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Ann. I. H. Poincaré – AN 35 (2018) 1119–1142

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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Well-posedness of the linearized Prandtl equation around a non-monotonic shear flow

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Received 5 October 2016; received in revised form 3 October 2017; accepted 7 November 2017
Available online 14 November 2017

Abstract

In this paper, we prove the well-posedness of the linearized Prandtl equation around a non-monotonic shear flow in Gevrey class $2-\theta$ for any $\theta > 0$. This result is almost optimal by the ill-posedness result proved by Gérard-Varet and Dormy, who construct a class of solution with the growth like $e^{\sqrt{kt}}$ for the linearized Prandtl equation around a non-monotonic shear flow.

Keywords: Prandtl equation; Gevrey class; Well-posedness

1. Introduction

In this paper, we study the Prandtl equation in $\mathbf{R}_+ \times \mathbf{R}^2_+$

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x p = 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0 \quad \text{and} \quad \lim\limits_{y\to+\infty} u(t,x,y) = U(t,x), \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where (u, v) denotes the tangential and normal velocity of the boundary layer flow, and $(U(t,x), p(t,x))$ are the values on the boundary of the tangential velocity and pressure of the outflow, which satisfies the Bernoulli's law

$$\partial_t U + U\partial_x U + \partial_x p = 0.$$

This system introduced by Prandtl [12] is the foundation of the boundary layer theory. It describes the first order approximation of the velocity field near the boundary in the zero viscosity limit of the Navier–Stokes equations with non-slip boundary condition. One may check [11] for an expanded introductions to the boundary layer theory.

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https://doi.org/10.1016/j.anihpc.2017.11.001
0294-1449/

To justify the zero viscosity limit, one of key step is to deal with the well-posedness of the Prandtl equation. Due to the lack of horizontal diffusion in (1.1), the nonlinear term $v\partial_y u$ will lead to one horizontal derivative loss in the process of energy estimate. Up to now, the question of whether the Prandtl equation with general data is well-posed in Sobolev spaces is still open except for some special cases:

• Under a monotonic assumption on the initial tangential velocity, Oleinik [11] proved the local existence and uniqueness of classical solutions to (1.1). With the additional favorable condition on the pressure, Xin and Zhang [14] obtained the global existence of weak solutions to (1.1).

• For the data which is analytic in x, y variables, Sammartino and Caflisch [13] established the local well-posedness of (1.1). Later, the analyticity in y variable was removed by Lombardo, Cannone and Sammartino [9]. Zhang and the third author [15] also established the long time well-posedness of (1.1) for small tangential analytic data.

Recently, Alexandre et al. [1] and Masmoudi and Wong [10] independently developed direct energy method to prove the well-posedness of the Prandtl equation for monotonic data in Sobolev spaces. Their works might shed some light on the zero viscosity limit problem in Sobolev spaces. See also [7] for the case with multiple monotonicity regions. Recently, we also present an elementary proof by using the paralinearized technique [2].

On the other hand, Gérard-Varet and Dormy [3] proved the ill-posedness in Sobolev spaces for the linearized Prandtl equation around non-monotonic shear flows. The nonlinear ill-posedness was also established in [5,6] in the sense of non-Lipschitz continuity of the flow. However, Gérard-Varet and Masmoudi [4] can prove the well-posedness of the Prandtl equation (1.1) for a class of data in Gevrey class $\frac{7}{4}$. In [4], the authors conjectured that their result should not be optimal. The analysis and numerics performed in [3] suggest that the optimal exponent may be $s = 2$. Indeed, Gérard-Varet and Dormy constructed a class of solution with the growth like $e^{\sqrt{k}t}$ for the linearized Prandtl equation around a non-monotonic shear flow, where k is the tangential frequency.

The goal of this paper is to prove the well-posedness of the linearized Prandtl equation around a non-monotonic shear flow in Gevrey class $2 - \theta$ for any $\theta > 0$. This result is almost optimal and in particular implies that the instability mechanism found in [3] should be severe. The same ideas can be applied to deal with nonlinear Prandtl equation. However, the proof is more involved technically. So, this will be presented in a separate paper in order to present our ideas more clearly here.

Let $u^s(t, y)$ be the solution of the heat equation

$$\begin{cases} \partial_t u^s - \partial_y^2 u^s = 0, \\ u^s|_{y=0} = 0 \quad \text{and} \quad \lim\limits_{y\to+\infty} u^s(t, y) = 1, \\ u^s|_{t=0} = u_0^s(y). \end{cases} \tag{1.2}$$

Obviously, $(u^s(t, y), 0)$ is a shear flow solution of the Prandtl equation (1.1). Let us assume that $\partial_y u_0(1) = 0$ and for some $c, \delta > 0$,

$$\begin{aligned} &\partial_y^2 u_0^s(y) \geq c \quad \text{for} \quad y \in \Big[\frac{1}{2}, 2\Big], \\ &\big|\partial_y u_0^s(y)\big| \geq c\delta e^{-y} \quad \text{for} \quad y \in [0, 1-\delta] \cup \big[1+\delta, +\infty). \end{aligned} \tag{1.3}$$

The linearized Prandtl equation around $(u^s, 0)$ takes as follows

$$\begin{cases} \partial_t u + u^s\partial_x u + v\partial_y u^s - \partial_y^2 u = 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0 \quad \text{and} \quad \lim\limits_{y\to+\infty} u(t, x, y) = 0, \\ u(0, x, y) = u_0(x, y). \end{cases} \tag{1.4}$$

The main result of this paper is stated as follows.

Theorem 1.1. *Let $\theta \in (0, \frac{1}{2}]$. Assume that $e^{\langle D_x\rangle^{\frac{1}{2}+2\theta}} u_0 \in H_\mu^{\frac{1}{2},1}$ with $\partial_y^k u_0|_{y=0} = 0$ for $k = 0, 2$. Then there exists $T > 0$ so that (1.4) has a unique solution u in $[0, T]$, which satisfies (7.2). In particular, we have*

$$u_\Phi \in L^\infty(0, T; H_\mu^{\frac{1}{4}+\theta,1}).$$

Here we denote

$$f_\Phi \triangleq \mathcal{F}^{-1}\big(e^{\Phi(t,\xi)}\widehat{f}(\xi)\big), \quad \Phi(t,\xi) \triangleq (1-\lambda t)\langle\xi\rangle^{\frac{1}{2}+2\theta},$$

and $H^{s,\sigma}_\mu$ is the weighted Sobolev space with $\mu = e^{\frac{y}{2}}$ which will be introduced later.

Remark 1.1. Li and Yang [8] proved the well-posedness of nonlinear Prandtl equation in Gevrey class 2 for data with non-degenerate critical point and polynomial decay in y. They used Gérard-Varet and Masmoudi's framework with an introduction of a new unknown $h_1 = \partial_y^2 u - \frac{\partial_y^3 u^s}{\partial_y^2 u^s}\partial_y u$, which is used to control the regularity of $\partial_y^2 u$. In the last section, we will explain how to obtain the well-posedness of (1.4) in Gevrey class 2 by using our framework and h_1. The two methods should be helpful to understand the complex structure of the Prandtl equation and provide evidence about the conjecture that the well-posedness in Gevrey class 2 is optimal.

Let us present some key ingredients of our proof.

1. Gevrey regularity estimate in monotonic domain. Motivated by [1], we will introduce the good unknown $w_1 = \partial_y\big(\frac{u}{\partial_y u^s}\big)$ to control the horizontal regularity of the solution in this domain, which satisfies

$$\partial_t w_1 + u^s\partial_x w_1 - \partial_y^2 w_1 = \partial_y F_1.$$

The key point is that the equation of w_1 does not lose the derivative.

2. Gevrey regularity estimate in non-monotonic domain. Because w_1 does not make sense in non-monotonic domain, motivated by [4], we introduce $h = d\partial_y u$, $d = (\partial_y^2 u^s)^{-\frac{1}{2}}$ to control the horizontal regularity of the solution in this domain, which satisfies

$$\partial_t h + u^s\partial_x h - \partial_y^2 h + d(v\partial_y^2 u^s) = (\partial_t d - \partial_y^2 d)\partial_y u - 2\partial_y d\partial_y^2 u.$$

All the terms in this equation are good except $d(v\partial_y^2 u^s)$. The key point is

$$\int_{\mathbf{R}^2_+} d(v\partial_y^2 u^s)h\,dx\,dy = 0. \tag{1.5}$$

So, this term is also good in the energy estimate. However, the localization in y variable will destroy the cancellation structure (1.5). In particular, the energy estimate in non-monotonic domain will give rise to a new trouble term

$$\big(\phi_3(y)\phi_3'(y)v, u\big)_{L^2}, \tag{1.6}$$

which can be reduced to control the terms like $(w_i, \partial_x u)_{L^2}, i = 1, 2$ modulus some lower order terms. Here $w_2 = \partial_y u^s\partial_y u - \partial_y^2 u^s u$ and $\phi_3(y)$ is a cut-off function supported in non-monotonic domain. To control them, we need to use the Gevrey regularity and the following.

3. Anisotropic regularity estimates. The unknowns w_i and h have to work in the functional spaces with different horizontal regularity. Roughly speaking,

$$\phi_3(y)h_\Phi \in L^2\big(0, T; H^{\frac{1}{4},0}\big), \quad \phi_1(y)(w_1)_\Phi, \phi_3(y)(w_2)_\Phi \in L^2(0, T; H^{\frac{3}{4},0}).$$

Here $\phi_1(y)$ is a cut-off function supported in monotonic domain.

4. The derivative gain of w_1 can be easily obtained by using Gevrey regularity and good structure of w_1. The unknown $w_2 = \partial_y u^s\partial_y u - \partial_y^2 u^s u$ satisfies an equation similar to w_1, but with a key trouble term in the $H^{\frac{1}{2},0}$ energy estimate, which takes

$$\big(\phi_3(y)\partial_y^2 u^s\partial_y^2 u_\Phi, \phi_3(y)(w_2)_\Phi\big)_{H^{\frac{1}{2},0}}. \tag{1.7}$$

The main difficulty is that one can not deduce $\phi_3(y)\partial_y^2 u_\Phi \in L^2(0, T; H^{\frac{1}{4},0})$ from $\phi_3(y)h_\Phi \in L^2\big(0, T; H^{\frac{1}{4},0}\big)$. However, one can prove $\phi_3(y)(w_2)_\Phi \in L^2(0, T; H^{\frac{5}{8},0})$ by using some key structures found in [4]. As we said above, this estimate is not enough to handle (1.6). On the other hand, one can prove the same regularity as w_1 in the framework of Gevrey class $\frac{7}{4}$. This may be the main reason why the work [4] can achieve the well-posedness in Gevrey class $\frac{7}{4}$.

5. Improved regularity estimate of w_2. Compared with w_1, w_2 lose $\frac{1}{8}$-order derivative. The reason is that following the argument in [4] of integration by parts to (1.7) gives rise to a boundary term (see A_5 in Lemma 5.2) at the critical point. To control it, one need to use Gevrey class $\frac{7}{4}$ regularity. However, we find that $\varphi^{1+\theta_1}\langle D_x\rangle^{\frac{3}{4}+\theta}(w_2)_\Phi \in L^2(0,T;L^2)$ if φ is a cut-off function vanishing at critical point. Compared with the work [4], this weighted estimate is completely new, and moreover is enough to handle (1.6). The price to pay is to use Gevrey $2-\theta$ regularity.
6. In our framework, if we use the unknown h_1, we can easily deduce $\phi_3(y)\partial_y^2 u_\Phi \in L^2(0,T;H^{\frac{1}{4},0})$, thus $\phi_3(y)(w_2)_\Phi \in L^2(0,T;H^{\frac{3}{4},0})$ and avoid the Gevrey regularity loss. This will be explained in the last section.

Let us conclude the introduction with the following notations. Let $\omega(y)$ be a nonnegative function in $\mathbf{R}^+$. We introduce the weighted L^p norm

$$\|f\|_{L^p_\omega} \stackrel{\text{def}}{=} \|\omega(y)f(x,y)\|_{L^p}, \quad \|f\|_{L^p_{y,\omega}} \stackrel{\text{def}}{=} \|\omega(y)f(y)\|_{L^p}.$$

The weighted anisotropic Sobolev space $H^{s,\ell}_\omega$ for $s=k+\sigma$ and $k,\ell\in\mathbf{N}, \sigma\in[0,1)$ consists of all functions $f\in L^2_\omega$ satisfying

$$\|f\|^2_{H^{s,\ell}_\omega} \stackrel{\text{def}}{=} \sum_{\alpha\le k}\sum_{\beta\le\ell}\|\partial_x^\alpha\langle D_x\rangle^\sigma\partial_y^\beta f\|^2_{L^2_\omega} < +\infty.$$

We denote by $H^\ell_{y,\omega}$ the weighted Sobolev space in $\mathbf{R}_+$, which consists of all functions $f\in L^2_{y,\omega}$ satisfying

$$\|f\|^2_{H^\ell_{y,\omega}} \stackrel{\text{def}}{=} \sum_{\beta\le\ell}\|\partial_y^\beta f\|^2_{L^2_{y,\omega}} < +\infty.$$

In the case when $\omega=1$, we denote $H^{k,\ell}_\omega$ by $H^{k,\ell}$, and $H^\ell_{y,\omega}$ by H^ℓ_y for the simplicity.

2. Basic estimates for the shear flow

Let $u^s(t,y)$ be the solution of the heat equation

$$\begin{cases} \partial_t u^s - \partial_y^2 u^s = 0, \\ u^s|_{y=0} = 0 \quad \text{and} \quad \lim\limits_{y\to+\infty} u^s(t,y) = 1, \\ u^s|_{t=0} = u^s_0(y). \end{cases} \tag{2.1}$$

Proposition 2.1. *Assume that $\partial_y u^s_0 \in H^3_{y,\mu}$ with $\mu = e^{\frac{y}{2}}$ and $u^s_0(0)=0, \partial_y^2 u^s_0(0)=0$. Then it holds that for any $t\in[0,+\infty)$,*

$$E^s(t) \stackrel{\text{def}}{=} \|\partial_y u^s(t)\|^2_{H^3_{y,\mu}} + \int_0^t \|\partial_y u^s(\tau)\|^2_{H^4_{y,\mu}}\,d\tau \le \|\partial_y u^s_0\|^2_{H^3_{y,\mu}} e^{Ct}.$$

Moreover, if for $k=0,1,2,3$,

$$|\partial_y^k\big(u^s_0-1\big)(y)| \le c^{-1}e^{-y} \qquad \text{for} \quad y\in[0,+\infty),$$

then we have

$$|\partial_y^k(u^s(t,y)-1)| \le Ce^{-y},$$

for $(t,y)\in[0,1]\times[0,+\infty)$.

Proof. Taking $L^2_{y,\mu}$ inner product between the first equation of (2.1) and u^s_t, we obtain

$$\frac{d}{d}t\|\partial_y u^s\|^2_{L^2_{y,\mu}} + \|u^s_t\|^2_{L^2_{y,\mu}} \le C\|\partial_y u^s\|^2_{L^2_{y,\mu}}.$$

Taking the time derivative to the first equation of (2.1), then taking $L^2_{y,\mu}$ inner product between the resulting equation and u^s_t, we get

$$\frac{d}{d}t\|u^s_t\|^2_{L^2_{y,\mu}} + \|\partial_y u^s_t\|^2_{L^2_{y,\mu}} \le C\|u^s_t\|^2_{L^2_{y,\mu}}.$$

And taking $L^2_{y,\mu}$ inner product between the resulting equation and $\partial_y^2 u^s_t$, we get

$$\frac{d}{dt}\|\partial_y\partial_t u^s\|_{L^2_{y,\mu}} + \|\partial_y^2\partial_t u^s\|^2_{L^2_{y,\mu}} \le C\|\partial_y\partial_t u^s\|^2_{L^2_{y,\mu}} + \frac{1}{2}\|\partial_t^2 u^s\|^2_{L^2_{y,\mu}}.$$

Taking the $\partial_t\partial_y$ to the first equation of (2.1), then taking $L^2_{y,\mu}$ inner product between the resulting equation and $\partial_t\partial_y^3 u^s$, we deduce that

$$\frac{d}{dt}\|\partial_y^2\partial_t u^s\|_{L^2_{y,\mu}} + \|\partial_t\partial_y^3 u^s\|^2_{L^2_{y,\mu}} \le C\|\partial_y^2\partial_t u^s\|^2_{L^2_{y,\mu}} + \frac{1}{2}\|\partial_y\partial_t^2 u^s\|^2_{L^2_{y,\mu}}.$$

Using $\partial_t u^s = \partial_y^2 u^s$, we deduce from Gronwall's inequality that

$$\|\partial_y u^s(t)\|^2_{L^2_{y,\mu}} + \|u^s_t(t)\|^2_{L^2_{y,\mu}} + \|\partial_y\partial_t u^s\|_{L^2_{y,\mu}} + \|\partial_y^2\partial_t u^s\|^2_{L^2_{y,\mu}} \le \|\partial_y u^s_0\|^2_{H^3_{y,\mu}} e^{Ct},$$

from which and $\partial_t u^s = \partial_y^2 u^s$, it follows that

$$E^s(t) \le \|\partial_y u^s_0\|^2_{H^3_{y,\mu}} e^{Ct}.$$

For the pointwise estimates, we need to use the representation formula of the solution

$$u^s(t,y) = \frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-y')^2}{4t}} - e^{-\frac{(y+y')^2}{4t}}\Big)u^s_0(y')dy'.$$

We write

$$\begin{aligned}
u^s(t,y) - 1 &= -\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{(y+y')^2}{4t}} u^s_0(y')dy' + \frac{1}{2\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{(y-y')^2}{4t}}(u^s_0(y') - 1)dy' \\
&\quad + \Big(\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{(y-y')^2}{4t}}dy' - 1\Big) \\
&\triangleq I_1 + I_2 + I_3.
\end{aligned}$$

The result is obvious for $|y| \le 4$. So, we assume $y \ge 4 \ge 4t$. Thanks to $|u^s_0(y)| \le C$, it follows that

$$\begin{aligned}
|I_1| &\le \frac{1}{2\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{y^2}{4t}}\, e^{-\frac{2yy'+(y')^2}{4t}}|u^s_0(y')|dy' \\
&\le \frac{1}{2\sqrt{\pi t}}e^{-\frac{y^2}{4t}}\int_0^{+\infty} e^{-\frac{(y')^2}{4t}}|u^s_0(y')|dy' \\
&\le Ce^{-y}.
\end{aligned}$$

Thanks to $|u_0^s(y)-1|\le c^{-1}e^{-y}$, we infer that

$$
\begin{aligned}
|I_2| &\le e^{-y}\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{(y-y')^2}{4t}}e^{y-y'}\,|u_0^s(y')-1|\,e^{y'}dy'\\
&\le Ce^{-y}\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty} e^{-\frac{y'^2}{4t}}e^{y'}dy'\\
&\le Ce^{-y}e^t\int_0^{+\infty} e^{-(\xi-\sqrt{t})^2}d\xi\\
&\le Ce^{-y}.
\end{aligned}
$$

For I_3, we have

$$
|I_3|\le\Big|\frac{1}{\sqrt{\pi}}\int_{-\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2}d\xi-1\Big|=\frac{1}{\sqrt{\pi}}\int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi^2}d\xi.
$$

If $2\sqrt{t}\le 1$, then

$$
|I_3|\le C\int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-\xi}d\xi\le Ce^{-\frac{y}{2\sqrt{t}}}\le Ce^{-y},
$$

and if $2\sqrt{t}\ge 1$ and $y\ge 4t$, then

$$
|I_3|\le C\int_{\frac{y}{2\sqrt{t}}}^{+\infty} e^{-2\sqrt{t}\xi}d\xi\le\frac{C}{2\sqrt{t}}e^{-y}\le Ce^{-y}.
$$

Putting the estimates of I_1-I_3 together, we deduce that

$$
|u^s(t,y)-1|\le Ce^{-y}.
$$

Thanks to $u_0^s(0)=0$ and $\partial_y^2u_0^s(0)=0$, we get by integration by parts that

$$
\begin{aligned}
\partial_y u^s(t,y)&=\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-y')^2}{4t}}+e^{-\frac{(y+y')^2}{4t}}\Big)\partial_y u_0^s(y')dy',\\
\partial_y^2 u^s(t,y)&=\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-y')^2}{4t}}-e^{-\frac{(y+y')^2}{4t}}\Big)\partial_y^2 u_0^s(y')dy',\\
\partial_y^3 u^s(t,y)&=\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\Big(e^{-\frac{(y-y')^2}{4t}}+e^{-\frac{(y+y')^2}{4t}}\Big)\partial_y^3 u_0^s(y')dy'.
\end{aligned}
$$

Then in the same derivation as in I_2, we have for $k=1,2,3$,

$$
|\partial_y^k u^s(t,y)|\le Ce^{-y}\quad\text{for }(t,y)\in[0,1]\times[0,+\infty).
$$

This finishes the proof of the proposition. □

Lemma 2.2. *Let $u_0^s(y)$ be as in Proposition 2.1. If $u_0^s(y)$ satisfies (1.3), then there exists $T_1 > 0$ so that for any $t \in [0, T_1]$,*

$$\partial_y^2 u^s(t,y) \geq \frac{c}{2} \quad \textit{for} \quad y \in \big[\frac{1}{2}, 2\big],$$
$$\partial_y u^s(t,y) \geq \frac{c}{2}\delta e^{-y} \quad \textit{for} \quad y \in [0, 1-\delta] \cup \big[1+\delta, +\infty).$$

Proof. We have

$$\partial_y u^s(t,y) = \partial_y u_0^s(y) + \int_0^t \partial_t \partial_y u^s(\tau, y) d\tau,$$

$$\partial_y^2 u^s(t,y) = \partial_y^2 u_0^s(y) + \int_0^t \partial_t \partial_y^2 u^s(\tau, y) d\tau.$$

Notice that

$$\Big| \int_0^t \partial_t \partial_y u^s(\tau, y) d\tau \Big| \leq C t^{\frac{1}{2}} \|\partial_y^3 u^s\|_{L_t^2 H_y^1},$$

$$\Big| \int_0^t \partial_t \partial_y^2 u^s(\tau, y) d\tau \Big| \leq C t^{\frac{1}{2}} \|\partial_y^4 u^s\|_{L_t^2 H_y^1}.$$

Then the lemma follows from Proposition 2.1 and (1.3). □

3. Introduction of good unknowns

An essential difficulty solving the Prandtl equations is the loss of one derivative in the horizontal direction x induced by the term $v\partial_y u^s$. To eliminate the trouble term $\partial_y u^s v$ in (1.4), it is natural to introduce a good unknown w_1 defined by

$$w_1 \stackrel{\text{def}}{=} \partial_y\Big(\frac{u}{\partial_y u^s}\Big),$$

which is motivated by the work [1]. Then a direct calculation gives

$$\begin{cases} \partial_t w_1 + u^s \partial_x w_1 - \partial_y^2 w_1 = \partial_y F_1, \\ \partial_y w_1|_{y=0} = 0 \quad \text{and} \quad \lim\limits_{y \to +\infty} w_1 = 0, \\ w_1|_{t=0} = w_0(x,y), \end{cases} \tag{3.1}$$

where F_1 is given by

$$F_1 = u\partial_t\Big(\frac{1}{\partial_y u^s}\Big) - \Big[\partial_y^2, \frac{1}{\partial_y u^s}\Big]u.$$

Here we used the fact that $\partial_y^2 u = 0$ on $y = 0$, which can be seen from (1.4).

Notice that w_1 is only well-defined in the monotonic domain. While, Lemma 2.2 tells us

$$|\partial_y u^s(t,y)| \geq \frac{c\delta}{2} e^{-y} \quad \text{for} \quad (t,y) \in [0, T_1] \times \big([0, 1-\delta] \cup [1+\delta, +\infty)\big). \tag{3.2}$$

Then it is natural to introduce a cut-off good known

$$\overline{w}_1 \triangleq e^{-\frac{y}{2}} \phi_1(y) \partial_y\Big(\frac{u}{\partial_y u^s}\Big) \triangleq \psi_1(y) \partial_y\Big(\frac{u}{\partial_y u^s}\Big),$$

where $\phi_1(y) \in C^\infty(\mathbb{R}_+)$ with the support included in $[0, 1-\delta] \cup [1+\delta, +\infty)$ and $\phi_1(y) = 1$ in $\big[0, 1-2\delta\big] \cup \big[1+2\delta, +\infty\big]$. A direct calculation shows

$$\partial_t \overline{w}_1 + u^s \partial_x \overline{w}_1 - \partial_y^2 \overline{w}_1 = [\psi_1(y), \partial_y^2] w_1 + \psi_1(y) \partial_y F_1. \tag{3.3}$$

To control the regularity of the solution in the non-monotonic domain, we need to use the non-degenerate condition

$$\partial_y^2 u^s(t, y) \geq \frac{c}{2} \quad \text{for} \quad (t, y) \in [0, T_1] \times [\frac{1}{2}, 2]. \tag{3.4}$$

Motivated by [4], we introduce a good unknown h defined by

$$h \stackrel{\text{def}}{=} d \partial_y u,$$

where $d(t, y) = \phi_3(y)(\partial_y^2 u^s)^{-1/2}$ and $\phi_3(y)$ is a cut-off function supported in $[\frac{1}{2}, 2]$ and $\phi_3(y) = 1$ as $y \in [\frac{3}{4}, \frac{7}{4}]$. Then h satisfies

$$\partial_t h + u^s \partial_x h - \partial_y^2 h + d(v \partial_y^2 u^s) = (\partial_t d - \partial_y^2 d) \partial_y u - 2 \partial_y d \partial_y^2 u. \tag{3.5}$$

To propagate the regularity of the solution from monotonic domain to non-monotonic domain, we need to introduce another good unknown $\overline{w}_2$

$$\overline{w}_2 \stackrel{\text{def}}{=} \psi_2(y)\big(\partial_y u^s \partial_y u - u \partial_y^2 u^s\big) \triangleq \psi_2(y) w_2, \tag{3.6}$$

where $\psi_2(y) \in C_0^\infty(\mathbb{R}_+)$ with the support included in $[1-3\delta, 1+3\delta]$ and $\psi_2(y) = 1$ in $\big[1-2\delta, 1+2\delta]$. It is easy to check that

$$\partial_t \overline{w}_2 + u^s \partial_x \overline{w}_2 - \partial_y^2 \overline{w}_2 = [\psi_2(y), \partial_y^2] w_2 + \psi_2(y) F_2, \tag{3.7}$$

where

$$F_2 = \partial_t \partial_y u^s \partial_y u + [\partial_y u^s, \partial_y^2] \partial_y u - u \partial_t \partial_y^2 u^s - [\partial_y^2 u^s, \partial_y^2] u.$$

In fact, w_1 and w_2 are basically equivalent in the monotonic domain by the relation

$$w_2 = (\partial_y u^s)^2 w_1.$$

This in particular implies that

Lemma 3.1. *It holds that*

$$\|1_{I_1}(y)(w_1)_\Phi\|_{H^{\frac{1}{2},0}} \leq C \|(\overline{w}_2)_\Phi\|_{H^{\frac{1}{2},0}},$$
$$\|1_{I_2}(y)(w_2)_\Phi\|_{H^{\frac{1}{2},0}} \leq C \|(\overline{w}_1)_\Phi\|_{H^{\frac{1}{2},0}},$$

where $I_1 = supp \phi_1'$ *and* $I_2 = supp \psi_2'$.

Let $a(t)$ be a critical point of $u^s(t, y)$, i.e.,

$$\partial_y u^s(t, a(t)) = 0.$$

Therefore, $a(t)$ satisfies

$$\partial_t a(t) = -\frac{\partial_t \partial_y u^s(t, a)}{\partial_y^2 u^s(t, a)}, \quad a(0) = 1.$$

By Proposition 2.1, there exists $T_2 > 0$ so that

$$|a(t) - 1| \leq 2\delta \quad \text{for} \quad t \in [0, T_2]. \tag{3.8}$$

Then u can be represented in terms of w_1, w_2. More precisely,

Lemma 3.2. *We can decompose u as* $u = u_1 + u_2$, *where*

$$u_1 = \begin{cases} \partial_y u^s \int_0^y \phi_1 w_1 dy' & \text{for} \quad y < 1 - 2\delta, \\ \partial_y u^s \Big(\int_0^{1-2\delta} \phi_1 w_1 dy' + \int_{1-2\delta}^{y'} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy' \Big) & \text{for} \quad 1 - 2\delta \le y < a(t), \\ \partial_y u^s \Big(\int_{1+2\delta}^{y} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy' + \int_2^{1+2\delta} \phi_1 w_1 dy' \Big) & \text{for} \quad a(t) < y < 1 + 2\delta, \\ \partial_y u^s \int_2^y \phi_1 w_1 dy' & \text{for} \quad y \ge 1 + 2\delta, \end{cases}$$

and

$$u_2 = \partial_y u^s \frac{u(t,x,2)}{\partial_y u^s(t,2)} 1_{\{y > a(t)\}}(y).$$

4. Gevrey regularity estimate of $\overline{w}_1$

In what follows, let us always assume that $T \le \min(T_1, T_2)$.

Proposition 4.1. *Let* $\overline{w}_1$ *be a smooth solution of (3.3) in* $[0, T]$. *Then it holds that for any* $t \in [0, T]$,

$$\begin{aligned} &\frac{d}{dt} \|(\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}} + (\lambda - C) \|(\overline{w}_1)_\Phi\|^2_{H^{\frac34+\theta,0}} + \|\partial_y (\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}} \\ &\quad \le C \Big(\|u_\Phi\|^2_{H^{\frac14,1}_\mu} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac12,0}} \Big). \end{aligned}$$

Let us begin with the estimates of source term F_1.

Lemma 4.2. *It holds that*

$$\|\psi_1(y) \partial_y (F_1)_\Phi\|_{H^{\frac14,0}} \le C \big(\|u_\Phi\|_{H^{\frac14,1}_\mu} + \|(\overline{w}_1)_\Phi\|_{H^{\frac12,1}} + \|(\overline{w}_2)_\Phi\|_{H^{\frac12,0}} \big).$$

Proof. An easy calculation gives

$$F_1 = -2 \frac{(\partial_y^2 u^s)^2}{(\partial_y u^s)^2} u + 2 \frac{\partial_y^2 u^s}{(\partial_y u^s)^2} \partial_y u.$$

By (3.2), we get

$$\|\psi_1(y) \partial_y (F_1)_\Phi\|_{H^{\frac{13}{4},0}} \le C \|u_\Phi\|_{H^{\frac14,1}_\mu} + C \|\psi_1(y) e^y (\partial_y^2 u)_\Phi\|_{H^{\frac14,0}}.$$

Notice that

$$\psi_1(y) \partial_y^2 u = \partial_y u^s \Big(\partial_y \overline{w}_1 - \psi_1' w_1 + 2\psi_1 \frac{\partial_y u \partial_y^2 u^s}{(\partial_y u^s)^2} + \psi_1 \partial_y \big(\frac{\partial_y^2 u^s}{(\partial_y u^s)^2} \big) u \Big),$$

which along with Lemma 2.2 implies that

$$\|\psi_1(y) e^y (\partial_y^2 u)_\Phi\|_{H^{\frac14,0}} \le C \big(\|u_\Phi\|_{H^{\frac14,1}_\mu} + \|(\overline{w}_1)_\Phi\|_{H^{\frac12,1}} + \|(\overline{w}_2)_\Phi\|_{H^{\frac12,0}} \big).$$

Putting the above estimates together, we conclude the lemma. □

Now we are in position to prove Proposition 4.1.

Proof. Applying $e^{\Phi(t,D_x)}$ to (3.3), we obtain

$$\partial_t(\overline{w}_1)_\Phi + \lambda\langle D_x\rangle^{\frac{1}{2}+2\theta}(\overline{w}_1)_\Phi + u^s\partial_x(\overline{w}_1)_\Phi - \partial_y^2(\overline{w}_1)_\Phi = [\psi_1(y), \partial_y^2](w_1)_\Phi + \psi_1(y)\partial_y(F_1)_\Phi. \tag{4.1}$$

Making $H^{\frac{1}{2},0}$ energy estimate to (4.1), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \lambda\|(\overline{w}_1)_\Phi\|^2_{H^{\frac{3}{4}+\theta,0}} - \big(\partial_y^2(\overline{w}_1)_\Phi, (\overline{w}_1)_\Phi\big)_{H^{\frac{1}{2},0}} + \big(u^s\partial_x(\overline{w}_1)_\Phi, (\overline{w}_1)_\Phi\big)_{H^{\frac{1}{2},0}}\\ &\quad = \big([\psi_1(y), \partial_y^2](w_1)_\Phi, (\overline{w}_1)_\Phi\big)_{H^{\frac{1}{2},0}} + \big(\psi_1(y)\partial_y(F_1)_\Phi, (\overline{w}_1)_\Phi\big)_{H^{\frac{1}{2},0}}.\end{aligned}$$

Thanks to $\partial_y(\overline{w}_1)_\Phi|_{y=0} = 0$, we get by integration by parts that

$$-\big(\partial_y^2(\overline{w}_1)_\Phi, (\overline{w}_1)_\Phi\big)_{H^{\frac{1}{2},0}} = \|\partial_y(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}}, \quad \big(u^s\partial_x(\overline{w}_1)_\Phi, (\overline{w}_1)_\Phi\big)_{H^{\frac{1}{2},0}} = 0.$$

We infer from Lemma 3.1 that

$$\begin{aligned}&\big([\psi_1(y), \partial_y^2](w_1)_\Phi, (\overline{w}_1)_\Phi\big)_{H^{\frac{1}{2},0}}\\ &\le 2\big|\big(\psi_1'(w_1)_\Phi, \partial_y(\overline{w}_1)_\Phi\big)_{H^{\frac{1}{2},0}}\big| + 2\big|\big(\psi_1''(w_1)_\Phi, (\overline{w}_1)_\Phi\big)_{H^{\frac{1}{2},0}}\big|\\ &\le C\|(\overline{w}_2)_\Phi\|_{H^{\frac{1}{2},0}}\big(\|\partial_y(\overline{w}_1)_\Phi\|_{H^{\frac{1}{2},0}} + \|(\overline{w}_1)_\Phi\|_{H^{\frac{1}{2},0}}\big)\\ &\le C\big(\|(\overline{w}_2)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}}\big) + \frac{1}{8}\|\partial_y(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}}.\end{aligned}$$

It follows from Lemma 4.2 that

$$\begin{aligned}\big(\psi_1(y)\partial_y(F_1)_\Phi, (\overline{w}_1)_\Phi\big)_{H^{\frac{1}{2},0}} &\le C\big(\|u_\Phi\|^2_{H_\mu^{\frac{1}{4},1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{1}{2},0}}\big)\\ &\quad + \frac{1}{8}\|\partial_y(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + C\|(\overline{w}_1)_\Phi\|^2_{H^{\frac{3}{4},0}}.\end{aligned}$$

Summing up all the estimates, we conclude the proposition. □

5. Gevrey regularity estimate of $\overline{w}_2$

First of all, we prove Gevrey regularity without weight.

Proposition 5.1. *Let $\overline{w}_2$ be a solution of (3.7) in $[0, T]$. There exists $\delta > 0$ small enough so that for any $t \in [0, T]$,*

$$\begin{aligned}&\frac{d}{dt}\|(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}} + (\lambda - C)\|(\overline{w}_2)_\Phi\|^2_{H^{\frac{5}{8}+\theta,0}} + \|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}}\\ &\quad \le C\Big(\|u_\Phi\|^2_{H^{\frac{1}{4},1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}}\Big).\end{aligned}$$

The proposition can be proved by following the proof of Proposition 4.1 and using the following lemma.

Lemma 5.2. *It holds that*

$$\begin{aligned}\big(\psi_2(y)(F_2)_\Phi, (\overline{w}_2)_\Phi\big)_{H^{\frac{3}{8},0}} &\le C\big(\|u_\Phi\|^2_{H^{\frac{1}{4},1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{5}{8},0}}\big)\\ &\quad + \frac{1}{2}\|\partial_y(\overline{w}_2)_\Phi\|_{H^{\frac{3}{8},0}}.\end{aligned}$$

Proof. Notice that

$$-u\partial_t\partial_y^2 u^s - [\partial_y^2 u^s, \partial_y^2]u = 2\partial_y^3 u^s \partial_y u, \tag{5.1}$$

therefore,

$$\big(\psi_2(u\partial_t\partial_y^2 u^s + [\partial_y^2 u^s, \partial_y^2]u)_\Phi, (\overline{w}_2)_\Phi\big)_{H^{\frac{3}{8},0}} \le C\|u_\Phi\|_{H_\mu^{\frac{1}{4},1}}\|(\overline{w}_2)_\Phi\|_{H^{\frac{5}{8},0}}.$$

Similarly, we have

$$\big(\psi_2(\partial_t\partial_y u^s\partial_y u + [\partial_y u^s, \partial_y^2]\partial_y u)_\Phi, (\overline{w}_2)_\Phi\big)_{H^{\frac{3}{8},0}} = -2\big(\psi_2\partial_y^2 u^s\partial_y^2 u_\Phi, (\overline{w}_2)_\Phi\big)_{H^{\frac{3}{8},0}}.$$

The estimate of this term is very tricky. The following argument was motivated by [4]. By Lemma 3.2, $\partial_y u$ can be written as $\partial_y u = \partial_y u_1 + \partial_y u_2$. Note that both $\partial_y u_1$ and $\partial_y u_2$ are discontinuous across $y = a(t)$. In particular, we have

$$\lim_{y\to a(t)-}\partial_y u_1 - \lim_{y\to a(t)+}\partial_y u_1 = \partial_y^2 u^s \frac{u(t,x,2)}{\partial_y u^s(t,2)} \triangleq J.$$

Then by integration by parts, we get

$$\begin{aligned}
-2\big(\psi_2(y)(\partial_y^2 u^s\partial_y^2 u)_\Phi, (\overline{w}_2)_\Phi\big)_{H^{\frac{3}{8},0}} &= -2\int\limits_{y>a(t)} \psi_2(y)\partial_y^2 u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y^2 u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\overline{w}_2)_\Phi dxdy\\
&\quad -2\int\limits_{y<a(t)} \psi_2(y)\partial_y^2 u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y^2 u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\overline{w}_2)_\Phi dxdy\\
&\quad -2\int\limits_{y>a(t)} \psi_2(y)\partial_y^2 u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y^2 u_2)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\overline{w}_2)_\Phi dxdy\\
&= 2\int\limits_{\mathbf{R}_+^2} \psi_2'(y)\partial_y^2 u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\overline{w}_2)_\Phi dxdy\\
&\quad +2\int\limits_{\mathbf{R}_+^2} \psi_2(y)\partial_y^3 u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\overline{w}_2)_\Phi dxdy\\
&\quad +2\int\limits_{\mathbf{R}_+^2} \psi_2(y)\partial_y^2 u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\partial_y\overline{w}_2)_\Phi dxdy\\
&\quad -2\int\limits_{y>a(t)} \psi_2(y)\partial_y^2 u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y^2 u_2)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\overline{w}_2)_\Phi dxdy\\
&\quad -2\int\limits_{y=a(t)} \partial_y^2 u^s\langle D_x\rangle^{\frac{3}{8}}J_\Phi\langle D_x\rangle^{\frac{3}{8}}(\overline{w}_2)_\Phi dx\\
&\triangleq A_1+\cdots+A_5.
\end{aligned}$$

Note that $\psi_2'(y)$ vanishes in a neighborhood of $y = a(t)$ so that $\partial_y u_1$ behaves like w_1 on the support of $\psi_2'(y)$. Thus,

$$A_1 \le C\|(\overline{w}_1)_\Phi\|_{H^{\frac{1}{2},0}}\|(\overline{w}_2)_\Phi\|_{H^{\frac{3}{8},0}}.$$

Thanks to $\partial_y^2 u_2 = \partial_y^3 u^s \frac{u(t,x,2)}{\partial_y u^s(t,2)}$, we obtain

$$A_4 \le C\|u_\Phi\|_{H^{\frac{1}{4},1}}\|(\overline{w}_2)_\Phi\|_{H^{\frac{5}{8},0}}.$$

Similarly, we have

$$A_2 \le C\|u_\Phi\|_{H^{\frac{1}{4},1}}\|(\overline{w}_2)_\Phi\|_{H^{\frac{5}{8},0}}.$$

For A_5, we get by Sobolev inequality that

$$\begin{aligned}A_5 &\le C\|u_\Phi\|_{H^{\frac{1}{4},1}}\|\langle D_x\rangle^{\frac{1}{2}}(\overline{w}_2)_\Phi\|_{L^\infty_y L^2_x} \le C\|u_\Phi\|_{H^{\frac{1}{4},1}}\|\partial_y(\overline{w}_2)_\Phi\|^{\frac{1}{2}}_{H^{\frac{3}{8},0}}\|(\overline{w}_2)_\Phi\|^{\frac{1}{2}}_{H^{\frac{5}{8},0}}\\ &\le C(\|u_\Phi\|^2_{H^{\frac{1}{4},1}_\mu} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{5}{8},0}}) + \frac{1}{16}\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}}.\end{aligned}$$

It remains to estimate A_3. One has

$$w_2 = \partial_y u^s\partial_y u - u\partial_y^2 u^s = \partial_y u^s\partial_y u_1 - u_1\partial_y^2 u^s,$$

which gives

$$\partial_y(\overline{w}_2) = \psi_2'(y)w_2 + \psi_2(y)(\partial_y u^s\partial_y^2 u_1 - u_1\partial_y^3 u^s).$$

Then we may write

$$\begin{aligned}A_3 &= 2\int_{\mathbf{R}^2_+}\psi_2(y)^2\partial_y^2u^s\partial_y u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\partial_y^2 u_1)_\Phi dxdy\\
&\quad - 2\int_{\mathbf{R}^2_+}\psi_2(y)^2\partial_y^2u^s\partial_y^3u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(u_1)_\Phi dxdy\\
&\quad + 2\int_{\mathbf{R}^2_+}\psi_2'(y)\partial_y^2u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\overline{w}_2)_\Phi dxdy\\
&= -\int_{\mathbf{R}^2_+}\psi_2(y)^2(\partial_y^2u^s)^2(\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi)^2 dxdy\\
&\quad - \int_{\mathbf{R}^2_+}\psi_2(y)^2\partial_y^3u^s\partial_y u^s(\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi)^2 dxdy\\
&\quad - 2\int_{\mathbf{R}^2_+}\psi_2'(y)\psi_2(y)\partial_y^2u^s\partial_y u^s(\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi)^2 dxdy\\
&\quad - 2\int_{\mathbf{R}^2_+}\psi_2(y)^2\partial_y u^s\partial_y^3u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi dxdy\\
&\quad + 2\int_{\mathbf{R}^2_+}\psi_2(y)\partial_y^3u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\overline{w}_2)_\Phi dxdy\\
&\quad + 2\int_{\mathbf{R}^2_+}\psi_2'(y)\partial_y^2u^s\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\langle D_x\rangle^{\frac{3}{8}}(\overline{w}_2)_\Phi dxdy\\
&\triangleq A_{31} + \cdots + A_{36}.\end{aligned}$$

Here we used integration by parts and $-u_1\partial_y^2u^s = -\partial_y u^s\partial_y u_1 + w_2$.

By Lemma 2.2, we have $\partial_y^2 u^s \ge \frac{c}{2} > 0$ on suppψ_2. So,

$$A_{31} \le -\frac{c^2}{4}\|\psi_2(y)\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\|^2_{L^2}.$$

On the other hand, $|\partial_y u^s| \le C_1\delta$ on suppψ_2 with C_1 independent of δ. So,

$$A_{32} + A_{34} \le C_1\delta \|\psi_2(y)\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\|_{L^2}^2,$$

and

$$A_{35} \le \delta\|\psi_2(y)\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\|_{L^2}^2 + C_\delta\|(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}}.$$

While, on the support of ψ_2', $\partial_y u_1$ behaves like $\overline{w}_1$. Similar to A_1, we have

$$A_{33} \le C_\delta\|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}},$$

$$A_{36} \le C\big(\|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}}\big).$$

This shows that

$$A_3 \le -\big(\frac{c^2}{4} - C_1\delta\big)\|\psi_2(y)\langle D_x\rangle^{\frac{3}{8}}(\partial_y u_1)_\Phi\|_{L^2}^2 + C\big(\|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}}\big).$$

Putting the above estimates together and taking δ small enough, we conclude our result. □

Next we prove Gevrey regularity estimate with weight. We introduce a weight function $\varphi(t,y) = \varphi(y - a(t))$, where $\varphi(0) = 0$ and $\varphi(y) = 0$ when $|y-1| > 2\delta$.

Proposition 5.3. *Let w_2 be a solution of (3.7) in $[0,T]$ and $\theta_1 > 0$ be a small constant determined later. There exists $\delta > 0$ small enough so that for any $t \in [0,T]$ and $\delta_2 > 0$,*

$$\begin{aligned}&\frac{d}{dt}\|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{1}{2},0}} + (\lambda - C)\|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{3}{4}+\theta,0}} + \|\partial_y(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{1}{2},0}}\\
&\le C\Big(\big(\|u_\Phi\|^2_{H^{\frac{1}{4},1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{5}{8},0}} + \|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{1}{2},0}}\Big) + \delta_2\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}}.\end{aligned}$$

Let us begin with the following estimate of source term.

Lemma 5.4. *It holds that for any $\delta_2 > 0$,*

$$\begin{aligned}\big((F_2)_\Phi, (w_2)_\Phi\varphi^{1+\theta_1}\big)_{H^{\frac{1}{2},0}} \le\ & C\Big(\|u_\Phi\|^2_{H^{\frac{1}{4},1}_\mu} + \|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{3}{4},0}}\\
&+ \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{5}{8},0}}\Big) + \delta_2\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}}.\end{aligned}$$

Proof. By (5.1), it is easy to show that

$$\big((u\partial_t\partial_y^2 u^s + [\partial_y^2 u^s, \partial_y^2]u)_\Phi, (w_2)_\Phi\varphi^{1+\theta_1}\big)_{H^{\frac{1}{2},0}} \le C\|u_\Phi\|_{H^{\frac{1}{4},1}_\mu}\|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|_{H^{\frac{3}{4},0}}.$$

Similar to the proof of Lemma 5.2, we have

$$\begin{aligned}&\big((\partial_t\partial_y u^s\partial_y u + [\partial_y u^s, \partial_y^2]\partial_y u)_\Phi, (w_2)_\Phi\varphi^{1+\theta_1}\big)_{H^{\frac{1}{2},0}}\\
&= -2\int_{\mathbf{R}^2_+}\varphi(t,y)^{1+\theta_1}\partial_y^2 u^s\langle D_x\rangle^{\frac{1}{2}}(\partial_y^2 u)_\Phi\langle D_x\rangle^{\frac{1}{2}}(w_2)_\Phi dxdy\\
&= -2\int_{y>a(t)}\varphi(t,y)^{1+\theta_1}\partial_y^2 u^s\langle D_x\rangle^{\frac{1}{2}}(\partial_y^2 u_1)_\Phi\langle D_x\rangle^{\frac{1}{2}}(w_2)_\Phi dxdy\\
&\quad - 2\int_{y<a(t)}\varphi(t,y)^{1+\theta_1}\partial_y^2 u^s\langle D_x\rangle^{\frac{1}{2}}(\partial_y^2 u_1)_\Phi\langle D_x\rangle^{\frac{1}{2}}(w_2)_\Phi dxdy\end{aligned}$$

$$
\begin{aligned}
&-2\int_{y>a(t)} \varphi(t,y)^{1+\theta_1}\partial_y^2 u^s \langle D_x\rangle^{\frac12}(\partial_y^2 u_2)_\Phi \langle D_x\rangle^{\frac12}(w_2)_\Phi dxdy\\
&=2(1+\theta_1)\int_{\mathbf{R}^2_+} \varphi^{\theta_1}\partial_y\varphi\partial_y^2 u^s \langle D_x\rangle^{\frac12}(\partial_y u_1)_\Phi \langle D_x\rangle^{\frac12}(w_2)_\Phi dxdy\\
&\quad+2\int_{\mathbf{R}^2_+} \varphi(t,y)^{1+\theta_1}\partial_y^3 u^s \langle D_x\rangle^{\frac12}(\partial_y u_1)_\Phi \langle D_x\rangle^{\frac12}(w_2)_\Phi dxdy\\
&\quad+2\int_{\mathbf{R}^2_+} \varphi(t,y)^{1+\theta_1}\partial_y^2 u^s \langle D_x\rangle^{\frac12}(\partial_y u_1)_\Phi \langle D_x\rangle^{\frac12}(\partial_y w_2)_\Phi dxdy\\
&\quad-2\int_{y>a(t)} \varphi(t,y)^{1+\theta_1}\partial_y^2 u^s \langle D_x\rangle^{\frac12}(\partial_y^2 u_2)_\Phi \langle D_x\rangle^{\frac12}(w_2)_\Phi dxdy\\
&\triangleq B_1+\cdots+B_4.
\end{aligned}
$$

Here integration by parts does not give rise to the boundary term due to $\varphi(t,a(t))=0$.

As $\partial_y^2 u_2=\partial_y^3 u^s \frac{u(t,x,2)}{\partial_y u^s(t,2)}$, we have

$$
B_2+B_4\le C\|u_\Phi\|_{H_\mu^{\frac14,1}}\|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|_{H^{\frac34,0}}.
$$

Thanks to $|\partial_y\varphi|\le C$ and $\mathrm{supp}\partial_y\varphi\subset[1-2\delta,1+2\delta]$, we get

$$
\begin{aligned}
B_1&\le C\Big|\int_{\mathbf{R}}\int_{1-2\delta}^{a(t)} \varphi^{\theta_1}\partial_y^2 u^s \langle D_x\rangle^{\frac12}(\partial_y u_1)_\Phi \langle D_x\rangle^{\frac12}(w_2)_\Phi dxdy\Big|\\
&\quad+C\Big|\int_{\mathbf{R}}\int_{a(t)}^{1+2\delta} \varphi^{\theta_1}\partial_y^2 u^s \langle D_x\rangle^{\frac12}(\partial_y u_1)_\Phi \langle D_x\rangle^{\frac12}(w_2)_\Phi dxdy\Big|\\
&\triangleq B_{11}+B_{12}.
\end{aligned}
$$

By Lemma 3.2, $\partial_y u_1$ can be expressed as

$$
\partial_y u_1=\begin{cases}
\partial_y^2 u^s\Big(\int_0^{1-2\delta}\phi_1 w_1 dy'+\int_{1-2\delta}^{y}\frac{\overline{w}_2}{(\partial_y u^s)^2}dy'\Big)+\frac{\overline{w}_2}{\partial_y u^s} & \text{for}\quad 1-2\delta\le y<a(t),\\
\partial_y^2 u^s\Big(\int_{1+2\delta}^{y}\frac{\overline{w}_2}{(\partial_y u^s)^2}dy'+\int_2^{1+2\delta}\phi_1 w_1 dy'\Big)+\frac{\overline{w}_2}{\partial_y u^s} & \text{for}\quad a(t)<y<1+2\delta.
\end{cases}
\tag{5.2}
$$

Then we have

$$
\begin{aligned}
B_{11}&\le C\Big|\int_{\mathbf{R}}\int_{1-2\delta}^{a(t)} \varphi^{\theta_1}(\partial_y^2 u^s)^2\langle D_x\rangle^{\frac12}\Big(\int_0^{1-2\delta}\phi_1 w_1 dy'\Big)_\Phi \langle D_x\rangle^{\frac12}(\overline{w}_2)_\Phi dxdy\Big|\\
&\quad+C\Big|\int_{\mathbf{R}}\int_{1-2\delta}^{a(t)} \varphi^{\theta_1}(\partial_y^2 u^s)^2\langle D_x\rangle^{\frac12}\Big(\int_{1-2\delta}^{y}\frac{\overline{w}_2}{(\partial_y u^s)^2}dy'\Big)_\Phi \langle D_x\rangle^{\frac12}(\overline{w}_2)_\Phi dxdy\Big|
\end{aligned}
$$

$$+C\Big|\int_{\mathbf{R}}\int_{1-2\delta}^{a(t)}\varphi^{\theta_1}\frac{\partial_y^2u^s}{\partial_yu^s}\langle D_x\rangle^{\frac12}(\overline{w}_2)_\Phi\langle D_x\rangle^{\frac12}(\overline{w}_2)_\Phi dxdy\Big|$$
$$\triangleq D_1+D_2+D_3.$$

It is easy to get

$$D_1\le C\|(\overline{w}_1)_\Phi\|_{H^{\frac12,0}}\|(\overline{w}_2)_\Phi\|_{H^{\frac58,0}}.$$

For $y\in[1-2\delta,a(t))$, φ and ∂_yu^s behaves like $|y-a(t)|$. So,

$$\begin{aligned}D_3&\le C\int_{1-2\delta}^{a(t)}\frac{\varphi^{\theta_1}}{\partial_yu^s}dy\|\langle D_x\rangle^{\frac12}(\overline{w}_2)_\Phi\|^2_{L^\infty_yL^2_x}\\&\le C\int_{1-2\delta}^{a(t)}\frac{1}{|y-a|^{1-\theta_1}}dy\|\langle D_x\rangle^{\frac38}\partial_y(\overline{w}_2)_\Phi\|_{L^2}\|\langle D_x\rangle^{\frac58}(\overline{w}_2)_\Phi\|_{L^2}\\&\le C\|\partial_y(\overline{w}_2)_\Phi\|_{H^{\frac38,0}}\|(\overline{w}_2)_\Phi\|_{H^{\frac58,0}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}D_2&\le C\int_{1-2\delta}^{a(t)}\varphi^{\theta_1}\int_{1-2\delta}^{y}\frac{1}{(\partial_yu^s)^2}dy'dy\|\langle D_x\rangle^{\frac12}(\overline{w}_2)_\Phi\|^2_{L^\infty_yL^2_x}\\&\le C\|\partial_y(\overline{w}_2)_\Phi\|_{H^{\frac38,0}}\|(\overline{w}_2)_\Phi\|_{H^{\frac58,0}}.\end{aligned}$$

Here we used

$$\begin{aligned}\int_{1-2\delta}^{a(t)}\varphi^{\theta_1}\int_{1-2\delta}^{y}\frac{1}{(\partial_yu^s)^2}dy'dy&\le C\int_{1-2\delta}^{a(t)}\varphi^{\theta_1}\int_{1-2\delta}^{y}\frac{1}{|y'-a|^2}dy'dy\\&\le C\int_{1-2\delta}^{a(t)}\frac{1}{|y-a|^{1-\theta_1}}dy\le C.\end{aligned}$$

This shows that for any $\delta_2>0$,

$$B_{11}\le C\big(\|(\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}}+\|(\overline{w}_2)_\Phi\|^2_{H^{\frac58,0}}\big)+\delta_2\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac38,0}}.$$

The same argument shows that

$$B_{12}\le C\big(\|(\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}}+\|(\overline{w}_2)_\Phi\|^2_{H^{\frac58,0}}\big)+\delta_2\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac38,0}}.$$

Thus, we obtain

$$B_1\le C\big(\|(\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}}+\|(\overline{w}_2)_\Phi\|^2_{H^{\frac58,0}}\big)+2\delta_2\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac38,0}}.$$

Now we deal with B_3. Similar to A_3 in Lemma 5.2, we obtain

$$\begin{aligned}B_3=&-\int_{\mathbf{R}^2_+}\varphi^{1+\theta_1}(\partial_y^2u^s)^2(\langle D_x\rangle^{\frac12}(\partial_yu_1)_\Phi)^2dxdy\\&-\int_{\mathbf{R}^2_+}\varphi^{1+\theta_1}\partial_y^3u^s\partial_yu^s(\langle D_x\rangle^{\frac12}(\partial_yu_1)_\Phi)^2dxdy\end{aligned}$$

$$
\begin{aligned}
&-(1+\theta_1)\int_{\mathbf{R}^2_+}\varphi^{\theta_1}\partial_y\varphi\partial_y^2u^s\partial_yu^s(\langle D_x\rangle^{\frac12}(\partial_yu_1)_\Phi)^2dxdy\\
&-2\int_{\mathbf{R}^2_+}\varphi^{1+\theta_1}\partial_yu^s\partial_y^3u^s\langle D_x\rangle^{\frac12}(\partial_yu_1)_\Phi\langle D_x\rangle^{\frac12}(\partial_yu_1)_\Phi dxdy\\
&+2\int_{\mathbf{R}^2_+}\varphi^{1+\theta_1}\partial_y^3u^s\langle D_x\rangle^{\frac12}(\partial_yu_1)_\Phi\langle D_x\rangle^{\frac12}(w_2)_\Phi dxdy\\
&\triangleq B_{31}+\cdots+B_{35}.
\end{aligned}
$$

Similar to $A_{31}, A_{32}, A_{34}, A_{35}$ in Lemma 5.2, we have

$$
B_{31}+B_{32}+B_{34}+B_{35}\le-\big(\frac{c^2}{4}-C_1\delta\big)\|(\partial_yu_1)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac12,0}}+C\|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac12,0}}.
$$

Similar to B_1, we have

$$
\begin{aligned}
B_{33}\le\ & C\Big|\int_{\mathbf{R}}\int_{1-2\delta}^{a(t)}\varphi^{\theta_1}\partial_yu^s(\langle D_x\rangle^{\frac12}(\partial_yu_1)_\Phi)^2dxdy\Big|\\
&+C\Big|\int_{\mathbf{R}}\int_{a(t)}^{1+2\delta}\varphi^{\theta_1}\partial_yu^s(\langle D_x\rangle^{\frac12}(\partial_yu_1)_\Phi)^2dxdy\Big|\triangleq E_1+E_2.
\end{aligned}
$$

We get by (5.2) that

$$
\begin{aligned}
E_1\le\ & C\Big|\int_{\mathbf{R}}\int_{1-2\delta}^{a(t)}\varphi^{\theta_1}\partial_yu^s\Big(\int_0^{1-2\delta}\langle D_x\rangle^{\frac12}(\phi_1w_1)_\Phi dy'\Big)^2dxdy\Big|\\
&+C\Big|\int_{\mathbf{R}}\int_{1-2\delta}^{a(t)}\varphi^{\theta_1}\partial_yu^s\Big(\int_{1-2\delta}^{y}\frac{\langle D_x\rangle^{\frac12}(\overline{w}_2)_\Phi}{(\partial_yu^s)^2}dy'\Big)^2dxdy\Big|\\
&+C\Big|\int_{\mathbf{R}}\int_{1-2\delta}^{a(t)}\varphi^{\theta_1}\frac{1}{\partial_yu^s}(\langle D_x\rangle^{\frac12}(\overline{w}_2)_\Phi)^2dxdy\Big|\\
\triangleq\ & E_{11}+E_{12}+E_{13}.
\end{aligned}
$$

It is easy to see that

$$
E_{11}\le C\|(\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}},
$$

and

$$
E_{13}\le C\int_{1-2\delta}^{a}\frac{\varphi^{\theta_1}}{\partial_yu^s}dy\|\langle D_x\rangle^{\frac12}(\overline{w}_2)_\Phi\|^2_{L^\infty_yL^2_x}\le C\|\partial_y(\overline{w}_2)_\Phi\|_{H^{\frac38,0}}\|(\overline{w}_2)_\Phi\|_{H^{\frac58,0}}.
$$

Similar to E_{13}, we have

$$
E_{12}\le C\|\partial_y(\overline{w}_2)_\Phi\|_{H^{\frac38,0}}\|(\overline{w}_2)_\Phi\|_{H^{\frac58,0}}.
$$

This shows that for any $\delta_1>0$,

$$
E_1\le C\big(\|(\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}}+\|(\overline{w}_2)_\Phi\|^2_{H^{\frac58,0}}\big)+\delta_2\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac38,0}}.
$$

Similarly, we have

$$E_2 \le C\big(\|(\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac58,0}}\big) + \delta_2\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac38,0}}.$$

Thus, we get

$$B_{33} \le C\big(\|(\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac58,0}}\big) + 2\delta_2\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac38,0}}.$$

Summing up, we obtain

$$B_3 \le -\big(\frac{c^2}{4} - C_1\delta\big)\|(\partial_y u_1)_\Phi \varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac12,0}} + C\big(\|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac12,0}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac12,0}}$$
$$+ \|(\overline{w}_2)_\Phi\|^2_{H^{\frac58,0}}\big) + 2\delta_2\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac38,0}}.$$

Summing up the estimates of B_1, B_2, B_3 and taking δ small enough, we conclude the lemma. □

Now we are in position to prove Proposition 5.3.

Proof. Recall that w_2 satisfies

$$\partial_t w_2 + u^s\partial_x w_2 - \partial_y^2 w_2 = F_2. \tag{5.3}$$

Applying $e^{\Phi(t,D_x)}$ to (5.3), we get

$$\partial_t (w_2)_\Phi + \lambda\langle D_x\rangle^{\frac12+2\theta}(w_2)_\Phi + u^s\partial_x(w_2)_\Phi - \partial_y^2(w_2)_\Phi = (F_2)_\Phi. \tag{5.4}$$

Taking $\langle D_x\rangle^{\frac72}$ on both sides of (5.4) and taking L^2 inner product with $\langle D_x\rangle^{\frac12}(w_2)_\Phi\varphi^{1+\theta_1}$, we obtain

$$\frac12\frac{d}{dt}\|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac12,0}} + \lambda\|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac34+\theta,0}} + \int_{\mathbf{R}^2_+} u^s\langle D_x\rangle^{\frac12}\partial_x(w_2)_\Phi\langle D_x\rangle^{\frac12}(w_2)_\Phi\varphi^{1+\theta_1}dxdy$$
$$- \int_{\mathbf{R}^2_+}\langle D_x\rangle^{\frac12}\partial_y^2(w_2)_\Phi\langle D_x\rangle^{\frac12}(w_2)_\Phi\varphi^{1+\theta_1}dxdy$$
$$\le \int_{\mathbf{R}^2_+}\varphi^{\theta_1}\partial_t\varphi|\langle D_x\rangle^{\frac12}(w_2)_\Phi|^2dxdy + \big((F_2)_\Phi, (w_2)_\Phi\varphi^{1+\theta_1}\big)_{H^{\frac12,0}}.$$

We get by integration by parts that

$$\int_{\mathbf{R}^2_+} u^s\langle D_x\rangle^{\frac12}\partial_x(w_2)_\Phi\langle D_x\rangle^{\frac12}(w_2)_\Phi\varphi^{1+\theta_1}dxdy = 0,$$

and

$$- \int_{\mathbf{R}^2_+}\langle D_x\rangle^{\frac12}\partial_y^2(w_2)_\Phi\langle D_x\rangle^{\frac12}(w_2)_\Phi\varphi^{1+\theta_1}dxdy$$
$$= \|\partial_y(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac12,0}} + (1+\theta_1)\int_{\mathbf{R}^2_+}\varphi^{\theta_1}\partial_y\varphi\langle D_x\rangle^{\frac12}\partial_y(w_2)_\Phi\langle D_x\rangle^{\frac12}(w_2)_\Phi dxdy$$
$$\ge \|\partial_y(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac12,0}} - C\|\partial_y(\overline{w}_2)_\Phi\|_{H^{\frac38,0}}\|(\overline{w}_2)_\Phi\|_{H^{\frac58,0}}$$
$$\ge \|\partial_y(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac12,0}} - C\|(\overline{w}_2)_\Phi\|^2_{H^{\frac58,0}} - \delta_2\|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac38,0}}.$$

Here we used $|\partial_y\varphi| \le C$ and $\psi_2(y) = 1$ for $y \in \mathrm{supp}\partial_y\varphi$. Similarly, we have

$$\int_{\mathbf{R}^2_+} \varphi^{\theta_1}\partial_t\varphi|\langle D_x\rangle^{\frac{1}{2}}(w_2)_\Phi|^2 dxdy \le C\|(\overline{w}_2)_\Phi\|^2_{H^{\frac{1}{2},0}}.$$

With the above estimates and using Lemma 5.4, we deduce our result. □

6. Gevrey regularity estimate of *h*

Proposition 6.1. *Let h be a solution of (3.5) in* $[0, T]$. *Then it holds that*

$$\frac{d}{dt}\|h_\Phi\|^2_{L^2} + \lambda\|h_\Phi\|^2_{H^{\frac{1}{4}+\theta,0}} + \|\partial_y h_\Phi\|^2_{L^2}$$
$$\le C\Big(\|h_\Phi\|^2_{L^2} + \|u_\Phi\|^2_{H^{\frac{1}{4},1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{3}{4},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{5}{8},0}} + \|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{3}{4},0}}\Big).$$

The proposition follows from the following two lemmas.

Lemma 6.2. *Let h be a smooth solution of (3.5) in* $[0, T]$. *Then it holds that*

$$\frac{d}{dt}\|h_\Phi\|^2_{L^2} + \lambda\|h_\Phi\|^2_{H^{\frac{1}{4}+\theta,0}} + \|\partial_y h_\Phi\|^2_{L^2}$$
$$\le C\big(\|h_\Phi\|^2_{L^2} + \|u_\Phi\|^2_{H^{\frac{1}{4},1}}\big) + 2\big(\phi_3'(y)v_\Phi, \phi_3(y)u_\Phi\big)_{L^2}.$$

Proof. Applying $e^{\Phi(t,D_x)}$ on (3.5) and making $H^{3,0}$ energy estimate, we obtain

$$\frac{d}{dt}\|h_\Phi\|^2_{L^2} + \lambda\|h_\Phi\|^2_{H^{\frac{1}{4}+\theta,0}} - \big((\partial_y^2 h)_\Phi, h_\Phi\big)_{L^2}$$
$$\le -\big(u^s\partial_x h_\Phi, h_\Phi\big)_{L^2} + \big((\partial_t d - \partial_y^2 d)(\partial_y u)_\Phi, h_\Phi\big)_{L^2} - 2\big((\partial_y d\partial_y^2 u)_\Phi, h_\Phi\big)_{L^2}$$
$$- \big(d(v\partial_y^2 u^s)_\Phi, h_\Phi\big)_{L^2}.$$

Thanks to $h_\Phi|_{y=0} = 0$, we get by integration by parts that

$$\big(u^s\partial_x h_\Phi, h_\Phi\big)_{L^2} = 0, \quad -\big((\partial_y^2 h)_\Phi, h_\Phi\big)_{L^2} = \|(\partial_y h)_\Phi\|^2_{L^2},$$

and

$$-2\big((\partial_y d\partial_y^2 u)_\Phi, h_\Phi\big)_{L^2} = 2\big((\partial_y^2 d\partial_y u)_\Phi, h_\Phi\big)_{L^2} + 2\big((\partial_y d\partial_y u)_\Phi, \partial_y h_\Phi\big)_{L^2}$$
$$\le C\big(\|u_\Phi\|^2_{H^{\frac{1}{4},1}} + \|h_\Phi\|^2_{L^2}\big) + \frac{1}{16}\|(\partial_y h)_\Phi\|^2_{L^2}.$$

After some calculations, we have

$$\partial_t d - \partial_y^2 d = \frac{\partial_y^3 u^s\phi_3'}{(\partial_y^2 u^s)^{\frac{3}{2}}} - \frac{\phi_3''}{(\partial_y^2 u^s)^{\frac{1}{2}}} - \frac{3}{4}\frac{(\partial_y^3 u^s)^2\phi_3}{(\partial_y^2 u^s)^{\frac{5}{2}}},$$

which gives

$$\big((\partial_t d - \partial_y^2 d)(\partial_y u)_\Phi, h_\Phi\big)_{L^2} \le C\|u_\Phi\|_{H^{\frac{1}{4},1}}\|h_\Phi\|_{L^2}.$$

Using $\partial_x u + \partial_y v = 0$, we get by integration by parts that

$$-\big(d(v\partial_y^2 u^s)_\Phi, h_\Phi\big)_{L^2} = -\big(\phi_3(y)(\partial_y^2 u^s)^{-1/2}\partial_y^2 u^s v_\Phi, \phi_3(y)(\partial_y^2 u^s)^{-1/2}(\partial_y u)_\Phi\big)_{L^2}$$
$$= -\big(\phi_3(y)v_\Phi, \phi_3(y)(\partial_y u)_\Phi\big)_{L^2}$$
$$= 2\big(\phi_3'(y)v_\Phi, \phi_3(y)u_\Phi\big)_{L^2} + \big(\phi_3(y)(\partial_y v)_\Phi, \phi_3(y)u_\Phi\big)_{L^2}$$
$$= 2\big(\phi_3'(y)v_\Phi, \phi_3(y)u_\Phi\big)_{L^2}.$$

This completes the proof of the lemma. □

The following lemma is devoted to the most trouble term $\big(\phi_3'(y)v_\Phi, \phi_3(y)u_\Phi\big)_{L^2}$. The argument is motivated by [4].

Lemma 6.3. *It holds that*

$$\big(\phi_3'(y)v_\Phi, \phi_3(y)u_\Phi\big)_{L^2} \le C\Big(\|u_\Phi\|^2_{H^{\frac14+\theta,1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac34,0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac58,0}} + \|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac34,0}}\Big).$$

Proof. Recall that $\mathrm{supp}\phi_3'$ is included in $E_1 \cup E_2$, where $E_1 = [\frac12, \frac34]$ and $E_2 = [\frac74, 2]$. Then we write

$$\begin{aligned}
\Big|\int_{\mathbf{R}^2_+} \phi_3\phi_3' v_\Phi u_\Phi dxdy\Big| &\le \Big|\int_{\mathbf{R}}\int_{E_1} \phi_3\phi_3' \int_0^y \partial_x u_\Phi dy' u_\Phi dxdy\Big| \\
&\quad + \Big|\int_{\mathbf{R}}\int_{E_2} \phi_3\phi_3' \int_0^y \partial_x u_\Phi dy' u_\Phi dxdy\Big| \\
&\triangleq J_1 + J_2.
\end{aligned}$$

In E_1, u can be expressed as $u = \partial_y u^s \int_0^y \overline{w}_1 dy'$ so that

$$\begin{aligned}
J_1 &\le \Big|\int_{\mathbf{R}}\int_{E_1} \phi_3\phi_3' \int_0^y \partial_x u_\Phi dy'\ \partial_y u^s \int_0^y (\overline{w}_1)_\Phi dy' dxdy\Big| \\
&\le C\|(\overline{w}_1)_\Phi\|_{H^{\frac34,0}}\|u_\Phi\|_{H^{\frac14,0}}
\end{aligned}$$

In E_2, u can be expressed as $u = \partial_y u^s \int_2^y \overline{w}_1 dy' + \partial_y u^s \frac{u(t,x,2)}{\partial_y u^s(t,2)}$ so that

$$\begin{aligned}
J_2 &\le \Big|\int_{\mathbf{R}}\int_{E_2} \phi_3\phi_3' \int_0^y \partial_x u_\Phi dy' \big(\partial_y u^s \int_2^y \overline{w}_1 dy'\big)_\Phi dxdy\Big| \\
&\quad + \Big|\int_{\mathbf{R}}\int_{E_2} \phi_3\phi_3' \int_0^y \partial_x u_\Phi dy' \partial_y u^s \frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)} dxdy\Big| \\
&\triangleq J_{21} + J_{22}.
\end{aligned}$$

Similar to J_1, we have

$$J_{21} \le C\|(\overline{w}_1)_\Phi\|_{H^{\frac34,0}}\|u_\Phi\|_{H^{\frac14,0}}.$$

Recall that $a(t)$ is a critical point of u^s. We decompose $\int_0^y \partial_x u_\Phi dy'$ into the following three parts

$$\int_0^y \partial_x u_\Phi dy' = \int_0^{a(t)} \partial_x u_\Phi dy' + \int_{a(t)}^{\frac74} \partial_x u_\Phi dy' + \int_{\frac74}^y \partial_x u_\Phi dy'.$$

Then we have

$$\begin{aligned}
J_{22} &\le \Big|\int_{\mathbf{R}}\int_{E_2} \phi_3\phi_3' \int_0^{a(t)} \partial_x u_\Phi dy'\ \partial_y u^s \frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)} dxdy\Big| \\
&\quad + \Big|\int_{\mathbf{R}}\int_{E_2} \phi_3\phi_3' \int_{a(t)}^{\frac74} \partial_x u_\Phi dy' \partial_y u^s \frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)} dxdy\Big|
\end{aligned}$$

$$+\Big|\int_{\mathbf{R}}\int_{E_2}\phi_3\phi_3'\int_{\frac{7}{4}}^{y}\partial_x u_\Phi dy'\partial_y u^s\frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)}dxdy\Big|$$
$$\triangleq K_1+K_2+K_3.$$

By Lemma 3.2, we get

$$K_1\le\Big|\int_{\mathbf{R}}\int_{E_2}\phi_3\phi_3'\int_0^{1-2\delta}\partial_x u_\Phi dy'\ \partial_y u^s\frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)}dxdy\Big|$$
$$+\Big|\int_{\mathbf{R}}\int_{E_2}\phi_3\phi_3'\int_{1-2\delta}^{a(t)}\partial_x\big(\partial_y u^s\int_0^{1-2\delta}\phi_1 w_1 dy''\big)_\Phi dy'\ \partial_y u^s\frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)}dxdy\Big|$$
$$+\Big|\int_{\mathbf{R}}\int_{E_2}\phi_3\phi_3'\int_{1-2\delta}^{a(t)}\partial_x(\partial_y u^s\int_{1-2\delta}^{y'}\frac{\overline{w}_2}{(\partial_y u^s)^2}dy'')_\Phi dy'\ \partial_y u^s\frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)}dxdy\Big|$$
$$\triangleq K_{11}+K_{12}+K_{13}.$$

As in J_1, we have

$$K_{11}+K_{12}\le C\|(\overline{w}_1)_\Phi\|_{H^{\frac{3}{4},0}}\|u_\Phi\|_{H^{\frac{1}{4},1}}.$$

For K_{13}, let us first estimate

$$\Big\|\int_{1-2\delta}^{a(t)}\langle D_x\rangle^{\frac{3}{4}-\theta}(\partial_y u^s\int_{1-2\delta}^{y'}\frac{\overline{w}_2}{(\partial_y u^s)^2}dy'')_\Phi dy'\Big\|_{L_x^2}$$
$$\le\int_{1-2\delta}^{a(t)}\partial_y u^s\int_{1-2\delta}^{y'}\frac{\|\langle D_x\rangle^{\frac{3}{4}-\theta}(\overline{w}_2)_\Phi\|_{L_x^2}}{(\partial_y u^s)^2}dy''dy'$$
$$\le\int_{1-2\delta}^{a(t)}\partial_y u^s\int_{1-2\delta}^{y'}\frac{\|\langle D_x\rangle^{\frac{5}{8}}(\overline{w}_2)_\Phi\|_{L_x^2}^{1-\alpha}\|\langle D_x\rangle^{\frac{3}{4}}(w_2)_\Phi\|_{L_x^2}^{\alpha}\varphi^{\frac{\alpha(1+\theta_1)}{2}}}{(\partial_y u^s)^2\varphi^{\frac{\alpha(1+\theta_1)}{2}}}dy''dy'$$
$$\le\int_{1-2\delta}^{a}\partial_y u^s\Big(\int_{1-2\delta}^{y'}\frac{1}{(\partial_y u^s)^4\varphi^{\alpha(1+\theta_1)}}dy''\Big)^{\frac{1}{2}}dy'\|\langle D_x\rangle^{\frac{5}{8}}(\overline{w}_2)_\Phi\|_{L^2}^{1-\alpha}\|\langle D_x\rangle^{\frac{3}{4}}(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|_{L^2}^{\alpha}$$
$$\le\int_{1-2\delta}^{a}|y'-a(t)|^{-\frac{1}{2}-\frac{\alpha(1+\theta_1)}{2}}dy'\|\langle D_x\rangle^{\frac{5}{8}}(\overline{w}_2)_\Phi\|_{L^2}^{1-\alpha}\|\langle D_x\rangle^{\frac{3}{4}}(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|_{L^2}^{\alpha}$$
$$\le C\|\langle D_x\rangle^{\frac{5}{8}}(\overline{w}_2)_\Phi\|_{L^2}^{1-\alpha}\|\langle D_x\rangle^{\frac{3}{4}}(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|_{L^2}^{\alpha}.$$

Here $\alpha=1-8\theta$ and take $\frac{\alpha(1+\theta_1)}{2}<\frac{1}{2}$ to ensure that $\int_{1-2\delta}^{a}|y'-a(t)|^{-\frac{1}{2}-\frac{\alpha(1+\theta_1)}{2}}dy'\le C$. As a result, we obtain

$$K_{13}\le C\big(\|u_\Phi\|_{H^{\frac{1}{4}+\theta,1}}^2+\|(\overline{w}_2)_\Phi\|_{H^{\frac{5}{8},0}}^2+\|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|_{H^{\frac{3}{4},0}}^2\big).$$

Then we have

$$K_1\le C\big(\|u_\Phi\|_{H^{\frac{1}{4}+\theta,1}}^2+\|(\overline{w}_1)_\Phi\|_{H^{\frac{3}{4},0}}^2+\|(\overline{w}_2)_\Phi\|_{H^{\frac{5}{8},0}}^2+\|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|_{H^{\frac{3}{4},0}}^2\big).$$

For K_2, by Lemma 3.2, we have

$$\begin{aligned}
K_2 \leq & \Big| \int_{\mathbf{R}} \int_{E_2} \phi_3 \phi_3' \int_{a(t)}^{1+2\delta} \partial_x \big(\partial_y u^s \int_{1+2\delta}^{y'} \frac{\overline{w}_2}{(\partial_y u^s)^2} dy''\big)_\Phi dy' \partial_y u^s \frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)} dx dy \Big| \\
& + \Big| \int_{\mathbf{R}} \int_{E_2} \phi_3 \phi_3' \int_{a(t)}^{1+2\delta} \partial_x \big(\partial_y u^s \int_{2}^{1+2\delta} \overline{w}_1 dy''\big)_\Phi dy' \partial_y u^s \frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)} dx dy \Big| \\
& + \Big| \int_{\mathbf{R}} \int_{E_2} \phi_3 \phi_3' \int_{a(t)}^{1+2\delta} \partial_y u^s \frac{(\partial_x u(t,x,2))_\Phi}{\partial_y u^s(t,2)} dy' \partial_y u^s \frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)} dx dy \Big| \\
& + \Big| \int_{\mathbf{R}} \int_{E_2} \phi_3 \phi_3' \int_{1+2\delta}^{\frac{7}{4}} \partial_x \big(\partial_y u^s \int_{2}^{y'} \overline{w}_1 dy''\big)_\Phi dy' \partial_y u^s \frac{\big(u(t,x,2)\big)_\Phi}{\partial_y u^s(t,2)} dx dy \Big| \\
& + \Big| \int_{\mathbf{R}} \int_{E_2} \phi_3 \phi_3' \int_{1+2\delta}^{\frac{7}{4}} \partial_y u^s \frac{(\partial_x u(t,x,2))_\Phi}{\partial_y u^s(t,2)} dy' \, \partial_y u^s \frac{(u(t,x,2))_\Phi}{\partial_y u^s(t,2)} dx dy \Big| \\
& \triangleq K_{21} + \cdots + K_{25}.
\end{aligned}$$

Similar to K_{13}, we have

$$K_{21} \leq C\big(\|u_\Phi\|^2_{H^{\frac{1}{4}+\theta,1}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{5}{8},0}} + \|(w_2)_\Phi \varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{3}{4},0}}\big).$$

Similar to K_{12}, we have

$$K_{22} + K_{24} \leq C\|(\overline{w}_1)_\Phi\|_{H^{\frac{3}{4},0}} \|u_\Phi\|_{H^{\frac{1}{4},1}}.$$

Integration by parts, we have

$$K_{23} + K_{25} = 0.$$

This shows that

$$K_2 \leq C\big(\|u_\Phi\|^2_{H^{\frac{1}{4}+\theta,1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{3}{4},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{5}{8},0}} + \|(w_2)_\Phi \varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{3}{4},0}}\big).$$

Similar to K_{24} and K_{25}, we have

$$K_3 \leq C\|(\overline{w}_1)_\Phi\|_{H^{\frac{3}{4},0}} \|u_\Phi\|_{H^{\frac{1}{4},1}}.$$

Summing up the estimates of K_1, K_2, K_3, we deduce that

$$J_{22} \leq C\big(\|u_\Phi\|^2_{H^{\frac{1}{4}+\theta,1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{3}{4},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{5}{8},0}} + \|(w_2)_\Phi \varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{3}{4},0}}\big).$$

Putting the above estimates together, we conclude the lemma. □

7. Proof of Theorem 1.1

Let us first recover the regularity of u from w_1 and h.

Lemma 7.1. *It holds that*

$$\|u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,1}} \leq C\big(\|(\overline{w}_1)_\Phi\|_{H^{\frac{1}{2},0}} + \|h_\Phi\|_{H^{\frac{1}{4}+\theta,0}}\big).$$

Proof. The proof is split into two steps.

Step 1. Estimate of $\|\partial_y u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}}$

First of all, we have

$$\begin{aligned}\|\partial_y u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}} &\le \|1_{[0,\frac{3}{4}]}(y)\partial_y u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}} + \|1_{[\frac{3}{4},\frac{7}{4}]}(y)\partial_y u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}} + \|1_{[\frac{7}{4},+\infty)}(y)\partial_y u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}}\\ &\triangleq I_1 + I_2 + I_3.\end{aligned}$$

For $y \in [0, \frac{3}{4}] \cup [\frac{7}{4}, +\infty)$, we have

$$\partial_y u(y) = \partial_y u^s \big(w_1 - \partial_y(\frac{1}{\partial_y u^s})u\big),$$

which gives

$$I_1 + I_3 \le C\|(\overline{w}_1)_\Phi\|_{H^{\frac{1}{2},0}} + C\|u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}}.$$

For $y \in [\frac{3}{4}, \frac{7}{4}]$, using $\partial_y u = h(\partial_y^2 u^s)^{-\frac{1}{2}}$, we get

$$I_2 \le C\|h_\Phi\|_{H^{\frac{1}{4}+\theta,0}}.$$

Step 2. Estimate of $\|u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}}$

We have

$$\begin{aligned}\|u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}} &\le \|1_{[0,\frac{3}{4}]}(y)u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}} + \|1_{[\frac{3}{4},\frac{7}{4}]}(y)u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}} + \|1_{[\frac{7}{4},+\infty)}(y)u_\Phi\|_{H_\mu^{\frac{1}{4}+\theta,0}}\\ &\triangleq I_4 + I_5 + I_6.\end{aligned}$$

For $y \in [0, \frac{3}{4}]$, we have

$$u(y) = \partial_y u^s \Big(\int_0^y w_1 dy'\Big),$$

which gives

$$I_4 \le C\|(\overline{w}_1)_\Phi\|_{H^{\frac{1}{2},0}}.$$

For $y \in [\frac{3}{4}, \frac{7}{4}]$, we have

$$\widetilde{u}(y) = \int_{\frac{3}{4}}^y \frac{h}{(\partial_y^2 u^s)^{\frac{1}{2}}} dy' + u(t, x, \frac{3}{4}), \tag{7.1}$$

from which and the estimate of I_4, we deduce that

$$I_5 \le C\big(\|(\overline{w}_1)_\Phi\|_{H^{\frac{1}{2},0}} + \|h_\Phi\|_{H^{\frac{1}{4}+\theta,0}}\big).$$

For $y \in [\frac{7}{4}, +\infty)$, we have

$$\widetilde{u}(y) = \partial_y u^s \Big(\int_{\frac{7}{4}}^y w_1 dy'\Big) + \partial_y u^s \frac{u(t, x, \frac{7}{4})}{\partial_y u^s(t, \frac{7}{4})}$$

from which and the estimate of I_5, we deduce that

$$I_6 \le C\big(\|(\overline{w}_1)_\Phi\|_{H^{\frac{1}{2},0}} + \|h_\Phi\|_{H^{\frac{1}{4}+\theta,0}}\big).$$

Now, the inequality follows by putting the estimates of $I_1 - I_6$ together. □

Now we are in position to prove Theorem 1.1.

Proof. The approximate solution can be easily constructed by adding the viscous term $-\epsilon^2\partial_x^2 u$ to (1.4). So, we just present the uniform estimate. For this end, we introduce

$$\mathcal{E}(t) \triangleq \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}} + \|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{1}{2},0}} + \|h_\Phi\|^2_{L^2},$$

$$\mathcal{D}(t) \triangleq \|\partial_y(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{8},0}} + \|\partial_y(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{1}{2},0}} + \|\partial_y h_\Phi\|^2_{L^2},$$

$$\mathcal{G}(t) \triangleq \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{3}{4}+\theta,0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{5}{8}+\theta,0}} + \|(w_2)_\Phi\varphi^{\frac{1+\theta_1}{2}}\|^2_{H^{\frac{3}{4}+\theta,0}} + \|h_\Phi\|^2_{H^{\frac{1}{4}+\theta,0}}.$$

Choosing λ large enough and δ_2 suitably small, we infer from Proposition 4.1, Proposition 5.1, Proposition 5.3, Proposition 6.1 and Lemma 7.1 that

$$\frac{d}{dt}\mathcal{E}(t) + \lambda\mathcal{G}(t) + \mathcal{D}(t) \le C\mathcal{E}(t).$$

Then Gronwall's inequality gives

$$\mathcal{E}(t) + \lambda\int_0^t \mathcal{G}(s)ds + \int_0^t \mathcal{D}(s)ds \le \mathcal{E}(0)e^{Ct} \tag{7.2}$$

for any $t \in [0, T]$. □

8. Note on well-posedness in Gevrey class 2

Let us explain how to use a new unknown $h_1 = \partial_y^2 u - \frac{\partial_y^3 u^s}{\partial_y^2 u^s}\partial_y u$ introduced in [8] to obtain the well-posedness of (1.4) in Gevrey class 2 in our framework. It is easy to verify that h_1 satisfies the following equation

$$\partial_t h_1 + u^s\partial_x h_1 + \partial_x w_2 - \partial_y^2 h_1 = \partial_t\Big(\frac{\partial_y^3 u^s}{\partial_y^2 u^s}\Big)\partial_y u + \Big[\frac{\partial_y^3 u^s}{\partial_y^2 u^s}, \partial_y^2\Big]\partial_y u.$$

The unknown h_1 is well-defined in non-monotonic domain. It is easy to show that

$$(\overline{h}_1)_\Phi \in L^\infty(0,T;L^2) \cap L^2(0,T;H^{\frac{1}{4},0}), \quad \overline{h}_1 = \phi_3(y)h_1,$$

if $(\overline{w}_2)_\Phi \in L^2(0,T;H^{\frac{3}{4},0})$. On the other hand, if we know that $(\overline{h}_1)_\Phi \in L^2(0,T;H^{\frac{1}{4},0})$ which will imply $\partial_y^2 u_\Phi \in L^2(0,T;H^{\frac{1}{4},0})$ because of $\partial_y u_\Phi \in L^2(0,T;H^{\frac{1}{4},0})$ by Lemma 7.1, we can show that $(\overline{w}_2)_\Phi \in L^2(0,T;H^{\frac{3}{4},0})$ by following the proof of Proposition 5.1. More precisely, we can deduce that

$$\frac{d}{dt}\|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + (\lambda - C)\|(\overline{w}_1)_\Phi\|^2_{H^{\frac{3}{4},0}} + \|\partial_y(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}}$$
$$\le C\Big(\|u_\Phi\|^2_{H_\mu^{\frac{1}{4},1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{1}{2},0}}\Big),$$

$$\frac{d}{dt}\|(\overline{w}_2)_\Phi\|^2_{H^{\frac{1}{2},0}} + (\lambda - C)\|(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{4},0}} + \|\partial_y(\overline{w}_2)_\Phi\|^2_{H^{\frac{1}{2},0}}$$
$$\le C\Big(\|u_\Phi\|^2_{H^{\frac{1}{4},1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{1}{2},0}} + \|(\overline{h}_1)_\Phi\|^2_{H^{\frac{1}{4},0}}\Big),$$

and

$$\frac{d}{dt}\|h_\Phi\|^2_{L^2} + \lambda\|h_\Phi\|^2_{H^{\frac{1}{4},0}} + \|\partial_y h_\Phi\|^2_{L^2}$$
$$\le C\Big(\|h_\Phi\|^2_{L^2} + \|u_\Phi\|^2_{H^{\frac{1}{4},1}} + \|(\overline{w}_1)_\Phi\|^2_{H^{\frac{3}{4},0}} + \|(\overline{w}_2)_\Phi\|^2_{H^{\frac{3}{4},0}}\Big),$$

$$\frac{d}{dt}\|(\overline{h}_1)_\Phi\|_{L^2}^2 + (\lambda - C)\|(\overline{h}_1)_\Phi\|_{H^{\frac{1}{4},0}}^2 + \|\partial_y(\overline{h}_1)_\Phi\|_{L^2}^2$$
$$\leq C\Big(\|h_\Phi\|_{L^2}^2 + \|u_\Phi\|_{H^{\frac{1}{4},1}}^2 + \|(\overline{w}_1)_\Phi\|_{H^{\frac{3}{4},0}}^2 + \|(\overline{w}_2)_\Phi\|_{H^{\frac{3}{4},0}}^2\Big).$$

Thus, we can close the energy estimates in Gevrey class 2.

Conflict of interest statement

The authors declare that there is no conflict of interest.

Acknowledgement

Z. Zhang is partially supported by NSF of China under Grant 11425103.

References

[1] R. Alexandre, Y. Wang, C.-J. Xu, T. Yang, Well-posedness of the Prandtl equation in Sobolev spaces, J. Am. Math. Soc. 28 (2015) 745–784.
[2] D. Chen, Y. Wang, Z. Zhang, Well-posedness of the Prandtl equation with monotonicity in Sobolev Spaces, submitted for publication.
[3] D. Gérard-Varet, E. Dormy, On the ill-posedness of the Prandtl equation, J. Am. Math. Soc. 23 (2010) 591–609.
[4] D. Gérard-Varet, N. Masmoudi, Well-posedness for the Prandtl system without analyticity or monotonicity, Ann. Sci. Éc. Norm. Supér. 48 (2015) 1273–1325.
[5] D. Gérard-Varet, T. Nguyen, Remarks on the ill-posedness of the Prandtl equation, Asymptot. Anal. 77 (2012) 71–88.
[6] Y. Guo, T. Nguyen, A note on Prandtl boundary layers, Commun. Pure Appl. Math. 64 (2011) 1416–1438.
[7] I. Kukavica, N. Masmoudi, V. Vicol, T.K. Wong, On the local well-posedness of the Prandtl and the hydrostatic Euler equations with multiple monotonicity regions, SIAM J. Math. Anal. 46 (2014) 3865–3890.
[8] W. Li, T. Yang, Well-posedness in Gevrey space for the Prandtl equations with non-degenerate critical points, arXiv:1609.08430.
[9] M.C. Lombardo, M. Cannone, M. Sammartino, Well-posedness of the boundary layer equations, SIAM J. Math. Anal. 35 (2003) 987–1004.
[10] N. Masmoudi, T.K. Wong, Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods, Commun. Pure Appl. Math. 68 (2015) 1683–1741.
[11] O.A. Oleinik, V.N. Samokhin, Mathematical Models in Boundary Layer Theory, Applied Mathematics and Mathematical Computation, vol. 15, Chapman & Hall/CRC, Boca Raton, FL, 1999.
[12] L. Prandtl, Über Flüssigkeitsbewegung bei sehr kleiner Reibung, in: Verhandlung des III Intern. Math.-Kongresses, Heidelberg, 1904, pp. 484–491.
[13] M. Sammartino, R.E. Caflisch, Zero viscosity limit for analytic solutions, of the Navier–Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, Commun. Math. Phys. 192 (1998) 433–461.
[14] Z. Xin, L. Zhang, On the global existence of solutions to the Prandtl's system, Adv. Math. 181 (2004) 88–133.
[15] P. Zhang, Z. Zhang, Long time well-posedness of Prandtl system with small and analytic initial data, J. Funct. Anal. 270 (2016) 2591–2615.