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Positive solutions of semilinear elliptic problems in unbounded domains with unbounded boundary ☆

Solutions positives de problèmes semi-linéaires elliptiques dans des domaines non bornes ayant frontière non borné

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Abstract

This paper is concerned with the existence and multiplicity of positive solutions of the equation $-\Delta u+u=u^{p-1}$, $2<p<2^*=\frac{2N}{N-2}$, with Dirichlet zero data, in an unbounded smooth domain $\Omega\subset\mathbb{R}^N$ having unbounded boundary. Under the assumptions:

(h_1) $\exists\tau_1,\tau_2,\dots,\tau_k\in\mathbb{R}^+\setminus\{0\}$, $1\leqslant k\leqslant N-2$, such that

$$(x_1,x_2,\dots,x_N)\in\Omega \iff (x_1,\dots,x_{i-1},x_i+\tau_i,\dots,x_N)\in\Omega,\ \forall i=1,2,\dots,k,$$

(h_2) $\exists R\in\mathbb{R}^+\setminus\{0\}$ such that $\mathbb{R}^N\setminus\Omega\subset\{(x_1,x_2,\dots,x_N)\in\mathbb{R}^N\colon \sum_{j=k+1}^N x_j^2\leqslant R^2\}$

the existence of at least $k+1$ solutions is proved.

Résumé

Dans cet article on étudie l'existence et la multiplicité de solutions positives pour l'équation $-\Delta u+u=u^{p-1}$, $2<p<2^*=\frac{2N}{N-2}$, avec la condition de Dirichlet $u=0$ sur $\partial\Omega$. Le domaine $\Omega\subset\mathbb{R}^N$ est non borné et $\partial\Omega$ est non borné aussi. En supposant que les conditions

(h_1) $\exists\tau_1,\tau_2,\dots,\tau_k\in\mathbb{R}^+\setminus\{0\}$, $1\leqslant k\leqslant N-2$, tels que

$$(x_1,x_2,\dots,x_N)\in\Omega \iff (x_1,\dots,x_{i-1},x_i+\tau_i,\dots,x_N)\in\Omega,\ \forall i=1,2,\dots,k,$$

(h_2) $\exists R\in\mathbb{R}^+\setminus\{0\}$ tel que $\mathbb{R}^N\setminus\Omega\subset\{(x_1,x_2,\dots,x_N)\in\mathbb{R}^N\colon \sum_{j=k+1}^N x_j^2\leqslant R^2\}$

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soient vérifiées, on démontre que le problème possède au moins $k+1$ solutions.

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1. Introduction and statement of the results

In this paper we are concerned with the existence and the multiplicity of solutions to

$$(P)\quad \begin{cases} -\Delta u + u = u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases}$$

where $2 < p < 2^* = 2N/(N-2)$ and $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is an unbounded smooth domain with $\partial\Omega$ unbounded.

Problem (P) has a variational structure: its solutions can be found looking for positive functions that are critical points of the functional

$$E(u) = \int_\Omega \left(|\nabla u|^2 + u^2\right) \mathrm{d}x$$

constrained to lie on the manifold

$$V = \left\{u \in H_0^1(\Omega)\colon\ |u|_{L^p(\Omega)} = 1\right\}.$$

However, the usual variational techniques (minimization, minimax methods) cannot be applied straightly, because of the lack of compactness, due to the unboundedness of Ω. Indeed, since the embedding $j : H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is continuous, but not compact, the manifold V is not closed for the weak H_0^1-topology and, moreover, the basic Palais–Smale condition is not satisfied by E at every energy level. Furthermore, as we shall see, the situations, one has to face, are strongly depending on the shape of the domain in which (P) is considered, so the corresponding technical difficulties can be considerably different.

When $\mathbb{R}^N \setminus \Omega$ is a ball the existence of a solution to (P) can be, quite easily, proved (see [13]), by minimizing E on the manifold $V_r = \{u \in H_r(\Omega)\colon\ |u|_{L^p(\Omega)} = 1\}$, $H_r(\Omega)$ being the subspace of $H_0^1(\Omega)$ consisting of spherically symmetric functions, that, as well known [21], embeds compactly in $L^p(\Omega)$. Analogous devices can be used when $\mathbb{R}^N \setminus \Omega$ is bounded and it is "nearly" spherical, in a suitable sense, or enjoys of some other kind of symmetry [10] and, moreover, when Ω is a "strip-like" domain [12].

The question becomes more difficult when $\mathbb{R}^N \setminus \Omega$ has no symmetry properties. Indeed a classical, by now, result [13] states that, for a very large class of unbounded domains, those satisfying the condition: $\exists \bar{x} \in \mathbb{R}^N$: $(\nu(x), \bar{x}) \geqslant 0$ $\forall x \in \partial\Omega$, $(\nu(x), \bar{x}) \not\equiv 0$, (P) admits only the trivial solution $u \equiv 0$. Moreover, even when $\mathbb{R}^N \setminus \Omega$ is bounded, problem (P) is not easy to handle and cannot be solved by minimization: the infimum of E on V equals the infimum of $\|u\|^2_{H^1(\mathbb{R}^N)}$ on the manifold $\{u \in H^1(\mathbb{R}^N)\colon\ |u|_{L^p(\mathbb{R}^N)} = 1\}$ and is not achieved [4]. Nevertheless, in this case, a careful analysis of the Palais–Smale sequences behaviour [4] has made possible an estimate of the energy levels in which the compactness is saved and to give some answers to the existence and multiplicity questions for (P). The existence of a positive nonminimizing solution to (P) has been proved (see [4,2]) by minimax methods, and furthermore multiplicity results have been obtained, by using subtle geometric and topological arguments, in [6–8,19].

When both Ω and $\mathbb{R}^N \setminus \Omega$ are unbounded the compactness situation can be even more complex. Indeed, when $\mathbb{R}^N \setminus \Omega$ is bounded, the above mentioned result states that a Palais–Smale sequence either converges strongly to its weak limit or differs from it by a finite number of sequences that are noting but normalized solutions of the limit problem

$$(P_\infty)\quad \begin{cases} -\Delta u + u = u^{p-1} & \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right) \end{cases}$$

"travelling to infinity" and infinitely far away each other. When Ω and $\mathbb{R}^N \setminus \Omega$ are unbounded and invariant with respect to a group of translations, it is not difficult to understand that, besides the above described behaviour, a noncompact Palais–Smale sequence can also look like a solution of (P) (normalized in $L^p(\Omega)$) "travelling to infinity", of course by means of translations that leave Ω invariant. This happens, for instance, when Ω is the complement of a cylinder: $\Omega = \mathbb{R}^N \setminus \{(x_1, \ldots, x_N) \in \mathbb{R}^N\colon \sum_{j=k+1}^N x_j^2 \leqslant R^2\}$, where $1 \leqslant k \leqslant N-2$ and $R > 0$; in fact, if u solves (P), then $u(x - y^i)/|u|_{L^p(\Omega)}$, with $y^i = (x_1^i, 0, \ldots, 0)$ and $x_1^i \underset{i\to+\infty}{\longrightarrow} +\infty$, is a noncompact Palais–Smale sequence at the same energy level of the critical point, of E on V, $u/|u|_{L^p(\Omega)}$. Moreover, we remark that even worse phenomena can occur, in fact unbounded domains with unbounded boundary exist such that the Palais–Smale condition for the related energy functional E can fail at every energy level (see [17]).

The question of the existence of solutions of (P) when Ω is an unbounded domain having unbounded boundary is still very partially investigated. Most of the known existence results concern domains bounded in some co-ordinates (strip-like, cylinders) (see [23] and references therein), while only recently some existence results have been proved under suitable condition at infinity on Ω and on $\partial\Omega$ [18,17].

The research, we present here, deals with problem (P) when Ω is an unbounded "exterior" domain having unbounded boundary. Precisely, we suppose that Ω satisfies the following assumptions:

(h_1) there exist k positive real numbers $\tau_1, \tau_2, \ldots, \tau_k$, $1 \leqslant k \leqslant N-2$, such that

$$(x_1, x_2, \ldots, x_N) \in \Omega \iff (x_1, \ldots, x_{i-1}, x_i + \tau_i, \ldots, x_N) \in \Omega,\ \forall i = 1, 2, \ldots, k,$$

(h_2) there exists $R \in \mathbb{R}$, $R > 0$, such that

$$\mathbb{R}^N \setminus \Omega \subset \left\{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N\colon \sum_{j=k+1}^N x_j^2 \leqslant R^2\right\}.$$

The main result we obtain is contained in the following

Theorem 1.1. *Let Ω be a smooth domain verifying conditions* (h_1) *and* (h_2). *Then problem* (P) *has at least* $(k+1)$ *solutions,* $u_1, u_2, \ldots, u_{k+1}$, *nonequivalent, in the sense that* $\forall i \neq j$, $i, j = 1, \ldots, k+1$, *does not exist* $(h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k$ *such that* $u_i(x_1, \ldots, x_N) = u_j(x_1 + h_1\tau_1, x_2 + h_2\tau_2, \ldots, x_k + h_k\tau_k, x_{k+1}, \ldots, x_N)$.

We stress the fact that, dropping assumption (h_1), Theorem 1.1 is not true and moreover (P) could not have any solution, as one easily understands considering $\Omega = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N\colon |(x_2, \ldots, x_N)| > f(x_1)\}$, where $f : \mathbb{R} \to \mathbb{R}$ is a bounded smooth function such that $0 < \inf_{\mathbb{R}} f < \sup_{\mathbb{R}} f < H$, $H \in \mathbb{R}$ and $f'(t) > 0$ $\forall t \in \mathbb{R}$. Ω is a domain satisfying the above mentioned nonexistence condition, our condition (h_2), but not condition (h_1). Moreover we point out that if in assumptions (h_1), (h_2) we have $k = N-1$ then Theorem 1.1 does not hold, in general. Consider, for example, $D = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N\colon |x_1| > R\}$, $R > 0$: since no solution exists on half-spaces, problem (P) has no solution on D. Indeed, the proof we carry on does not work when $k = N-1$, because in such a case Lemma 4.7 is not true, with $q = 2$.

On the contrary, a simple example of a domain to which Theorem 1.1 applies, giving the existence of at least two solutions, is obtained considering $\Omega = \{x \in \mathbb{R}^3\colon \operatorname{dist}(x, \mathcal{E}) > H\}$, $H \in \mathbb{R}^+ \setminus \{0\}$, with $\mathcal{E} = \{(x_1, \cos x_1, \sin x_1)\colon x_1 \in \mathbb{R}\}$.

We, also, must observe that, in domains verifying the assumptions of Theorem 1.1, but having richer symmetry properties (as the complement of a cylinder), solutions nonequivalent in the sense of Theorem 1.1 can be identified by a different kind of translation. Because of this we explicitly state the following corollary, for which, on the other hand, we have an independent proof, simpler than that of Theorem 1.1.

Theorem 1.2. *Let Ω be a smooth domain verifying conditions* (h_1) *and* (h_2). *Then problem* (P) *has at least one solution.*

The paper is organized as follows: Sections 2 and 3 are devoted to build the variational framework for the study, namely, in Section 2, after some remarks, the necessary equivariant, Ljusternik–Schnirelmann type, theory is exposed, while in Section 3 the compactness question is studied. Section 4 contains some basic asymptotic estimates and in Section 5 the proofs of Theorems 1.1 and 1.2 are displayed.

2. Notations, preliminary remarks, equivariant theory recalls

Throughout the paper we make use of the following notations:

- $L^p(\mathcal{D})$, $1 \leqslant p < +\infty$, $\mathcal{D} \subseteq \mathbb{R}^N$, denotes a Lebesgue space; the norm in $L^p(\mathcal{D})$ is denoted by $|\cdot|_{p,\mathcal{D}}$;
- $H_0^1(\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^N$, and $H^1(\mathbb{R}^N) \equiv H_0^1(\mathbb{R}^N)$ denote the Sobolev spaces obtained, respectively, as closure of $\mathcal{C}_0^\infty(\mathcal{D})$, and $\mathcal{C}_0^\infty(\mathbb{R}^N)$, with respect to the norms
$$\|u\|_{\mathcal{D}} = \left[\int_{\mathcal{D}} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x\right]^{1/2}, \qquad \|u\|_{\mathbb{R}^N} = \left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x\right]^{1/2};$$
- if $\mathcal{D}_1 \subset \mathcal{D}_2 \subseteq \mathbb{R}^N$ and $u \in H_0^1(\mathcal{D}_1)$, we denote also by u its extension to $\mathcal{D}_2$ obtained setting $u \equiv 0$ outside $\mathcal{D}_1$. Hence for all $u \in H_0^1(\Omega)$
$$E(u) = \int_{\Omega} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x \equiv \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x;$$
- the generic point $x = (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_N) \in \mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$ is denoted by $(x_1, x_2, \ldots, x_k, x')$ where $x' = (x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-k}$, k being the number that appears in assumption (h_1), we put also $|x'| = (\sum_{j=k+1}^N x_j^2)^{1/2}$;
- $B(y, r)$ denotes the open ball of $\mathbb{R}^N$, having radius r and centred at y.

We set
$$m := \inf\left\{\|u\|_{\mathbb{R}^N}^2\colon u \in H^1\left(\mathbb{R}^N\right),\ |u|_{p,\mathbb{R}^N} = 1\right\}. \tag{2.1}$$
The infimum in (2.1) is achieved (see [21,5]) by a positive function ω, that is unique modulo translation [16] and radially symmetric about the origin, decreasing when the radial co-ordinate increases and such that
$$\lim_{|x|\to+\infty} \left|D^s\omega(x)\right| |x|^{(N-1)/2} \mathrm{e}^{|x|} = d_s > 0, \quad s = 0, 1 \tag{2.2}$$
(see [15] and [5]).

The following proposition shows that, on the contrary, (P) cannot be solved by minimization.

Proposition 2.1. *Setting*
$$m_\Omega := \inf\left\{E(u)\colon u \in V\right\} \tag{2.3}$$
the relation
$$m_\Omega = m \tag{2.4}$$
holds and the minimization problem (2.3) *has no solution.*

Proof. Since we may consider $H_0^1(\Omega)$ as a subspace of $H^1(\mathbb{R}^N)$,
$$m_\Omega \geqslant m.$$
To prove that the equality holds, let us consider the sequence $(\omega_{y_n})_{n\in\mathbb{N}}$ defined by
$$\omega_{y_n}(x) := \frac{\varphi(x)\omega(x - y_n)}{|\varphi(x)\omega(x - y_n)|_{p,\Omega}}, \quad x \in \Omega,$$
where, $\forall n \in \mathbb{N}$, $y_n = ((y_n)_1, (y_n)_2, \ldots, (y_n)_k, y_n') \in \Omega$, $\lim_{n\to+\infty} |y_n'| = +\infty$, ω is the function realizing (2.1) and $\varphi \in \mathcal{C}^\infty(\mathbb{R}^N, [0,1])$ is a cut-off function defined by $\varphi(x_1, x_2, \ldots, x_k, x') = \tilde{\varphi}(|x'|)$, $\tilde{\varphi}\colon \mathbb{R}^+ \to [0,1]$ being a $\mathcal{C}^\infty$ non-decreasing function such that $\tilde{\varphi}(s) = 0$ $\forall s \leqslant R$, $\tilde{\varphi}(s) = 1$ $\forall s \geqslant R + 1$ (where R is the number defined in assumption (h_2)). Using (2.2) it is not difficult to verify that
$$\begin{cases} \text{(a) } \lim_{n\to+\infty} \left|\varphi(x)\omega(x - y_n) - \omega(x - y_n)\right|_{p,\mathbb{R}^N} = 0, \\ \text{(b) } \lim_{n\to+\infty} \left\|\varphi(x)\omega(x - y_n) - \omega(x - y_n)\right\|_{\mathbb{R}^N} = 0 \end{cases} \tag{2.5}$$

hence

$$\lim_{n\to+\infty} E(\omega_{y_n}) = m. \tag{2.6}$$

Let us now assume $u^* \in V$ exists so that $E(u^*) = m$, then by the uniqueness of the family of functions realizing (2.1)

$$u^*(x) = \omega(x - y^*) \quad \text{for some } y^* \in \mathbb{R}^N.$$

This is impossible because $\omega(x) > 0$ $\forall x \in \mathbb{R}^N$ and $\mathbb{R}^N \setminus \overline{\Omega} \neq \emptyset$. □

From the above result we can deduce, also, some useful estimates on the L^p norm of a critical point and a lower bound for the energy of a changing sign critical point of E on V.

Corollary 2.2. *Let $\bar{u}$ be a nontrivial solution of*

$$(P_\mu) \quad \begin{cases} -\Delta u + u = \mu |u|^{p-2} u & \text{in } \mathcal{D}, \\ u \in H_0^1(\mathcal{D}) \end{cases}$$

with either $\mathcal{D} = \Omega$ or $\mathcal{D} = \mathbb{R}^N$. Then

$$|\bar{u}|_{p,\mathcal{D}} \geqslant \left(\frac{m}{\mu}\right)^{1/(p-2)}. \tag{2.7}$$

Proof. By (2.1), (2.3) and (2.4) we have

$$m|\bar{u}|^2_{p,\mathcal{D}} \leqslant \|\bar{u}\|^2_{\mathcal{D}},$$

and being $\bar{u}$ solution of (P_μ)

$$\|\bar{u}\|^2_{\mathcal{D}} = \mu |\bar{u}|^p_{p,\mathcal{D}}.$$

Thus

$$|\bar{u}|^{p-2}_{p,\mathcal{D}} \geqslant \frac{m}{\mu}$$

and (2.7) follows. □

Corollary 2.3. *Let $\bar{u}$ be a critical point of E on V. If $E(\bar{u}) \in (m, 2^{1-2/p} m)$ then $\bar{u}$ does not change sign.*

Proof. Let us assume $E(\bar{u}) = \mu$, $|\bar{u}|_{p,\Omega} = 1$, $\bar{u} = \bar{u}^+ - \bar{u}^-$ and $\bar{u}^+ \not\equiv 0$, $\bar{u}^- \not\equiv 0$. Then, taking into account that $\bar{u}$ solves (P_μ) in Ω and using (2.3),(2.4), we obtain

$$m\left|\bar{u}^\pm\right|^2_{p,\Omega} \leqslant \left\|\bar{u}^\pm\right\|^2_\Omega = \mu\left|\bar{u}^\pm\right|^p_{p,\Omega}$$

hence

$$1 = |\bar{u}|^p_{p,\Omega} = \left|\bar{u}^+\right|^p_{p,\Omega} + \left|\bar{u}^-\right|^p_{p,\Omega} \geqslant 2\left(\frac{m}{\mu}\right)^{p/(p-2)}$$

that implies

$$\mu \geqslant 2^{1-2/p} m. \qquad \square$$

Remark 2.4. Obviously, the same conclusion holds true for every changing sign critical point of $\|u\|^2_{\mathbb{R}^N}$ on $\{u \in H^1(\mathbb{R}^N)\colon |u|_{p,\mathbb{R}^N} = 1\}$ and every normalized changing sign solution of (P_μ) in Ω or in $\mathbb{R}^N$.

We recall, now, some facts about equivariant critical points theory, that is the needful topological framework for our research.

Let X be a normed space and G a topological group. The *action of G on X is a continuous map*

$$G \times X \xrightarrow{[\,]} X, \quad [g, u] \mapsto gu$$

verifying the conditions

(i) $1 \cdot u = u$,
(ii) $(gh)u = g(hu)$,
(iii) $u \mapsto gu$ is linear.

The *action of G on X is* said *isometric* if

$$\|gu\| = \|u\| \quad \forall g \in G \ \forall u \in X.$$

A *set $A \subseteq X$* is *G-invariant* if $gA = A$ for all $g \in G$.

Two elements $w, z \in X$ are G-equivalent if $g(w) = z$ for some $g \in G$. We denote by $[w]$ the *orbit* of w, i.e. the subspace $\{gw\colon\ g \in G\}$, and by X/G the *orbit space*, i.e. the quotient space obtained by identifying each orbit to a point.

Two sequences $(w_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$, $w_n, z_n \in X$ are G-equivalent if, for all $n \in \mathbb{N}$, w_n is G-equivalent to z_n, in other words, a sequence $(g_n)_{n\in\mathbb{N}}$ exists so that $g_n \in G$ and $g_n w_n = z_n$.

A *map* $f : Y \to T$, $Y \subseteq X$ G-invariant set, T normed space, is *G-invariant* if

$$f \circ g(y) = f(y) \quad \forall g \in G \ \forall y \in Y.$$

A *map* $f : Y \to X$, $Y \subseteq X$ G-invariant set, is *G-equivariant* if

$$g \circ f = f \circ g \quad \forall g \in G.$$

Definition 2.5. Let A, B, Y, $B \subset A \subseteq Y$ be closed G-invariant subsets of a normed space X on which a topological group G acts. The *G-equivariant category of A in Y, relative to B*, denoted by $\mathrm{cat}_Y^G(A, B)$, is the least integer l such that there exist $(l + 1)$ closed G-invariant subsets of Y, $C_0, C_1, \ldots, C_l$ and $(l + 1)$ maps $h_j \in \mathcal{C}(C_j \times [0, 1], Y)$, such that

$$\text{(a)}\ A \subseteq \bigcup_{j=0}^{l} C_j, \qquad B \subseteq \mathcal{C}_0;$$

$$\text{(b)}\ \begin{cases} \text{(i) } h_j(\cdot, t) \text{ is } G\text{-equivariant } \forall t \in [0, 1],\ j = 0, 1, \ldots, l; \\ \text{(ii) } h_j(c, 0) = c \quad \forall c \in C_j,\ j = 0, 1, \ldots, l; \\ \text{(iii) } h_0(c, 1) \in B \quad \forall c \in C_0; \quad h_0(B, t) \subset B \quad \forall t \in [0, 1]; \\ \text{(iv) } \forall j = 1, 2, \ldots, l\ \exists w_j \in Y \text{ such that } h_j(c, 1) \in [w_j]\ \forall c \in C_j. \end{cases}$$

If such a number does not exist, we say that the G-equivariant category of A in Y relative to B is $+\infty$.

Definition 2.6. Let M be a $\mathcal{C}^1$, G-invariant manifold embedded in an Hilbert space H on which the topological group G acts isometrically. Let $F \in \mathcal{C}^1(M, \mathbb{R})$ a G-invariant functional. The functional F satisfies the G-Palais–Smale condition, briefly $(\mathrm{PS})^G$, at the level c if for every sequence $(u_n)_n$, $u_n \in M$, such that

$$F(u_n) \underset{n\to+\infty}{\longrightarrow} c, \qquad \nabla F(u_n) \underset{n\to+\infty}{\longrightarrow} 0$$

there exists a sequence $(v_n)_n$, G-equivalent to $(u_n)_n$, relatively compact.

The following Ljusternik–Schnirelmann type theorem provides a lower bound for the number of critical points of an invariant functional in suitable ranges of its values.

Theorem 2.7. *Let H be an Hilbert space on which the topological group G acts isometrically. Let be $M \subseteq H$ a G-invariant $\mathcal{C}^{1,1}$- manifold and $F \in \mathcal{C}^{1,1}(M, \mathbb{R})$ a G-invariant functional. Put for any $c \in \mathbb{R}$*

$$F^c = \{u \in M\colon\ F(u) \leqslant c\},$$
$$K^c = \{u \in M\colon\ F(u) = c,\ (\nabla F)(u) = 0\}.$$

Consider $-\infty < a < b < +\infty$, and assume $K^a = \emptyset = K^b$ and that F satisfies the $(\mathrm{PS})^G$ *for all $c \in [a, b]$. Then F has at least* $\mathrm{cat}_{F^b}^G(F^b, F^a)$ *critical points that are not G-equivalent and to which there correspond critical levels lying in (a, b).*

Proof. Since F is G-invariant, we have for all $u \in M$, $\forall g \in G$

$$F'(gu)[v] = \lim_{t \to 0} \frac{F(u + tg^{-1}v) - F(u)}{t} = F'(u)[g^{-1}v].$$

Thus, being the action of G isometric, we obtain

$$\big(\nabla F(gu), v\big) = \big(\nabla F(u), g^{-1}v\big) = \big(g\nabla F(u), v\big) \quad \forall g \in G,\ \forall u \in M$$

and we deduce that $\nabla F : M \to H$ is an equivariant map. Furthermore, because of the $(\mathrm{PS})^G$ condition, if F has not critical values in the interval $[\alpha, \beta] \subseteq [a, b]$, there exists a positive number $\delta > 0$ for which

$$\big\|\nabla F(u)\big\| > \delta \quad \forall u \in F^{-1}\big([\alpha, \beta]\big). \tag{2.8}$$

Indeed, if (2.8) were false, a sequence $(u_n)_n$, $u_n \in M$, would exist so that

$$F(u_n) \underset{n\to+\infty}{\longrightarrow} c \in [\alpha, \beta],$$
$$\nabla F(u_n) \underset{n\to+\infty}{\longrightarrow} 0.$$

Thus, a sequence $(g_n)_n$, $g_n \in G$, would exist so that, passing eventually to a subsequence, $g_n u_n \underset{n\to+\infty}{\longrightarrow} v \in M$. Hence we infer

$$F(v) = \lim_{n\to+\infty} F(g_n u_n) = \lim_{n\to+\infty} F(u_n) = c,$$
$$\big\|\nabla F(v)\big\| = \lim_{n\to+\infty} \big\|\nabla F(g_n u_n)\big\| = \lim_{n\to+\infty} \big\|g_n \nabla F(u_n)\big\| = \lim_{n\to+\infty} \big\|\nabla F(u_n)\big\| = 0$$

contradicting the nonexistence of critical values in $[\alpha, \beta]$.

Therefore, using well known methods (see e.g. [24] Lemmas 1.14 and 3.1), a number $\varepsilon > 0$ and a continuous deformation $\eta \in \mathcal{C}([0, 1] \times M, M)$ can be constructed so that

(i) $\eta(t, \cdot)$ $\forall t \in [0, 1]$ is a G-equivariant homeomorphism of M;

(ii) $\eta(0, u) = u$, $\forall u \in M$;

(iii) $\eta(t, u) = u$, $\forall t \in [0, 1]$ if $u \notin F^{-1}[\alpha - \varepsilon, \beta + \varepsilon]$;

(iv) $\eta(1, F^{\beta}) \subset F^{\alpha}$;

(v) $F\big(\eta(\cdot, u)\big)$ is nonincreasing $\forall u \in M$.

Then, the conclusion follows, by applying classical arguments of the generalized Ljusternik–Schnirelmann theory (see [24] Theorem 5.19). □

Remark 2.8. The same result could be also proved supposing H Banach, instead of Hilbert, space and under weaker regularity assumptions on F.

We end this section pointing out that, in our setting, *a noncompact group of translations, G, acting on $\mathbb{R}^N$ and, in turn, on $H^1(\mathbb{R}^N)$ and $H_0^1(\Omega)$ is considered*. Namely, for all $h \equiv (h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k$, we define $T_h : \mathbb{R}^N \to \mathbb{R}^N$, by

$$T_h(x) = T_h(x_1, x_2, \ldots, x_k, x') := (x_1 + \tau_1 h_1, x_2 + \tau_2 h_2, \ldots, x_k + \tau_k h_k, x')$$

hence we say that $x, y \in \mathbb{R}^N$ *are equivalent* if and only if there exists $(h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k$ such that $y = (x_1 + \tau_1 h_1, x_2 + \tau_2 h_2, \ldots, x_k + \tau_k h_k, x')$.

Clearly Ω is invariant under the action of G.

Analogously, for all $h \equiv (h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k$, we define $\mathcal{T}_h : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$, by

$$\mathcal{T}_h(u)(x) := u\big(T_h(x)\big) = u(x_1 + \tau_1 h_1, x_2 + \tau_2 h_2, \ldots, x_k + \tau_k h_k, x')$$

and we say that $u, v \in H^1(\mathbb{R}^N)$ *are equivalent* if and only if there exists $(h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k$ such that

$$v(x_1, x_2, \ldots, x_k, x') = u(x_1 + \tau_1 h_1, x_2 + \tau_2 h_2, \ldots, x_k + \tau_k h_k, x').$$

We remark that the action on $H^1(\mathbb{R}^N)$ of the group $\mathcal{G} := \{\mathcal{T}_h\colon h \in \mathbb{Z}^k\}$ is isometric, that $H_0^1(\Omega)$ can be seen as an invariant subspace of $H^1(\mathbb{R}^N)$, V is an invariant manifold in $H_0^1(\Omega)$ and the functional E is invariant.

3. A compactness result

The purpose of this section is to show that there exists an energy interval in which the compactness of the functional E is saved.

The result we prove is stated in the following

Proposition 3.1. *The functional E satisfies the $\mathcal{G}$-Palais–Smale condition on V at every level $c \in (m, 2^{1-2/p}m)$.*

Proof. Let $(u_n)_n$ be a Palais–Smale sequence for E constrained on V, e.g.

$$\begin{cases} \text{(a) } |u_n|_{p,\Omega} = 1, \\ \text{(b) } \lim_{n\to+\infty} E(u_n) = c, \\ \text{(c) } \lim_{n\to+\infty} \nabla E_{|V}(u_n) = 0 \end{cases} \tag{3.1}$$

and assume

$$c \in \bigl(m, 2^{1-2/p}m\bigr). \tag{3.2}$$

By definition of E, (3.1)(b) implies that $(u_n)_n$ is bounded in $H_0^1(\Omega)$, so there exists $u_0 \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{cases} \text{(a) } u_n \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega) \text{ and in } L^p(\Omega), \\ \text{(b) } u_n(x) \to u_0(x) \quad \text{a.e. in } \Omega. \end{cases} \tag{3.3}$$

By (3.1)(c), there exists a sequence $(\mu_n)_n$, $\mu_n \in \mathbb{R}$, such that

$$\bigl(\nabla E_{|V}(u_n), w\bigr) = \int_\Omega \bigl[(\nabla u_n, \nabla w) + u_n w\bigr]\,\mathrm{d}x - \mu_n \int_\Omega |u_n|^{p-2} u_n w = \mathrm{o}(1)\|w\|_\Omega \quad \forall w \in H_0^1(\Omega) \tag{3.4}$$

and, in view of (3.1)(a), (b), setting in (3.4) $w = u_n$, we deduce

$$\lim_{n\to+\infty} \mu_n = c. \tag{3.5}$$

Hence u_0 solves

$$\begin{cases} -\Delta u + u = c|u|^{p-2}u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{3.6}$$

Set now

$$v_n(x) = \begin{cases} (u_n - u_0)(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then, by (3.3)(a),

$$v_n \rightharpoonup 0 \quad \text{weakly in } H^1\bigl(\mathbb{R}^N\bigr) \text{ and } L^p\bigl(\mathbb{R}^N\bigr) \tag{3.7}$$

and

$$\|v_n\|_{\mathbb{R}^N}^2 = \|u_n\|_\Omega^2 - \|u_0\|_\Omega^2 + \mathrm{o}(1). \tag{3.8}$$

Furthermore, by (3.3)(b), the Brezis–Lieb theorem can be applied and it gives

$$|v_n|_{p,\mathbb{R}^N}^p = |u_n|_{p,\Omega}^p - |u_0|_{p,\Omega}^p + \mathrm{o}(1). \tag{3.9}$$

Let us suppose, now, $\|v_n\|_{\mathbb{R}^N} \not\to 0$ strongly (otherwise we are done). So, up to a subsequence, $\|v_n\|_{\mathbb{R}^N} \geqslant k_0 > 0$ $\forall n \in \mathbb{N}$, for some $k_0 \in \mathbb{R}$. Then, using (3.4), (3.6), (3.8), (3.9), we deduce that $k_1 \in \mathbb{R}$ exists such that $|v_n|_{p,\mathbb{R}^N}^p \geqslant k_1 > 0$.

Let us decompose, now, $\mathbb{R}^N$ into N-dimensional hypercubes Q_l, having unitary sides and vertices with integer co-ordinates, and put for all $n \in \mathbb{N}$

$$d_n = \max_{l\in\mathbb{N}} |v_n|_{p,Q_l}.$$

We claim that $\gamma \in \mathbb{R}$, $\gamma > 0$, exists such that (up to a subsequence)

$$d_n \geqslant \gamma > 0 \quad \forall n \in \mathbb{N}. \tag{3.10}$$

Indeed

$$\begin{aligned} 0 < k_1 &\leqslant |v_n|^p_{p,\mathbb{R}^N} = \sum_{l\in\mathbb{N}} |v_n|^p_{p,Q_l} \\ &\leqslant \max_{l\in\mathbb{N}} |v_n|^{p-2}_{p,Q_l} \sum_{l\in\mathbb{N}} |v_n|^2_{p,Q_l} \\ &\leqslant d_n^{p-2} k_2 \sum_{l\in\mathbb{N}} \|v_n\|^2_{Q_l} \leqslant d_n^{p-2} k_2 \|v_n\|^2_{\mathbb{R}^N}, \end{aligned}$$

$k_2 \in \mathbb{R}^+ \setminus \{0\}$ independent of i. Thus, in view of (3.1) and (3.8), (3.10) follows.

Let us call, for all $n \in \mathbb{N}$, y_n the centre of an hypercube Q_n in which $|v_n|_{p,Q_n} = d_n$.

If $(y_n)_n$ were bounded, then, passing eventually to a subsequence, we could assume that the y_n, for all n, belong to the same cube $\widehat{Q}$, and, hence, that they coincide. Thus, in $\widehat{Q}$ we would have, for all n, $|v_n|_{p,\widehat{Q}} \geqslant \gamma > 0$ and, on the other hand, $\|v_n\|_{\widehat{Q}} \leqslant \|v_n\|_{\mathbb{R}^N} \leqslant k_3$; as a consequence, by the Rellich Theorem, $(v_n)_n$ would converge strongly in $L^p(\widehat{Q})$ to a nonzero function, contradicting (3.7). Therefore

$$|y_n| \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Let us, now, call $\tilde{v}_0$ the weak limit, in $H^1(\mathbb{R}^N)$, of the sequence $\tilde{v}_n(x) := v_n(x + y_n)$. Arguing as before in the hypercube $\widetilde{Q}$ centred at the origin and having unitary sides, we conclude that $\tilde{v}_0 \not\equiv 0$. Moreover, as a consequence of (3.4), (3.5), $\tilde{v}_0$ is a weak solution, on its domain $\mathcal{D}$, of $-\Delta u + u = c|u|^{p-2}u$ and, since $|y_n| \to +\infty$ and Ω satisfies (h_2), we deduce $\mathcal{D} = \mathbb{R}^N$, when $\operatorname{dist}(y_n, \mathbb{R}^N \setminus \Omega) \underset{n\to+\infty}{\longrightarrow} +\infty$, $\mathcal{D} = \Omega$ (up to a translation) when $\operatorname{dist}(y_n, \mathbb{R}^N \setminus \Omega)$ is bounded.

Now, we claim that

$$\begin{cases} \text{(a) } u_0 = 0, \\ \text{(b) } \tilde{v}_0 \text{ does not change sign,} \\ \text{(c) } \tilde{v}_n \underset{n\to+\infty}{\longrightarrow} \tilde{v}_0 \text{ strongly in } H^1_0(\mathcal{D}). \end{cases} \tag{3.11}$$

Equality (3.11)(a) follows by observing that (3.8) implies

$$\|u_n\|^2_\Omega \geqslant \|u_0\|^2_\Omega + \|\tilde{v}_0\|^2_{\mathcal{D}} + \mathrm{o}(1), \tag{3.12}$$

thus, if u_0 were not zero, taking into account that $\tilde{v}_0 \not\equiv 0$ and that Corollary 2.2 applies to both u_0 and $\tilde{v}_0$, we would infer

$$E(u_n) = \|u_n\|^2_\Omega \geqslant 2m \cdot \left(\frac{m}{c}\right)^{2/(p-2)} + \mathrm{o}(1)$$

and, then

$$c \geqslant 2^{1-2/p} m$$

contradicting (3.2).

Assertion (3.11)(b) is a direct consequence of Remark 2.4 and of the arguments of Corollary 2.3.

Let us prove, then, (3.11)(c). Let us assume, by contradiction, that $\tilde{v}_n \nrightarrow \tilde{v}_0$ strongly. Then, setting $w_n(x) := (\tilde{v}_n - \tilde{v}_0)(x)$, $w_n(x) \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$, and $w_n(x) \nrightarrow 0$ strongly in $H^1(\mathbb{R}^N)$. So, we can repeat step by step the argument before applied to $(v_n)_n$, concluding that a sequence of points $(z_n)_n$, $z_n \in \mathbb{R}^N$, $|z_n| \underset{n\to+\infty}{\longrightarrow} +\infty$, and a nonzero function, $\widetilde{w}_0$, exist such that

$$\widetilde{w}_n(x) := w_n(x + z_n) \rightharpoonup \widetilde{w}_0(x) \quad \text{weakly in } H^1_0\big(\widehat{\mathcal{D}}\big),$$

$\widehat{\mathcal{D}}$ being either $\mathbb{R}^N$ or Ω, and $\widetilde{w}_0$ being a solution of $-\Delta u + u = c|u|^{p-2}u$ in $\widehat{\mathcal{D}}$. Furthermore the inequality

$$\|\widetilde{w}_n\|^2_{\mathbb{R}^N} = \|w_n\|^2_{\mathbb{R}^N} = \|u_n\|^2_\Omega - \|\tilde{v}_0\|^2_{\mathcal{D}} + \mathrm{o}(1)$$

holds, thus

$$\|u_n\|^2_{\Omega} \geqslant \|\tilde{v}_0\|^2_{\mathcal{D}} + \|\widetilde{w}_0\|^2_{\widetilde{\mathcal{D}}} + \mathrm{o}(1)$$

and, then,

$$c \geqslant 2^{1-2/p} m$$

follows, contradicting (3.2) and giving (3.11)(c). Now, (3.11) and (3.9) imply $|\tilde{v}_0|_{p,\mathcal{D}} = 1$ and, by (3.11)(b), we can suppose $\tilde{v}_0 \geqslant 0$ on $\mathcal{D}$. Hence, if $\mathcal{D} = \mathbb{R}^N$, the uniqueness of the positive regular solutions to $-\Delta u + u = c|u|^{p-2}u$ in $\mathbb{R}^N$ implies $c = E(\tilde{v}_0) = m$, contradicting (3.2). Therefore, $\mathcal{D} = \Omega$ and $0 < \mathrm{dist}(y_n, \mathbb{R}^N \setminus \Omega) < H$ for some $H \in \mathbb{R}^+ \setminus \{0\}$. Let us consider the sequence $(h_n)_n = (h_{n,1}, h_{n,2}, \dots, h_{n,k}) \in \mathbb{Z}^k$ such that

$$\tau_i h_{n,i} \leqslant y_{n,i} < \tau_i (h_{n,i} + 1), \quad i = 1, 2, \dots, k,\ n \in \mathbb{N},$$

and define

$$u_n^*(x_1, x_2, \dots, x_k, x') := u_n(x_1 + \tau_1 h_{n,1}, \dots, x_k + \tau_k h_{n,k}, x').$$

The sequence $(u_n^*)_n$ is $\mathcal{G}$-equivalent to $(u_n)_n$ and converges strongly in $H^1_0(\Omega)$ to a function u^* that is a nontrivial critical point of E on V. □

4. Useful tools and basic estimates

For what follows we need to introduce a barycenter type function. For all $u \in L^p(\mathbb{R}^N)$ we set

$$\tilde{u}(x) = \frac{1}{|B(x,1)|} \int_{B(x,1)} |u(y)| \, \mathrm{d}y \quad \forall x \in \mathbb{R}^N,$$

$|B(x,1)|$ denoting the Lebesgue measure of $B(x,1)$, and

$$\hat{u}(x) = \left[\tilde{u}(x) - \frac{1}{2} \max_{\mathbb{R}^N} \tilde{u}(x)\right]^+ \quad \forall x \in \mathbb{R}^N;$$

we, then, define $\beta : L^p(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$\beta(u) = \frac{1}{|\hat{u}|^p_{p,\mathbb{R}^N}} \int_{\mathbb{R}^N} x \big(\hat{u}(x)\big)^p \, \mathrm{d}x. \tag{4.1}$$

We point out that β is well defined for all $u \in L^p(\mathbb{R}^N) \setminus \{0\}$, because $\hat{u} \not\equiv 0$ and has compact support, that β is continuous and

$$\begin{cases} \text{(a) } \beta\big(u(x-y)\big) = \beta\big(u(x)\big) + y \quad \forall u \in L^p\big(\mathbb{R}^N\big) \setminus \{0\},\ \forall y \in \mathbb{R}^n, \\ \text{(b) } \beta\big(\omega(x)\big) = 0. \end{cases} \tag{4.2}$$

Remark 4.1. We stress the fact that the above barycenter map has been introduced some years ago by the first and the third author ([9], pages 265–266).

This map has been useful in many situations; in fact, it has been used also in [6,17,18] and recently, with a very slight modification, in [3] (actually, in [3] the definition of barycenter map is introduced as a new one and the definition given in [9] is not quoted).

We set, for all $r \in \mathbb{R}^+$,

$$D_r := \big\{(x_1, x_2, \dots, x_k, x') \in \mathbb{R}^N\colon\ |x'| < r\big\}$$

and

$$\mathcal{B}_r := \inf\big\{E(u)\colon\ u \in V,\ \beta(u) \in D_r\big\}, \tag{4.3}$$

so, in particular,

$$D_0 := \big\{(x_1, x_2, \dots, x_k, x') \in \mathbb{R}^N\colon\ x' = 0\big\}$$

and

$$\mathcal{B}_0 := \inf\{E(u):\ u \in V,\ (\beta(u))' = ((\beta(u))_{k+1}, (\beta(u))_{k+2}, \dots, (\beta(u))_N) = 0\}. \tag{4.4}$$

We remark that

$$\mathcal{B}_0 \geqslant \mathcal{B}_r \quad \forall r > 0. \tag{4.5}$$

In what follows, for every $y = (y_1, y_2, \dots, y_k, y') \in \mathbb{R}^N$, we set

$$z_y = (y_1, y_2, \dots, y_k, 1, 0, \dots, 0) \in \mathbb{R}^N$$

and we denote by S, Σ and Λ, respectively, the sets

$$S := \{(x_1, x_2, \dots, x_k, x') \in \mathbb{R}^N:\ |x'| = 2\}, \tag{4.6}$$

$$\Sigma := S + z_0, \tag{4.7}$$

$$\Lambda := \{\sigma y + (1-\sigma) z_y:\ y \in \Sigma,\ \sigma \in [0,1]\}. \tag{4.8}$$

For every $\rho > 0$ we define the operator

$$\Psi_\rho : \Sigma \times [0,1] \longrightarrow V$$

by

$$\Psi_\rho[y,\sigma](x) = \frac{\varphi(x)[(1-\sigma)\omega(x-\rho y) + \sigma\omega(x-\rho z_y)]}{|\varphi(x)[(1-\sigma)\omega(x-\rho y) + \sigma\omega(x-\rho z_y)]|_{p,\Omega}}, \tag{4.9}$$

where $\varphi \in \mathcal{C}^\infty(\mathbb{R}^N, [0,1])$ is the cut-off function introduced in Proposition 2.1.

We note that

$$\Psi_\rho[y,0](x) = \frac{\varphi(x)\omega(x-\rho y)}{|\varphi(x)\omega(x-\rho y)|_{p,\Omega}} = \omega_{\rho y}(x). \tag{4.10}$$

For every $\rho > 0$ we consider, also, the map

$$\xi_\rho : \Sigma \times [0,1] \longrightarrow \rho\Lambda$$

defined by

$$\xi_\rho[y,\sigma] = \rho((1-\sigma)y + \sigma z_y). \tag{4.11}$$

Proposition 4.2. *Let $\mathcal{B}_r$ be the numbers defined in* (4.3). *Then for all $r \in \mathbb{R}^+$, there exists $\mu_r \in \mathbb{R}$, such that*

$$\mathcal{B}_r \geqslant \mu_r > m. \tag{4.12}$$

Proof. Clearly, for all $r \geqslant 0$, $\mathcal{B}_r \geqslant m$; to prove (4.12) we argue by contradiction and we assume that $\mathcal{B}_{\hat r} = m$ for some $\hat r \geqslant 0$. Hence, a sequence $(u_n)_n$ must exist such that $u_n \in V$ and

$$\begin{cases} \text{(i) } \beta(u_n) \in D_{\hat r} \quad \forall n \in \mathbb{N}, \\ \text{(ii) } \lim_{n\to+\infty} E(u_n) = m. \end{cases} \tag{4.13}$$

Then, by the uniqueness of the minimizers family of (2.1), a sequence of points $(y_n)_n$, $y_n \in \mathbb{R}^N$, and a sequence of functions $(\chi_n)_n$, $\chi_n \in H^1(\mathbb{R}^N)$ exist so that, passing eventually to a subsequence, still denoted by $(u_n)_n$,

$$\begin{cases} \text{(i) } u_n(x) = \omega(x - y_n) + \chi_n(x) \quad \forall x \in \mathbb{R}^N, \\ \text{(ii) } \lim_{n\to+\infty} \chi_n(x) = 0 \quad \text{in } H^1(\mathbb{R}^N) \text{ and in } L^p(\mathbb{R}^N) \end{cases} \tag{4.14}$$

(see also [4] Lemma 3.1). Therefore, by (4.2), (4.14) (i) and the continuity of β, the relation

$$|\beta(u_n) - y_n| \underset{n\to+\infty}{\longrightarrow} 0$$

holds and, together with (4.13)(i), implies that the sequence $(y_n')_n$ is bounded. Hence, either the sequence $(y_n)_n$ is bounded, or it is unbounded, but, in view of the assumption (h_1), it can be replaced by an equivalent sequence, still denoted by $(y_n)_n$, contained in a bounded set; so, passing eventually to a subsequence, we conclude that $y_n \underset{n\to+\infty}{\longrightarrow} \bar{y}$. Thus, either $(u_n)_n$ or a $\mathcal{G}$-equivalent sequence, still denoted by $(u_n)_n$, satisfies

$$\lim_{n\to+\infty} u_n(x) = \omega(x - \bar{y}).$$

Then, by (4.13)(ii),

$$m = \lim_{n\to+\infty} E(u_n) = \int_{\Omega} \left[\left|\nabla\omega(x-\bar{y})\right|^2 + \left(\omega(x-\bar{y})\right)^2\right] \mathrm{d}x$$

that is impossible, because $\omega(x)$ realizes (2.1), $\omega > 0$ in $\mathbb{R}^N$ and $\mathbb{R}^N \setminus \overline{\Omega} \neq \emptyset$, so the statement follows. □

Lemma 4.3. *Let $\Sigma, \Psi_\rho, \mathcal{B}_0$ be as defined, respectively, in* (4.7), (4.9), (4.4). *Then there exists $\tilde{\rho} \in \mathbb{R}$ such that for all $\rho \geqslant \tilde{\rho}$*

$$\mathcal{B}_0 \leqslant \max_{\Sigma\times[0,1]} E\left(\Psi_\rho[y,\sigma]\right). \tag{4.15}$$

Proof. Taking into account (2.2), (4.2), (4.10), (2.5)(a), it is not difficult to verify that

$$\lim_{\rho\to+\infty} \left|\beta \circ \Psi_\rho[y,0] - \rho y\right|_{\mathbb{R}^N} = 0 \quad \forall y \in \Sigma. \tag{4.16}$$

Thus, for ρ large enough, $\beta \circ \Psi_\rho(\Sigma \times \{0\})$ is homotopically equivalent in $\mathbb{R}^N \setminus D_0$ to $\rho\Sigma$ and, then, there exists $(\hat{y}, \hat{\sigma}) \in \Sigma \times [0,1]$ such that $(\beta \circ \Psi_\rho)[\hat{y}, \hat{\sigma}] \in D_0$, so

$$\mathcal{B}_0 \leqslant E\left(\Psi_\rho[\hat{y},\hat{\sigma}]\right) \leqslant \max_{\Sigma\times[0,1]} E\left(\Psi_\rho[y,\sigma]\right),$$

as desired. □

Corollary 4.4. *Let Σ, Ψ_ρ and $\tilde{\rho}$ as in Lemma* 4.3. *Let $\mathcal{B}_r$, $r \in \mathbb{R}^+$, as defined* (4.3). *Then for all $\rho \geqslant \tilde{\rho}$*

$$\mathcal{B}_r \leqslant \max_{\Sigma\times[0,1]} E\left(\Psi_\rho[y,\sigma]\right). \tag{4.17}$$

Proof. Inequality (4.17) is an immediate consequence of (4.15) and (4.5). □

Next step is to establish some, crucial, asymptotic estimates on the energy of $\Psi_\rho(\Sigma \times \{0\})$ and of $\Psi_\rho(\Sigma \times [0,1])$. To this end, we need, first, to recall some known results:

Lemma 4.5. *For all $a, b \in \mathbb{R}^+$, for all $p \geqslant 2$, the relation*

$$(a+b)^p \geqslant a^p + b^p + (p-1)\left(a^{p-1}b + ab^{p-1}\right)$$

holds true.

Lemma 4.6. *Let $g \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $h \in \mathcal{C}(\mathbb{R}^N)$ be radially symmetric functions satisfying for some $\alpha \geqslant 0$, $b \geqslant 0$, $\gamma \in \mathbb{R}$*

$$\lim_{|x|\to+\infty} g(x)\exp\left(\alpha|x|\right)|x|^b = \gamma,$$

$$\int_{\mathbb{R}^N} \left|h(x)\right| \exp\left(\alpha|x|\right)\left(1+|x|^b\right) \mathrm{d}x < +\infty.$$

Then

$$\lim_{|y|\to+\infty} \left(\int_{\mathbb{R}^N} g(x+y)h(x)\,\mathrm{d}x\right) \exp\left(\alpha|y|\right)|y|^b = \gamma \int_{\mathbb{R}^N} h(x)\exp(-\alpha x_1)\,\mathrm{d}x$$

holds.

The proof of Lemma 4.5 can be found in [8], while the proof of Lemma 4.6 is in [1].

Now, we state, in the following lemma, a basic, preliminary, asymptotic relation.

Lemma 4.7. *Let be $k \in \mathbb{N}$, $1 \leqslant k \leqslant N-2$ and $h \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ such that $h(x_1, x_2, \dots, x_k, x') = \tilde{h}(x')$ with $\tilde{h} \in \mathcal{C}_0(\mathbb{R}^{N-k}, \mathbb{R})$, then the relation*

$$\lim_{\rho\to+\infty} \sup_{\Sigma\times[0,1]} \left[\int_{\mathbb{R}^N} h(x)\big[(1-\sigma)\omega(x-\rho y) + \sigma\omega(x-\rho z_y)\big]^q \,\mathrm{d}x \right] \cdot \rho^{(N-1)/2} \exp(2\rho) = 0 \tag{4.18}$$

holds for all $q \geqslant 2$.

Proof. Since $\tilde{h}$ has compact support, there exists $\widehat{R} > 0$ such that $\operatorname{supp} \tilde{h} \subset \{w \in \mathbb{R}^{N-k}\colon |w| < \widehat{R}\} = B_{N-k}(0, \widehat{R})$, so $h(x) = 0$ if $x' > \widehat{R}$, furthermore $\max_{\mathbb{R}^N} h = \max_{R^{N-k}} \tilde{h} < +\infty$.

In what follows we can, also, suppose $\rho > \max(1, \widehat{R})$, hence, observing that, for all $y \in \Sigma$, $|y| \geqslant 1$ and that ω is radially decreasing when the radial co-ordinate increases, we deduce that $\forall x \in \mathbb{R}^N$, for which $|x'| < \widehat{R}$, $\omega(x - \rho\frac{y}{|y|}) \geqslant \omega(x-\rho y)$.

Thus we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} h(x)\big[(1-\sigma)\omega(x-\rho y) + \sigma\omega(x-\rho z_y)\big]^q \,\mathrm{d}x \right| \\ &\quad \leqslant c_1 \max_{\mathbb{R}^N} |h| \int_{\{x\in\mathbb{R}^N\colon |x'|<\widehat{R}\}} \left(\left[\omega\left(x - \rho\frac{y}{|y|}\right)\right]^q + \big[\omega(x-\rho z_y)\big]^q \right) \mathrm{d}x, \end{aligned}$$

$c_1 \in \mathbb{R}^+ \setminus \{0\}$. So we must show

$$\lim_{\rho\to+\infty} \left[\int_{\mathbb{R}^k} \left(\int_{B_{N-k}(0,\widehat{R})} \big(\omega(x-\rho v)\big)^q \,\mathrm{d}x_{k+1}\cdots\mathrm{d}x_N \right) \mathrm{d}x_1\cdots\mathrm{d}x_k \right] \cdot \rho^{(N-1)/2} \exp(2\rho) = 0 \tag{4.19}$$

with both $v = \frac{y}{|y|}$, $y \in \Sigma$ and $v = z_y$, $y \in \Sigma$.

Let us evaluate (4.19) when $v = \frac{y}{|y|}$, $y \in \Sigma$.

Without any loss of generality, we can assume $v = (0, 0, \dots, 0, v')$, $|v'| = 1$. Taking, again, advantage of the behaviour of ω and of its asymptotic decay, we infer, for large values of ρ,

$$\begin{aligned} &\int_{\mathbb{R}^k} \left(\int_{B_{N-k}(0,\widehat{R})} \big(\omega(x-\rho v)\big)^q \,\mathrm{d}x_{k+1}\cdots\mathrm{d}x_N \right) \mathrm{d}x_1\cdots\mathrm{d}x_k \\ &\quad \leqslant c_2 \int_{\mathbb{R}^k} \big(\omega\big(x_1, x_2, \dots, x_k, \big(\widehat{R}-\rho\big)v'\big)\big)^q \,\mathrm{d}x_1\cdots\mathrm{d}x_k \\ &\quad \leqslant c_3 \int_{\mathbb{R}^k} \left[\frac{1}{[((\rho-\widehat{R})^2 + \sum_{i=1}^k x_i^2)^{1/2}]^{(N-1)/2}} \cdot \frac{1}{\mathrm{e}^{[(\rho-\widehat{R})^2 + \sum_{i=1}^k x_i^2]^{1/2}}} \right]^q \mathrm{d}x_1\cdots\mathrm{d}x_k, \end{aligned}$$

$c_2, c_3 \in \mathbb{R}^+ \setminus \{0\}$.

So, setting $\hat{x} = (x_1, x_2, \dots, x_k)$, to obtain (4.19), we have to show that, for all $q \geqslant 2$

$$\lim_{\rho\to+\infty} \int_{\mathbb{R}^k} \left(\frac{\rho}{[(\rho-\widehat{R})^2 + |\hat{x}|^2]^{q/2}} \right)^{(N-1)/2} \cdot \frac{1}{\mathrm{e}^{q[(\rho-\widehat{R})^2+|\hat{x}|^2]^{1/2}-2\rho}} \,\mathrm{d}\hat{x} = 0. \tag{4.20}$$

When $q > 2$, (4.20) follows at once because

$$\left(\frac{\rho}{[(\rho-\widehat{R})^2 + |\hat{x}|^2]^{q/2}} \right)^{(N-1)/2} \cdot \frac{1}{\mathrm{e}^{q[(\rho-\widehat{R})^2+|\hat{x}|^2]^{1/2}-2\rho}} \leqslant c_4 \frac{1}{\mathrm{e}^{c_5[\rho^2+|\hat{x}|^2]^{1/2}}}, \quad c_4, c_5 \in \mathbb{R}^+ \setminus \{0\}.$$

Let us, then, consider the case $q=2$. Clearly, being $\widehat{R}$ fixed, (4.20) easily comes once we show that

$$\lim_{\rho\to+\infty}\int_{\mathbb{R}^k}\left(\frac{\rho}{\rho^2+|\hat{x}|^2}\right)^{(N-1)/2}\cdot\frac{\mathrm{e}^{2\rho}}{\mathrm{e}^{2(\rho^2+|\hat{x}|^2)^{1/2}}}\,\mathrm{d}\hat{x}=0. \tag{4.21}$$

Now

$$\begin{aligned}\int_{\mathbb{R}^k}\left(\frac{\rho}{\rho^2+|\hat{x}|^2}\right)^{(N-1)/2}\cdot\frac{\mathrm{e}^{2\rho}}{\mathrm{e}^{2(\rho^2+|\hat{x}|^2)^{1/2}}}\,\mathrm{d}\hat{x}&=\frac{1}{\rho^{(N-1)/2}}\int_{\mathbb{R}^k}\left(\frac{1}{1+|\hat{x}/\rho|^2}\right)^{(N-1)/2}\cdot\frac{1}{\mathrm{e}^{2\rho[(1+|\hat{x}/\rho|^2)^{1/2}-1]}}\,\mathrm{d}\hat{x}\\&=\int_{\mathbb{R}^k}\left(\frac{1}{1+|\tilde{x}|^2}\right)^{(N-1)/2}\cdot\frac{\rho^{(2k+1-N)/2}}{\mathrm{e}^{2\rho[(1+|\tilde{x}|^2)^{1/2}-1]}}\,\mathrm{d}\tilde{x}\end{aligned}$$

and for all $\tilde{x}\neq 0$

$$\lim_{\rho\to+\infty}\left(\frac{1}{1+|\tilde{x}|^2}\right)^{(N-1)/2}\cdot\frac{\rho^{(2k+1-N)/2}}{\mathrm{e}^{2\rho[(1+|\tilde{x}|^2)^{1/2}-1]}}=0.$$

Furthermore, when $k\leqslant\frac{N-1}{2}$

$$\left(\frac{1}{1+|\tilde{x}|^2}\right)^{(N-1)/2}\cdot\frac{\rho^{(2k+1-N)/2}}{\mathrm{e}^{2\rho[(1+|\tilde{x}|^2)^{1/2}-1]}}<\left(\frac{1}{1+|\tilde{x}|^2}\right)^{(N-1)/2} \tag{4.22}$$

for large ρ, while, when $k>\frac{N-1}{2}$, taking into account that

$$\max_{t\in\mathbb{R}^+}\frac{t^\alpha}{\mathrm{e}^{ct}}=\left(\frac{\alpha}{\mathrm{e}}\right)^\alpha\frac{1}{c^\alpha},\quad \alpha>0,$$

we deduce

$$\begin{aligned}&\left(\frac{1}{1+|\tilde{x}|^2}\right)^{(N-1)/2}\cdot\frac{\rho^{(2k+1-N)/2}}{\mathrm{e}^{2\rho[(1+|\tilde{x}|^2)^{1/2}-1]}}\\&\quad\leqslant\left(\frac{2k+1-N}{4\mathrm{e}}\right)^{(2k+1-N)/2}\left(\frac{1}{(1+|\tilde{x}|^2)^{1/2}-1}\right)^{(2k+1-N)/2}\left(\frac{1}{1+|\tilde{x}|^2}\right)^{(N-1)/2}\end{aligned} \tag{4.23}$$

and, since $k\leqslant N-2$, the right-hand sides of (4.22) and (4.23) belong to $L^1(\mathbb{R}^k)$. Thus, by the Lebesgue Theorem, (4.21) is true. As a consequence, (4.19) holds true when $v=\frac{y}{|y|}$, $y\in\Sigma$. The argument when $v=z_y$, $y\in\Sigma$ is quite analogous, so the statement is proved. □

We are, now, ready to state and prove the main energy asymptotic estimates.

Proposition 4.8. *The relations*

$$\lim_{\rho\to+\infty}\max_{\Sigma}E\big(\Psi_\rho[y,0]\big)=m, \tag{4.24}$$

$$\max_{\Sigma\times[0,1]}E\big(\Psi_\rho[y,\sigma]\big)<2^{1-2/p}m\quad\text{for }\rho\text{ large enough} \tag{4.25}$$

hold.

Proof. In view of (4.7) and of the fact that, $\forall y\in\Sigma$, $\lim_{\rho\to+\infty}|\rho y'|=+\infty$, arguing as for proving (2.6) in Proposition 2.1, it is easy to show that

$$\lim_{\rho\to+\infty}E\big(\Psi_\rho[y,0]\big)=m\quad\forall y\in\Sigma,$$

so (4.24) follows.

In order to prove (4.25), let us put

$$N_\rho[y,\sigma] = \left\|\varphi(x)\left[(1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right]\right\|^2_{\mathbb{R}^N}$$
$$D_\rho[y,\sigma] = \left|\varphi(x)\left[(1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right]\right|^p_{p,\mathbb{R}^N}$$

for all $\rho > 0$, for all $(y,\sigma) \in \Sigma \times [0,1]$.

We have

$$\begin{aligned}
N_\rho[y,\sigma] &= \int_{\mathbb{R}^N} \left(\varphi(x)\right)^2\left[\left|\nabla\left((1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right)\right|^2\right.\\
&\qquad \left.+\left((1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right)^2\right]\mathrm{d}x\\
&\qquad + \int_{\mathbb{R}^N} \left|\nabla\varphi(x)\right|^2\left[(1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right]^2\mathrm{d}x\\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^N}\left(\nabla\left(\varphi(x)\right)^2, \nabla\left((1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right)^2\right)\mathrm{d}x\\
&\leqslant \int_{\mathbb{R}^N}\left[\left|\nabla\left((1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right)\right|^2+\left((1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right)^2\right]\mathrm{d}x\\
&\qquad + \int_{\mathbb{R}^N}\left(\left|\nabla\varphi(x)\right|^2-\frac{1}{2}\Delta\left(\varphi(x)\right)^2\right)\left[(1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right]^2\mathrm{d}x\\
&= \left[(1-\sigma)^2+\sigma^2\right]\int_{\mathbb{R}^N}\left[\left|\nabla\omega(x)\right|^2+\left(\omega(x)\right)^2\right]\mathrm{d}x + 2m\sigma(1-\sigma)\int_{\mathbb{R}^N}\left[\omega(x-\rho y)\right]^{p-1}\omega(x-\rho z_y)\,\mathrm{d}x\\
&\qquad - \int_{\mathbb{R}^N}(\varphi\Delta\varphi)\left[(1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right]^2\mathrm{d}x. \qquad (4.26)
\end{aligned}$$

Now, setting

$$\varepsilon_\rho := \int_{\mathbb{R}^N}\left[\omega(x-\rho y)\right]^{p-1}\omega(x-\rho z_y)\,\mathrm{d}x = \int_{\mathbb{R}^N}\omega(x-\rho y)\left[\omega(x-\rho z_y)\right]^{p-1}\mathrm{d}x, \qquad (4.27)$$

in view of (2.2), by applying Lemma 4.6, we get

$$\lim_{\rho\to+\infty}\varepsilon_\rho\left[(2\rho)^{(N-1)/2}\exp(2\rho)\right] = \tilde{c} > 0. \qquad (4.28)$$

Therefore, using Lemma 4.7, we obtain

$$N_\rho[y,\sigma] \leqslant \left[(1-\sigma)^2+\sigma^2\right]m + 2\sigma(1-\sigma)m\varepsilon_\rho + \mathrm{o}(\varepsilon_\rho).$$

On the other hand, using Lemmas 4.5 and 4.7, we deduce

$$\begin{aligned}
D_\rho[y,\sigma] &= \int_{\mathbb{R}^N}\left[(1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right]^p\mathrm{d}x\\
&\qquad + \int_{\mathbb{R}^N}\left(\varphi^p-1\right)\left[(1-\sigma)\omega(x-\rho y)+\sigma\omega(x-\rho z_y)\right]^p\mathrm{d}x\\
&\geqslant \left[(1-\sigma)^p+\sigma^p\right]|\omega|^p_{p,\mathbb{R}^N} + (p-1)\left[(1-\sigma)^{p-1}\sigma+\sigma^{p-1}(1-\sigma)\right]\varepsilon_\rho + \mathrm{o}(\varepsilon_\rho)
\end{aligned}$$

where ε_ρ is defined in (4.27).

Hence

$$
\begin{aligned}
E\big(\Psi_\rho[y,\sigma]\big) &= \frac{N_\rho(y,\sigma)}{[D_\rho(y,\sigma)]^{2/p}} \\
&\leqslant \frac{[(1-\sigma)^2+\sigma^2]m + 2\sigma(1-\sigma)m\varepsilon_\rho + \mathrm{o}(\varepsilon_\rho)}{([(1-\sigma)^p+\sigma^p] + (p-1)[(1-\sigma)^{p-1}\sigma + \sigma^{p-1}(1-\sigma)]\varepsilon_\rho + \mathrm{o}(\varepsilon_\rho))^{2/p}} \\
&\leqslant \frac{(1-\sigma)^2+\sigma^2}{((1-\sigma)^p+\sigma^p)^{2/p}} m + 2m\gamma(\sigma)\varepsilon_\rho + \mathrm{o}(\varepsilon_\rho),
\end{aligned}
$$

where

$$
\gamma(\sigma) = \frac{(1-\sigma)\sigma}{[(1-\sigma)^p+\sigma^p]^{2/p}} \left\{ 1 - \frac{p-1}{p} \frac{(1-\sigma)^2+\sigma^2}{(1-\sigma)^p+\sigma^p} \big[(1-\sigma)^{p-2} + \sigma^{p-2}\big] \right\}.
$$

Since $\gamma(1/2) < 0$, denoting by $I(1/2)$ a neighbourhood of $1/2$ in which $\gamma(\sigma) < c < 0$, for all σ, and taking into account (4.28), we obtain, for large ρ,

$$
\sup_{\Sigma\times I(1/2)} E\big(\Psi_\rho[y,\sigma]\big) < 2^{1-2/p} m.
$$

On the other hand

$$
\lim_{\rho\to+\infty} \sup_{\Sigma\times([0,1]\setminus I(1/2))} E\big(\Psi_\rho[y,\sigma]\big) \leqslant m \cdot \sup_{[0,1]\setminus I(1/2)} \frac{(1-\sigma)^2+\sigma^2}{[(1-\sigma)^p+\sigma^p]^{2/p}} < 2^{1-2/p} m,
$$

completing the proof of (4.25). □

5. Proof of the results

In what follows for all $c \in \mathbb{R}$ we set

$$
\begin{aligned}
&E^c = \{u \in V:\ E(u) \leqslant c\}, \\
&(E^c)^+ = \{u \in E^c:\ u \geqslant 0 \text{ a.e. in } \Omega\}, \\
&(E^c)^- = \{u \in E^c:\ u \leqslant 0 \text{ a.e. in } \Omega\}.
\end{aligned}
$$

We, first, give the proof of Theorem 1.2. Clearly, it can be obtained as a straight corollary of Theorem 1.1, nevertheless we believe interesting to exhibit the following independent argument, because it is considerably simpler than that of Theorem 1.1 and it need neither strong tools of $\mathcal{G}$-equivariant Ljusternik–Schnirelmann generalized theory, neither delicate topological invariants.

Proof of Theorem 1.2. By Propositions 4.2, 4.8, and Lemma 4.3, a $\hat{\rho} > 0$ exists such that for all $\rho > \hat{\rho}$ the inequalities

$$
m < \max_{\Sigma} E\big(\Psi_\rho[y,0]\big) < \mathcal{B}_0 \leqslant \max_{\Sigma\times[0,1]} E\big(\Psi_\rho[y,\sigma]\big) < 2^{1-2/p} m \tag{5.1}
$$

hold and, moreover, $\beta \circ \Psi_\rho[\Sigma \times \{0\}]$ is homotopically equivalent in $\mathbb{R}^N \setminus D_0$ to $\rho\Sigma$. So let us fix $\rho > \hat{\rho}$ and set

$$
\begin{aligned}
&\mathcal{A} := \max_{\Sigma} E\big(\Psi_\rho[y,0]\big), \\
&\mathcal{L} := \max_{\Sigma\times[0,1]} E\big(\Psi_\rho[y,\sigma]\big).
\end{aligned}
$$

We claim that there exists a critical level $c^* \in [\mathcal{B}_0, \mathcal{L}]$. Arguing by contradiction, let us assume

$$
\{u \in V:\ E(u) \in [\mathcal{B}_0, \mathcal{L}],\ \nabla E_{|V}(u) = 0\} = \emptyset.
$$

Then, by Proposition 3.1, using standard arguments (as displayed in Theorem 2.7), a positive number $\delta \in (0, \mathcal{B}_0 - \mathcal{A})$ and a continuous function

$$
\eta : E^{\mathcal{L}} \longrightarrow E^{\mathcal{B}_0-\delta} \tag{5.2}
$$

can be found so that

$$
\eta(u) = u \quad \forall u \in E^{\mathcal{B}_0-\delta}. \tag{5.3}
$$

Now, let us define $\mathcal{H}: \Sigma \times [0,1] \to \mathbb{R}^N$ by

$$\mathcal{H}\big([y,\sigma]\big) := \beta \circ \eta\big(\Psi_\rho[y,\sigma]\big).$$

By (5.3) and the choice of δ, $\mathcal{H}([y,0]) = \beta(\Psi_\rho[y,0])$, thus, by the choice of $\rho > \hat{\rho}$, $\mathcal{H}([\Sigma \times \{0\}])$ is homotopically equivalent in $\mathbb{R}^N \setminus D_0$ to $\rho\Sigma$; moreover $\mathcal{H}$ is continuous, so a point $(\bar{y},\bar{\sigma}) \in \Sigma \times [0,1]$ must exist so that $\beta \circ \eta(\Psi_\rho[\bar{y},\bar{\sigma}]) \in D_0$.

This is impossible, because, by (5.2)

$$\mathcal{H}\big(\Sigma \times [0,1]\big) \cap D_0 = \emptyset.$$

Therefore, the claim is proved and there exists a critical point u^*, of E on V, such that $E(u^*) = c^*$. By Corollary 2.3 we can assume $u^* \geqslant 0$, so $v^* = (c^*)^{1/(p-2)}u^* \geqslant 0$ solves (P) and, by the maximum principle $v^* > 0$. □

Proof of Theorem 1.1. By Propositions 4.2, 4.8, Corollary 4.4 and (4.16) of Lemma 4.3, a $\bar{\rho} \in \mathbb{R}$, $\bar{\rho} > 0$, exists such that for all $\rho > \bar{\rho}$

$$m < \max_{\Sigma} E\big(\Psi_\rho[y,0]\big) < \mathcal{B}_{3R} \leqslant \max_{\Sigma\times[0,1]} E\big(\Psi_\rho[y,\sigma]\big) < 2^{1-2/p}m, \tag{5.4}$$

R being as in assumption (h_2), and

$$\big|\beta \circ \Psi_\rho(y,0)\big| > \frac{\rho}{2} \quad \forall y \in \Sigma. \tag{5.5}$$

So, let us fix $\rho > \max(\bar{\rho}, 6R)$ and set

$$\mathcal{A} := \max_{\Sigma} E\big(\Psi_\rho[y,0]\big),$$
$$\mathcal{L} := \max_{\Sigma\times[0,1]} E\big(\Psi_\rho[y,\sigma]\big).$$

Let us choose then $\hat{\mathcal{A}} \in [\mathcal{A}, \mathcal{B}_{3R})$ and $\hat{\mathcal{L}} \in [\mathcal{L}, 2^{1-2/p}m)$ such that

$$\big\{u \in V\colon\ E(u) = \hat{\mathcal{A}},\ \nabla E_{|V}(u) = 0\big\} = \emptyset,$$
$$\big\{u \in V\colon\ E(u) = \hat{\mathcal{L}},\ \nabla E_{|V}(u) = 0\big\} = \emptyset.$$

Let us remark that, if one of this choices were not possible, we would be done, because the functional E constrained on V would have infinitely many critical values, to which there would correspond infinitely many solutions of (P).

Now, taking into account Proposition 3.1, we can apply Theorem 2.7 to the functional E, on V, subject to the action of the group $\mathcal{G}$. So, we deduce that E possesses at least $\operatorname{cat}^{\mathcal{G}}_{E^{\hat{\mathcal{L}}}}(E^{\hat{\mathcal{L}}}, E^{\hat{\mathcal{A}}})$ not $\mathcal{G}$-equivalent critical points, to which there correspond critical values lying in $(\hat{\mathcal{A}}, \hat{\mathcal{L}})$.

Now, to conclude the proof, we just need to show that

$$\operatorname{cat}^{\mathcal{G}}_{E^{\hat{\mathcal{L}}}}\big(E^{\hat{\mathcal{L}}}, E^{\hat{\mathcal{A}}}\big) \geqslant 2\operatorname{cat}^{G}_{\rho\Lambda}(\rho\Lambda, \rho\Sigma). \tag{5.6}$$

Indeed, being true (5.6), we can infer the existence of at least $2\operatorname{cat}^{G}_{\rho\Lambda}(\rho\Lambda, \rho\Sigma)$ not $\mathcal{G}$-equivalent critical points of E on V, u_i, that, since $E(u_i) \in (\hat{\mathcal{A}}, \hat{\mathcal{L}}) \subset (m, 2^{1-2/p}m)$, by Corollary 2.3, do not change sign. Hence, in view of the maximum principle, the existence of at least $\operatorname{cat}^{G}_{\rho\Lambda}(\rho\Lambda, \rho\Sigma)$ positive not $\mathcal{G}$-equivalent solutions of (P) follows. The argument is, then, completed by observing that, by applying Corollary 7.6(ii) in [11] (see also [14]), we obtain

$$\operatorname{cat}^{G}_{\rho\Lambda}(\rho\Lambda, \rho\Sigma) = k+1 \tag{5.7}$$

(for the reader's convenience, a proof of (5.7) is given in Appendix A).

Let us prove now (5.6). To this end, we show that

$$\begin{cases} \text{(a)}\ \operatorname{cat}^{\mathcal{G}}_{(E^{\hat{\mathcal{L}}})^+}\big((E^{\hat{\mathcal{L}}})^+, (E^{\hat{\mathcal{A}}})^+\big) \geqslant \operatorname{cat}^{G}_{\rho\Lambda}(\rho\Lambda, \rho\Sigma), \\ \text{(b)}\ \operatorname{cat}^{\mathcal{G}}_{(E^{\hat{\mathcal{L}}})^-}\big((E^{\hat{\mathcal{L}}})^-, (E^{\hat{\mathcal{A}}})^-\big) \geqslant \operatorname{cat}^{G}_{\rho\Lambda}(\rho\Lambda, \rho\Sigma), \end{cases} \tag{5.8}$$

we remark, in fact, that $(E^{\hat{\mathcal{L}}})^+$ and $(E^{\hat{\mathcal{L}}})^-$ belong to disjoint connected components of $E^{\hat{\mathcal{L}}}$, because $E^{\hat{\mathcal{L}}} < 2^{1-2/p}m$.

Obviously (5.8)(a) is true when $\mathrm{cat}^{\mathcal{G}}_{(E^{\hat{\mathcal{L}}})^+}((E^{\hat{\mathcal{L}}})^+, (E^{\hat{\mathcal{A}}})^+) = +\infty$, hence let us assume

$$\mathrm{cat}^{\mathcal{G}}_{(E^{\hat{\mathcal{L}}})^+}\big((E^{\hat{\mathcal{L}}})^+, (E^{\hat{\mathcal{A}}})^+\big) = l \in \mathbb{N};$$

this means l is the least number for which there exist $(l+1)$ closed $\mathcal{G}$-invariant sets $\mathcal{T}_i \subset (E^{\hat{\mathcal{L}}})^+$, $i = 0, 1, \dots, l$, $(l+1)$ continuous maps $\theta_i : \mathcal{T}_i \times [0,1] \to (E^{\hat{\mathcal{L}}})^+$, $i = 0, 1, \dots, l$, and l points $w_i \in (E^{\hat{\mathcal{L}}})^+$, $i = 1, 2, \dots, l$, such that

$$\begin{cases} \big(E^{\hat{\mathcal{L}}}\big)^+ = \bigcup_{i=0}^{l} \mathcal{T}_i, \quad \big(E^{\hat{\mathcal{A}}}\big)^+ \subseteq \mathcal{T}_0, \\ \theta_i(\cdot, t) \text{ is } \mathcal{G}\text{-equivariant } \forall t \in [0,1], \quad i = 0, 1, \dots, l, \\ \theta_i(u, 0) = u \quad \forall u \in \mathcal{T}_i,\ i = 0, 1, \dots, l, \\ \theta_i(u, 1) \in [w_i] \quad \forall u \in \mathcal{T}_i,\ i = 1, \dots, l, \\ \theta_0(u, 1) \in \big(E^{\hat{\mathcal{A}}}\big)^+ \quad \forall u \in \mathcal{T}_0, \\ \theta_0(u, t) \in \big(E^{\hat{\mathcal{A}}}\big)^+ \quad \forall u \in \big(E^{\hat{\mathcal{A}}}\big)^+,\ \forall t \in [0,1]. \end{cases} \tag{5.9}$$

Now, we consider

$$\mathcal{K}_i := \big(\xi_\rho \circ \Psi_\rho^{-1}\big)(\mathcal{T}_i), \quad i = 0, 1, \dots, l,$$

Ψ_ρ and ξ_ρ being the maps defined in (4.10) and (4.11) respectively. We remark that $\mathcal{K}_i$ are G-invariant subsets of $\mathbb{R}^N$ and, by (5.4) and (5.9),

$$\mathcal{K}_i \subset \rho\Lambda, \quad \bigcup_{i=0}^{l} \mathcal{K}_i = \rho\Lambda, \quad \rho\Sigma \subset \mathcal{K}_0. \tag{5.10}$$

Let us denote for all $x \in \mathbb{R}^N \setminus \mathcal{D}_0$, $x = (x_1, x_2, \dots, x_k, x')$, by $\Pi(x)$ the unique point belonging to $\rho\Sigma$ and to the half line containing x and having origin at $(x_1, x_2, \dots, x_k, 0)$.

We define, then, for $i = 1, 2, \dots, l$

$$\lambda_i : \mathcal{K}_i \times [0,1] \longrightarrow \rho\Lambda$$

by

$$\lambda_i(x, t) = \begin{cases} (1-2t)x + 2th \circ \beta\big(\Psi_\rho \circ \xi_\rho^{-1}(x)\big), & 0 \leqslant t \leqslant \frac{1}{2}, \\ h \circ \beta \circ \theta_i\big(\Psi_\rho \circ \xi_\rho^{-1}(x), 2t-1\big), & \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

where

$$h(x_1, x_2, \dots, x_k, x') = \begin{cases} (x_1, x_2, \dots, x_k, x') & \text{if } |x'| \leqslant \big|\big(\Pi(x)\big)'\big| \text{ or } x' = 0 \\ \Pi(x) & \text{if } |x'| \geqslant \big|\big(\Pi(x)\big)'\big| \end{cases}$$

and we define

$$\lambda_0 : \mathcal{K}_0 \times [0,1] \longrightarrow \rho\Lambda$$

by

$$\lambda_0(x, t) = \begin{cases} (1-3t)x + 3t\tilde{h}(x), & 0 \leqslant t \leqslant \frac{1}{3}, \\ \tilde{h}\big(3tx + (3t-1)\beta\big(\Psi_\rho \circ \xi_\rho^{-1}(x)\big)\big), & \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ \tilde{h} \circ \beta \circ \theta_0\big(\Psi_\rho \circ \xi_\rho^{-1}(x), 3t-2\big), & \frac{2}{3} \leqslant t \leqslant 1, \end{cases}$$

where $\tilde{h} : \mathbb{R}^N \to \mathbb{R}^N$ is the map defined by

$$\tilde{h}(x_1, \dots, x_k, x') = \begin{cases} (x_1, \dots, x_k, x') & \text{if } |x'| \leqslant R, \\ \left(x_1, \dots, x_k, \left[R + \frac{|(\Pi(x))'| - R}{R}\big(|x'| - R\big)\right]\frac{x'}{|x'|}\right) & \text{if } R \leqslant |x'| \leqslant 2R, \\ \Pi(x) & \text{if } |x'| \geqslant 2R. \end{cases}$$

We remark that the maps λ_i, $i = 0, 1, \dots, l$, are well defined: indeed, even if ξ_ρ^{-1} can contain more than one element, $\Psi_\rho \circ \xi_\rho^{-1}(x)$ is uniquely determined; moreover the λ_i, $i = 0, 1, \dots, l$, turn out to be continuous maps having the following properties

$$\begin{cases} \lambda_i(\cdot, t) \text{ is } G\text{-equivariant } \forall t \in [0,1], \ i = 0, 1, \dots, l, \\ \lambda_i(x, 0) = x \quad \forall x \in \mathcal{K}_i, \ i = 0, 1, \dots, l, \\ \lambda_i(x, 1) \in [h \circ \beta(w_i)] \quad \forall x \in \mathcal{K}_i, \ i = 1, 2, \dots, l, \\ \lambda_0(x, 1) \in \rho\Sigma \quad \forall x \in \mathcal{K}_0, \\ \lambda_0(x, t) \in \rho\Sigma \quad \forall x \in \rho\Sigma, \ \forall t \in [0,1]. \end{cases} \tag{5.11}$$

Relations (5.10) and (5.11) imply

$$\operatorname{cat}^G_{\rho\Lambda}(\rho\Lambda, \rho\Sigma) \leqslant l,$$

so (5.8)(a) is proved. An analogous argument gives (5.8)(b), completing the proof. □

Acknowledgements

We are indebted with Dr P. Salvatore for the proof of Proposition A.1.

Appendix A

Proposition A.1. *Let Σ and Λ as in* (4.7) *and* (4.8)*, respectively. Then, for every $\rho > 0$,*

$$\operatorname{cat}^G_{\rho\Lambda}(\rho\Lambda, \rho\Sigma) = k + 1. \tag{A.1}$$

Proof. If we set $B = (S^1)^k \times D_{N-k}$ and $S = (S^1)^k \times S^{N-k-1}$, then (A.1) is equivalent to $\operatorname{cat}(B, S) = k + 1$.

It is well known that $\operatorname{cat}(B) = k + 1$ (see [20] or [22]), so, by definition of category, $\operatorname{cat}(B, S) \leqslant \operatorname{cat}(B) = k + 1$ follows.

In order to show that the reverse inequality holds, let us observe that the cohomology algebra $H^*(B)$ is an exterior algebra on k one-dimensional generators and, moreover, the relative cohomology algebra $H^*(B, S)$ is a free $H^*(B)$-module on a $(N - k)$-dimensional generator. Thus,

$$\bigl(\overline{H}^*(B)\bigr)^k H^*(B, S) \neq 0,$$

where $\overline{H}^*(B)$ is the reduced cohomology, $(\overline{H}^*(B))^k$ stands for the n-th power and the product is the cup product. As a consequence, by using Corollary 7.6(ii) in [11] (see, also, [14]) $\operatorname{cat}(B, S) \geqslant k + 1$ follows. □

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