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Global solvability of massless Dirac–Maxwell systems

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Abstract

We consider the Cauchy problem for massless Dirac–Maxwell equations on an asymptotically flat background and give a global existence and uniqueness theorem for initial values small in an appropriate weighted Sobolev space. The result can be extended via analogous methods to Dirac–Higgs–Yang–Mills theories.

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1. Introduction

Let (M^n, g) be a globally hyperbolic spin manifold endowed with a trivial $U(1)$-principal bundle $\pi: E \to M$. Let A be a connection one-form on π, or equivalently, a $U(1)$-invariant $i\mathbb{R}$-valued one-form on E. We will assume in the following that M is simply-connected and will regard A as a real-valued one-form on M. We denote the standard spinor bundle of (M, g) by $\sigma: \Sigma \to M$, by $\langle \cdot, \cdot \rangle$ the pointwise Hermitian inner product on σ and by "$\cdot$" the pointwise Clifford multiplication by vector fields or forms on σ. Recall that the Levi-Civita connection ∇ on TM induces a metric covariant derivative on σ that we also denote by ∇. That covariant derivative together with A define a new covariant derivative ∇^A on σ via $\nabla^A_X(\psi) := \nabla_X\psi + iA(X)\psi$ for any vector field X on M. By definition, the Dirac operator associated to A is the Clifford-trace of ∇^A, that is, for any local orthonormal frame $(e_j)_{1\le j\le n}$ of TM, we have $D^A := i\sum_{j=0}^n \epsilon_j e_j \cdot \nabla^A_{e_j}$, where $\epsilon_j = g(e_j, e_j) = \pm 1$. Alternatively, we can write $D^A = D - A\cdot$, where D is the standard Dirac operator of (M, g) and is obtained as the Clifford-trace of ∇.

The **Dirac–Maxwell Lagrangian density $\mathcal{L}_{DM}$ for N particles of masses $m_1, \dots, m_N$ and charges** $\operatorname{sgn}(\mu_1)\sqrt{|\mu_1|}, \dots, \operatorname{sgn}(\mu_N)\sqrt{|\mu_N|}$ is defined by

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$$\mathcal{L}_{DM}(\psi \oplus A) := \frac{1}{4} tr(F^A \wedge F^A) + \sum_{l=1}^{N} \frac{1}{2}(\langle D^{\mu_l A}\psi^l, \psi^l\rangle + \langle \psi^l, D^{\mu_l A}\psi^l\rangle) - \sum_{l=1}^{N} m_l \langle \psi^l, \psi^l\rangle,$$

where $\psi = (\psi^1, \dots, \psi^N)$ is a section of $\bigoplus_{l=1}^N \sigma$ and A is a real one-form on M. The critical points of the Lagrangian are exactly the preimages of zero under the operator P_{DM} given by

$$P_{DM}(\psi^1 \oplus ... \oplus \psi^N \oplus A) = (D^{\mu_1 A}\psi^1 - m_1\psi^1, \dots, D^{\mu_N A}\psi^N - m_N\psi^N, d^*dA - J_\psi),$$

where $J_\psi(X) := \sum_{l=1}^N \mu_l \cdot j_{ll}(X)$ and $j_{kl}(X) := \langle X \cdot \psi^k, \psi^l\rangle$. If ψ^k and ψ^l have equal mass and charge, then it is easy to see that $d^* j_{kl} = 0$, thus in particular J_ψ is divergence-free for $(\psi, A) \in P_{DM}^{-1}(0)$. In the sequel, we shall call a pair $(\psi = (\psi^1, \dots, \psi^N), A)$ as above a solution to the **Dirac–Maxwell equation** if $(\psi, A) \in P_{DM}^{-1}(0)$, that is, if

$$D^{\mu_l A}\psi^l = m_l \psi^l, \ l = 1, \dots, N \qquad \text{and} \qquad d^*dA = J_\psi.$$

The *massless* Dirac–Maxwell equation is the Dirac–Maxwell equation with $m_1 = \ldots = m_N = 0$.

Let us first shortly review the state of the art on this subject. Considering the fact that the massless Dirac–Maxwell equation is in dimension 4 conformally invariant, Christodoulou and Choquet-Bruhat [4] show existence of solutions of Dirac–Yang–Mills–Higgs solutions on four-dimensional Minkowski space with initial values small in weighted Sobolev spaces, the weights being induced by rescaling via the conformal Penrose embedding Minkowski space into the Einstein cylinder. Trying to transfer their result to Maxwell–Dirac Theory, firstly we notice that their method is *not applicable in any spacetime (M, g) satisfying the timelike convergence condition but not isometric to the Minkowski spacetime*. The reason is that in this case the positive mass theorem inplies that there is no conformal compactification of (M, g), i.e. no sufficiently smooth open conformal embedding with precompact image, and the latter is an important ingredient of their proof (see [16]). So some essentially new method is needed here. Secondly, in the next paragraph we will see that the resulting statement is only nonempty if we extend their setting to a system of finitely many massless particles whose total charge is zero. Psarelli [17], in contrast, treats the question of Dirac–Maxwell equations with or without mass on $\mathbb{R}^{1,3}$, not in terms of connections modelling potentials, but in terms of curvature tensors modelling field strength,[1] with results of the form: If C is any compact subset of a Cauchy surface S of $\mathbb{R}^{1,3}$ then there is a number a depending on C such that, if some initial values I with (among others) spinor part supported in C have Sobolev norm smaller than a, then there is a global solution with initial values I. In the massless case, this result is of course strictly weaker than the weighted Sobolev result.

Flato, Simon and Taflin [11] were the first to show global existence for *massive* Dirac–Maxwell equations on $\mathbb{R}^{1,3}$ via the construction of explicit approximate solutions and for suitable initial data that are not easy to handle. For initial data sufficiently small in some weighted Sobolev norm in $\mathbb{R}^{1,3}$, it is Georgiev [13] who established the first global existence result for massless or massive Maxwell–Dirac equations. The core idea of Georgiev's proof is a gauge in which the potential one-form A satisfies $tA_0 + \sum_{i=1}^3 x^j A_j = 0$ in canonical coordinates of Minkowski space, implying that after the usual transformation to a Maxwell–Klein–Gordon problem the equations satisfy Klainerman's null condition. The entire construction uses canonical coordinates of Minkowski space, and whereas it seems likely that the proof can be generalized to spacetime geometries decaying to Minkowski spacetimes in an appropriate sense, the question of global existence in other spacetime geometries remains completely open. Let us mention however that, using the complete null structure for Dirac–Maxwell equations from [8], D'Ancona and Selberg can prove [9] global existence and well-posedness for Dirac–Maxwell equations on $\mathbb{R}^{1,2}$. The analysis of Dirac–Maxwell equations also includes refining decay estimates, see for instance [3] where the authors show peeling estimates for non-zero-charge Dirac–Klein–Gordon equations with small initial data on $\mathbb{R}^{1,3}$.

The aim of the present article is to generalize Georgiev's results to the much more general case of so-called *conformally extendible* spacetimes. This latter notion, explained in greater detail in the next section, is located between asymptotic simplicity and weak asymptotic simplicity and does not require any asymptotics of the curvature tensor along hypersurfaces. Actually, it is easy to construct examples by hand of conformally extendible manifolds that are not asymptotically flat. Conversely, maximal Cauchy developments of initial values in a weighted Sobolev neighbourhood of initial values are known to possess conformal extensions due to criteria developed by Friedrich and Chruściel.

[1] Recall, however, that the Aharanov–Bohm effect shows that rather than the electromagnetic fields, the potentials play the more fundamental role in electrodynamics.

Our main result is the well-posedness of the Cauchy problem for small Lorenz-gauge constrained initial values for massless Dirac–Maxwell systems of vanishing total charge. A precise formulation is given in Section 3. Our method also applies to other field equations, as long as they display an appropriate conformal behaviour and are gauge-equivalent to a semilinear symmetric hyperbolic system admitting a global solution. In particular, Dirac–Higgs–Yang–Mills systems as in Choquet-Bruhat's and Christodoulou's article can be handled similarly. The main ingredient in our method — a special kind of "causal induction" — can be found in Section 5 and seems to be completely new.

In a subsequent work, we will furthermore examine the question whether the solutions of the constraint equations of fixed regularity intersected with any open ball around 0 always form an infinite-dimensional Banach manifold.

The article is structured as follows: Section 2 reviews the main facts we need about local and global existence in time for symmetric hyperbolic systems of first or second order on curved backgrounds. Section 3 introduces the concept of conformal extendibility and gives a detailed account of the main result. Section 4 recalls well-known facts on transformations under which the Dirac–Maxwell equations display some sort of covariance, proves Proposition 4.3 and derives the constraint equations used in Theorem 3.1, which is proved in Section 5.

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2. Brief review of symmetric hyperbolic systems

Let (M^n, g) be any globally hyperbolic spacetime and $S \subset M$ be any spacelike Cauchy hypersurface with induced Riemannian metric g_S. That is, S is intersected exactly once by any C^0-inextendible causal curve. Let $E \xrightarrow{\pi} M$ be any vector bundle. A **differential operator** P **of order** $k \in \mathbb{N}$ on π is a fibre-bundle-morphism from the kth jet bundle $J^k\pi$ of π to π. It is called **semilinear** if $[\dots[P, f\cdot], f\cdot, \dots, f\cdot] =: \sigma_P(df)$ is a vector bundle endomorphism for all scalar functions f on M, where f appears k times in the brackets. Generalizing [2, Definition 5.1] to the nonlinear case, we define a **semilinear symmetric hyperbolic operator** of first order acting π as a semilinear first-order-differential operator P acting on sections of π such that, denoting by $\sigma_P : T^*M \to \mathrm{End}(E)$ its principal symbol, there is an (definite or indefinite) inner product $\langle\cdot,\cdot\rangle$ on E such that for any $\xi \in T^*M$, the endomorphism $\sigma_P(\xi)$ of E is symmetric/Hermitian and positive-definite in case ξ is future-directed causal.

We first recall the following local existence and uniqueness result for nonsmooth initial data, see e.g. [14] for a proof:

Theorem 2.1. *Let (M^n, g) be any globally hyperbolic spacetime and $S \subset M$ be any spacelike Cauchy hypersurface with induced Riemannian metric g_S. Let $E \xrightarrow{\pi} M$ be any vector bundle with (definite or indefinite) inner product and P be any semilinear symmetric hyperbolic operator of first order acting on sections of π. Let $k \in \mathbb{N}$ with $k > \frac{n-1}{2} + 1$. Then for any f in the Sobolev space $H^{k,2}(S, g_S)$, there exists an open neighbourhood U of S in M such that a unique solution $u \in \Gamma_{C^1}(U, E)$ to $Pu = 0$ with $u_{|S} = f$ exists.*

On the global scale, the only kind of result one may generally hope for is an **estimate on lifetime** for solutions to symmetric hyperbolic systems, see e.g. [14] for a proof:

Theorem 2.2. *Let (M^n, g) be any globally hyperbolic spacetime with compact Cauchy hypersurface $S \subset M$. Let $k \in \mathbb{N}$ with $k > \frac{n-1}{2} + 1$. Let $E \xrightarrow{\pi} M$ be any vector bundle with (definite or indefinite) inner product and P be any C^k semilinear symmetric hyperbolic operator of first order acting on sections of π with $P = L + h$, where L is linear and h is of order zero with $h(0) = 0$. Then for each $T > 0$, there is an $\varepsilon > 0$ such for all initial values u_0 on S with H^k-norm smaller than ε, the lifetime for the solution u of $Pu = 0$ with $u_{|S} = u_0$ is greater than T.*

Symmetric hyperbolic operators of second order on $E \xrightarrow{\pi} M$ are defined as follows: a differential operator P of second order on π is called symmetric hyperbolic if there exists a symmetric hyperbolic operator of first order Q — called the first prolongation of P — acting on sections of $\pi \oplus T^*M \otimes \pi$ such that $Pu = Q(u, \nabla u)$ for every section u of π. This fits to the restriction to charts — there, ∇u is expressed as $\partial u + \Gamma$ where Γ is an algebraic (actually, linear) expression in the u variable. Therefore a representation by Q as above entails an analogous expression in each chart. Furthermore, common textbook knowledge assures that every operator of the form

$$Pu = -\partial_t^2 u + \sum_{i,j=1}^{m} A_{ij}(t,x) \cdot \nabla_{ij} u + \sum_{i=1}^{m} B_i(t,x) \cdot \nabla_i u + c \cdot \partial_t u + d \cdot u$$

(with A_{ij} symmetric and uniformly positive) can be presented as $Pu = Q(u, \nabla u)$ as above, and the Laplace–d'Alembert equation on a compact subset can be brought into the form $Pu = 0$ for P as above. If P is semilinear, so is Q; if $P = P_0 + p$ with P_0 linear and p of zeroth order with $p(0) = 0$, then $Q = Q_0 + q$ with Q_0 linear, q of zeroth order and $q(0) = 0$. The local-in-time existence result for second-order symmetric hyperbolic systems is based on Theorem 2.1. It is important to note that, if P has C^k coefficients, then so has Q. However, as the new operator Q includes a derivative of u, we loose one order of regularity for u, but as we do not care much for the weakest possible regularity condition on the initial values anyway, we treat the semilinear operator Q just like a quasilinear operator. Notice that there is a folklore theorem mentioned in Taylor's books stating that semilinear symmetric hyperbolic systems of first order have a C^0-extension criterion, therefore we could avoid the loss of one derivative of u and obtain sharper statements for the necessary regularity of the initial values.

3. The notion of conformal extendibility and the precise statement of the result

Let (M, g) and (N, h) be globally hyperbolic Lorentzian manifolds, where g, h are supposed to be C^k metrics for some $k \in \mathbb{N} \setminus \{0\}$ (this reduced regularity is essential for our purposes!). An open conformal embedding $f \in C^k(M, N)$ is said to **C^k-extend g conformally** or to be a **C^k-conformal extension of (M, g)** if and only if $\overline{f(M)}$ is causally convex (its intersection with any causal curve is connected) and future compact (its intersection with the causal future of any point is compact). A globally hyperbolic manifold (M, g) is, called **C^k-extendible** for $k \in \mathbb{N} \cup \{\infty\}$ if and only if there is a C^k-conformal extension of (M, g) into a globally hyperbolic manifold.

Whereas Choquet-Bruhat and Christodoulou work with the Penrose embedding which is a C^∞-conformal extension of the entire spacetime, it turns out that, in order to generalize the result by Choquet-Bruhat and Christodoulou, we have to generalize our notion of conformal compactification in a twofold way. First, only the timelike future of a Cauchy surface will be conformally embeddable with open image; furthermore, we have to relax the required regularity of the metric of the target manifolds from C^∞ to C^k. The reason for the second generalization is that we want to include maximal Cauchy developments (g, Φ) of initial values for Einstein–Klein–Gordon theories that satisfy decay conditions at spatial infinity only for finitely many derivatives (controlled by a single weighted Sobolev norm). Thus one cannot control higher derivatives at future null infinity. This is why we need a version of the standard existence and uniqueness theorems for solutions to symmetric hyperbolic systems of *finite* regularity presented in Section 2.

The second need for modification comes from the fact that the extension via the Penrose embedding into the Einstein cylinder can, of course, be generalized in a straightforward manner to every compact perturbation of the Minkowski metric. But compact perturbations of Minkowski metric are physically rather unrealistic, as (with interactions like Maxwell theory satisfying the dominant energy condition) a nonzero energy–momentum tensor necessarily entails a positive mass of the metric. A positive mass of the metric, in turn, is an obstacle to a smooth extension at spacelike infinity i_0, for a discussion see [15, pp. 180–181]. Thus we necessarily have a singularity in the surrounding metric at i_0, so that we have to restrict to the timelike future of a fixed Cauchy surface.

Results by Anderson and Chruściel (cf. [1, Theorems 5.2, 6.1 & 6.2]), improving earlier results by Friedrich [12] imply that, apart from the — physically less interesting — class of compact perturbations of Minkowski space, there is a rich and more realistic class of manifolds which is C^4-extendible in the sense above, namely the class of all static initial values with Schwarzschildian ends and small initial values in an appropriate Sobolev space — see also Corvino's article on this topic [5]. This space of initial values is quite rich, which can be seen by the conformal gluing technique of Corvino and Schoen [6]. This holds in any even dimension. And in the case of a four-dimensional spacetime, there is, in fact, an even larger class of initial values satisfying the conditions of our global existence theorem which is given by a smallness condition to the Einstein initial values in a weighted Sobolev space encoding a good asymptotic decay towards Schwarzschild initial data, cf. the remark following Theorem 6.2 in [1] and the remarks following Theorem 2.6 in [7]. The maximal Cauchy development of any such initial data set carries even a

Cauchy temporal function t such that, for all level sets $S_a := t^{-1}(\{a\})$ of t, both $I^{\pm}(S_a)$ are C^4-extendible and thus satisfy even the stronger assumption of Theorem 3.2 below.[2]

The central insight presented in this article is that the above mentioned weakened notion of conformal extension suffices to establish — however slightly less explicit — weighted Sobolev spaces of initial values allowing for a global solution. In particular, we do not impose asymptotic flatness: the theorem is, e.g., applicable to any precompact open subset of de Sitter spacetime whose closure is causally convex. In order to formulate the main theorem, we need to introduce the constraint equations arising from the transformation of the Dirac–Maxwell equations into a symmetric hyperbolic system. Since we shall consider *conformal* embeddings of an open subset of the original spacetime (M, g) into another spacetime (N, h), we must fix a Cauchy hypersurface S of N as well as a Cauchy time function t on N with $t^{-1}(\{0\}) = S$. Denoting by $h = -\beta dt^2 + g_t$ the induced metric splitting and by $S_\tau := t^{-1}(\{\tau\})$, we let $A_0, A_1 \in \Gamma(T^*M_{|S_0})$ and $\psi_0^l \in \Gamma(\sigma_{|S_0})$, $1 \le l \le N$, be initial data for the Dirac–Maxwell equations. We call *constraint equations* for A_0, A_1, ψ_0^l the following identities:

$$0 = \frac{1}{\beta} A_1(\frac{\partial}{\partial t}) - \sum_{j=1}^{3} (\nabla_{e_j} A_0)(e_j) \tag{3.1}$$

and

$$\begin{aligned} 0 = &-(\nabla^{\tan})^* \nabla^{\tan} A_0(\frac{\partial}{\partial t}) - \sum_{j=1}^{3} \nabla_{e_j} A_1(e_j) - \frac{1}{2\beta} \mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t}) A_1(\frac{\partial}{\partial t}) + \frac{1}{\beta} A_1(\mathrm{grad}_{g_t}(\beta(t,\cdot))) \\ &+ \frac{1}{2\beta} \nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))} A_0(\frac{\partial}{\partial t}) + \frac{1}{2} g_t(\nabla^{\tan} A_0, \frac{\partial g_t}{\partial t}) + \mathrm{ric}^M(\frac{\partial}{\partial t}, A_0^{\sharp}) + \sum_{l=1}^{N} \mu_l j_{\psi_0^l}(\frac{\partial}{\partial t}), \end{aligned} \tag{3.2}$$

where $(e_j)_j$ is a local h-orthonormal basis of TM, $(\nabla^{\tan})^* \nabla^{\tan} := \sum_{j=1}^{n-1} \nabla_{\nabla^{S_t}_{e_j} e_j} - \nabla_{e_j} \nabla_{e_j}$ and the spinors for two conformally related metrics are identified as usual.

Every solution in Lorenz gauge, when restricted to a Cauchy hypersurface, satisfies the constraint equation (see Proposition 4.2). Our main theorem is that, conversely, small constrained initial values can be extended to global solutions:

Theorem 3.1 *(Main theorem). Let (M, g) be a 4-dimensional globally hyperbolic spacetime with a Cauchy hypersurface S' such that $I^+(S')$ is C^4-extendible in a globally hyperbolic spacetime (N, h). Let P_{DM} be the massless Dirac–Maxwell operator for a finite number of fermion fields. Then, for any Cauchy hypersurface $S \subset I^+(S')$ of (M, g), there is a weighted $W^{4,\infty}$-neighbourhood U of 0 in $\pi|_S$ such that for every initial value $\left(A_0 = A_{|S_0}, A_1 = \frac{\nabla A}{\partial t}{}_{|S_0}, \psi_0^l = \psi^l_{|S_0}\right)$ in U with zero total charge w.r.t. S and satisfying the constraint equations (3.1) and (3.2) there is a solution (ψ, A) of $P_{DM}(\psi, A) = 0$ in all of $I^+(S)$. The weight is explicitly computable from the geometry.*

Remark 1. The result and its proof still work if we replace the Dirac–Maxwell system by a general Dirac–Higgs–Yang–Mills systems in the sense of Choquet-Bruhat and Christodoulou, if the Yang–Mills group G is a product of a compact semisimple group and an abelian group and if the Yang–Mills G-principal bundle is trivial.

Remark 2. In case $\beta = 1$, which can be assumed without loss of generality by the existence of Fermi coordinates w.r.t. h in a neighbourhood of S, the constraint equations (3.1) and (3.2) simplify to

$$0 = \frac{\partial}{\partial t}\left(A(\frac{\partial}{\partial t})\right) + d_S^*(A_S) + (n-1) H \cdot A(\frac{\partial}{\partial t})$$

[2] This is a remarkable fact as it is a first approach to the question whether *Einstein–Dirac–Maxwell theory is stable around zero*, as the stability theorems imply that Einstein–Maxwell theory is stable around zero initial values for given small Dirac fields, and our main result implies that Maxwell–Dirac Theory is stable around zero for maximal Cauchy developments of small Einstein initial values.

$$0 = -\Delta_S\left(A(\frac{\partial}{\partial t})\right) + d_S^*\left(\frac{\nabla A}{\partial t}_S\right) - 3g_t(\nabla^S A_S, W) + A(d_S^* W) + 2|W|^2 A(\frac{\partial}{\partial t}) + \mathrm{ric}^M\left(\frac{\partial}{\partial t}, A^\sharp\right)$$
$$+ \sum_{l=1}^N \mu_l j_{\psi_0^l}(\frac{\partial}{\partial t}),$$

where $A_S := \iota_S^* A \in \Gamma(T^*S)$, $\frac{\nabla A}{\partial t}_S := \iota_S^* \frac{\nabla A}{\partial t} \in \Gamma(T^*S)$, $W := \frac{1}{2} g_t^{-1} \frac{\partial g_t}{\partial t}$ is the Weingarten map of $\iota_S : S \hookrightarrow M$, $H := \frac{1}{n-1}\mathrm{tr}(W)$ is its mean curvature and $\psi_0^l := \psi^l_{|S} \in \Gamma(\sigma_{|S})$.

Remark 3. An inspection of the proof shows that the assumption of C^4-extendibility of $I^+(S)$ could be replaced by the weaker assumption of weak C^4-extendibility, defined as follows: A globally hyperbolic manifold (A, k) is **weakly C^l-extendible** if there is a sequence of smooth spacelike hypersurfaces (not necessarily Cauchy) of (A, k) such that $S_n \subset I^+(S_{n+1})$, $A = \bigcup_{i \in \mathbb{N}} I^+(S_n)$ and $I^+(S_n)$ is C^l-extendible, for all $n \in \mathbb{N}$. This generalization could be interesting applied to $(A, k) = I_M^+(S)$ for an asymptotically flat spacetime M and hyperboloidal subsets S_n.

We can derive as an immediate corollary for the case that M has a Cauchy temporal function t all of whose level sets are "extendible in both directions". Here it is important to note that every conformal extension $\mathcal{I}$ induces a pair of constraint equations $C_{\mathcal{I}}$ as above. Then we obtain:

Theorem 3.2. *Let (M, g) be a 4-dimensional globally hyperbolic manifold with a Cauchy temporal function t such that for all level sets $S_a := t^{-1}(\{a\})$ of t, $I^\pm(S_a)$ are both C^4-extendible by a conformal extension $\mathcal{I}^\pm(a)$. Then for every Cauchy surface S such that $t|_S$ is bounded, and for any initial values satisfying the neutrality and the constraint equations $C_{\mathcal{I}^-(e)}, C_{\mathcal{I}^+(f)}$ for $e > \sup t(S)$, $f < \inf t(S)$ and small in the respective Sobolev spaces, there is a global solution on M to the massless Dirac–Maxwell system above extending those initial values.* □

For the physically interested reader, we make a little more precise what would have to be done to connect our setting to proper QED. First of all, one should build up the n-particle space as the vector space generated by exterior products of classical solutions that are totally antisymmetric under permutations of different spinor fields of equal mass and charge to obtain the usual fermionic commutation relations. Expanding in a basis of $\mathrm{Span}(\psi^1, \ldots, \psi^N)$ orthonormal w.r.t. the conserved L^2-scalar product $(\psi, \phi) := \int_S j_{\psi,\phi}(\nu)$ (where ν is the normal vector field to a Cauchy surface S), we see we can assume that the spinor fields form a $(\cdot,\cdot)$-orthogonal system. If we have initial values on S in appropriate Sobolev spaces satisfying this condition, so will the restrictions of the solution to any other Cauchy surface due to the divergence-freeness of the $j_{\psi,\phi}$. The neutrality condition $\int_S J_\psi(\nu) = 0$ is in the case of an orthonormal system of spinors equivalent to the condition $\sum_{l=1}^N \mu_l = 0$. Moreover, in that case, J_ψ can be seen as the expectation value of the quantum-mechanical Dirac current operator, cf. [10, Sec. 3]. In the end, one would also need to quantize the bosonic potential A. Furthermore, one should consider the sum of all n-particle spaces to include phenomena like particle creation, particle annihilation, and also possibly the Dirac sea.

4. Invariances of the Dirac–Maxwell equations

We first recall the well-known invariances of the Dirac–Maxwell equation, see e.g. [14] for a proof:

Lemma 4.1. *Let (ψ, A) be a solution of the Dirac–Maxwell equations on a spin spacetime (M^n, g).*

1. *(Gauge invariance) For any $f \in C^\infty(M, \mathbb{R})$, the pair $(\psi' := (e^{-i\mu_1 f}\psi^1, \ldots, e^{-i\mu_N f}\psi^N), A' := A + df)$ solves again the Dirac–Maxwell equations on (M^n, g).*
2. *(Conformal invariance) If $n = 4$, then for any $u \in C^\infty(M, \mathbb{R})$, the pair $(\overline{\varphi} := e^{-\frac{3}{2}u}\overline{\psi}, A)$ solves $D_{\overline{g}}^{\mu_l A}\overline{\varphi^l} = m_l e^{-u}\overline{\varphi^l}$ and $d_{\overline{g}}^* dA = \sum_{l=1}^n \mu_l j_{\overline{\varphi^l}}$ on $(M^n, \overline{g} := e^{2u}g)$, where $\psi \mapsto \overline{\psi}$, $S_g M \otimes E \to S_{\overline{g}} M \otimes E$, denotes the natural unitary isomorphism induced by the conformal change of metric. In particular, in dimension 4, the Dirac–Maxwell equations are scaling-invariant and the massless Dirac–Maxwell equations are even conformally invariant.*

The **Dirac-wave operator** P_{DW} is defined by

$$P_{DW}(\psi^1 \oplus ... \oplus \psi^N \oplus A) := (D^A\psi^1 - m_1\psi^1, \dots, D^A\psi^N - m_N\psi^N, \Box A - J_\psi),$$

and the **Dirac-wave equation** is just the equation $P_{DW}(\psi, A) = 0$, where $\Box := dd^* + d^*d$.

Proposition 4.2 *(Lorenz gauge). Let (M, g) be as above.*

i) For any solution (ψ, A) of the Dirac-wave equation, $\Box(d^*A) = 0$ holds on M. In particular $d^*A = 0$ on M if and only if $(d^*A)_{|S_0} = 0 = \big(\frac{\partial}{\partial t}d^*A\big)_{|S_0}$.

ii) Given any solution (ψ, A) to the Dirac-wave equation, the equations $(d^*A)_{|S_0} = 0 = \big(\frac{\partial}{\partial t}d^*A\big)_{|S_0}$ are equivalent to

$$0 = \frac{1}{\beta}A_1(\frac{\partial}{\partial t}) - \sum_{j=1}^{3}(\nabla_{e_j}A_0)(e_j) \tag{4.1}$$

$$\begin{aligned}0 = &-(\nabla^{\tan})^*\nabla^{\tan}A_0(\frac{\partial}{\partial t}) - \sum_{j=1}^{3}\nabla_{e_j}A_1(e_j) - \frac{1}{2\beta}\mathrm{tr}_{g_t}(\frac{\partial g_t}{\partial t})A_1(\frac{\partial}{\partial t}) + \frac{1}{\beta}A_1(\mathrm{grad}_{g_t}(\beta(t,\cdot)))\\ &+ \frac{1}{2\beta}\nabla_{\mathrm{grad}_{g_t}(\beta(t,\cdot))}A_0(\frac{\partial}{\partial t}) + \frac{1}{2}g_t(\nabla^{\tan}A_0, \frac{\partial g_t}{\partial t}) + \mathrm{ric}^M(\frac{\partial}{\partial t}, A_0^\sharp) + \sum_{l=1}^{N}\mu_l j_{\psi_0^l}(\frac{\partial}{\partial t}),\end{aligned} \tag{4.2}$$

where $A_0 := A_{|S_0} \in \Gamma(T^*M_{|S_0})$, $A_1 := \frac{\nabla A}{\partial t}_{|S_0} \in \Gamma(T^*M_{|S_0})$ and $\psi_0^l := \psi^l_{|S_0} \in \Gamma(\sigma_{|S_0})$.

For a proof, we refer to [14]. Another important property of solutions to the Dirac–Maxwell equation is charge conservation, see e.g. [14] for a proof:

Proposition 4.3. *Let $(\psi = (\psi^1, \dots, \psi^N), A)$ be any classical solution to the Dirac–Maxwell equation such that, along a given (smooth, spacelike) Cauchy hypersurface S with future-directed unit normal ν, the 1-form $dA(\nu, \cdot)$ is compactly supported. Then $\int_{S'} J_\psi(\nu') = 0$ for all Cauchy hypersurfaces S' of M with future unit normal vector ν'. In particular, for $N = 1$ and $\mu_1 \neq 0$, we can conclude $\psi_1 = 0$.*

Proposition 4.3 implies that if the initial data allow for a conformal extension and are not pure Maxwell theory, then the system has vanishing total charge.

5. Proof of the main theorem

In a first geometric step, we choose a C^k extension F of $(I^+(S'), g)$ to a globally hyperbolic manifold (N, h) and consider the chosen Cauchy surface $S \subset I^+(S')$. Note that $U := N \setminus J^-(S)$ is a future subset of N and thus globally hyperbolic; let us choose a Cauchy temporal function T on U, and consider a sequence of Cauchy hypersurfaces $S_n := T^{-1}(r_n)$ of (U, h). The exact values of the r_i will be specified later. Note that the S_n are never Cauchy hypersurfaces of $F(I^+(S'))$. In the following we adopt the convention of denoting different spatio-temporal regularities related to the splitting induced by the temporal function T. The term C^lH^k in this notation refers to an object which is C^l regular in the time coordinate and H^k-regular in spatial direction.

The general strategy in the following is to find appropriate bounds on the initial values in different subsets of $F(S)$ (or, equivalently, corresponding bounds on S) implying that there is a global solution of a certain regularity. In our main theorem, we assume the initial Lorenz gauge condition on $F(S)$ (see Proposition 4.2) and therefore can use the first prolongation (for the definition, see end of Appendix, after Corollary 2.2) $\tilde{P}_{\mathrm{DW}}$ of the Dirac-wave operator P_{DW} in N instead of P_{DM}. We are first interested in regularity C^1H^4, as the degree of the operator P_{DW} is 2 and as the critical regularity of the associated symmetric hyperbolic operator defined as a first prolongation is $k = 4$ satisfying $\frac{k-1}{2} = 3/2$. Due to the lifetime estimate in Theorem 2.2, which is a generalization of the well-known extension/breakdown criterion for smooth coefficients, there is a positive number δ such that for initial values u_1 on

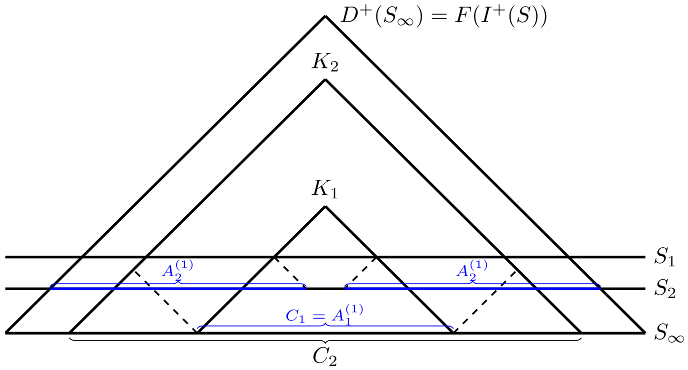

Fig. 1. Construction of the sequence $A_i^{(n)}$.

S_1 with $\|u_1\|_{H^s(S_1,h)} < \delta$ there is a global solution on $D^+(S_1) \cap F(I^+(S))$ in N. In a second step, we have to manage the "initial jump" from $S_\infty := F(S)$ to S_1, that is, we have to define sufficient conditions on S_∞ such that initial values satisfying those conditions induce solutions u reaching S_1 and satisfying $\|u\|_{H^s(S_1,h)} < \delta$ there, so we get a global solution on $D^+(F(S))$, where D^+ is the future domain of dependence. In the end, by conformally back-transforming the solution, we obtain a solution on $J^+(S)$ for the given initial values on S.

Due to the unavoidable divergence of the conformal structure, we have to "avoid spatial infinity" in all computations, in the following sense: We transport sufficient H^4 bounds from S_1 down to S_∞ in regions of a certain distance from the boundary of $D^+(F(S)) \subset N$, while closer to the boundary we only transport them "halfway down" from one hypersurface S_n to the next hypersurface S_{n+1}. More exactly, we choose a compact exhaustion of S_∞, i.e. a sequence of open sets C_n in S_∞ such that $\overline{C_n}$ is compact, such that $\overline{C_n} \subset C_{n+1}$ and $\bigcup_{i=1}^\infty C_i = S_\infty$. Furthermore, we define $K_n := D^+(C_n)$ as their future domains of dependence. We choose $r_1 < \sup(T(D^+(C_1)))$. Inductively, by compactness of the possibly empty subset

$$V_n := J^+(\overline{C_n}) \cap \partial K_{n+1},$$

we find $\tau_n := \min\{T(x)|x \in V_n\} > -\infty$ and define $r_{n+1} := \min\{r_n - 1, \tau_n\}$ and $S_{n+1} := T^{-1}(r_{n+1})$. With this choice, $\lim_{n\to\infty} r_n = -\infty$ and

$$J^-(S_{n+1} \setminus K_{n+1}) \cap C_n = \emptyset. \tag{5.1}$$

Now we construct inductively a locally finite family of subsets A_j of $F(S)$ and a sequence b such that if u_∞ is an initial value on S_∞ with $\|u_\infty|_{A_j}\|_{C^4} < b_j$ then there is a global C^1 solution u on $D^+(S_\infty)$ of $\tilde{P}_{DW} u = 0$ with $u|_{S_\infty} = u_\infty$. This sequence b will be constructed via a corresponding sequence a for the H^4 norms, which in turn is constructed as a limit of finite sequences $a^{(m)} \in \mathbb{R}^{m+1}$ that are stable in the sense that $a_n^{(m)} = a_n^{(m')}$ whenever $n \leq m-2, m'-2$, so that, for n fixed, the sequence $m \mapsto a_n^{(m)}$ is eventually constant, thus we will, indeed, be able to define $a_i := \lim_{m\to\infty} a_i^{(m)} > 0$.

We define, for $n \geq 1$, a finite set of subsets $\{A_1^{(n)}, \ldots A_{n+1}^{(n)}\}$ of $D^+(F(S))$ by (see Fig. 1)

$$A_1^{(n)} := C_1, \qquad A_{i+1}^{(n)} := J^-(S_i \setminus K_i) \cap C_{i+1} \qquad \forall 1 \leq i \leq n-1, \qquad A_{n+1}^{(n)} := J^-(S_n \setminus K_n) \cap S_{n+1}.$$

Note that the first n subsets are in S_∞ whereas the last one is in S_{n+1}. Note furthermore that the sequence stabilizes in the sense that $A_i^{(n)} = A_i^{(m)}$ if $m, n > i+1$, and the limit sequence is $A_1 := C_1$, $A_{i+1} := J^-(S_i \setminus K_i) \cap C_{i+1}$ $\forall i > 1$.

Let us call a finite positive sequence $a_1^{(n)}, \ldots, a_{n+1}^{(n)}$ a **control sequence at step** n iff every C^1H^4 solution u of $\tilde{P}_{DW} u = 0$ in $J^+(S_\infty) \cap J^-(S_{n+1})$ with $\|u|_{A_i^{(n)}}\|_{H^4} < a_i^{(n)}$ for all $i \in \mathbb{N} \cap [0, n+1]$ extends to a global C^1H^4 solution on $D^+(S_\infty) = F(I^+(S))$.

Lemma 1. *For every $n \geq 2$, there is a control sequence $a_i^{(n)}$ at step n, and the sequences stabilize in the sense that $a_i^{(n)} = a_i^{(m)}$ if $m, n > i+1$.*

Proof of the lemma. Obviously, for $n = 1$, we only have to ensure that $||u||_{H^4(S_1)} \leq \delta$. The lifetime estimate of Theorem 2.2 in the region K_1 implies that there is a positive constant $a_1^{(2)}$ such that $||u||_{H^4(C_1)} < a_1^{(2)}$ ensures that u extends up to $S_1 \cap K_1$ and $||u||_{H^4(S_1 \cap K_1)} < \delta/2$. Moreover, the lifetime estimate in $I^+(S_2) \cap I^-(S_1 \setminus K_1)$ implies that there is a second constant $a_2^{(2)}$ such that $||u||_{H^4(S_2 \cap J^-(S_1 \setminus K_1))} < a_2^{(2)}$ implies $||u||_{H^4(S_1 \setminus K_1)} < \delta/2$. Then it is straightforward to show that if both conditions are satisfied, the solution u fulfills $||u||_{H^4(S_1)} < \delta$, and therefore the solution extends to all of $F(J^+(S))$.

Each induction step is again done by applying the lifetime estimate in two regions. Now assume that there is a control sequence at step n. We have to look for an appropriate sequence of H^4 bounds $(a_1^{(n+1)}, \dots, a_{n+2}^{(n+2)})$ on $A_1^{(n+1)}, \dots, A_{n+2}^{(n+1)}$. First we define

$$a_i^{(n+1)} := a_i^{(n)} \qquad \forall 1 \leq i \leq n$$

To ensure the H^4-bound on $A_{n+1}^{(n)}$, we divide $A_{n+1}^{(n)}$ into its inner part $I_{n+1}^{(n)} := A_{n+1}^{(n)} \cap K_{n+1}$ and its outer part $O_{n+1}^{(n)} := A_{n+1}^{(n)} \setminus K_{n+1} = S_{n+1} \setminus K_{n+1}$. We want to ensure the H^4-bound $a_{n+1}^{(n)}$ on both parts. To guarantee the H^4 bound $a_{n+1}^{(n)}$ on the inner part there is a sufficient H^4 bound $a_{n+1}^{(n+1)}$ on

$$J^-(I_{n+1}^{(n)}) \cap S_\infty = J^-(S_n \setminus K_n) \cap C_{n+1} = A_{n+1}^{(n+1)},$$

whereas for the H^4 bound $a_{n+1}^{(n)}$ on the outer part, an H^4 bound $a_{n+2}^{(n+1)}$ on

$$J^-(O_{n+1}^{(n)}) \cap S_{n+2} = J^-(S_{n+1} \setminus K_{n+1}) \cap S_{n+2}^+ = A_{n+2}^{(n+1)}$$

is sufficient. Thus $a_i^{(n+1)}$ is a control sequence at step $n + 1$, and indeed the sequences stabilize in the sense above by definition. □

As the sequences stabilize, we can define the (infinite, positive) limit sequence a_i. Now there are $b_i > 0$ such that $\|u_0\|_{H^4(A_i)} < a_i$ is satisfied if $\|u_0\|_{C^4(A_i)} < b_i$. Now, the condition (5.1) ensures that for the annular regions $D_i := C_{i+1} \setminus C_i$, with $D_0 := C_0$ and for every $i \in \mathbb{N}$ we have $D_i \cap A_j \neq \emptyset$ only if $j = i$ or $j = i + 1$. So on every D_i we have to satisfy only two C^4 bounds b_i for all control sequences to be satisfied; let $\underline{b}_i$ be the minimum of those two bounds. For initial values u_0 with

$$||u_0||_{C^4(D_i)} < \underline{b}_i, \tag{5.2}$$

and given any point $q \in F(M)$, we want to show that q is contained in a domain of definition for a C^1 solution u of $\tilde{P}_{DW} u = 0$ with $u|_{S_\infty} = u_0$. To that purpose, we choose an i such that $q \in K_i$ and choose $f_i \in C^\infty(S_\infty, [0, 1])$ with $f_i(C_i) = \{1\}$ and $\mathrm{supp}(f_i) \subset S_\infty \setminus C_{i+1}$. Then we solve the initial value problem for $u^{(i)} = f_i \cdot u_0$. Applying the ith step in the induction above, we get a solution $u^{[i]}$ on a domain of definition including q. Locality implies that any local solution with initial value u_0 coincides with $u^{[i]}$ on K_i. This is, the domain of definition of a maximal solution includes q. Note that Eq. (5.2) corresponds to a bound in a weighted C^4-space on S.

As usual, we show higher regularity by bootstrapping, i.e. considering the differentiated equation (which is a linear equation in the highest derivatives again). Consider the highest derivatives in a Sobolev Hilbert space as independent variables and show that they are in the same Sobolev Hilbert space as the coefficients, thereby gaining one order of (weak) differentiability. Finally we use Sobolev embeddings in the usual way. □

Conflict of interest statement

There is no conflict of interest.

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