_1$ and upper bounds for $\mathcal{E}_2, \mathcal{E}_3$.

For the lower bound observe

$$2^{-kM} \le x_2 - y_2 \le 2^{2-kM} \text{ for } x_2 \in J_{k,n}^L, \, y_2 \in J_{k,n}^L.$$

Thus

$$\int_{I_k^L \times J_{k,n}^L} \int_{I_k^R \times J_{k,n}^L} K_{|x_1-y_1|}(x_2 - y_2) dx \, dy$$

$$= \iiiint_{\substack{(x_1,y_1) \in [2^{-kM-1}, 2^{-kM}] \\ (Mn+2)2^{-kM} \le x_2 \le (Mn+3)2^{-kM} \\ Mn2^{-kM} \le y_2 \le (Mn+1)2^{-kM}}} \frac{x_2 - y_2}{((x_1 + y_1)^2 + (x_2 - y_2)^2)^2} dy_2 \, dx_2 \, dy_1 \, dx_1$$

$$\ge \frac{2^{-kM}}{1000}$$

and thus

$$\mathcal{E}_1 \ge 10^{-3} \sum_{k=1}^{L-1} \sum_{0 \le n_L \le \frac{2^{kM}}{M+1}} 2^{-kM} \ge \frac{L-1}{10^3(M+1)}.$$

If $n_L \ne n_R$ we have

$$\int_{I_k^L \times J_{k,n_L}^L} \int_{I_k^R \times J_{k,n_R}^L} |K_{|x_1-y_1|}(x_2 - y_2)| dx \, dy \lesssim \frac{2^{-kM}}{M^3 |n_L - n_R|^3}$$

and thus

$$|\mathcal{E}_2| \leq \sum_{1 \leq k < L} \sum_{0 \leq n_L \leq \frac{2kM}{M+1}} \sum_{n_R \neq n_L} \frac{2^{-kM}}{M^3 |n_L - n_R|^3} \leq C \frac{L}{M^4}.$$

Next, set $g(s) = |s|(1 + s^2)^{-2}$, and

$$G(x_1, y_1) = \iint\limits_{-1 \leq x_2, y_2 \leq 1} \frac{1}{|x_1 - y_1|} |g\left(\frac{|x_2 - y_2|}{|x_1 - y_1|}\right)| dy_2 dx_2$$

so that $G(x_1, y_1)$ is nonnegative and uniformly bounded. We have $|\mathcal{E}_3| \leq \mathcal{E}_{3,1} + \mathcal{E}_{3,2}$ where

$$\mathcal{E}_{3,1} = \sum_{1 \leq k_L < k_R \leq L-1} \iint\limits_{I^L_{k_L} \times I^R_{k,R}} |x_1 - y_1|^{-2} G(x_1, y_1) \, dy_1 dx_1$$

and $\mathcal{E}_{3,2}$ is the corresponding term with the (k_L, k_R) summation extended over $1 \leq k_R < k_L \leq L - 1$. The two terms are symmetric and it suffices to estimate $\mathcal{E}_{3,1}$.

Now $|x_1 - y_1| \approx 2^{-k_L M}$ if $x_1 \in I^L_{k_L}$, $y_1 \in I^L_{k_R}$, and $k_L < k_R$. Therefore

$$\mathcal{E}_{3,1} \lesssim \sum_{1 \leq k_L < k_R \leq L-1} 2^{2k_L M} |I^L_{k_L}| |I^R_{k,R}| \lesssim \sum_{1 \leq k_L < k_R \leq L-1} 2^{(k_L - k_R)M} \lesssim L 2^{-M}$$

and similarly we also get $\mathcal{E}_{3,2} \lesssim L 2^{-M}$. Combining the estimates we get

$$\mathcal{U}(2^{-LM}) \geq \mathcal{E}_1 - |\mathcal{E}_2| - |\mathcal{E}_3| \geq 10^{-3} \frac{L-1}{M+1} - C_1 L M^{-3} - C_2 L 2^{-M}$$

and the assertion follows by choosing M sufficiently large. $\square$

6.3. A discrete problem

The counterexample suggests that to make progress towards the resolution of the L^1-conjecture, we need to first understand the effects of shear flows such as the vector field b above. To highlight this particular difficulty, we propose a simple discrete problem reminiscent of the Rubik's cube.

We mix the discrete torus $\Omega_n = \mathbb{Z}^2 / 2n\mathbb{Z}^2$ by applying a sequence of sliding moves. The goal is to transform the initial set

$$A_0 = [1, n] \times [1, 2n] + 2n\mathbb{Z}^2$$

into the final set

$$A_1 = \{(x, y) \in \mathbb{Z}^2 : (-1)^{x+y} = 1\}.$$

For integers $0 < b - a < 2n$, consider the periodic strips $S \subseteq \mathbb{Z}^2$ given by

$$S = \mathbb{Z} \times ([a, b] + 2n\mathbb{Z})$$

and the permutation $P : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by

$$P(x, y) = (x, y) + (1, 0) 1_S(x, y).$$

Such permutations, when composed with an arbitrary number of 90° rotations, are the allowed sliding moves.

For this simplified problem, a positive answer to the Bressan's mixing conjecture would imply that it takes at least $cn \log n$ sliding moves to transform A_0 into A_1. It is clear from looking at the Cayley graph of the group generated by the finite set of sliding moves, that the diameter of the set of reachable configurations is much larger than $n \log n$. However, Bressan's conjecture in this context is a statement about the minimal distance between two particular configurations A_0 and A_1.

7. A toy problem on $\mathbb{T}^2$

Consider the problem of mixing $\mathbb{T}^2$ by a finite sequence of $90°$ rotations of squares. Given $x \in \mathbb{T}^2$ and $r \in (0, 1/4)$, let $R_{x,r} : \mathbb{T}^2 \to \mathbb{T}^2$ be the map which rotates the square $(x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r)$ by $90°$ counter-clockwise:

$$R_{x,r}(y) := \begin{cases} (x_1 + x_2 - y_2, x_2 - x_1 + y_1) & \text{if } y - x \in (-r, r)^2, \\ y & \text{otherwise.} \end{cases}$$

We assign the *cost* r^2 to the rotation $R_{x,r}$. To motivate this definition observe that we can write $R_{0,r}(x) = X_r(1, x)$ where $X_r : [0, 1] \times \mathbb{T}^2 \to \mathbb{T}^2$ is the incompressible flow that satisfies

$$D_t X_r(t, x) = \begin{cases} (0, 2x_1) & \text{if } |x_2| < |x_1| < r, \\ (-2x_2, 0) & \text{if } |x_1| < |x_2| < r, \\ (0, 0) & \text{otherwise,} \end{cases}$$

in the coordinates $(-1/2, 1/2)^2$ for the torus $\mathbb{T}^2$. The vector field $D_t X_r(t, \cdot)$ is the weakly divergence free square vortex:
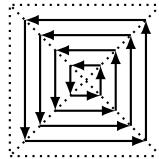
Let $M(\mathbb{T}^d)$ be the space of Borel measures on $\mathbb{T}^2$. Since

$$\int_0^1 \|D_x D_t X_r(t, x)\|_{M(\mathbb{T}^2)} dt = Cr^2$$

our choice for the cost is natural. The following result can therefore be considered to solve a discrete toy version of Bressan's conjecture.

Theorem 7.1. *If $R_{x_1, r_1} \circ \cdots \circ R_{x_n, r_n}(0, 1/2)^2$ is mixed to scale $\varepsilon \in (0, 1/2)$, then*
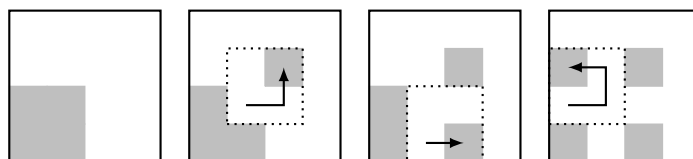
$$\sum_{i=1}^n r_i^2 \geq C^{-1} \log \varepsilon^{-1}, \tag{51}$$

with a universal constant $C > 0$.

To see the sharpness of the result consider the composition

$$R^3_{(\frac{1}{4}, \frac{1}{2}), \frac{1}{4}} \circ R_{(\frac{1}{2}, \frac{1}{4}), \frac{1}{4}} \circ R^2_{(\frac{1}{2}, \frac{1}{2}), \frac{1}{4}}$$

which divides $(0, 1/2)^2$ into four smaller squares, at cost $6r^2$:

Applying this idea recursively, we see that we can mix to scale 2^{-n} at cost Cnr^2.

Proof of Theorem 7.1. We use the Bianchini semi-norm defined in §1.3.

Lemma 7.2. *If $u : \mathbb{T}^d \to \mathbb{T}^d$ is measure preserving, $A \subseteq \mathbb{T}^d$, and $\|\mathbb{1}_A\|_{\mathcal{B}}$ is finite, then*

$$\|\mathbb{1}_{u(A)}\|_{\mathcal{B}} - \|\mathbb{1}_A\|_{\mathcal{B}} \leq \int\limits_0^{1/4} \frac{1}{r|B_r(0)|} \int\limits_{\mathbb{T}^d} |u(B_r(x)) \triangle B_r(u(x))| \, dx \, dr. \tag{52}$$

Proof. We compute $\|\mathbb{1}_{u(A)}\|_{\mathcal{B}} - \|\mathbb{1}_A\|_{\mathcal{B}}$ as

$$\int\limits_0^{1/4} \frac{1}{r} \int\limits_{\mathbb{T}^d} \left| \chi_{u(A)}(x) - \fint\limits_{B_r(x)} \chi_{u(A)}(y) \, dy \right| dx \, dr$$

$$- \int\limits_0^{1/4} \frac{1}{r} \int\limits_{\mathbb{T}^d} \left| \chi_A(x) - \fint\limits_{B_r(x)} \chi_A(y) \, dy \right| dx \, dr$$

$$= \int\limits_0^{1/4} \frac{1}{r} \int\limits_{\mathbb{T}^d} \left| \chi_A(x) - \fint\limits_{u^{-1}(B_r(u(x)))} \chi_A(y) \, dy \right| dx \, dr$$

$$- \int\limits_0^{1/4} \frac{1}{r} \int\limits_{\mathbb{T}^d} \left| \chi_A(x) - \fint\limits_{B_r(x)} \chi_A(y) \, dy \right| dx \, dr$$

$$\leq \int\limits_0^{1/4} \frac{1}{r} \int\limits_{\mathbb{T}^d} \left| \fint\limits_{u^{-1}(B_r(u(x)))} \chi_A(y) \, dy - \fint\limits_{B_r(x)} \chi_A(y) \, dy \right| dx \, dr$$

$$\leq \int\limits_0^{1/4} \frac{1}{r|B_r(0)|} \int\limits_{\mathbb{T}^d} |u(B_r(x)) \triangle B_r(u(x))| \, dx \, dr,$$

using the fact that u is measure preserving to change variables. □

Lemma 7.3. *There is a constant $C > 0$ such that*

$$\int\limits_0^{1/4} \frac{1}{r|B_r(0)|} \int\limits_{\mathbb{T}^2} \left| R_{x,s}(B_r(y)) \triangle B_r(R_{x,s}(y)) \right| dy \, dr \leq C s^2, \tag{53}$$

for all $x \in \mathbb{T}^2$ and $s \in (0, 1/4)$.

Proof. By scaling, observe that

$$\int\limits_0^s \frac{1}{r|B_r(0)|} \int\limits_{\mathbb{T}^2} \left| R_{x,s}(B_r(y)) \triangle B_r(R_{x,s}(y)) \right| dy \, dr$$

$$\leq s^2 \int\limits_0^{1/4} \frac{1}{r|B_r(0)|} \int\limits_{\mathbb{T}^2} \left| R_{x,1/4}(B_r(y)) \triangle B_r(R_{x,1/4}(y)) \right| dy \, dr = C s^2.$$

Next, observe that if $r \geq s$ and

$$\left| R_{x,s}(B_r(y)) \triangle B_r(R_{x,s}(y)) \right| > 0,$$

then either

$$\left| R_{x,s}(B_r(y)) \triangle B_r(R_{x,s}(y)) \right| \leq Cs^2 \quad \text{and} \quad r - \sqrt{2}s \leq |y - x| \leq r + \sqrt{2}s,$$

or

$$\left| R_{x,s}(B_r(y)) \triangle B_r(R_{x,s}(y)) \right| \leq Csr \quad \text{and} \quad |y - x| < \sqrt{2}s.$$

In particular, we may estimate

$$\int\limits_s^{1/4} \frac{1}{r|B_r(0)|} \int\limits_{\mathbb{T}^2} \left| R_{x,s}(B_r(y)) \triangle B_r(R_{x,s}(y)) \right| dy \, dr$$

$$\leq \int\limits_s^{1/4} \frac{1}{r|B_r(0)|} Cs^3 r \, dr \leq Cs^2.$$

Putting these two estimates together gives (53). $\quad\square$

Proof of Theorem 7.1, conclusion. If A is mixed to scale $\varepsilon \in (0, \kappa)$, with mixing constant κ then the average of $\mathbb{1}_A$ over $B_r(x)$ lies between $\kappa |A|$ and $(1 - \kappa)|A|$ when $r \geq \varepsilon$. Thus

$$\|\mathbb{1}_A\|_{\mathcal{B}} \geq \kappa \int\limits_\varepsilon^\kappa \frac{1}{r} \min\{|A|, (1 - |A|)\} \geq \frac{1}{C} \min\{|A|, 1 - |A|\} \log \varepsilon^{-1}.$$

Combine this with (52), and (53) to conclude the proof. $\quad\square$

Remark. This L^1-type Bressan result for the toy problem is possible since the natural scale s for the rotation $R_{y,s}$ is linked in the proof with the scale r in the Bianchini semi-norm, with maximal contributions for $r \approx s$.

Conflict of interest statement

There is no conflict of interest.

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