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Stationary solutions to the compressible Navier–Stokes system with general boundary conditions

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Abstract

We consider the stationary compressible Navier–Stokes system supplemented with general inhomogeneous boundary conditions. Assuming the pressure to be given by the standard hard sphere EOS we show existence of weak solutions for arbitrarily large boundary data.

Keywords: Compressible Navier–Stokes system; Stationary solution; Inhomogeneous boundary conditions

1. Introduction

The boundary behavior of fluids influences essentially the motion inside their natural physical domain. Basically all real world applications in fluid mechanics contain boundary conditions as the main factor determining the behavior of the fluid. We consider the problem of identifying the stationary motion of a compressible viscous fluid driven by general in/out flux boundary conditions. Specifically, the mass density $\varrho = \varrho(x)$ and the velocity $\mathbf{u} = \mathbf{u}(x)$ of the fluid satisfy the Navier–Stokes system,

$$\mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{1.1}$$

$$\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}), \tag{1.2}$$

$$\mathbb{S}(\nabla_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u}\right) + \lambda\mathrm{div}_x\mathbf{u}\mathbb{I},\ \mu > 0,\ \lambda \geq 0, \tag{1.3}$$

in $\Omega \subset R^N$, $N = 2, 3$, where $p = p(\varrho)$ is the barotropic pressure. We consider rather general boundary conditions,

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$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B,\ \varrho|_{\Gamma_{\rm in}} = \varrho_B, \tag{1.4}$$

where

$$\Gamma_{\rm in} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_B \cdot \mathbf{n} < 0 \right\},\ \Gamma_{\rm out} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_B \cdot \mathbf{n} > 0 \right\}. \tag{1.5}$$

We concentrate on the inflow/outflow phenomena, we have therefore deliberately omitted the contribution of external forces $\varrho\mathbf{f}$. Nevertheless, all results of this paper remain valid also in the presence of external forces.

Investigation and better insight to the equations in this setting is important for many real world applications. In fact this is a natural and basic abstract setting for flows in pipelines, wind tunnels, turbines to name a few concrete examples. In spite of this fact the problem resists to all attempts of its solution for decades. To the best of our knowledge, this is the first work ever treating this system for large boundary data.

Indeed, the only results available in setting (1.1)–(1.5) are those on existence of strong solutions for small boundary data perturbations of an equilibrium state or of a specific given flow in a particular geometry (as e.g. the Poiseuille flow in a cylinder) in isentropic regime, see e.g. Plotnikov, Ruban, Sokolowski [19], Mucha, Piasecki [14], Piasecki [17], Piasecki and Pokorny [18] among others. The only available results on existence of weak solutions for large flows treat system (1.1)–(1.3) with large external force at the right hand side of equation (1.2) and with homogeneous Dirichlet boundary condition for velocity ($\mathbf{u}|_{\partial\Omega} = 0$) or with Navier slip boundary conditions ($\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega=0}$, $(\mathbb{S}(\nabla_x \mathbf{u})\mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0$). For the examples of latter relevant results, see Lions [13], Brezina, Novotny [2], Frehse, Steinhauer, Weigant [9], Plotnikov, Sokolowski [22], Jiang, Zhou [12] among others.

Our goal is to establish the existence of a weak solution $[\varrho, \mathbf{u}]$ to problem (1.1)–(1.5) for general *large* boundary data ϱ_B, $\mathbf{u}_B$. Our approach is based on two physically grounded hypotheses:

- **Molecular hypothesis (hard sphere model).** The specific volume of the fluid is bounded below away from zero. Equivalently, the fluid density cannot exceed a limit value $\overline{\varrho} > 0$. Accordingly, the pressure $p = p(\varrho)$ satisfies

$$\lim_{\varrho \to \overline{\varrho}} p(\varrho) = \infty. \tag{1.6}$$

- **Positive compressibility.** The pressure $p = p(\varrho)$ is a non-decreasing function of the density, more precisely

$$p \in C[0, \overline{\varrho}) \cap C^1(0, \overline{\varrho}),\ \ p(0) = 0,\ \ p'(\varrho) \geq 0 \text{ for } \varrho \geq 0. \tag{1.7}$$

The reader may consult Carnahan and Starling [3] for the physical background of hypotheses (1.6), (1.7). Although apparently satisfied by any *real* fluid, condition (1.6) eliminates the more standard equations of state $p(\varrho) = a\varrho^\gamma$ used for the isentropic gases.

Clearly, the fact that the density is *a priori* expected to be confined to a bounded interval $[0, \overline{\varrho}]$ facilitates the analysis. On the other hand the presence of non-zero boundary data makes the analysis more difficult. The proper construction of solutions in this setting is far from being obvious. We use a method relying on a suitable elliptic regularization based on adding artificial diffusion terms to both continuity and momentum equations, along with nonlinear flux-type boundary conditions for density. Note that the more standard approximation based on solving directly the transport equation (1.1), see e.g. Plotnikov, Ruban, Sokolowski [19], [20], [21] is unlikely to work for the large data problems. The proof contains a construction of a suitable extension of the boundary velocity field in the class of functions with positive divergence that may be of independent interest. Compactness of the family of approximate solutions is established by means of compensated compactness combined with the monotone operator theory in the spirit of Lions' work [13].

The paper is organized in the following way. In Section 2, we introduce the concept of weak solution to problem (1.1)–(1.5) and state our main result (Theorem 2.1 and Remark 2.3). Section 3 contains preliminary material concerning extension of vector fields given on $\partial\Omega$. The approximate solutions are constructed in Section 4. After having established the necessary uniform bounds in Section 5, the limit in the family of approximate solutions is performed in Section 7.

2. Main result

In order to avoid additional technicalities, we suppose the $\Omega \subset R^N$, $N = 2, 3$ is a bounded simply connected domain with a boundary of class $C^{2,1}$. The boundary data satisfy

$$\mathbf{u}_B \in C^2(\partial\Omega; R^N), \ \varrho_B \in C(\partial\Omega). \tag{2.1}$$

We say that $[\varrho, \mathbf{u}]$ is a weak solution of problem (1.1)–(1.4) if:

- $$\varrho \in L^\infty(\Omega), \ 0 \le \varrho < \overline{\varrho} \text{ a.a. in } \Omega, \ \mathbf{u} \in W^{1,2}(\Omega; R^N), \ \mathbf{u}|_{\partial\Omega} = \mathbf{u}_B;$$
- the integral identity
$$-\int_\Omega \varrho\mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x + \int_{\partial\Omega} \varrho_B \mathbf{u}_B \cdot \mathbf{n}\varphi \ \mathrm{d}S_x = 0$$
holds for any $\varphi \in C^1(\overline{\Omega})$, $\varphi|_{\Gamma_{\rm out}} = 0$;
- $p(\varrho) \in L^2(\Omega)$, and the integral identity
$$\int_\Omega \left[(\varrho\mathbf{u} \otimes \mathbf{u}) : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x = \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x$$
holds for any $\boldsymbol{\varphi} \in C^1_c(\Omega; R^N)$.

Our main result is the following theorem.

Theorem 2.1. *Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded simply connected domain of class $C^{2,1}$. Let the boundary data $\mathbf{u}_B$, ϱ_B satisfy (2.1), and*

$$0 < \min \varrho_B \le \max \varrho_B < \overline{\varrho}, \ \int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \ \mathrm{d}S_x \ge 0. \tag{2.2}$$

Let the pressure satisfy hypotheses (1.6), (1.7).

Then problem (1.1)–(1.4) possesses at least one weak solution $[\varrho, \mathbf{u}]$.

Remark 2.2. Hypothesis

$$\int_\Omega \mathbf{u}_B \cdot \mathbf{n} \ \mathrm{d}x \ge 0$$

is not optimal in view of the available *small* data results, see e.g. Plotnikov, Ruban, Sokolowski [20]. On the other hand, it implies $\Gamma_{\rm out} \ne \emptyset$ whenever $\Gamma_{\rm in} \ne \emptyset$. It is easy to see that the problem is not solvable if $\Gamma_{\rm out} = \emptyset$ unless $\Gamma_{\rm in} = \emptyset$. In the latter case, the total mass of the fluid

$$\int_\Omega \varrho \ \mathrm{d}x = M$$

must be prescribed to avoid the trivial solution $\varrho = 0$, see [4]. In general, as the density is bounded above by $\overline{\varrho}$, the problem may not be solvable for *arbitrary* ϱ_B if

$$\int_\Omega \mathbf{u}_B \cdot \mathbf{n} \ \mathrm{d}x < 0.$$

Remark 2.3. For the sake of simplicity, we treat the case of smooth boundary with smooth boundary data. The method can be easily adapted to more complex geometries as the case of a bounded cylinder handled in [18]. More precisely, a direct inspection of the proof presented in the remaining part of this paper yields the conclusion of Theorem 2.1 provided Ω is a bounded Lipschitz domain, the boundary $\partial\Omega$ of which consists of a finite number of components of class $C^{2,1}$, specifically,

$$\partial\Omega = \left(\cup_{i=1}^I \overline{\Gamma}_{i,\rm in} \right) \cup \left(\cup_{j=1}^J \overline{\Gamma}_{j,\rm out} \right) \cup \overline{\Gamma}_0,$$

where $\Gamma_{i,\rm in}$, $\Gamma_{j,\rm out}$, Γ_0 are of class $C^{2,1}$, $\overline{\Gamma}_{i,\rm in}$, $\overline{\Gamma}_{j,\rm out}$ are mutually disjoint, $\Gamma_{i,\rm in}$, $\Gamma_{j,\rm out}$ simply connected, such that

$$\mathbf{u}_B \cdot \mathbf{n} < 0 \text{ on } \Gamma_{i,\rm in},\ \mathbf{u}_B \cdot \mathbf{n} > 0 \text{ on } \Gamma_{j,\rm out},\ i = 1, \dots, I,\ j = 1, \dots, J,\ \mathbf{u}_B \cdot \mathbf{n} = 0 \text{ on } \Gamma_0.$$

Remark 2.4. For the sake of simplicity, the action of external forces is omitted in (1.2). However Theorem 2.1 remains valid if a driving force $\varrho \mathbf{f}$, $\mathbf{f} \in L^\infty(\Omega)$ is added to (1.2). The proof requires only minor modifications.

3. Preliminaries

Without loss of generality, we will assume that $\Gamma_{\rm in} \neq \emptyset$; whence, in accordance with hypothesis (2.2), $\overline{\Gamma}_{\rm in} \neq \partial\Omega$ and $\Gamma_{\rm out} \neq \emptyset$. The reader may consult [4], for the discussion of the problem with tangential boundary data $\mathbf{u}_B$.

3.1. Renormalization

Lemma 3.1. *Suppose that $\Omega \subset R^N$ is a bounded simply connected domain of class $C^{2,1}$,*

$$\mathbf{u}_B \in W^{2-\frac{1}{q},q}(\partial\Omega; R^N) \text{ for some } q > N,\ \overline{\Gamma}_{\rm in} \neq \partial\Omega.$$

Then there exists $\tilde{\mathbf{u}}_B \in W^{1,\infty}(R^N \setminus \Omega; R^N)$ such that

$$\mathrm{div}_x \tilde{\mathbf{u}}_B = 0,\ \tilde{\mathbf{u}}_B|_{\Gamma_{\rm in}} = \mathbf{u}_B|_{\Gamma_{\rm in}},\ \tilde{\mathbf{u}}_B \cdot \mathbf{n}|_{\partial\Omega\setminus\Gamma_{\rm in}} \geq 0.$$

Proof. As $\Omega \setminus \overline{\Gamma}_{\rm in}$ is a non-empty open subset of $\partial\Omega$. We can extend the field $\mathbf{u}_B$ outside $\Gamma_{\rm in}$ in such a way that

$$\mathbf{u}_B \cdot \mathbf{n}|_{\partial\Omega\setminus\Gamma_{\rm in}} \geq 0,\ \int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n}\, \mathrm{d}S_x = 0.$$

The existence of the solenoidal extension $\tilde{\mathbf{u}}_B$ is then standard, see [10, Chapter IX, Section IX.4]. More precisely, we construct the field $\tilde{\mathbf{u}}_B$ in the class $W^{2,q}$ and use the Sobolev embedding $W^{2,q} \hookrightarrow W^{1,\infty}$. □

Consider now the flow generated by the field $-\tilde{\mathbf{u}}_B$, where $\tilde{\mathbf{u}}_B$ is the extension of the boundary data constructed in Lemma 3.1,

$$\mathbf{X}'(t, \mathbf{x}_0) = -\tilde{\mathbf{u}}_B(\mathbf{X}(t, \mathbf{x}_0)),\ \mathbf{X}(0) = \mathbf{x}_0 \in \partial\Omega. \tag{3.1}$$

Let

$$\tilde{\Omega}_\delta = \left\{ x \in R^N \setminus \overline{\Omega} \,\middle|\, x \in \mathbf{X}(\tau, \mathbf{x}_0) \text{ for a certain } 0 < \tau < \delta,\ \mathbf{x}_0 \in \partial\Omega,\ \cup_{0<t\leq\tau} \mathbf{X}(t, \mathbf{x}_0) \subset R^N \setminus \overline{\Omega} \right\}$$

As solutions of (3.1) depend continuously on the initial data, the set $\tilde{\Omega}_\delta$ is open. Moreover, as $\Gamma_{\rm in} \neq \emptyset$, we have $\tilde{\Omega}_\delta \neq \emptyset$. In addition, if $x \in \tilde{\Omega}_\delta$, $x = \mathbf{X}(\tau, \mathbf{x}_0)$, then $x_0 \in \Gamma_{\rm in}$ and we may extend the boundary data ϱ_B to $\tilde{\Omega}_\delta$ setting $\tilde{\varrho}_B(\mathbf{X}(\tau, x_0)) = \varrho_B(x_0)$. Accordingly, as $\tilde{\mathbf{u}}_B$ is solenoidal,

$$\mathrm{div}_x(\tilde{\varrho}_B \tilde{\mathbf{u}}_B) = 0 \text{ in } \tilde{\Omega}_\delta,\ \tilde{\varrho}_B|_{\Gamma_{\rm in}} = \varrho_B. \tag{3.2}$$

Lemma 3.2. *Suppose that $\Omega \subset R^N$ is a bounded simply connected domain of class $C^{2,1}$,*

$$\mathbf{u}_B \in W^{2-\frac{1}{q},q}(\partial\Omega; R^N) \text{ for some } q > N,\ \overline{\Gamma}_{\rm in} \neq \partial\Omega.$$

Let $[\varrho, \mathbf{u}]$, $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B$ be a weak solution to the problem

$$\mathrm{div}_x(\varrho \mathbf{u}) = 0 \text{ in } \Omega,\ \varrho|_{\Gamma_{\rm in}} = \varrho_B,$$

specifically

$$\varrho \in L^\infty(\Omega),\ \mathbf{u} \in W^{1,2}(\Omega; R^N),$$
$$-\int_\Omega \varrho \mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x + \int_{\partial\Omega} \varrho_B \mathbf{u}_B \cdot \mathbf{n} \varphi \ \mathrm{d}S_x = 0$$

for any $\varphi \in C^1(\overline{\Omega})$, $\varphi|_{\Gamma_{\rm out}} = 0$.

Then $[\varrho, \mathbf{u}]$ *is also a renormalized solution, meaning*

$$-\int_\Omega b(\varrho)\mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x + \int_\Omega \varphi \left(b'(\varrho)\varrho - b(\varrho)\right) \mathrm{div}_x \mathbf{u} \ \mathrm{d}x + \int_{\partial\Omega} b(\varrho_B)\mathbf{u}_B \cdot \mathbf{n}\varphi \ \mathrm{d}S_x = 0 \tag{3.3}$$

for any $\varphi \in C^1(\overline{\Omega})$, $\varphi|_{\Gamma_{\rm out}} = 0$, *and any continuously differentiable* b.

Proof. We may extend $[\varrho, \mathbf{u}]$ as $[\tilde{\varrho}_B, \tilde{\mathbf{u}}_B]$ in $\Omega \cup \Gamma_{\rm in} \cup \tilde{\Omega}_\delta$ so that the extended functions satisfy the equation of continuity in the domain $B_\delta = \Omega \cup \Gamma_{\rm in} \cup \tilde{\Omega}_\delta$.

Next, we use the regularization procedure due to DiPerna and Lions [5] applying convolution with a family of regularizing kernels obtaining

$$\mathrm{div}_x([\varrho]_\varepsilon [\mathbf{u}]_\varepsilon) = R_\varepsilon \text{ in } B_{\varepsilon,\delta} = \left\{ x \in B_\delta \ \middle|\ \mathrm{dist}[x, \partial B_\delta] > \varepsilon \right\},\ R_\varepsilon \to 0 \text{ in } L^2(B_\delta) \text{ as } \varepsilon \to 0 \tag{3.4}$$

for the regularized functions $[\varrho]_\varepsilon$, $[\mathbf{u}]_\varepsilon$.

Multiplying equation (3.4) on $b'([\varrho]_\varepsilon)$, we get

$$\mathrm{div}_x(b([\varrho]_\varepsilon)[\mathbf{u}]_\varepsilon) + \left(b'([\varrho]_\varepsilon)[\varrho]_\varepsilon - b([\varrho]_\varepsilon)\right) \mathrm{div}_x \mathbf{u} = b'([\varrho]_\varepsilon) R_\varepsilon$$

or

$$-\int_{B_\delta} b([\varrho]_\varepsilon)[\mathbf{u}]_\varepsilon \cdot \nabla_x \varphi \ \mathrm{d}x + \int_{B_\delta} \varphi \left(b'([\varrho]_\varepsilon)[\varrho]_\varepsilon - b([\varrho]_\varepsilon)\right) \mathrm{div}_x [\mathbf{u}]_\varepsilon \ \mathrm{d}x = \int_{B_\delta} \varphi b'([\varrho]_\varepsilon) R_\varepsilon$$

for any $\varphi \in C^1_c(B_\delta)$. Thus, letting $\varepsilon \to 0$ we get

$$-\int_{B_\delta} b(\varrho)\mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x + \int_{B_\delta} \varphi \left(b'(\varrho)\varrho - b(\varrho)\right) \mathrm{div}_x \mathbf{u} \ \mathrm{d}x = 0$$

for any $\varphi \in C^1_c(B_\delta)$.

Seeing that $\varrho = \tilde{\varrho}_B$, $\mathbf{u} = \tilde{\mathbf{u}}_B$ are continuous on $\overline{\tilde{\Omega}}_\delta$, we deduce that

$$-\int_{B_\delta} b(\varrho)\mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x + \int_{B_\delta} \varphi \left(b'(\varrho)\varrho - b(\varrho)\right) \mathrm{div}_x \mathbf{u} \ \mathrm{d}x + \int_{\partial B_\delta} b(\tilde{\varrho}_B)\tilde{\mathbf{u}}_B \cdot \mathbf{n}\varphi \ \mathrm{d}S_x = 0$$

for any $\varphi \in C^1(\overline{B_\delta})$, $\varphi|_{\Gamma_{\rm out}} = 0$. Finally, we let $\delta \to 0$ to obtain the desired conclusion. □

3.2. *Extension of the boundary velocity inside* Ω

Our goal is to find a suitable extension of the boundary field $\mathbf{u}_B$ into Ω so that a suitable norm of this extension is "small". Such a result is intimately related to the so-called Leray's inequality in the context of incompressible fluids, see e.g. Galdi [10]. We start with an auxiliary result.

Lemma 3.3. *Let* $\Omega \subset R^N$, $N = 2, 3$ *be a bounded simply connected domain of class* $C^{2+\nu}$. *Let* $K > 0$ *and* $\varepsilon > 0$ *be given.*

Then there exists a vector field $\mathbf{V} \in C^2(\overline{\Omega}; R^N)$ *enjoying the following properties:*

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \ \mathrm{d}S_x = K, \tag{3.5}$$

$$\mathrm{div}_x \mathbf{V} \ge 0 \text{ in } \Omega, \tag{3.6}$$

$$\|\mathbf{V}\|_{L^4(\Omega)} < \varepsilon. \tag{3.7}$$

Proof. Let $d_\Omega(x)$ denote the distance of $x \in \overline{\Omega}$ to $\partial\Omega$. As Ω is of class $C^{2+\nu}$, there is an open neighborhood $\mathcal{U}$ of $\partial\Omega$ in Ω such that d_Ω belongs to the same class, see e.g. Foote [8]. Moreover, for any $x \in \mathcal{U}$ there exists a unique point $x_B = x_B(x) \in \partial\Omega$ such that $d_\Omega(x) = |x - x_B|$. Next, as Ω is bounded simply connected, there exists a non-empty open subset $\mathcal{C} \subset \partial\Omega$ such that the set

$$\left\{ x \in \mathcal{U} \ \middle| \ x_B(x) \in \mathcal{C} \right\} \text{ is convex.}$$

Finally, as the boundary distance functions of convex domains are superharmonic, see e.g. Armitage and Kuran [1], we may infer that there exists a non-empty open connected subset $B \subset \mathcal{C} \subset \partial\Omega$ such that

$$\Delta_x d_\Omega(x) \le 0 \text{ whenever } x \in \mathcal{U},\ x_B(x) \in B. \tag{3.8}$$

We consider a function $\Lambda \in C^2(\partial\Omega)$ such that

$$\Lambda \in C^2_c(B),\ \Lambda \ge 0,\ \int_{\partial\Omega} \Lambda \ \mathrm{d}S_x = \int_{\partial\Omega} \Lambda \mathbf{n} \cdot \mathbf{n} \, \mathrm{d}S_x = K.$$

Thus our ultimate goal is to extend the vector field $\Lambda\mathbf{n}$ inside Ω with a small L^4-norm. To this end, we set

$$\mathbf{V}(x) = -\Lambda(x_B(x)) h_\delta(d_\Omega(x)) \nabla_x d_\Omega(x),\ x \in \Omega.$$

The function h_δ is chosen in such a way that

$$h_\delta \in C^\infty_c[0,\infty),\ h' \le 0,\ h(0) = 1,\ h(y) = 0 \text{ whenever } y \ge \delta.$$

In particular, taking $\delta > 0$ small enough, $h_\delta(d_\Omega(x)) = 0$ for $x \in \Omega \setminus \mathcal{U}$.

It is easy to check that $\mathbf{V} = \Lambda\mathbf{n}$ on $\partial\Omega$ and as such satisfies (3.5). Moreover, taking $\delta = \delta(K) > 0$ small enough we achieve (3.7). Finally, we compute

$$\begin{aligned} \mathrm{div}_x \mathbf{V}(x) &= -\Lambda(x_B(x)) h'_\delta(d_\Omega(x)) |\nabla_x d_\Omega(x)|^2 - \Lambda(x_B(x)) h_\delta(d_\Omega(x)) \Delta_x d_\Omega(x) \\ &\quad - h_\delta(d_\Omega(x)) \nabla_x(\Lambda(x_B(x))) \cdot \nabla_x d_\Omega(x) \ge -h_\delta(d_\Omega(x)) \nabla_x(\Lambda(x_B(x))) \cdot \nabla_x d_\Omega(x). \end{aligned}$$

As the function $\Lambda(x_B(x))$ is constant on each segment $[x, x_B(x)]$ and $\nabla_x d_\Omega(x)$ is parallel to this segment, we get

$$\nabla_x(\Lambda(x_B(x))) \cdot \nabla_x d_\Omega(x) = 0$$

and (3.6) follows. □

We are ready to establish the main result of the present section.

Proposition 3.4. *Suppose that $\Omega \subset R^N$ is a bounded simply connected domain of class $C^{2,1}$,*

$$\mathbf{u}_B \in W^{2-\frac{1}{q},q}(\partial\Omega; R^N) \textit{ for some } q > N,\ \overline{\Gamma}_{\mathrm{in}} \ne \partial\Omega.$$

Then, given $\omega > 0$, $\mathbf{u}_B$ can be extended as a function in $W^{1,\infty}(\Omega; R^N)$ such that

$$\mathrm{div}_x \mathbf{u}_B \ge 0 \textit{ a.a. in } \Omega.$$

$$\int_\Omega |\mathbf{v} \cdot \nabla_x \mathbf{v} \cdot \mathbf{u}_B| \ \mathrm{d}x \le \omega \|\nabla_x \mathbf{v}\|^2_{L^2(\Omega; R^{N\times N})} \textit{ for any } \mathbf{v} \in W^{1,2}_0(\Omega; R^N). \tag{3.9}$$

Proof. It is well known that the result holds if the total flux over the boundary vanishes, meaning

$$\int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \, \mathrm{d}S_x = 0.$$

In such a case, the desired extension can be constructed to be solenoidal $\mathrm{div}_x \mathbf{u}_B = 0$, see Finn [7], Hopf [11], Galdi [10].

If

$$\int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \, \mathrm{d}S_x = K > 0$$

we may write $\mathbf{u}_B = \mathbf{u}_B - \mathbf{V} + \mathbf{V}$, where $\mathbf{V}$ is the vector field constructed in Lemma 3.3. Seeing that

$$\int_{\partial\Omega} (\mathbf{u}_B - \mathbf{V}) \cdot \mathbf{n} \, \mathrm{d}S_x = 0$$

we may apply to above mentioned result to extend $\mathbf{u}_B - \mathbf{V}$. The desired inequality (3.9) is then achieved by adjusting $\mathbf{V}$ to satisfy (3.7) with a sufficiently small $\varepsilon > 0$. □

4. Approximate problems

Our goal is to construct solutions the existence of which is claimed in Theorem 2.1. To this end, we adopt the approximation scheme based on pressure regularization and adding artificial viscosity terms to both (1.1) and (1.2). Although the scheme is fairly similar to that used in [4] for the tangential velocity $\mathbf{u}_B$, the presence of the in/out flux boundary terms requires a non-trivial modification of some arguments presented in [4]. The approximate problems read:

$$-\delta \Delta_x \varrho + \delta \varrho + \mathrm{div}_x (T(\varrho)\mathbf{u}) = 0, \tag{4.1}$$

$$(-\delta \nabla_x \varrho + T(\varrho)\mathbf{u}) \cdot \mathbf{n}|_{\partial\Omega} = \begin{cases} T(\varrho_B)\mathbf{u}_B \cdot \mathbf{n} \text{ if } \mathbf{u}_B \cdot \mathbf{n} \le 0, \\ T(\varrho)\mathbf{u}_B \cdot \mathbf{n} \text{ if } \mathbf{u}_B \cdot \mathbf{n} > 0, \end{cases} \tag{4.2}$$

$$\mathrm{div}_x (T(\varrho)\mathbf{u} \otimes \mathbf{u}) + \nabla_x p_{\varepsilon,\delta}(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \delta \Delta_x (\varrho \mathbf{u}) - \delta \varrho \mathbf{u}, \tag{4.3}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B, \tag{4.4}$$

with positive parameters $\varepsilon > 0$, $\delta > 0$, where we have denoted

$$T(\varrho) = \left\{ \begin{array}{l} 0 \text{ if } \varrho \le 0 \\ \varrho \text{ if } 0 \le \varrho \le \overline{\varrho} \\ \overline{\varrho} \text{ if } \varrho \ge \overline{\varrho} \end{array} \right\}, \quad p_{\varepsilon,\delta}(\varrho) = p_\varepsilon(\varrho) + \sqrt{\delta}\varrho,$$

$$p_\varepsilon(\varrho) = \left\{ \begin{array}{l} p(\varrho) \text{ if } 0 \le \varrho \le \overline{\varrho} - \varepsilon \\ p(\overline{\varrho} - \varepsilon) + p'(\overline{\varrho} - \varepsilon)(\varrho - \overline{\varrho} + \varepsilon) \text{ if } \varrho > \overline{\varrho} - \varepsilon \end{array} \right\}.$$

4.1. Solvability of the approximate problems

We adopt the nowadays standard procedure based on computing the approximate density ϱ in terms of $\mathbf{u}$ in (4.1), (4.2) and applying a fixed point argument. We start by recalling the weak formulation of (4.1), (4.2),

$$\int_{\Omega} \left[\delta \nabla_x \varrho \cdot \nabla_x \varphi + \delta \varrho \varphi - T(\varrho)\mathbf{u} \cdot \nabla_x \varphi \right] \, \mathrm{d}x = - \int_{\Gamma_{\mathrm{in}}} T(\varrho_B)\mathbf{u}_B \cdot \mathbf{n} \varphi \mathrm{d}S_x - \int_{\Gamma_{\mathrm{out}}} T(\varrho)\mathbf{u}_B \cdot \mathbf{n} \varphi \mathrm{d}S_x \tag{4.5}$$

for any $\varphi \in W^{1,2}(\Omega)$.

Lemma 4.1. *Suppose that $\mathbf{u} \in W^{1,2}(\Omega)$ complies with the boundary conditions (4.4). Let $\varrho_1, \varrho_2 \in W^{1,2}(\Omega)$ be two functions satisfying*

$$\int_{\Omega} \left[\delta \nabla_x \varrho_1 \cdot \nabla_x \varphi + \delta \varrho_1 \varphi - T(\varrho_1)\mathbf{u} \cdot \nabla_x \varphi \right] \, \mathrm{d}x \ge - \int_{\Gamma_{\mathrm{in}}} T(\varrho_B)\mathbf{u}_B \cdot \mathbf{n} \varphi \mathrm{d}S_x - \int_{\Gamma_{\mathrm{out}}} T(\varrho_1)\mathbf{u}_B \cdot \mathbf{n} \varphi \mathrm{d}S_x \tag{4.6}$$

for any $\varphi \in W^{1,2}(\Omega)$, $\varphi \ge 0$,

$$\int_\Omega \left[\delta\nabla_x\varrho_2\cdot\nabla_x\varphi+\delta\varrho_2\varphi-T(\varrho_2)\mathbf{u}\cdot\nabla_x\varphi\right]\,\mathrm{d}x\le-\int_{\Gamma_{\rm in}} T(\varrho_B)\mathbf{u}_B\cdot\mathbf{n}\varphi\mathrm{d}S_x-\int_{\Gamma_{\rm out}} T(\varrho_2)\mathbf{u}_B\cdot\mathbf{n}\varphi\mathrm{d}S_x \tag{4.7}$$

for any $\varphi\in W^{1,2}(\Omega)$, $\varphi\ge 0$, respectively.

Then

$$\varrho_1\ge\varrho_2 \text{ a.a. in } \Omega.$$

Corollary 4.2. *For a given $\mathbf{u}\in W^{1,2}(\Omega)$, problem (4.5) admits at most one solution $\varrho\in W^{1,2}(\Omega)$. Any solution $\varrho\in W^{1,2}(\Omega)$ of (4.5) satisfies $\varrho\ge 0$.*

Proof. Let $\Phi: R\to R$ be twice continuously differentiable *convex* function such that $\Phi'(Z)=-1$ if $Z\le -1$, $\Phi(Z)=0$ if $Z\ge 0$. Accordingly, $\varphi=-\Phi'(\varrho_1-\varrho_2)\ge 0$ can be used as a test function in (4.6), (4.7), respectively to obtain

$$\delta\int_\Omega\left[\Phi''(\varrho_1-\varrho_2)|\nabla_x(\varrho_1-\varrho_2)|^2+(\varrho_1-\varrho_2)\Phi'(\varrho_1-\varrho_2)\right]\,\mathrm{d}x$$
$$\le-\int_{\Gamma_{\rm out}}(T(\varrho_1)-T(\varrho_2))\,\Phi'(\varrho_1-\varrho_2)\mathbf{u}_B\cdot\mathbf{n}\mathrm{d}S_x+\int_\Omega (T(\varrho_1)-T(\varrho_2))\mathbf{u}\cdot\nabla_x(\varrho_1-\varrho_2)\Phi''(\varrho_1-\varrho_2)\,\mathrm{d}x$$

Seeing that (i) Φ is convex, (ii) $(T(\varrho_1)-T(\varrho_2))\Phi'(\varrho_1-\varrho_2)\ge 0$, $\mathbf{u}_B\cdot\mathbf{n}\ge 0$ on $\Gamma_{\rm out}$, we deduce that

$$\delta\int_\Omega(\varrho_1-\varrho_2)\Phi'(\varrho_1-\varrho_2)\,\mathrm{d}x$$
$$\le\int_\Omega(T(\varrho_1)-T(\varrho_2))\,\mathbf{u}\cdot\nabla_x(\varrho_1-\varrho_2)\Phi''(\varrho_1-\varrho_2)\,\mathrm{d}x$$

Finally, we have

$$\int_\Omega(T(\varrho_1)-T(\varrho_2))\,\mathbf{u}\cdot\nabla_x(\varrho_1-\varrho_2)\Phi''(\varrho_1-\varrho_2)\,\mathrm{d}x\le\int_\Omega|\mathbf{u}\cdot\nabla_x(\varrho_1-\varrho_2)|\,|\varrho_1-\varrho_2|\Phi''(\varrho_1-\varrho_2)\,\mathrm{d}x$$

Consequently, approximating $[Z]^-=\max\{-Z,0\}$ by a family of convex functions Φ we obtain the desired conclusion

$$\int_\Omega[\varrho_1-\varrho_2]^-\,\mathrm{d}x=0.\qquad\Box$$

Seeing that *existence* of a solution $\varrho=\varrho[\mathbf{u}]$ to problem (4.1), (4.2) for a given $\mathbf{u}\in W^{1,2}(\Omega)$ can be established by the classical method of monotone operators, see e.g. Nittka [15], we may define an operator

$$\mathbf{u}\mapsto\varrho[\mathbf{u}]$$

where ϱ is the unique solution of (4.1), (4.2). In addition, the standard elliptic regularity estimates imply that $\varrho\in W^{2,2}(\Omega)$, in particular, (4.1) holds a.a. in Ω and the boundary conditions (4.2) are satisfied in the sense of traces. Under these circumstances, the existence of approximate solutions $[\varrho,\mathbf{u}]$ to problem (4.1)–(4.4) can be shown in a similar way as in [16, Chapter 4].

5. Uniform bounds for the approximate problems

As our ultimate goal is to establish convergence of the sequence of approximate solutions, uniform bounds independent of the parameters ε, δ must be established. In the following text we use the notation

$$a\stackrel{<}{\sim}b \text{ for } a\le cb,$$

where c is a positive constant independent of the approximation parameters. Similarly, we define $a \gtrsim b$, and $a \approx b$ whenever $-a \lesssim -b$, and $a \lesssim b$, $b \lesssim a$, respectively. In accordance with Proposition 3.4, we suppose that the field $\mathbf{u}_B$ is defined on the whole domain Ω and enjoys the following properties:

$$\mathbf{u}_B \in W^{1,\infty}(\Omega; R^N), \tag{5.1}$$

$$\mathrm{div}_x \mathbf{u}_B \geq 0 \text{ a.a. in } \Omega, \tag{5.2}$$

$$\int_\Omega |\mathbf{v} \cdot \nabla_x \mathbf{v} \cdot \mathbf{u}_B| \ \mathrm{d}x \leq \omega \|\nabla_x \mathbf{v}\|^2_{L^2(\Omega; R^{N\times N})} \text{ for any } \mathbf{v} \in W_0^{1,2}(\Omega; R^N), \tag{5.3}$$

for a given $\omega > 0$ specified below.

Taking $\mathbf{u} - \mathbf{u}_B \in W_0^{1,2}(\Omega; R^N)$ as a test function in (4.3), we obtain

$$\begin{aligned} &\int_\Omega \left[T(\varrho)\mathbf{u} \otimes \mathbf{u} : \nabla_x(\mathbf{u} - \mathbf{u}_B) + p_{\varepsilon,\delta}(\varrho)\mathrm{div}_x \mathbf{u} \right] \ \mathrm{d}x \\ &= \int_\Omega \left[\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x(\mathbf{u} - \mathbf{u}_B) + p_{\varepsilon,\delta}(\varrho)\mathrm{div}_x \mathbf{u}_B + \delta \nabla_x(\varrho \mathbf{u}) : \nabla_x(\mathbf{u} - \mathbf{u}_B) + \delta \varrho \mathbf{u} \cdot (\mathbf{u} - \mathbf{u}_B) \right] \ \mathrm{d}x \end{aligned} \tag{5.4}$$

We proceed via several steps:

- By virtue of Poincaré's inequality and hypothesis (5.2),

$$\begin{aligned} &\int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x(\mathbf{u} - \mathbf{u}_B) + p_{\varepsilon,\delta}(\varrho)\mathrm{div}_x \mathbf{u}_B \ \mathrm{d}x \\ &\geq \int_\Omega \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{u}_B) : \nabla_x(\mathbf{u} - \mathbf{u}_B) \ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_B) : \nabla_x(\mathbf{u} - \mathbf{u}_B) \ \mathrm{d}x \\ &\gtrsim \|\mathbf{u} - \mathbf{u}_B\|^2_{W^{1,2}(\Omega; R^N)} - \|\nabla_x \mathbf{u}_B\|^2_{L^2(\Omega; R^{N\times N})} \gtrsim \|\mathbf{u}\|^2_{W^{1,2}(\Omega; R^N)} - 2\|\mathbf{u}_B\|^2_{W^{1,2}(\Omega; R^N)} \end{aligned} \tag{5.5}$$

- Next, we recall the renormalized form of (4.1), namely,

$$\delta G''(\varrho)|\nabla_x \varrho|^2 + \delta G'(\varrho)\varrho - \delta \mathrm{div}_x(G'(\varrho)\nabla_x \varrho) + \mathrm{div}_x(H(\varrho)\mathbf{u}) + \left[G'(\varrho)T(\varrho) - H(\varrho)\right]\mathrm{div}_x \mathbf{u} = 0, \tag{5.6}$$

  where

$$H'(\varrho) = G'(\varrho)T'(\varrho).$$

  We choose $G = G_{\varepsilon,\delta}$ such that

$$G''_{\varepsilon,\delta}(\varrho) = \frac{p'_{\varepsilon,\delta}(\varrho)}{T(\varrho)} \text{ or } G'_{\varepsilon,\delta}(\varrho)T(\varrho) - H_{\varepsilon,\delta}(\varrho) = p_{\varepsilon,\delta}(\varrho). \tag{5.7}$$

  Consequently, we may use relation (5.6) to express the pressure term in (5.4):

$$\begin{aligned} &\int_\Omega p_{\varepsilon,\delta}(\varrho)\mathrm{div}_x \mathbf{u} \ \mathrm{d}x \\ &= -\delta \int_\Omega \left[G''_{\varepsilon,\delta}(\varrho)|\nabla_x \varrho|^2 + \varrho G'_{\varepsilon,\delta}(\varrho) \right] \ \mathrm{d}x + \int_{\partial\Omega} \left[\delta G'_{\varepsilon,\delta}(\varrho)\nabla_x \varrho \cdot \mathbf{n} - H_{\varepsilon,\delta}(\varrho)\mathbf{u}_B \cdot \mathbf{n}\right] \mathrm{d}S_x. \end{aligned}$$

Furthermore, in view of the boundary conditions (4.2),

$$\int_{\partial\Omega} \left[\delta G'_{\varepsilon,\delta}(\varrho)\nabla_x\varrho\cdot\mathbf{n} - H_{\varepsilon,\delta}(\varrho)\mathbf{u}_B\cdot\mathbf{n}\right]\mathrm{d}S_x$$
$$= \int_{\Gamma_{\rm in}} \left[\delta G'_{\varepsilon,\delta}(\varrho)\nabla_x\varrho\cdot\mathbf{n} - H_{\varepsilon,\delta}(\varrho)\mathbf{u}_B\cdot\mathbf{n}\right]\mathrm{d}S_x + \int_{\Gamma_{\rm out}} \left[\delta G'_{\varepsilon,\delta}(\varrho)\nabla_x\varrho\cdot\mathbf{n} - H_{\varepsilon,\delta}(\varrho)\mathbf{u}_B\cdot\mathbf{n}\right]\mathrm{d}S_x$$
$$= -\int_{\Gamma_{\rm in}} \left(p_{\varepsilon,\delta}(\varrho) - G'_{\varepsilon,\delta}(\varrho)T(\varrho_B)\right)|\mathbf{u}_B\cdot\mathbf{n}|\mathrm{d}S_x - \int_{\Gamma_{\rm out}} H_{\varepsilon,\delta}(\varrho)\mathbf{u}_B\cdot\mathbf{n}\,\mathrm{d}S_x.$$

Thus we conclude that

$$-\int_\Omega p_{\varepsilon,\delta}(\varrho)\mathrm{div}_x\mathbf{u}\ \mathrm{d}x = \delta\int_\Omega \left[G''_{\varepsilon,\delta}(\varrho)|\nabla_x\varrho|^2 + \varrho G'_{\varepsilon,\delta}(\varrho)\right]\ \mathrm{d}x$$
$$+\int_{\Gamma_{\rm in}} \left(p_{\varepsilon,\delta}(\varrho) - G'_{\varepsilon,\delta}(\varrho)T(\varrho_B)\right)|\mathbf{u}_B\cdot\mathbf{n}|\mathrm{d}S_x + \int_{\Gamma_{\rm out}} H_{\varepsilon,\delta}(\varrho)\mathbf{u}_B\cdot\mathbf{n}\,\mathrm{d}S_x. \tag{5.8}$$

- Next,

$$\int_\Omega T(\varrho)\mathbf{u}\otimes\mathbf{u} : \nabla_x(\mathbf{u}-\mathbf{u}_B)\ \mathrm{d}x$$
$$= \int_\Omega T(\varrho)\mathbf{u}\otimes(\mathbf{u}-\mathbf{u}_B) : \nabla_x(\mathbf{u}-\mathbf{u}_B)\ \mathrm{d}x + \int_\Omega T(\varrho)\mathbf{u}\otimes\mathbf{u}_B : \nabla_x(\mathbf{u}-\mathbf{u}_B)\ \mathrm{d}x$$
$$= \int_\Omega T(\varrho)\mathbf{u}\cdot\frac{1}{2}\nabla_x|\mathbf{u}-\mathbf{u}_B|^2\ \mathrm{d}x + \int_\Omega T(\varrho)\mathbf{u}_B\otimes\mathbf{u}_B : \nabla_x(\mathbf{u}-\mathbf{u}_B)\ \mathrm{d}x$$
$$+\int_\Omega T(\varrho)(\mathbf{u}-\mathbf{u}_B)\otimes\mathbf{u}_B : \nabla_x(\mathbf{u}-\mathbf{u}_B)\ \mathrm{d}x,$$

where, in accordance with (5.1),

$$\left|\int_\Omega T(\varrho)\mathbf{u}_B\otimes\mathbf{u}_B : \nabla_x(\mathbf{u}-\mathbf{u}_B)\ \mathrm{d}x\right|$$
$$\leq \overline{\varrho}\|\mathbf{u}_B\|^2_{L^4(\Omega;R^N)}\|\nabla_x(\mathbf{u}-\mathbf{u}_B)\|_{L^2(\Omega;R^{N\times N})} \lesssim \|\nabla_x(\mathbf{u}-\mathbf{u}_B)\|_{L^2(\Omega;R^{N\times N})}. \tag{5.9}$$

Using equation (4.1) we compute

$$\int_\Omega T(\varrho)\mathbf{u}\cdot\frac{1}{2}\nabla_x|\mathbf{u}-\mathbf{u}_B|^2\ \mathrm{d}x = -\frac{1}{2}\int_\Omega |\mathbf{u}-\mathbf{u}_B|^2\mathrm{div}_x(T(\varrho)\mathbf{u})\ \mathrm{d}x$$
$$= \frac{1}{2}\int_\Omega |\mathbf{u}-\mathbf{u}_B|^2\left(-\delta\Delta_x\varrho + \delta\varrho\right)\ \mathrm{d}x$$
$$= \delta\int_\Omega \nabla_x\varrho\cdot\frac{1}{2}\nabla_x|\mathbf{u}-\mathbf{u}_B|^2\ \mathrm{d}x + \frac{1}{2}\int_\Omega \delta\varrho|\mathbf{u}-\mathbf{u}_B|^2\ \mathrm{d}x$$

On the other hand, computing the remaining integral on the right-hand side of (5.4), we get

$$
\begin{aligned}
&\int_\Omega \delta \nabla_x(\varrho \mathbf{u}) : \nabla_x(\mathbf{u}-\mathbf{u}_B) + \delta \varrho \mathbf{u}\cdot(\mathbf{u}-\mathbf{u}_B) \ \mathrm{d}x \\
&= \int_\Omega \delta \nabla_x(\varrho(\mathbf{u}-\mathbf{u}_B)) : \nabla_x(\mathbf{u}-\mathbf{u}_B) + \delta \varrho(\mathbf{u}-\mathbf{u}_B)\cdot(\mathbf{u}-\mathbf{u}_B) \ \mathrm{d}x \\
&+ \int_\Omega \delta \nabla_x(\varrho \mathbf{u}_B) : \nabla_x(\mathbf{u}-\mathbf{u}_B) + \delta \varrho \mathbf{u}_B\cdot(\mathbf{u}-\mathbf{u}_B) \ \mathrm{d}x \\
&= \int_\Omega \delta \varrho \left[|\nabla_x(\mathbf{u}-\mathbf{u}_B)|^2 + |\mathbf{u}-\mathbf{u}_B|^2 \right] \ \mathrm{d}x + \delta \int_\Omega \nabla_x \varrho \cdot \frac{1}{2}\nabla_x |\mathbf{u}-\mathbf{u}_B|^2 \ \mathrm{d}x \\
&+ \int_\Omega \delta \nabla_x(\varrho \mathbf{u}_B) : \nabla_x(\mathbf{u}-\mathbf{u}_B) + \delta \varrho \mathbf{u}_B\cdot(\mathbf{u}-\mathbf{u}_B) \ \mathrm{d}x.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\int_\Omega \delta \nabla_x(\varrho \mathbf{u}) : \nabla_x(\mathbf{u}-\mathbf{u}_B) + \delta \varrho \mathbf{u}\cdot(\mathbf{u}-\mathbf{u}_B) \ \mathrm{d}x - \int_\Omega T(\varrho)\mathbf{u}\cdot \frac{1}{2}\nabla_x|\mathbf{u}-\mathbf{u}_B|^2 \ \mathrm{d}x \\
&= \int_\Omega \delta \varrho \left[|\nabla_x(\mathbf{u}-\mathbf{u}_B)|^2 + \frac{1}{2}|\mathbf{u}-\mathbf{u}_B|^2 \right] \ \mathrm{d}x \\
&+ \int_\Omega \delta \nabla_x(\varrho \mathbf{u}_B) : \nabla_x(\mathbf{u}-\mathbf{u}_B) + \delta \varrho \mathbf{u}_B\cdot(\mathbf{u}-\mathbf{u}_B) \ \mathrm{d}x.
\end{aligned}
\tag{5.10}
$$

Summing up (5.5)–(5.9) and going back to (5.4) we may infer that

$$
\begin{aligned}
&\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^N)} + \delta \int_\Omega \left[G''_{\varepsilon,\delta}(\varrho)|\nabla_x\varrho|^2 + \varrho G'_{\varepsilon,\delta}(\varrho) \right] \ \mathrm{d}x + \delta \int_\Omega \varrho \left[|\nabla_x(\mathbf{u}-\mathbf{u}_B)|^2 + \frac{1}{2}|\mathbf{u}-\mathbf{u}_B|^2 \right] \ \mathrm{d}x \\
&+ \int_{\Gamma_{\rm in}} \left(p_{\varepsilon,\delta}(\varrho) - G'_{\varepsilon,\delta}(\varrho)T(\varrho_B) \right) |\mathbf{u}_B\cdot\mathbf{n}| \mathrm{d}S_x + \int_{\Gamma_{\rm out}} H_{\varepsilon,\delta}(\varrho)\mathbf{u}_B\cdot\mathbf{n} \ \mathrm{d}S_x \\
&\lesssim 1 + \delta \int_\Omega \left[\nabla_x(\varrho\mathbf{u}_B) : \nabla_x(\mathbf{u}-\mathbf{u}_B) + \varrho \mathbf{u}_B\cdot(\mathbf{u}-\mathbf{u}_B) \right] \ \mathrm{d}x + \int_\Omega |(\mathbf{u}_B-\mathbf{u})\otimes\mathbf{u}_B : \nabla_x(\mathbf{u}_B-\mathbf{u})| \ \mathrm{d}x.
\end{aligned}
\tag{5.11}
$$

Now, observe that

$$
\left(p_{\varepsilon,\delta}(\varrho) - G'_{\varepsilon,\delta}(\varrho)T(\varrho_B) \right)' = p'_{\varepsilon,\delta}(\varrho) - G''_{\varepsilon,\delta}(\varrho)T(\varrho_B) = p'_{\varepsilon,\delta}(\varrho)\left(1 - \frac{T(\varrho_B)}{T(\varrho)}\right);
$$

whence the function $\varrho \mapsto p_{\varepsilon,\delta}(\varrho) - G'_{\varepsilon,\delta}(\varrho)T(\varrho_B)$ attains its minimum at $\varrho = \varrho_B$ therefore

$$
p_{\varepsilon,\delta}(\varrho) - G'_{\varepsilon,\delta}(\varrho)T(\varrho_B) \geq p_{\varepsilon,\delta}(\varrho_B) - G'_{\varepsilon,\delta}(\varrho_B)T(\varrho_B) = -H_{\varepsilon,\delta}(\varrho_B). \tag{5.12}
$$

Finally, the integral

$$
\int_\Omega |(\mathbf{u}_B-\mathbf{u})\otimes\mathbf{u}_B : \nabla_x(\mathbf{u}_B-\mathbf{u})| \ \mathrm{d}x
$$

is controlled by the left-hand side of (5.11) if $\omega > 0$ in (5.3) has been chosen small enough. We conclude that

$$\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^N)} + \delta \int_\Omega \left[G''_{\varepsilon,\delta}(\varrho)|\nabla_x \varrho|^2 + \varrho G'_{\varepsilon,\delta}(\varrho) \right] \ \mathrm{d}x \le c \tag{5.13}$$

where the constant is independent of ε and δ.

6. Limit $\varepsilon \to 0$

The first step in the limit passage is to let $\varepsilon \to 0$ in the pressure regularization to recover the pressure $p_\delta(\varrho) = p(\varrho) + \sqrt{\delta}\varrho$ in the momentum equation along with the uniform bounds on the density

$$0 \le \varrho < \overline{\varrho} \text{ a.a. in } \Omega.$$

Denote $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ the solutions of the approximate problem (4.1)–(4.4) for a fixed $\delta > 0$. As $\delta > 0$ is kept fixed, the standard elliptic estimates imply

$$\|\varrho_\varepsilon\|_{W^{2,2}(\Omega)} \le c\left(\delta, \|\mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega;R^N)}\right).$$

In particular, as $N = 2, 3$, we get

$$\varrho_\varepsilon \to \varrho \text{ in } C(\overline{\Omega}),\ \varrho \ge 0, \tag{6.1}$$

passing to a suitable subsequence as the case may be.

The crucial observation is that the limit density satisfies

$$0 \le \varrho < \overline{\varrho} \text{ a.a. in } \Omega. \tag{6.2}$$

Indeed take

$$F_Z(\varrho) = \min\left\{1; \frac{2}{\overline{\varrho} - Z}[\varrho - Z]^+\right\}, \text{ where } 0 < Z < \overline{\varrho} \text{ is chosen so that } G'_{\varepsilon,\delta}(Z) \ge 1.$$

By virtue of (5.13), we get

$$\int_\Omega F_Z(\varrho_\varepsilon)\ \mathrm{d}x \le \frac{1}{G'_{\varepsilon,\delta}(Z)} \int_{\{\varrho_\varepsilon \ge Z\}} G'_{\varepsilon,\delta}(Z)\,\mathrm{d}x \le \frac{1}{ZG'_{\varepsilon,\delta}(Z)} \int_{\{\varrho_\varepsilon \ge Z\}} \varrho_\varepsilon G'_{\varepsilon,\delta}(\varrho_\varepsilon)\ \mathrm{d}x.$$

In view of the uniform bounds (5.13), letting $\varepsilon \to 0$ yields

$$\int_\Omega F_Z(\varrho)\ \mathrm{d}x \le \frac{c}{ZG'(Z)}, \text{ where } G''(\varrho) = \frac{p'(\varrho)}{T(\varrho)}.$$

As G is convex, we have $G'(Z) \to \infty$ for $Z \to \overline{\varrho}$ and (6.2) follows.

Finally, we claim that the pressure is bounded in suitable spaces. It follows from the approximate momentum equation (4.3) and the estimate (6.2) that

$$\|\nabla_x p_{\varepsilon,\delta}(\varrho_\varepsilon)\|_{W^{-1,2}(\Omega;R^N)} \le c(\delta). \tag{6.3}$$

In addition, as $\int_\Omega \varrho_\varepsilon G'_{\varepsilon,\delta}(\varrho_\varepsilon)\ \mathrm{d}x$ is uniformly bounded, we deduce that

$$\|p_{\varepsilon,\delta}(\varrho_\varepsilon)\|_{L^1(\Omega)} \le c(\delta). \tag{6.4}$$

Thus, in accordance with Nečas' lemma (see e.g. [6, Lemma 10.10]), we may suppose

$$p_\varepsilon(\varrho_\varepsilon) \to \overline{p(\varrho)} \text{ weakly in } L^2(\Omega),$$

where, in view of (6.1), (6.2),

$$\overline{p(\varrho)} = p(\varrho).$$

Letting $\varepsilon \to 0$ in (4.2)–(4.4) we obtain the following system of equations:

$$-\delta\Delta_x\varrho + \delta\varrho + \operatorname{div}_x(\varrho\mathbf{u}) = 0, \tag{6.5}$$

$$(-\delta\nabla_x\varrho + \varrho\mathbf{u})\cdot\mathbf{n}|_{\partial\Omega} = \begin{cases} \varrho_B\mathbf{u}_B\cdot\mathbf{n} \text{ if } \mathbf{u}_B\cdot\mathbf{n} \le 0, \\ \varrho\mathbf{u}_B\cdot\mathbf{n} \text{ if } \mathbf{u}_B\cdot\mathbf{n} > 0, \end{cases} \tag{6.6}$$

$$\operatorname{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho) + \sqrt{\delta}\nabla_x\varrho = \operatorname{div}_x\mathbb{S}(\nabla_x\mathbf{u}) + \delta\Delta_x(\varrho\mathbf{u}) - \delta\varrho\mathbf{u}, \tag{6.7}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B, \tag{6.8}$$

with positive parameter $\delta > 0$.

7. Limit $\delta \to 0$

Let $[\varrho_\delta, \mathbf{u}_\delta]$ be the approximate solutions of system (6.5)–(6.8) for $\delta > 0$. Our ultimate goal is to pass to the limit $\delta \to 0$.

7.1. Limit in the field equations

By virtue of the uniform bounds (5.13), we get

$$\delta^{3/2}\|\nabla_x\varrho_\delta\|^2_{L^2(\Omega;R^N)} \le c, \text{ and } 0 \le \varrho_\delta < \overline{\varrho} \text{ a.a. in } \Omega. \tag{7.1}$$

Consequently, in view of (6.2) and compactness of the embedding

$$W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega),\ 1 \le q < 6 \text{ if } N = 3,\ 1 \le q \text{ arbitrary finite if } N = 2,$$

it is easy to perform the limit $\delta \to 0$ in the approximate equation of continuity

$$-\int_\Omega \varrho_\delta\mathbf{u}_\delta\cdot\nabla_x\varphi\ \mathrm{d}x + \int_{\Gamma_{\rm in}} \varphi\varrho_B\mathbf{u}_B\cdot\mathbf{n}\,\mathrm{d}S_x = -\delta\int_\Omega \varphi\varrho_\delta\ \mathrm{d}x - \delta\int_\Omega \nabla_x\varrho_\delta\cdot\nabla_x\varphi\ \mathrm{d}x \tag{7.2}$$

to obtain

$$-\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\varphi\ \mathrm{d}x + \int_{\Gamma_{\rm in}} \varphi\varrho_B\mathbf{u}_B\cdot\mathbf{n}\,\mathrm{d}S_x = 0 \tag{7.3}$$

for any

$$\varphi \in C^\infty(\overline{\Omega}),\ \varphi|_{\Gamma_{\rm out}} = 0.$$

In addition, by virtue of Lemma 3.2, equation (7.16) is satisfied also in the renormalized form, in particular,

$$-\int_\Omega \varrho\log(\varrho)\mathbf{u}\cdot\nabla_x\varphi\ \mathrm{d}x + \int_\Omega \varrho\operatorname{div}_x\mathbf{u}\varphi\ \mathrm{d}x + \int_{\Gamma_{\rm in}} \varphi\varrho_B\log(\varrho_B)\mathbf{u}_B\cdot\mathbf{n}\,\mathrm{d}S_x = 0, \tag{7.4}$$

for any

$$\varphi \in C^\infty(\overline{\Omega}),\ \varphi|_{\Gamma_{\rm out}} = 0.$$

Our final observation is that

$$\frac{1}{|\Omega|}\int_\Omega \varrho_\delta\ \mathrm{d}x \to \frac{1}{|\Omega|}\int_\Omega \varrho\ \mathrm{d}x < \overline{\varrho}. \tag{7.5}$$

Indeed, as $\varrho_\delta \le \overline{\varrho}$, we would get

$$\frac{1}{|\Omega|}\int_\Omega \varrho\ \mathrm{d}x = \overline{\varrho} \Rightarrow \varrho_\delta \to \overline{\varrho} \text{ (strongly) in } L^q(\Omega) \text{ for any } 1 \le q < \infty,$$

in particular,

$$-\int_{\Omega} \overline{\varrho}\mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x + \int_{\Gamma_{\rm in}} \varphi \varrho_B \mathbf{u}_B \cdot \mathbf{n} \ \mathrm{d}S_x = 0$$

for any

$$\varphi \in C^\infty(\overline{\Omega}), \ \varphi|_{\Gamma_{\rm out}} = 0$$

yielding

$$\varrho_B = \overline{\varrho} \text{ on } \Gamma_{\rm in}$$

in contrast with (2.2).

In order to pass to the limit in the momentum equation (6.7) we need uniform estimates on the pressure. To this end, we multiply (6.7) by $\mathbf{B}[\varrho_\delta]$,

$$\mathrm{div}_x \mathbf{B}[\varrho_\delta] = \varrho_\delta - \frac{1}{|\Omega|} \int_{\Omega} \varrho_\delta \ \mathrm{d}x,$$

where $\mathbf{B}$ is the so-called Bogovskii operator, see e.g. Galdi [10, Chapter 3]. As $\mathbf{B}$ vanishes on $\partial\Omega$, we may integrate by parts obtaining

$$\begin{aligned} &\int_{\Omega} \left(p(\varrho_\delta) + \sqrt{\delta}\varrho_\delta \right) \varrho_\delta \ \mathrm{d}x = \int_{\Omega} \left(p(\varrho_\delta) + \sqrt{\delta}\varrho_\delta \right) \frac{M_\delta}{|\Omega|} \ \mathrm{d}x - \int_{\Omega} \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla_x \mathbf{B}[\varrho_\delta] \ \mathrm{d}x \\ &+ \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{B}[\varrho_\delta] \ \mathrm{d}x + \delta \int_{\Omega} \nabla_x (\varrho_\delta \mathbf{u}_\delta) \cdot \nabla_x \mathbf{B}[\varrho_\delta] \ \mathrm{d}x - \delta \int_{\Omega} \varrho_\delta \mathbf{u}_\delta \cdot \mathbf{B}[\varrho_\delta] \ \mathrm{d}x, \end{aligned} \tag{7.6}$$

where we have set

$$M_\delta = \int_{\Omega} \varrho_\delta \ \mathrm{d}x.$$

Since the Bogovskii operator $\mathbf{B} : L^q(\Omega) \mapsto W_0^{1,q}(\Omega)$, $1 < q < \infty$, is bounded, we may use the uniform bounds (5.13), (7.1) to find that

$$\int_{\Omega} p(\varrho_\delta)\varrho_\delta \ \mathrm{d}x \le c + \int_{\Omega} p(\varrho_\delta) \frac{M_\delta}{|\Omega|} \ \mathrm{d}x.$$

Next, by virtue of (7.5) there exists $\lambda > 1$ such that

$$\limsup_{\delta \to 0} \lambda \frac{M_\delta}{|\Omega|} < \overline{\varrho}.$$

Consequently,

$$\begin{aligned} &\int_{\Omega} p(\varrho_\delta)\varrho_\delta \ \mathrm{d}x \le c + \int_{\Omega} p(\varrho_\delta) \frac{M_\delta}{|\Omega|} \ \mathrm{d}x \\ &\le c + \int_{\{\varrho_\delta \le \lambda \frac{M_\delta}{|\Omega|}\}} p(\varrho_\delta) \frac{M_\delta}{|\Omega|} \ \mathrm{d}x + \int_{\{\varrho_\delta > \lambda \frac{M_\delta}{|\Omega|}\}} p(\varrho_\delta) \frac{M_\delta}{|\Omega|} \le c + M_\delta p\left(\lambda \frac{M_\delta}{|\Omega|} \right) + \frac{1}{\lambda} \int_{\Omega} \varrho_\delta p(\varrho_\delta) \ \mathrm{d}x; \end{aligned}$$

whence

$$\int_{\Omega} \varrho_\delta p(\varrho_\delta) \ \mathrm{d}x \le c. \tag{7.7}$$

Having established the bound (7.7), we may repeat the same procedure with the multiplier $\mathbf{B}[p^\alpha(\varrho_\delta)]$, where $\alpha > 0$ will be fixed below. Similarly to (7.6), we have

$$\begin{aligned}\int_\Omega \left(p(\varrho_\delta) + \sqrt{\delta}\varrho_\delta\right) p^\alpha(\varrho_\delta)\ \mathrm{d}x &= \frac{1}{|\Omega|}\int_\Omega \left(p(\varrho_\delta) + \sqrt{\delta}\varrho_\delta\right)\ \mathrm{d}x \int_\Omega p^\alpha(\varrho_\delta)\ \mathrm{d}x \\ &- \int_\Omega \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla_x \mathbf{B}[p^\alpha(\varrho_\delta)]\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{B}[p^\alpha(\varrho_\delta)]\ \mathrm{d}x \\ &+ \delta \int_\Omega \nabla_x(\varrho_\delta \mathbf{u}_\delta)\cdot \nabla_x \mathbf{B}[p^\alpha(\varrho_\delta)]\ \mathrm{d}x - \delta \int_\Omega \varrho_\delta \mathbf{u}_\delta \cdot \mathbf{B}[p^\alpha(\varrho_\delta)]\ \mathrm{d}x.\end{aligned} \tag{7.8}$$

By virtue of (7.7),

$$\|p^\alpha(\varrho_\delta)\|_{L^{\frac{1}{\alpha}}(\Omega)} \le c(\alpha) \text{ for any } 0 < \alpha \le 1,$$

therefore all integrals on the right-hand side of (7.8) remain bounded uniformly for $\delta \to 0$ for a suitably small $\alpha > 0$. Accordingly

$$\|p(\varrho_\delta)\|_{L^{\alpha+1}(\Omega)} \le c \text{ for a certain } \alpha > 0, \tag{7.9}$$

and

$$p(\varrho_\delta) \to \overline{p(\varrho)} \text{ weakly in } L^{\alpha+1}(\Omega). \tag{7.10}$$

Letting $\delta \to 0$ we may infer that

$$\int_\Omega \left[\varrho \mathbf{u}\otimes\mathbf{u} : \nabla_x \varphi + \overline{p(\varrho)}\mathrm{div}_x \varphi\right]\ \mathrm{d}x = \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi\ \mathrm{d}x \tag{7.11}$$

for any $\varphi \in C^1_c(\Omega)$.

7.2. *Compactness of the pressure*

To complete the proof of Theorem 2.1, we have to show that

$$\overline{p(\varrho)} = p(\varrho). \tag{7.12}$$

We evoke the nowadays standard method based on the weak continuity of the effective viscous flux developed by Lions [13]. Specifically, we show that

$$\lim_{\delta\to 0} \int_\Omega p(\varrho_\delta)\varrho_\delta\ \mathrm{d}x = \int_\Omega \overline{p(\varrho)}\varrho\ \mathrm{d}x. \tag{7.13}$$

Relation (7.13) implies strong convergence of ϱ_δ,

$$\varrho_\delta \to \varrho \text{ a.a. in } \Omega \text{ (up to a subsequence)}$$

if $p'(\varrho) > 0$ for all $\varrho \in (0, \overline{\varrho})$ or

$$p(\varrho_\delta) \to p(\varrho) \text{ weakly in } L^{\alpha+1}(\Omega).$$

Obviously, this yields (7.12) in both cases.

We start with the renormalized version of the approximate equation of continuity (5.6):

$$\delta H''(\varrho_\delta)|\nabla_x \varrho_\delta|^2 + \delta H'(\varrho_\delta)\varrho_\delta - \delta \mathrm{div}_x(H'(\varrho_\delta)\nabla_x \varrho_\delta) + \mathrm{div}_x(H(\varrho_\delta)\mathbf{u}_\delta) + \left[H'(\varrho_\delta)\varrho_\delta - H(\varrho_\delta)\right]\mathrm{div}_x \mathbf{u}_\delta = 0.$$

In particular, for $H(\varrho) \equiv L(\varrho) = \varrho\log(\varrho)$ we get

$$\delta \log(\varrho_\delta)\varrho_\delta - \delta \mathrm{div}_x(L'(\varrho_\delta)\nabla_x \varrho_\delta) + \mathrm{div}_x(\varrho_\delta \log(\varrho_\delta)\mathbf{u}_\delta) + \varrho_\delta \mathrm{div}_x \mathbf{u}_\delta \le 0. \tag{7.14}$$

Passing to the weak formulation we obtain

$$
\begin{aligned}
&-\int_\Omega \varrho_\delta \log(\varrho_\delta)\mathbf{u}_\delta \cdot \nabla_x \varphi \ \mathrm{d}x + \int_\Omega \varrho_\delta \mathrm{div}_x \mathbf{u}_\delta \varphi \ \mathrm{d}x \\
&\qquad + \int_{\Gamma_{\rm in}} \varphi \varrho_\delta \log(\varrho_\delta)\mathbf{u}_B \cdot \mathbf{n} \,\mathrm{d}S_x - \delta \int_{\Gamma_{\rm in}} \varphi L'(\varrho_\delta)\nabla_x \varrho_\delta \cdot \mathbf{n}\, \mathrm{d}S_x \\
&\qquad \leq 0 + o(\delta)
\end{aligned}
\tag{7.15}
$$

for any

$$\varphi \in C^\infty_c(\overline{\Omega}),\ \varphi \geq 0,\ \varphi|_{\Gamma_{\rm out}} = 0.$$

Finally, we use the boundary conditions (4.2) obtaining

$$
\begin{aligned}
&-\int_\Omega \varrho_\delta \log(\varrho_\delta)\mathbf{u}_\delta \cdot \nabla_x \varphi \ \mathrm{d}x + \int_\Omega \varrho_\delta \mathrm{div}_x \mathbf{u}_\delta \varphi \ \mathrm{d}x \\
&\qquad + \int_{\Gamma_{\rm in}} \varphi L(\varrho_\delta)\mathbf{u}_B \cdot \mathbf{n} \,\mathrm{d}S_x + \int_{\Gamma_{\rm in}} \varphi L'(\varrho_\delta)(\varrho_B - \varrho_\delta)\mathbf{u}_B \cdot \mathbf{n}\, \mathrm{d}S_x \\
&\qquad \leq 0 + o(\delta)
\end{aligned}
\tag{7.16}
$$

for any

$$\varphi \in C^\infty_c(\overline{\Omega}),\ \varphi \geq 0,\ \varphi|_{\Gamma_{\rm out}} = 0.$$

Subtracting (7.4) from (7.16) we get

$$
\begin{aligned}
&\int_\Omega \varrho \log(\varrho)\mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x - \int_\Omega \varrho_\delta \log(\varrho_\delta)\mathbf{u}_\delta \cdot \nabla_x \varphi \ \mathrm{d}x + \int_\Omega \varrho_\delta \mathrm{div}_x \mathbf{u}_\delta \varphi \ \mathrm{d}x - \int_\Omega \varrho \mathrm{div}_x \mathbf{u} \varphi \ \mathrm{d}x \\
&\qquad + \int_{\Gamma_{\rm in}} \varphi \left[L(\varrho_B) - L'(\varrho_\delta)(\varrho_B - \varrho_\delta) - L(\varrho_\delta) \right] |\mathbf{u}_B \cdot \mathbf{n}| \,\mathrm{d}S_x \leq 0 + o(\delta)
\end{aligned}
$$

for any

$$\varphi \in C^\infty_c(\overline{\Omega}),\ \varphi \geq 0,\ \varphi|_{\Gamma_{\rm out}} = 0.$$

As L is convex, we deduce

$$\int_\Omega \varrho \log(\varrho)\mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x - \int_\Omega \varrho_\delta \log(\varrho_\delta)\mathbf{u}_\delta \cdot \nabla_x \varphi \ \mathrm{d}x + \int_\Omega \varrho_\delta \mathrm{div}_x \mathbf{u}_\delta \varphi \ \mathrm{d}x - \int_\Omega \varrho \mathrm{div}_x \mathbf{u} \varphi \ \mathrm{d}x \leq 0 + o(\delta);$$

whence letting $\delta \to 0$,

$$\int_\Omega \left[\varrho \log(\varrho) - \overline{\varrho \log(\varrho)} \right] \mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x + \int_\Omega \overline{\varrho \mathrm{div}_x \mathbf{u}} \varphi \ \mathrm{d}x - \int_\Omega \varrho \mathrm{div}_x \mathbf{u} \varphi \ \mathrm{d}x \leq 0 \tag{7.17}$$

for any

$$\varphi \in C^\infty_c(\overline{\Omega}),\ \varphi \geq 0,\ \varphi|_{\Gamma_{\rm out}} = 0.$$

Since the boundary $\partial\Omega$ is of class C^2, there is $h > 0$ such that the map $(t, x_0) \mapsto x_0 + t\mathbf{n}(x_0)$ is a diffeomorphism from $(0, h) \times \partial\Omega$ onto $\mathcal{U}^-(\partial\Omega) \equiv \{x \in \Omega \,|\, 0 < \mathrm{dist}(x, \partial\Omega) < h\}$. We first extend $\mathbf{u}_B$ to $\mathcal{U}^-(\partial\Omega)$ by setting $\overline{\mathbf{u}}(x_0 - t\mathbf{n}(x_0)) = \mathbf{u}_B(x_0)$, and then to Ω by setting

$$\tilde{\mathbf{u}}_B(x) = \overline{\mathbf{u}}_B(x) w(\mathrm{dist}(x, \partial\Omega)),$$

where $w \in C[0,\infty)$, $0 \le w \le 1$, $w(s) = 1$ if $s \in [0, h/2]$, $w(s) = 0$ if $s \ge h$. In particular, if $x_0 \in \Gamma_{\rm out}$, then

$$\tilde{\mathbf{u}}_B(x) \cdot \nabla_x {\rm dist}(x, \partial\Omega) < 0, \ \text{ if } x \in \mathcal{U}^-(\partial\Omega). \tag{7.18}$$

Consider a family of Lipschitz test functions in Ω,

$$\varphi_\varepsilon(x) = \begin{cases} 1 \text{ if } {\rm dist}(x, \Gamma_{\rm out}) > \varepsilon \\ \frac{1}{\varepsilon}{\rm dist}(x, \Gamma_{\rm out}) \text{ if } {\rm dist}(x, \Gamma_{\rm out}) \le \varepsilon \end{cases}$$

By Lebesgue theorem,

$$\int_\Omega \left[\varrho \log(\varrho) - \overline{\varrho \log(\varrho)} \right] (\mathbf{u} - \tilde{\mathbf{u}}_B) \cdot \nabla_x \varphi_\varepsilon \ {\rm d}x \to 0 \text{ as } \varepsilon \to 0,$$

while, in accordance with (7.18),

$$\liminf_{\varepsilon \to 0} \int_\Omega \left[\varrho \log(\varrho) - \overline{\varrho \log(\varrho)} \right] \tilde{\mathbf{u}}_B \cdot \nabla_x \varphi_\varepsilon \ {\rm d}x \ge 0.$$

Thus, using φ_ε as a test function in (7.17) and performing the limit $\varepsilon \to 0$, we conclude that

$$\int_\Omega \left[\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} \right] \ {\rm d}x \le 0. \tag{7.19}$$

The next step is to multiply the approximate momentum equation (6.7) on

$$\varphi \nabla_x \Delta_x^{-1}[\varphi \varrho_\delta], \ \varphi \in C^\infty_c(\Omega),$$

where Δ_x^{-1} denotes the inverse of the Laplacian on R^3, specifically a pseudodifferential operator with Fourier symbol $-\frac{1}{|\xi|^2}$. After a bit tedious but straightforward computation we obtain

$$\begin{aligned}
&\int_{R^N} \varphi^2 p(\varrho_\delta)\varrho_\delta \ {\rm d}x - \int_{R^N} \varphi \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi \varrho_\delta] \ {\rm d}x \\
&+ \int_{R^N} \varphi (\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi \varrho_\delta] \ {\rm d}x = - \int_{R^N} \Big((\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) \cdot \nabla_x \varphi \Big) \cdot \nabla_x \Delta_x^{-1} [\varphi \varrho_\delta] \ {\rm d}x \\
&- \int_{R^N} \Big(\mathbb{S}(\nabla_x \mathbf{u}_\delta) \cdot \nabla_x \varphi \Big) \cdot \nabla_x \Delta_x^{-1} [\varphi \varrho_\delta] \ {\rm d}x - \int_{R^N} p(\varrho_\delta) \nabla_x \varphi \cdot \nabla_x \Delta_x^{-1} [\varphi \varrho_\delta] \ {\rm d}x \\
&- \sqrt{\delta} \int_{R^N} \varrho_\delta \nabla_x \varphi \cdot \nabla_x \Delta_x^{-1} [\varphi \varrho_\delta] \ {\rm d}x - \sqrt{\delta} \int_{R^N} \varphi^2 \varrho_\delta^2 \ {\rm d}x \\
&\delta \int_{R^N} \nabla_x \varphi \cdot \nabla_x (\varrho \mathbf{u}) \cdot \nabla_x \Delta_x^{-1} [\varphi \varrho_\delta] \ {\rm d}x + \delta \int_{R^N} \varrho_\delta \mathbf{u}_\delta \varphi \nabla_x \Delta_x^{-1} [\varphi \varrho_\delta] \ {\rm d}x \\
&- \delta \int_{R^N} \varphi \nabla_x (\varrho \mathbf{u}) : \nabla_x \Delta_x^{-1} [\varphi \varrho_\delta] \ {\rm d}x.
\end{aligned} \tag{7.20}$$

Now, similarly, consider

$$\varphi \nabla_x \Delta_x^{-1} [\varphi \varrho]$$

as a test function in the limit equation (7.11):

$$
\begin{aligned}
&\int_{R^N} \varphi^2 \overline{p(\varrho)}\varrho \ \mathrm{d}x - \int_{R^N} \varphi \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho] \ \mathrm{d}x \\
&+ \int_{R^N} \varphi(\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho] \ \mathrm{d}x = - \int_{R^N} (\varrho\mathbf{u}\otimes\mathbf{u}) \cdot \nabla_x \varphi \cdot \nabla_x \Delta_x^{-1} [\varphi\varrho] \ \mathrm{d}x \\
&- \int_{R^N} \mathbb{S}(\nabla_x \mathbf{u}) \cdot \nabla_x\varphi \cdot \nabla_x \Delta_x^{-1}[\varphi\varrho] \ \mathrm{d}x - \int_{R^N} \overline{p(\varrho)} \nabla_x \varphi \cdot \nabla_x \Delta_x^{-1} [\varphi\varrho] \ \mathrm{d}x.
\end{aligned}
\tag{7.21}
$$

Observe that **(i)** all δ-dependent term in (7.20) vanish in the asymptotic limit $\delta \to 0$, **(ii)** in view of the compactification effect $\nabla_x \Delta_x^{-1} : L^q \to W^{1,q}_{\rm loc}$, all integrals on the right-hand side of (7.20) converge to their counterparts in (7.21). Therefore we have shown

$$
\begin{aligned}
&\int_{R^N} \varphi^2 p(\varrho_\delta)\varrho_\delta \ \mathrm{d}x - \int_{R^N} \varphi \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho_\delta] \ \mathrm{d}x + \int_{R^N} \varphi(\varrho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho_\delta] \ \mathrm{d}x \\
&\to \\
&\int_{R^N} \varphi^2 \overline{p(\varrho)}\varrho \ \mathrm{d}x - \int_{R^N} \varphi \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho] \ \mathrm{d}x + \int_{R^N} \varphi(\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho] \ \mathrm{d}x
\end{aligned}
\tag{7.22}
$$

as $\delta \to 0$.

Now we have

$$
\int_{R^N} \varphi(\varrho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho_\delta] \ \mathrm{d}x = \int_{R^N} \sum_{j=1}^N \varphi u^j_\delta \sum_{i=1}^N \left(\varrho_\delta u^i_\delta \partial_{x_i} \Delta_x^{-1} [\partial_{x_j}(\varphi\varrho_\delta)] \right) \ \mathrm{d}x,
$$

where, in accordance with

$$
\mathrm{div}_x(\varrho_\delta \mathbf{u}_\delta) = \delta \Delta_x \varrho_\delta - \delta\varrho_\delta \in \text{precompact subset of } W^{-1,2}(\Omega).
$$

Moreover, as $\mathbf{curl}[\nabla_x \Delta_x^{-1}[\partial_{x_j}(\varphi\varrho_\delta)] = 0$, we may apply the celebrated Div–Curl lemma of Murat and Tartar [23] to conclude that

$$
\int_{R^N} \varphi(\varrho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho_\delta] \ \mathrm{d}x \to \int_{R^N} \varphi(\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho] \ \mathrm{d}x.
$$

Thus relation (7.22) reduces to

$$
\begin{aligned}
&\int_{R^N} \varphi^2 p(\varrho_\delta)\varrho_\delta \ \mathrm{d}x - \int_{R^N} \varphi \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho_\delta] \ \mathrm{d}x \\
&\to \\
&\int_{R^N} \varphi^2 \overline{p(\varrho)}\varrho \ \mathrm{d}x - \int_{R^N} \varphi \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho] \ \mathrm{d}x
\end{aligned}
\tag{7.23}
$$

as $\delta \to 0$.

Finally, we check that

$$
\begin{aligned}
&\lim_{\delta\to 0} \int_{R^N} \varphi \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho_\delta] \ \mathrm{d}x - \int_{R^N} \varphi \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \Delta_x^{-1} \nabla_x [\varphi\varrho] \ \mathrm{d}x \\
&= \lim_{\delta\to 0} \int_{R^N} \nabla_x \Delta_x^{-1} \nabla_x [\varphi \mathbb{S}(\nabla_x \mathbf{u}_\delta)] \varphi\varrho_\delta \ \mathrm{d}x - \int_{R^N} \nabla_x \Delta_x^{-1} \nabla_x [\varphi \mathbb{S}(\nabla_x \mathbf{u})] \varphi\varrho \ \mathrm{d}x \\
&= (\lambda + 2\mu) \int_{R^N} \varphi^2 \mathrm{div}_x \mathbf{u}_\delta \varrho_\delta \ \mathrm{d}x - (\lambda + 2\mu) \int_{R^N} \varphi^2 \mathrm{div}_x \mathbf{u} \varrho \ \mathrm{d}x;
\end{aligned}
$$

whence, in combination with (7.23),

$$\int_{R^N} \varphi^2 \left[\overline{p(\varrho)\varrho} - \overline{p(\varrho)}\varrho \right] \, \mathrm{d}x = (\lambda + 2\mu) \int_{R^N} \varphi^2 \left[\overline{\mathrm{div}_x \mathbf{u}\varrho} - \mathrm{div}_x \mathbf{u}\varrho \right] \, \mathrm{d}x. \tag{7.24}$$

Thus, using (7.19), we deduce that

$$\int_{\Omega} \left[\overline{p(\varrho)\varrho} - \overline{p(\varrho)}\varrho \right] \, \mathrm{d}x \le 0.$$

As p is non-decreasing, the standard Minty's trick yields the desired conclusion (7.12). Theorem 2.1 has been proved.

Conflict of interest statement

There is no conflict of interests.

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