

# The gradient flow motion of boundary vortices

## Mouvement par flot de gradient de vortex de frontière

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### Abstract

We consider the gradient flow of a family of energy functionals describing the formation of boundary vortices in thin magnetic films. We obtain motion laws for the singularities in all time scalings by using the method of  $\Gamma$ -convergence of gradient flows.

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### Résumé

On considère le flot de gradient d'une famille de fonctionnelles d'énergie qui décrivent la formation de vortex dans les films magnétiques minces. Ces singularités se forment à la frontière, et nous obtenons leurs équations du mouvement, pour tous les scalings de temps, en utilisant la méthode de la  $\Gamma$ -convergence des flots de gradient.

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## 1. Introduction and main results

In the absence of an external field, the stable magnetization patterns of a soft ferromagnetic sample can be found as local minimizers of the micromagnetic energy

$$d^2 \int_G |\nabla m|^2 + \int_{\mathbb{R}^3} |\nabla U|^2 \quad (1.1)$$

among maps  $m \in H^1(G; S^2)$ . Here  $G$  is a domain in  $\mathbb{R}^3$  corresponding to the sample,  $m$  the magnetization vector,  $d$  is the exchange length, a material constant, and  $U$  is related to  $m$  via  $\Delta U = \operatorname{div}(m \chi_G)$ . The second term in (1.1) corresponds to the energy of the magnetic field created by  $m$  in all of space and makes the problem nonlocal. It is also nonconvex by the constraint  $|m| = 1$ .

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In the interesting special case where  $G = \Omega \times (0, t)$ , with  $t \ll 1 = \text{diam } \Omega$ , is a thin film, asymptotic limits of (1.1) in different regimes have been considered by various authors, see Gioia and James [12], Carbou [8], DeSimone et al. [9], Kohn and Slastikov [14], and Moser [21,20].

Kohn and Slastikov [14] considered the asymptotic behavior for  $t \rightarrow 0$  and  $\frac{t}{d^2} \log \frac{1}{t} \rightarrow \frac{1}{2\pi\varepsilon}$ . The energy divided by  $4\pi\varepsilon t \log \frac{1}{t}$  then  $\Gamma$ -converges to the limit functional

$$\frac{1}{2} \int_{\Omega} |\nabla m|^2 + \frac{1}{2\varepsilon} \int_{\partial\Omega} (m \cdot \nu)^2 \quad (1.2)$$

defined on maps  $m \in H^1(\Omega, S^1)$ . Here  $\nu$  is a unit normal to  $\partial\Omega$ . The Kohn–Slastikov theorem shows that for this special scaling, the nonlocal contribution arising as the energy of the induced field reduces to a local term charging the boundary.

The asymptotic behavior as  $\varepsilon \rightarrow 0$  of (1.2) on a simply connected domain  $\Omega$  was studied in [16], where it was shown that the energy of minimizers diverges logarithmically, and critical points satisfying a logarithmic energy bound converge to harmonic maps with boundary singularities, similar to results by Moser [21,20] for a related thin-film limit of micromagnetics. In [16], the position of the singularities, called “boundary vortices”, was shown for some classes of critical points including minimizers to be governed by a renormalized energy. These results correspond to those of Bethuel, Brezis and Hélein [5] for the Ginzburg–Landau functional, which is perhaps not surprising if one considers that the singularities in [5] and in [16] arise from the same topological phenomenon, see [16] for some more discussion.

The renormalized energy theorems of [5] and [16] both make a connection between an infinite dimensional nonlinear energy and a finite dimensional energy that can be calculated by solving linear boundary value problems.

In this article, we focus on the motion law for the boundary vortices of [16], more specifically, on the gradient flow, and we will show that the motion described by appropriately time-rescaled equations can in a certain sense be reduced to the motion of the boundary vortices by the gradient flow of the renormalized energy. Other rescalings lead to trivial motion laws for the boundary vortices. Related results for a different model for boundary vortices were found by Moser [22], but without the exact motion law. Moser actually studies the Landau–Lifshitz–Gilbert (LLG) equations instead of the gradient flow, which is the correct model for the evolution of magnetic structures. However, in the thin-film approximation of [14], where the magnetization is forced to be in the film plane, the gyromagnetic term in the LLG equations disappears and the evolution reduces to the gradient flow. For some related scalings, this was rigorously proved by Kohn and Slastikov [15].

There are strong similarities between our results for the motion of boundary vortices and those in the theory of gradient flow motion of interior vortices as studied by Jerrard and Sonner [13] and Lin [18,19]. Their proofs rely on PDE methods to study the gradient flow. We will use the method of  $\Gamma$ -convergence of gradient flows developed by Sandier and Serfaty [23], which allows us to work mostly with energy estimates. The method is similar in spirit to proving convergence of minimizers via  $\Gamma$ -convergence, where one has to first establish a lower bound inequality and then show that this lower bound is essentially optimal, which is usually done by a construction. Sandier and Serfaty [23] give similar criteria consisting again of a lower bound inequality and a construction that allow one to deduce that gradient flows converge to gradient flows.

The application of this abstract result to the Ginzburg–Landau functional relies on a “product estimate” due to Sandier and Serfaty [24] which helps to separate space- and time-variables. We prove an analogous result by somewhat different methods. One main ingredient is the extension of a compactness theorem proved by Alberti, Bouchitté and Seppecher [3] in the case of a coercive two-well potential to the noncoercive case of a periodic potential. Another proof of the compactness result has been recently given by Garroni and Müller [11]. Our proof reduces the problem to the one-dimensional case that has been treated in [17].

We expect that our main results for the gradient flow carry over to the renormalized energy of Cabré and Cónsul [6], where other penalty terms than those in [16] can be treated, thanks to the uniqueness result of Cabré and Solà-Morales [7].

As in [16], we will use the fact that maps  $m \in H^1(\Omega; S^1)$  can be lifted via  $m = e^{iu}$  to  $u \in H^1(\Omega)$ . Using this lifting, we can rewrite the energy (1.2) as

$$\mathcal{F}^\varepsilon(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin^2(u - g). \quad (1.3)$$

Here  $g$  is related to  $v$  by  $v = ie^{ig}$ , and can be chosen as smooth as  $v$  except for one jump of height  $-2\pi$ . We will examine the more general case where  $g$  is a function with a single jump of height  $-2\pi D$ , with  $D \geq 0$ , corresponding to the map  $e^{ig}$  of degree  $D$ . For regularity, we assume that  $\partial\Omega \in C^{2,\alpha}$  and  $e^{ig} \in C^1$ . By  $(\partial\Omega)_*^N$ , we denote the set of  $N$ -tuples  $(a_1, \dots, a_N) \in (\partial\Omega)^N$  such that  $a_i \neq a_j$  for  $i \neq j$ .

**Definition 1.1.** For  $\vec{a} = (a_1, \dots, a_N) \in (\partial\Omega)_*^N$  and  $\vec{d} = (d_1, \dots, d_N) \in \mathbb{Z}^N$  with  $\sum_{i=1}^N d_i = 2D$ , we let  $u_* = u_*(\vec{a}, \vec{d})$  denote the harmonic function on  $\Omega$  that satisfies  $\sin(u_* - g) = 0$  on  $\partial\Omega$  and jumps by  $-d_i\pi$  at  $a_i$ .

The compactness result of [17] suggests the following definition for the “sense of convergence” necessary for the application of the theory of  $\Gamma$ -convergence of gradient flows:

**Definition 1.2.** We say that a sequence  $(u_\varepsilon)$  of functions in  $H^1(\Omega)$  converges in singularities to  $(\vec{a}, \vec{d})$  with  $\vec{a} \in (\partial\Omega)_*^N$  and  $\vec{d} \in \mathbb{Z}^N$  if the boundary traces satisfy  $u_\varepsilon \rightarrow u_*(\vec{a}, \vec{d})$  in  $L^2(\partial\Omega)$ . We will write  $u_\varepsilon \xrightarrow{S} (\vec{a}, \vec{d})$ . Note that since the  $a_i$  are distinct,  $(\vec{a}, \vec{d})$  is uniquely determined by  $u_*(\vec{a}, \vec{d})$ .

**Definition 1.3.** For  $\vec{a} \in (\partial\Omega)_*^N$  and  $\vec{d} \in \mathbb{Z}^N$  we define the renormalized energy as

$$W(\vec{a}, \vec{d}) = \frac{1}{2} \lim_{\rho \rightarrow 0} \left( \int_{\Omega_\rho} |\nabla u_*|^2 - \pi \sum_{i=1}^N d_i^2 \log \frac{1}{\rho} \right), \tag{1.4}$$

where  $\Omega_\rho = \Omega \setminus \bigcup_{i=1}^N B_\rho(a_i)$ .

The renormalized energy can be expressed via the solution of a linear boundary value problem for the Laplacian, see [16, Proposition 7.1].

We can now state our main result:

**Theorem 1.4.** Let  $0 < T \leq \infty$  and let  $(u_\varepsilon)$  be a sequence of solutions of

$$\lambda_\varepsilon \partial_t u_\varepsilon = \Delta u_\varepsilon \quad \text{in } \Omega \times (0, T), \tag{1.5}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = -\frac{1}{2\varepsilon} \sin 2(u_\varepsilon - g) \quad \text{on } \partial\Omega \times (0, \infty). \tag{1.6}$$

For the initial conditions we assume that  $u_\varepsilon(0) \xrightarrow{S} (\vec{a}, \vec{d})$  with  $\vec{a} = (a_1, \dots, a_N) \in (\partial\Omega)_*^N$  and  $\vec{d} = (d_1, \dots, d_N) \in \{\pm 1\}^N$ . Furthermore,  $u_\varepsilon$  is supposed to be initially well-prepared, meaning that

$$\mathcal{F}^\varepsilon(u_\varepsilon(0)) - \frac{\pi N}{2} \log \frac{1}{\varepsilon} - \frac{\pi N}{2} (1 - \log 2) \leq W(\vec{a}, \vec{d}) + o(1) \tag{1.7}$$

as  $\varepsilon \rightarrow 0$ .

Depending on the asymptotic behavior of  $\lambda_\varepsilon$ , we then have:

- (i) If  $\lambda_\varepsilon = \frac{1}{\log(1/\varepsilon)}$ , then there exists a time  $T^* > 0$  such that for all  $t \in [0, T^*)$ , there holds  $u_\varepsilon(t) \xrightarrow{S} (\vec{a}(t), \vec{d}(0))$ . Furthermore, the  $\vec{a}(t)$  satisfy the motion law

$$\frac{da_i}{dt} = -\frac{2}{\pi} \frac{\partial}{\partial a_i} W(\vec{a}(t), \vec{d}(0)) \tag{1.8}$$

in the tangent space at  $a_i$  to  $\partial\Omega$ . If  $T^* < T$  is the maximal time with these properties, then as  $t \rightarrow T^*$ , there exist  $i \neq j$  such that  $a_i(t)$  and  $a_j(t)$  converge to the same point.

The energy of  $u_\varepsilon(t)$  satisfies the expansion

$$\mathcal{F}^\varepsilon(u_\varepsilon(t)) = \frac{\pi N}{2} \log \frac{1}{\varepsilon} + \frac{\pi N}{2} (1 - \log 2) + W(\vec{a}(t), \vec{d}) + o(1) \tag{1.9}$$

as  $\varepsilon \rightarrow 0$ .

- (ii) If  $\lambda_\varepsilon \log \frac{1}{\varepsilon} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , then for almost every  $t \in [0, T)$  we have  $u_\varepsilon(t) \xrightarrow{S} (\vec{a}(0), \vec{d}(0))$ , so there is no motion.
- (iii) If  $\lambda_\varepsilon \log \frac{1}{\varepsilon} \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ , then for almost every  $t \in [0, \infty)$  we have  $u_\varepsilon(t) \xrightarrow{S} (\vec{b}, \vec{d})$  with  $\nabla W(\vec{b}, \vec{d}) = 0$ , so the system instantaneously jumps into a critical point.

The proof is based on the technique of  $\Gamma$ -convergence of gradient flows [23] that we will apply to the functionals

$$\mathcal{E}^\varepsilon(u) = \mathcal{F}^\varepsilon(u) - \frac{\pi N}{2} \left( \log \frac{1}{\varepsilon} + 1 - \log 2 \right) \tag{1.10}$$

and the limit functional

$$\mathcal{E}(\vec{a}) = W(\vec{a}, \vec{d}). \tag{1.11}$$

The PDE with the nonlinear boundary condition is the gradient flow of  $\mathcal{E}^\varepsilon$  with respect to the norm  $\sqrt{\lambda_\varepsilon} \|\cdot\|_{L^2}$ , which we will use as the spaces  $X_\varepsilon$  in the terminology of [23]. The functionals  $\mathcal{E}^\varepsilon$  are defined on  $\mathcal{M} = H^1(\Omega)$ .

With  $\langle \cdot, \cdot \rangle$  denoting the  $L^2(\Omega)$  scalar product, (1.5), (1.6) is the strong form of  $\lambda_\varepsilon \langle \partial_t u, \varphi \rangle = -d\mathcal{E}^\varepsilon(u)(\varphi)$ , which is the condition for being the gradient flow.

The limit functional is defined on  $\mathcal{N} = (\partial\Omega)_*^N$ , which is an open subset of the (flat) Riemannian manifold  $(\partial\Omega)^N$ . The approach of [23] for Euclidean limit spaces carries over to this situation without changes. As the limiting norm on the tangent spaces  $T_u\mathcal{N}$  which are identified with  $\mathbb{R}^N$  we use the constant Riemannian metric  $\sqrt{\frac{\pi}{2}} \|\cdot\|_{\mathbb{R}^N}$ .

**Definition 1.5.** We say that functionals  $\mathcal{E}^\varepsilon$   $\Gamma$ -converge to  $\mathcal{E}$  along the trajectory  $u_\varepsilon(t)$  with respect to the convergence  $\xrightarrow{S}$  if there exist  $u(t)$  and a subsequence such that for all  $t$ ,  $u_\varepsilon(t) \xrightarrow{S} u(t)$  and

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}^\varepsilon(u_\varepsilon(t)) \geq \mathcal{E}(u(t)). \tag{1.12}$$

The energy excess  $D_\varepsilon(t)$  and the limiting energy excess  $D(t)$  for a sequence  $u_\varepsilon(t)$  are defined via

$$D_\varepsilon(t) = \mathcal{E}^\varepsilon(u_\varepsilon(t)) - \mathcal{E}(u(t)), \quad D(t) = \limsup_{\varepsilon \rightarrow 0} D_\varepsilon(t). \tag{1.13}$$

If  $u_\varepsilon(t)$  are solutions to the gradient flow for  $\mathcal{E}^\varepsilon$  that satisfy  $D(0) = 0$ , they are said to be *initially well-prepared*.

We will use the following version of Sandier and Serfaty’s theorem on the  $\Gamma$ -convergence of gradient flows:

**Theorem 1.6** (Sandier and Serfaty [23]). Assume  $\mathcal{E}^\varepsilon \in C^1(\mathcal{M})$  and  $\mathcal{E} \in C^1(\mathcal{N})$ . Let  $u_\varepsilon$  be a sequence of solutions of the gradient flow for  $\mathcal{E}^\varepsilon$  on  $[0, T)$  with respect to the metric structure  $X_\varepsilon$  that satisfy

$$\mathcal{E}^\varepsilon(u_\varepsilon(0)) - \mathcal{E}^\varepsilon(u_\varepsilon(t)) = \int_0^t \|\partial_t u_\varepsilon(s)\|_{X_\varepsilon}^2 ds. \tag{1.14}$$

Assume  $u_\varepsilon(0) \xrightarrow{S} u_0$ , that  $\mathcal{E}^\varepsilon$   $\Gamma$ -converges to  $\mathcal{E}$  along the trajectory  $u_\varepsilon(t)$ , and that  $(u_\varepsilon)$  is initially well-prepared. Furthermore, assume that (LB) and (CON) hold:

(LB) For a subsequence such that  $u_\varepsilon(t) \xrightarrow{S} u(t)$ , we have  $u \in H^1((0, T); \mathcal{N})$  and there exists  $f \in L^1(0, T)$  such that for every  $s \in [0, T)$  there holds

$$\liminf_{\varepsilon \rightarrow 0} \int_0^s \|\partial_t u_\varepsilon(t)\|_{X_\varepsilon}^2 dt \geq \int_0^s (\|\partial_t u\|_{T_{u(t)}\mathcal{N}}^2 - f(t)D(t)) dt. \tag{1.15}$$

(CON) If  $u_\varepsilon(t) \xrightarrow{S} u(t)$ , there exists a locally bounded function  $g$  on  $[0, T)$  such that for any  $t_0 \in [0, T)$  and any  $v$  defined in a neighborhood of  $t_0$  that satisfies  $v(t_0) = u(t_0)$  and  $\partial_t v(t_0) = -\nabla_{T_{u(t_0)}\mathcal{N}} \mathcal{E}(u(t_0))$ , there exists a sequence  $v_\varepsilon(t)$  such that  $v_\varepsilon(t_0) = u_\varepsilon(t_0)$  and the following inequalities hold:

$$\limsup_{\varepsilon \rightarrow 0} \|\partial_t v_\varepsilon(t_0)\|_{X_\varepsilon}^2 \leq \|\partial_t v(t_0)\|_{T_{v(t_0)}\mathcal{N}}^2 + g(t_0)D(t_0), \tag{1.16}$$

$$\liminf_{\varepsilon \rightarrow 0} \left( -\frac{d}{dt} \Big|_{t=t_0} \mathcal{E}^\varepsilon(v_\varepsilon(t)) \right) \geq -\frac{d}{dt} \Big|_{t=t_0} \mathcal{E}(v(t)) - g(t_0)D(t_0). \tag{1.17}$$

Then  $u_\varepsilon(t) \xrightarrow{S} u(t)$  which is the solution of the gradient flow for  $\mathcal{E}$  with respect to the structure of  $T\mathcal{N}$ .

To carry out the program of [23] and prove Theorem 1.4, we thus need to prove compactness of logarithmically bounded sequences and a lower bound for the energies in space variables only for every time  $t$  as in (1.12), which will be done in Section 2. Then we need to show (LB), i.e. prove that the vortices move  $H^1$  in time, and show that the time-derivative of the vortex motion is a lower bound in  $L^2$  for the rescaled time-derivatives of the solutions  $u_\varepsilon$ . This is achieved in Section 5. Finally, we need to prove (CON), which means to construct for a given vortex motion an approximating sequence  $u_\varepsilon$  corresponding to this motion and satisfying some limiting inequalities, which will be the content of Section 6.

With these preparations, our Theorem 1.4 then follows from Theorem 1.6 above and (for the other time scalings) Proposition 1.5 in [23] just as Theorem 1.6 from [23] does: The result can first be shown to hold for small time and then continues to hold until the vortices collide. This argument is carried out in Section 7.

## 2. The lower bound in space

In this section, we restate some results of [17] and sketch how we can generalize some results of [16] from the case of critical points to more general sequences by a localization and regularization technique.

**Theorem 2.1.** *If  $(u_\varepsilon)$  is a sequence of functions with  $\mathcal{F}^\varepsilon(u_\varepsilon) \leq M \log(1/\varepsilon)$ , then there exists a sequence  $(z_\varepsilon)$  in  $\pi\mathbb{Z}$  such that  $v_\varepsilon = u_\varepsilon - z_\varepsilon$  has a subsequence that converges in singularities to some  $(\vec{a}, \vec{d})$  with  $\vec{a} \in (\partial\Omega)_*^N$  and  $\vec{d} \in \mathbb{Z}^N$ .*

**Proof.** By the results of [17],  $u_\varepsilon$  is precompact up to translation in all  $L^p(\partial\Omega)$ . The accumulation points  $v_*$  satisfy  $v_* - g \in BV(\partial\Omega, \pi\mathbb{Z})$ , hence can be written as  $v_* = u_*(\vec{a}, \vec{d})$  for some  $N \in \mathbb{N}$ ,  $\vec{d} \in \mathbb{Z}^N$  and  $\vec{a} \in (\partial\Omega)_*^N$ .  $\square$

The following theorem can be seen as a kind of second order  $\Gamma$ -convergence result. For minimizers and some classes of critical points, it was proved in [16].

**Theorem 2.2.** *If  $(u_\varepsilon)$  are functions with  $\mathcal{F}^\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$  that converge in singularities to  $(\vec{a}, \vec{d})$  with  $a_i \in (\partial\Omega)_*^N$  and  $\vec{d} \in \{\pm 1\}^N$ , then*

$$\liminf_{\varepsilon \rightarrow 0} \left( \mathcal{F}^\varepsilon(u_\varepsilon) - \frac{\pi N}{2} \left( \log \frac{1}{\varepsilon} + 1 - \log 2 \right) \right) \geq W(\vec{a}, \vec{d}). \tag{2.1}$$

The main step in the proof of Theorem 2.2 is to determine the local behavior of the energy near a singularity. We introduce a local version of the energy by setting for any  $A \subset \Omega$

$$\mathcal{F}^\varepsilon(v; A) := \frac{1}{2} \int_A |\nabla v|^2 + \frac{1}{2\varepsilon} \int_{\vec{A} \cap \partial\Omega} \sin^2(v - g). \tag{2.2}$$

We will fix one of the singularities  $a := a_i$  for some  $i$ , and set  $\omega_\rho := \Omega \cap B_\rho(a)$  for  $\rho > 0$ . To estimate  $\mathcal{F}^\varepsilon(u_\varepsilon; \omega_\rho)$ , we will use a regularization technique related to the Yosida transform, similar to the approach used by Almeida and Bethuel [4] and Serfaty [25,26] in the context of Ginzburg–Landau vortices. More specifically, we will assume (without loss of generality, via approximation)  $u_\varepsilon \in H^2(\Omega)$  and define for  $0 < \beta < 1$  a new functional  $\mathcal{F}_\beta^\varepsilon$  by

$$\mathcal{F}_\beta^\varepsilon(v) = \mathcal{F}^\varepsilon(v; \omega_\rho) + \frac{1}{2\varepsilon^\beta} \int_{\Gamma_\rho} |v - u_\varepsilon|^2, \tag{2.3}$$

where  $\Gamma_\rho = \partial\Omega \cap B_\rho(a)$ . Let  $v_\varepsilon = v_\varepsilon^\rho$  be a minimizer of  $F_\beta^\varepsilon$ . Observe that

$$\mathcal{F}^\varepsilon(v_\varepsilon; \omega_\rho) \leq \mathcal{F}_\beta^\varepsilon(v_\varepsilon) \leq \mathcal{F}_\beta^\varepsilon(u_\varepsilon) = \mathcal{F}^\varepsilon(u_\varepsilon; \omega_\rho), \tag{2.4}$$

so any lower bound for  $\mathcal{F}^\varepsilon(v_\varepsilon; \omega_\rho)$  yields a lower bound for  $\mathcal{F}^\varepsilon(u_\varepsilon; \omega_\rho)$ . The advantage of  $v_\varepsilon$  over  $u_\varepsilon$  lies in the fact that we can use the Euler–Lagrange equations for  $v_\varepsilon$ , which read as follows:

$$\Delta v_\varepsilon = 0 \quad \text{in } \omega_\rho, \tag{2.5}$$

$$\frac{\partial v_\varepsilon}{\partial \nu} = -\frac{1}{2\varepsilon} \sin 2(v_\varepsilon - g) - \frac{1}{\varepsilon\beta}(v_\varepsilon - u_\varepsilon) \quad \text{on } \Gamma_\rho, \tag{2.6}$$

$$\frac{\partial v_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial B_\rho \cap \Omega. \tag{2.7}$$

Since it is easy to see that  $v_\varepsilon \in H^2(\Omega)$ , these equations are actually satisfied in the strong sense.

We can now adapt the analysis of [16] to this new situation. In particular, we have

**Proposition 2.3.** *There is a  $N > 0$  such that the approximate vortex set  $S_\varepsilon = \{z \in \Gamma_\rho : \sin^2(v_\varepsilon(z) - g(z)) \geq \frac{1}{4}\}$  can be covered by  $N$  balls of radius  $\varepsilon$ , such that the  $\varepsilon/5$  balls with the same centers are disjoint.*

Considering  $\varepsilon$ -scale blowups of  $v_\varepsilon$ , we have the following result:

**Proposition 2.4.** *Let  $b \in \Gamma_\rho$  and  $G$  be a local harmonic continuation of  $g$ . Let  $\Psi : \omega_\rho \rightarrow \mathbb{R}_+^2$  be a local flattening map taking  $b$  to 0. Then setting  $w_\varepsilon = (u_\varepsilon - G) \circ \Psi^{-1}$  and  $V_\varepsilon(z) = w_\varepsilon(\varepsilon z)$  there holds  $V_\varepsilon \rightarrow V$  in  $H_{\text{loc}}^1(\mathbb{R}_+^2)$ , where  $V$  is a nonperiodic solution of*

$$\Delta V = 0 \quad \text{in } \mathbb{R}_+^2, \tag{2.8}$$

$$\frac{\partial V}{\partial \nu} = -\frac{1}{2} \sin 2V \quad \text{on } \mathbb{R}. \tag{2.9}$$

If  $b$  is a corner point of  $\omega_\rho$ , the analogous result holds with  $V$  being a solution to quarter-space problem, satisfying additionally  $\frac{\partial V}{\partial \nu} = 0$  on the other part of the boundary.

**Proof.** This follows as in Section 6 of [16], since the  $\varepsilon^\beta$  terms disappear in the limit. The corner version follows with the appropriate changes, i.e. the flattening map should map into a quarter-space instead of a half-space.  $\square$

**Proposition 2.5.** *There is a  $\sigma > 0$  such that  $S_\varepsilon$  can be covered balls  $B_{\sigma\varepsilon}(b_j^\varepsilon)$  with  $b_j^\varepsilon \rightarrow a$  as  $\varepsilon \rightarrow 0$ .*

**Proof.** Apart from the convergence to  $a$ , this follows as Corollary 6.3 in [16] from the convergence in Proposition 2.4 and the classification of solutions of (2.8), (2.9) due to Toland [27]: Solutions are constant or of the form

$$\pm \arctan \frac{x+c}{y+1} + \pi \left( k + \frac{1}{2} \right)$$

with  $k \in \mathbb{Z}$ . (There are also periodic solutions, but these can be excluded here as in [16].) In the case of a corner, the quarter-space solutions can be extended to half-space solutions by reflection.

However, since  $v_\varepsilon \rightarrow u_*$  in  $L^2(\Gamma_\rho)$ , the  $b_j^\varepsilon$  (which are now the points near which  $v_\varepsilon$  makes a transition from one well of  $\sin^2$  to the next) can only converge to  $a$ . Moreover, one such ball suffices due to the minimality of  $v_\varepsilon$ , by an argument similar to the one in [16].  $\square$

**Proposition 2.6.** *The functions  $v_\varepsilon$  converge for any  $s < \rho$  in  $W^{1,p}(\omega_\rho \setminus B_s(a))$  for  $p < 2$  and weakly in  $H^1(\omega_\rho \setminus B_s(a))$  to a harmonic function  $v_*$  that satisfies  $v_* = u_*$  on  $\Gamma_\rho$  and  $\frac{\partial v_*}{\partial \nu} = 0$  on  $\partial B_\rho(a) \cap \Omega$ .*

**Proof.** This can be deduced either by directly constructing upper bounds for the energy as in Section 5 of [16] or, if we assume  $\mathcal{F}^\varepsilon(u_\varepsilon; \omega_\rho) \leq \frac{\pi}{2} \log \frac{1}{\varepsilon} + M$  by a comparison method using upper and lower bounds as in Section 4 of that

paper. The assumption on  $\mathcal{F}^\varepsilon(u_\varepsilon; \omega_\rho)$  can be made in the context of the proof of Proposition 2.9 since otherwise that proposition is trivially true.  $\square$

We can now calculate the energy of  $v_\varepsilon$  and thus estimate that of  $u_\varepsilon$ .

**Proposition 2.7.** *For any function  $f$  with  $0 < f(\rho) < \rho$  for  $\rho > 0$ , we have*

$$\liminf_{\rho \rightarrow 0} \left( \liminf_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{\omega_\rho \setminus \omega_{f(\rho)}} |\nabla v_\varepsilon^\rho|^2 - \frac{\pi}{2} \log \frac{\rho}{f(\rho)} \right) \geq 0. \tag{2.10}$$

**Proof.** By Proposition 2.6 above and the lower semicontinuity of the Dirichlet integral, we have

$$\liminf_{\varepsilon \rightarrow 0} \int_{\omega_\rho \setminus \omega_{f(\rho)}} |\nabla v_\varepsilon^\rho|^2 \geq \int_{\omega_\rho \setminus \omega_{f(\rho)}} |\nabla v_*^\rho|^2,$$

where  $v_\varepsilon^\rho$  is the harmonic function satisfying  $v_*^\rho = u_*$  and  $\frac{\partial v_*^\rho}{\partial \nu} = 0$  on  $\partial B_\rho \cap \Omega$ . Blowing up these functions at scale  $\rho$ , we have that  $v_*^\rho(\frac{\cdot}{\rho})$  converges in  $C^1$  away from 0 to a translate of the argument function or its negative, from which we deduce (2.10).  $\square$

**Proposition 2.8.** *There is a sequence of points  $b_\varepsilon \rightarrow a$  such that for any small  $\rho$*

$$\lim_{s \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left( \mathcal{F}^\varepsilon(v_\varepsilon; B_s(b_\varepsilon)) - \frac{\pi}{2} \left( \log \frac{s}{\varepsilon} + 1 - \log 2 \right) \right) = 0. \tag{2.11}$$

**Proof.** This follows from an argument similar to the one in Section 8 of [16] by comparison with the rescaled solution of the half-space problem, which has an energy expansion like (2.11).  $\square$

**Proposition 2.9.** *There holds*

$$\liminf_{\rho \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \left( \mathcal{F}^\varepsilon(v_\varepsilon^\rho; \omega_\rho) - \frac{\pi}{2} \left( \log \frac{\rho}{\varepsilon} + 1 - \log 2 \right) \right) \geq 0. \tag{2.12}$$

**Proof.** This follows from combining the last two propositions with appropriately chosen radii, namely using for a sequence  $\rho_k \rightarrow 0$  and  $s_k \rightarrow 0$  the radii  $f(\rho_k) = s_k(1 + s_k)$ .  $\square$

**Proof of Theorem 2.2.** On  $\Omega_\rho = \Omega \setminus \bigcup B_\rho(a_i)$ , we can use the lower semicontinuity of the Dirichlet integral to obtain

$$\int_{\Omega_\rho} |\nabla u_*|^2 \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\rho} |\nabla u_\varepsilon|^2.$$

In the  $\omega_\rho(a_i)$ , we can use Proposition 2.9 and use that the energy for  $v_\varepsilon$  cannot be lower that of  $u_\varepsilon$ . Adding up, we obtain the statement of the theorem.  $\square$

We will later also need the following results about the local energy:

**Lemma 2.10.** *Assume that  $u_\varepsilon \xrightarrow{S} \vec{a}$ ,  $\mathcal{E}^\varepsilon(u_\varepsilon) \leq W(\vec{a}, \vec{d}) + D_\varepsilon$ ,  $D_\varepsilon$  bounded.*

*Then for  $\rho > 0$  such that  $B_\rho(a_i)$  are disjoint and setting as usual  $\Omega_\rho = \Omega \setminus \bigcup B_\rho(a_i)$ , we have, with  $O(1)$  and  $o(1)$  denoting quantities that stay bounded or tend to 0 as  $\varepsilon \rightarrow 0$ , respectively:*

$$\frac{1}{2} \int_{B_\rho(a_i) \cap \Omega} |\nabla u_\varepsilon|^2 = \frac{\pi}{2} \log \frac{1}{\varepsilon} + O(1), \tag{2.13}$$

$$\frac{1}{2\varepsilon} \int_{\partial\Omega \cap \partial\Omega_\rho} \sin^2(u_\varepsilon - g) \leq D_\varepsilon + o(1), \tag{2.14}$$

$$\frac{1}{2} \int_{\Omega_\rho} |\nabla u_\varepsilon - \nabla u_*|^2 \leq D_\varepsilon + o(1), \tag{2.15}$$

$$\frac{1}{2\varepsilon} \int_{B_\rho(a_i) \cap \partial\Omega} \sin^2(u_\varepsilon - g) \leq \frac{\pi}{2} \log \frac{1}{\rho} + O(1). \tag{2.16}$$

**Proof.** We have by assumption, with  $C_0 = \frac{\pi}{2}(1 - \log 2)$ ,

$$\frac{1}{2} \int_{\Omega} |\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin^2(u_\varepsilon - g) \leq \frac{\pi N}{2} \log \frac{1}{\varepsilon} + NC_0 + W(\vec{a}) + D_\varepsilon. \tag{2.17}$$

From the proof of Theorem 2.2 we have the

$$\liminf_{\rho \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \left( \frac{1}{2} \int_{\Omega_\rho} |\nabla u_\varepsilon|^2 - \frac{\pi N}{2} \log \frac{1}{\rho} + W(\vec{a}) \right) \geq 0 \tag{2.18}$$

and

$$\liminf_{\rho \rightarrow 0} \liminf_{\varepsilon \rightarrow 0} \left( \frac{1}{2} \int_{\Omega \cap B_\rho(a_i)} |\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon} \int_{\partial\Omega \cap B_\rho(a_i)} \sin^2(u_\varepsilon - g) - \frac{\pi}{2} \log \frac{\rho}{\varepsilon} + C_0 \right) \geq 0. \tag{2.19}$$

Combining these, we see that

$$\limsup_{\rho \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \left( \frac{1}{2\varepsilon} \int_{\partial\Omega_\rho \cap \partial\Omega} \sin^2(u_\varepsilon - g) - D_\varepsilon \right) \leq 0. \tag{2.20}$$

Since the  $\varepsilon$ -limit is a decreasing function of  $\rho$ , we obtain that (2.20) holds without the  $\rho$ -limit, which shows (2.14).

We similarly see that for fixed  $\rho$  and with  $D_\varepsilon = O(1)$ ,

$$\frac{1}{2} \int_{\Omega \cap B_\rho(a_i)} |\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon} \int_{\partial\Omega \cap B_\rho(a_i)} \sin^2(u_\varepsilon - g) = \frac{\pi}{2} \log \frac{1}{\varepsilon} + O(1), \tag{2.21}$$

hence (2.13). Comparing with (2.19), we obtain (2.16).

For (2.15), we need that — similar to the discussion in Chapter I of [5]—another definition of  $W$  can be given by using instead of  $u_*$  the function  $\tilde{u}_\rho$  which is harmonic, equal to  $u_*$  on  $\partial\Omega \cap \partial\Omega_\rho$  and has  $\frac{\partial \tilde{u}_\rho}{\partial \nu} = 0$  on  $\partial B_\rho(a_i) \cap \Omega$ . Now we can calculate

$$\int_{\Omega_\rho} |\nabla u_\varepsilon - \nabla \tilde{u}_\rho|^2 = \int_{\Omega_\rho} |\nabla u_\varepsilon|^2 + |\nabla \tilde{u}_\rho|^2 - 2\nabla u_\varepsilon \cdot \nabla \tilde{u}_\rho. \tag{2.22}$$

Now

$$\int_{\Omega_\rho} |\nabla \tilde{u}_\rho|^2 = \int_{\partial\Omega_\rho} \tilde{u}_\rho \frac{\partial \tilde{u}_\rho}{\partial \nu} \quad \text{and} \quad \int_{\Omega_\rho} \nabla u_\varepsilon \cdot \nabla \tilde{u}_\rho = \int_{\partial\Omega_\rho} u_\varepsilon \frac{\partial \tilde{u}_\rho}{\partial \nu} \longrightarrow \int_{\partial\Omega_\rho} \tilde{u}_\rho \frac{\partial \tilde{u}_\rho}{\partial \nu},$$

hence

$$\int_{\Omega_\rho} |\nabla u_\varepsilon - \nabla \tilde{u}_\rho|^2 = \int_{\Omega_\rho} |\nabla u_\varepsilon|^2 - |\nabla \tilde{u}_\rho|^2 + o(1). \tag{2.23}$$

It follows that

$$\limsup_{\varepsilon \rightarrow 0} \left( \frac{1}{2} \int_{\Omega_\rho} |\nabla u_\varepsilon - \nabla \tilde{u}_\rho|^2 + |\nabla \tilde{u}_\rho|^2 - |\nabla u_*|^2 - D_\varepsilon \right) \leq 0. \tag{2.24}$$



Letting  $\rho \rightarrow 0$  and using  $\lim_{\rho \rightarrow 0} \int_{\Omega_\rho} (|\nabla u_*|^2 - |\nabla \tilde{u}_\rho|^2) = 0$  and (2.23), this leads to

$$\limsup_{\rho \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \left( \frac{1}{2} \int_{\Omega_\rho} |\nabla u_\varepsilon - \nabla u_*|^2 - D_\varepsilon \right) \leq 0. \tag{2.25}$$

Since the integral and hence the  $\varepsilon$ -limit is again decreasing in  $\rho$ , this result also holds for fixed  $\rho$ , showing (2.15).  $\square$

### 3. Compactness in 3D

In this section, we prove that sequences of functions on three-dimensional domains satisfying a logarithmic energy bound have compact boundary traces. These results are adaptations of corresponding results in the work of Alberti, Bouchitté, and Seppecher [3], with changes resulting from our use of the compactness theory for noncoercive periodic potentials from [17] instead of that for coercive potentials from [2]. Other proofs of these results were given in a different context and by somewhat different methods by Garroni and Müller [11]. We will later apply these theorems to domains that are products of a two-dimensional space domain and a time interval.

Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain with  $C^1$  boundary. For  $B \subset \mathbb{R}^3$ ,  $C \subset \partial B$  we define the following functional:

$$F_\varepsilon(u; B; C) = \frac{1}{2} \int_B |\nabla u|^2 + \frac{1}{2\varepsilon} \int_C V(u) \, d\mathcal{H}^2, \tag{3.1}$$

where  $V: \mathbb{R} \rightarrow [0, \infty)$  is a  $\pi$ -periodic, continuous function with  $V^{-1}(0) = \pi\mathbb{Z}$ . In our applications, we will use  $V(t) = \sin^2 t$ . In the proofs, we will often make use of the fact that also  $V^\mu = \mu V$  for  $\mu > 0$  satisfies the same assumptions.

**Theorem 3.1.** *Let  $(u_\varepsilon)$  be a sequence in  $H^1(\Omega)$  such that  $F_\varepsilon(u_\varepsilon; \Omega; \partial\Omega) \leq M \log \frac{1}{\varepsilon}$ . Then the boundary traces of  $u_\varepsilon$  are bounded in  $L^2(\partial\Omega)$  and precompact in  $L^1(\partial\Omega)$ , with every cluster point belonging to  $BV(\partial\Omega, \pi\mathbb{Z})$ . If  $u_\varepsilon \rightarrow u$  in  $L^1(\partial\Omega)$ , then*

$$\int_{\partial\Omega} |Du| \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{\Omega} |\nabla u_\varepsilon|^2. \tag{3.2}$$

**Remark 3.2.** It follows from Theorem 3.1 that  $u_\varepsilon$  is in fact precompact in all  $L^p(\partial\Omega)$  with  $1 \leq p < 2$ .

To prove Theorem 3.1, we will locally flatten the boundary and then reduce the statement to the one-dimensional case by slicing. We start by stating the corresponding one-dimensional results:

For  $I \subset \mathbb{R}$  an interval set

$$G_\varepsilon(u; I) := \frac{1}{4\pi \log(1/\varepsilon)} \int_{I \times I} \left| \frac{u(x) - u(x')}{x - x'} \right|^2 \, dx \, dx' + \frac{1}{2\varepsilon \log(1/\varepsilon)} \int_I V(u). \tag{3.3}$$

**Definition 3.3.** For a measurable function  $u$  on a set  $A$  we define the distribution function  $\lambda_u$  by

$$\lambda_u(t) = |\{x \in A: |u(x)| > t\}| \tag{3.4}$$

and the median  $m(u)$  (with respect to  $\pi\mathbb{Z}$ ) by

$$m(u) = \max \left\{ q \in \pi\mathbb{Z}: |\{u - q > 0\}| \geq \frac{|A|}{2} \right\}. \tag{3.5}$$

It is clear that

$$\frac{|A|}{2} |m(u)|^2 \leq \int_{\{u > m(u)\}} u^2 \leq \|u\|_{L^2(A)}^2. \tag{3.6}$$

**Lemma 3.4.** *There exist constants  $C_1, C_2 > 0$  and  $\varepsilon_1 > 0$  such that for  $\varepsilon < \varepsilon_1$ , any  $u \in L^1(I)$  such that  $G_\varepsilon(u; I) < \infty$  satisfies*

$$\lambda_{u-m(u)}(t) \leq C_1 e^{-C_2 t (\frac{1}{\sqrt{G_\varepsilon(u; I)}} \wedge 1)} (|I| \vee 1). \quad (3.7)$$

**Proof.** For  $\varepsilon$  sufficiently small, this follows from reexamining the proof of Proposition 2.11 in [17] and tracking the dependency on  $I$  and  $G$  (the functional used in [17] is for small  $|I|$  equivalent to the one considered here).  $\square$

**Proposition 3.5.** *Let  $(u_\varepsilon)$  be a sequence in  $L^1(I)$  such that  $G_\varepsilon(u_\varepsilon; I) \leq M < \infty$  and such that  $(u_\varepsilon)$  is bounded in some  $L^p$  for  $p > 1$ . Then  $(u_\varepsilon)$  is relatively compact in  $L^1(I)$ , every cluster point belongs to  $BV(I, \pi\mathbb{Z})$ , and the following inequality holds for every sequence  $(u_\varepsilon)$  with  $u_\varepsilon \rightarrow u$  in  $L^1(I)$ :*

$$\int_I |Du| \leq 2 \liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon; I). \quad (3.8)$$

We can even find sets  $A_\varepsilon \subset I$  with  $|A_\varepsilon| \rightarrow 0$  such that

$$\int_I |Du| \leq 2 \liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon; A_\varepsilon). \quad (3.9)$$

**Proof.** The first statements are basically the content of Proposition 2.13 of [17]. To see that we even have (3.9), we need to take a closer look at the proof of that result, and track the dependency on  $I$ : We choose shrinking intervals  $I_\varepsilon$ , symmetric around a point in  $S_u$ , such that  $|I_\varepsilon \cap \{u = \alpha_i\}| = \frac{1}{2}|I_\varepsilon|$  for values  $\alpha_1 \neq \alpha_2$  of  $u$ . Then the argument in the proof of Proposition 2.13 of [17] shows that

$$\int_{I_\varepsilon} |Du| \leq 2G_\varepsilon(u; I_\varepsilon) + o(1) - \frac{\log |I_\varepsilon|}{\log(1/\varepsilon)}. \quad (3.10)$$

We now choose  $I_\varepsilon$  of such that  $|I_\varepsilon| \rightarrow 0$  and  $\frac{\log |I_\varepsilon|}{\log(1/\varepsilon)} \rightarrow 0$ , e.g.  $|I_\varepsilon| = e^{-\sqrt{\log(1/\varepsilon)}}$ . Defining  $A_\varepsilon$  as the union of such  $I_\varepsilon$  near each of the jump points, we obtain (3.9).  $\square$

**Corollary 3.6.** *By replacing  $V$  with  $V^\mu = \mu V$  and letting  $\mu \rightarrow 0$ , it follows that, while an energy bound with the penalty term is necessary to derive (3.8) and (3.9), these equations actually hold true without the presence of the penalty term on their right-hand sides.*

**Lemma 3.7.** *Let  $Q \in \mathbb{R}^2$  be a square with edges parallel to the coordinate axes,  $u: Q \rightarrow \mathbb{R}$  be such that  $\|u(x, y) - f(x)\|_{L^2(Q)} \leq A$  and  $\|u(x, y) - g(y)\|_{L^2(Q)} \leq A$ , for some functions  $f$  and  $g$  that depend only on one variable. Then there exists a  $z \in \mathbb{R}$  such that  $\|u(x, y) - z\|_{L^2(Q)} \leq 3A$ .*

**Proof.** We may assume  $Q$  to be the unit square. By the triangle inequality,  $\|f(x) - g(y)\|_{L^2(Q)} \leq 2A$ . Setting  $G = \int_0^1 g(y) dy$ , we have

$$\int_Q |f(x) - G|^2 = \int_Q f^2(x) - 2f(x)G + G^2. \quad (3.11)$$

Now by Hölder's inequality,  $G^2 \leq \int_0^1 g^2(y) dy = \int_Q g^2$ , hence

$$\int_Q |f(x) - G|^2 \leq \int_Q f^2(x) - 2f(x)g(y) + g^2(y) \leq 4A^2, \quad (3.12)$$

and using the triangle inequality again we obtain the claim.  $\square$

We define for  $r > 0$  the sets  $D_r := B_r(0) \cap \{x_3 > 0\} \subset \mathbb{R}^3$  and  $E_r := B_r(0) \cap \{x_3 = 0\}$ .

**Proposition 3.8.** *Let  $u_\varepsilon \in H^1(D_r)$  be a sequence of functions satisfying the energy bound*

$$F_\varepsilon(u_\varepsilon; D_r; E_r) \leq M \log \frac{1}{\varepsilon}. \tag{3.13}$$

*Then there exists  $z_\varepsilon \in \pi\mathbb{Z}$  such that the traces of  $u_\varepsilon - z_\varepsilon$  on  $E_r$  are bounded in  $L^2(E_{\gamma r})$  and precompact in  $L^1(E_{\gamma r})$ , and every cluster point belongs to  $BV(E_{\gamma r}, \pi\mathbb{Z})$ , where  $\gamma = \frac{1}{\sqrt{3}}$ .*

*Furthermore, if  $u_\varepsilon \rightarrow u$  in  $L^1(E_{\gamma r})$ , then*

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{D_{\gamma r}} F_\varepsilon(u_\varepsilon; D_{\gamma r}; E_{\gamma r}) \geq \frac{1}{2} \left| \int_{E_{\gamma r}} Du \right|. \tag{3.14}$$

**Proof.** Let  $C$  be a maximal cube inscribed in  $B_r$  and  $H = C \cap D_r$ . Let  $Q = C \cap E_r$ . From the geometrical setup we see that the maximal circle in  $P = \{x_3 = 0\}$  inscribed in  $Q$  has radius  $\frac{r}{\sqrt{3}}$ . Let  $e \in P$  be a unit vector parallel to a side of  $Q$ . Let  $M$  denote the orthogonal complement of  $e$  in  $P$  and  $p$  the projection of  $\mathbb{R}^3$  onto  $M$ . We set  $Q_e = p(Q)$ . For every  $y \in Q_e$ , we let  $Q^y = p^{-1}(y) \cap Q$  and  $H^y = p^{-1}(y) \cap H$ . Just as in [3], we can use Fubini’s theorem and some facts on slicing of Sobolev functions found in the appendix of that paper to show that the traces  $u^y$  of  $u$  satisfy

$$\begin{aligned} \frac{1}{\log(1/\varepsilon)} F_\varepsilon(u; H; Q) &\geq \int_{Q_e} \left( \frac{1}{4\pi \log(1/\varepsilon)} \int_{Q^y \times Q^y} \left| \frac{u^y(x) - u^y(x')}{x - x'} \right|^2 dx dx' + \frac{1}{2\varepsilon \log(1/\varepsilon)} \int_{Q^y} V(u^y) \right) dy \\ &= \int_{Q_e} G_\varepsilon(u^y; Q^y) dy. \end{aligned}$$

From this we obtain the  $L^2$  bound as follows: By Fubini’s theorem and since  $2 \int_0^\infty t \lambda_u(t) = \int_{Q^y} u^2$ , we can calculate  $\|u_\varepsilon - m(u_\varepsilon^y)\|_{L^2(Q)}^2$  as

$$\|u_\varepsilon - m(u_\varepsilon^y)\|_{L^2(Q)}^2 = \int_{Q_e} \int_{Q^y} |u_\varepsilon^y - m(u_\varepsilon^y)|^2 = \int_{Q_e} 2 \int_0^\infty t \lambda_{u_\varepsilon^y - m(u_\varepsilon^y)}(t) dt dy. \tag{3.15}$$

We estimate the integrand in the  $y$ -integral. To avoid clutter, we write  $G$  for  $G_\varepsilon(u_\varepsilon^y; Q^y)$ . We have, using (3.7)

$$2 \int_0^\infty t \lambda_{u_\varepsilon^y - m(u_\varepsilon^y)}(t) dt \leq C \int_0^\infty e^{-Ct(G^{-1/2} \wedge 1)} \tag{3.16}$$

$$\leq C(1 \vee G). \tag{3.17}$$

Integrating over  $y$ , this shows with  $m_\varepsilon^\varepsilon(x, y) = m(u_\varepsilon^y)$  that

$$\|u_\varepsilon - m_\varepsilon^\varepsilon\|_{L^2(Q)}^2 \leq C + C \int_{Q_e} G_\varepsilon(u_\varepsilon^y; Q^y) dy \leq C + CM. \tag{3.18}$$

By choosing  $e^\perp$  instead of  $e$ , we obtain the same bound for  $u_\varepsilon - m_{\varepsilon^\perp}^\varepsilon$ , so  $u_\varepsilon$  is  $L^2$ -bounded by a constant away from a function of one variable in each of two orthogonal directions. Lemma 3.7 now shows the existence of an appropriate  $z_\varepsilon$  such that  $u_\varepsilon - z_\varepsilon$  is bounded in  $L^2$ .

To show the precompactness of  $(u_\varepsilon - z_\varepsilon)$  in  $L^1(Q)$ , we use Theorem 3.10. From now on, we assume that  $z_\varepsilon = 0$ . The approximating family of functions will be given slice-wise by

$$w_{\varepsilon, \delta}^y = \begin{cases} u_\varepsilon^y & \text{for } y \in Q_e \text{ with } G_\varepsilon(u_\varepsilon^y, Q^y) \leq C_\delta \text{ and } m(u_\varepsilon^y) < C_\delta, \\ 0 & \text{else,} \end{cases} \tag{3.19}$$

for some  $C_\delta$  to be chosen below. Using Hölder’s inequality we see that

$$\begin{aligned} \int_Q |u_\varepsilon - w_{\varepsilon,\delta}| &\leq \int_{\{y \in Q_e, G_\varepsilon(u_\varepsilon^y, e^y) > C_\delta\}} |u_\varepsilon^y| \, dy + \int_{\{y \in Q_e, m(u_\varepsilon^y) > C_\delta\}} |u_\varepsilon^y| \, dy \\ &\leq \|u_\varepsilon\|_{L^2} \left( |\{y \in Q_e, G_\varepsilon(u_\varepsilon^y, e^y) > C_\delta\}|^{1/2} + |\{y \in Q_e, m(u_\varepsilon^y) > C_\delta\}|^{1/2} \right). \end{aligned} \tag{3.20}$$

By the “weak- $L^1$ ” bound

$$|\{y \in Q_e, G_\varepsilon(u_\varepsilon^y, e^y) > C_\delta\}| \leq \frac{M}{C_\delta}, \tag{3.21}$$

and a similar bound resulting from (3.6) for  $m(u_\varepsilon^y)$ , we see from (3.20) and the  $L^2$  bound that we can choose  $C_\delta$  such that  $\|u_\varepsilon - w_{\varepsilon,\delta}\| \leq \delta$ .

The functions  $w_{\varepsilon,\delta}^y$  now satisfy  $G_\varepsilon(w_{\varepsilon,\delta}^y, Q^y) \leq C_\delta$  for every  $y \in Q_e$ , and the one-dimensional theory applies: From Lemma 3.4 we see that  $(w_{\varepsilon,\delta}^y - m(w_{\varepsilon,\delta}^y))$  is bounded in all  $L^p$ , so we can use the boundedness of  $m(w_{\varepsilon,\delta}^y)$  and Proposition 3.5 to obtain that for every  $\delta$ , the family  $\varepsilon \mapsto w_{\varepsilon,\delta}^y$  is compact in  $L^1(Q^y)$ . By Theorem 3.10, this shows that in fact  $(u_\varepsilon)$  is compact in  $L^1(Q)$ .

It remains to prove that if  $u_\varepsilon \rightarrow u$  in  $L^1(E)$  for  $E = E_{\gamma r}$ , then  $u \in BV(E, \pi\mathbb{Z})$ , and inequality (3.14) holds. Slicing again (using  $E_e$  and  $E^y$  as we did  $Q_e$  and  $Q^y$  above) and using Fatou’s lemma, we obtain that

$$M \geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} F_\varepsilon(u_\varepsilon; D; E) \geq \int_{E_e} \liminf_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon^y; E^y) \, dy. \tag{3.22}$$

We now finish as in the proof of Proposition 4.7 of [3]: Since  $u_\varepsilon \rightarrow u$  in  $L^1(E)$ , we have (possibly for a subsequence) that  $u_\varepsilon^y \rightarrow u^y$  in  $L^1(E^y)$  for a.e.  $y \in E^e$ . From Proposition 3.5 we obtain  $u^y \in BV(E^y, \pi\mathbb{Z})$  and

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} F_\varepsilon(u_\varepsilon; D; E) \geq \frac{1}{2} \int_{E_e} |Du^y| \, dy. \tag{3.23}$$

Using Proposition 6.9 of [3], generalized from characteristic functions to  $\pi\mathbb{Z}$ -valued functions or using Section 5.10 of [10], we obtain that  $u \in BV(E, \pi\mathbb{Z})$ , and

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} F_\varepsilon(u_\varepsilon; D; E) \geq \frac{1}{2} \int_E \langle Du, e \rangle, \tag{3.24}$$

from which (3.14) follows for  $e$  parallel to  $\int_E Du$ .  $\square$

**Lemma 3.9.** *For  $u \in BV(\partial\Omega, \pi\mathbb{Z})$ , the jump set  $S_u$  is countably  $\mathcal{H}^1$ -rectifiable, and both the jump  $[u]$  and the normal  $\nu_{S_u}$  are approximately continuous  $\mathcal{H}^1$ -a.e. on  $S_u$ .*

**Proof.** It suffices to show this for characteristic functions that are in  $BV$ . For these, the conclusion follows from the countable rectifiability of  $S_u$  (see e.g. [10, Section 5.9, Theorem 1]).  $\square$

**Proof of Theorem 3.1.** As in [3, p. 26], we cover the compact set  $\partial\Omega$  by finitely many balls  $B_i$  centered on  $\partial\Omega$  such that  $\Omega \cap B_i$  is the image of a half-ball under a map  $\Psi^i$  with isometry defect less than 1. From here we obtain the  $L^2(\partial\Omega)$  boundedness and  $L^1(\partial\Omega)$  compactness results by Proposition 3.8. To prove (3.2), we can again proceed as in [3] (see (4.29), (4.30) there), just using Lemma 3.9 and replacing the use of  $\nu_{S_u}$  by that of  $Du$ .

Since the estimate with  $V$  is valid for all  $V$ , we can use  $V^\mu = \mu V$  and let  $\mu \rightarrow 0$  to obtain that the bound holds for the Dirichlet integral alone, as remarked in [1].  $\square$

We have used the following version of a compactness theorem found in [3]. We will show that the result stated there remains true without an a priori  $L^\infty$  bound if we have a better control on the approximation.

We consider functions in  $L^1(A)$ , where  $A$  is a bounded subset of  $\mathbb{R}^N$ . Take a unit vector  $e \in \mathbb{R}^N$ . Let  $M = e^\perp$  be its orthogonal complement. Let  $A_e$  be the projection of  $A$  onto  $M$ . For every  $y \in M$ , set  $A_e^y := \{t \in \mathbb{R}: y + te \in A\}$ . For a function  $u$ , we denote its trace on  $A_e^y$  by  $u_e^y$ , so  $u_e^y(t) = u(y + te)$ .

**Theorem 3.10.** *Let  $(v^n)$  be a sequence of functions in  $L^1(A)$ . Assume that for every  $\delta > 0$ , there exists a sequence of functions  $w_\delta^n$  that satisfies for  $N$  linearly independent unit vectors  $e = e_N$  the properties  $\|w_\delta^n\|_{L^1(A_e^y)} \leq \|v^n\|_{L^1(A_e^y)}$  and  $\|w_\delta^n - v^n\|_{L^1(A)} \leq \delta$ . Assume furthermore that  $(w_\delta^n)_e^y$  is precompact in  $L^1(A_e^y)$  for  $\mathcal{H}^{N-1}$ -a.e.  $y \in A_e$ , and that  $\|w_\delta^n\|_{L^1(A_e^y)} \leq C(e, \delta)$  for all  $y$ .*

*Then  $(v^n)$  is precompact in  $L^1(A)$ .*

**Proof.** By the assumptions,  $\sup_n \|w_\delta^n\|_{L^1(A)} \leq C(\delta)$ , hence  $\sup_n \|v^n\|_{L^1(A)} < \infty$ .

Without loss of generality, we assume  $|A_e^y| \leq 1$  for all  $y, e$ . We extend all functions defined on  $A$  to functions on  $\mathbb{R}^N$  by 0, and similarly all functions defined on  $A_e^y$  to functions on  $\mathbb{R}$ .

Fix a unit vector  $e$  so that the condition of the theorem holds. For  $y \in A_e$  and  $s > 0$  define

$$\omega_\delta^y(s) = \sup \left\{ \int_{\mathbb{R}} |(w_\delta^n)_e^y(t+h) - (w_\delta^n)_e^y(t)| dt : n \in \mathbb{N}, h \in [-s, s] \right\}. \tag{3.25}$$

By assumption, this is bounded by  $2C(e, \delta)$ . The Fréchet–Kolmogorov theorem shows by the precompactness of  $((w_\delta^n)_e^y)$  that  $\omega_\delta^y(s) \rightarrow 0$  as  $s \rightarrow 0$ .

We now calculate

$$\begin{aligned} \int_{\mathbb{R}^N} |v^n(x+he) - v^n(x)| dx &\leq 2\delta + \int_{\mathbb{R}^N} |w_\delta^n(x+he) - w_\delta^n(x)| dx \\ &= 2\delta + \int_{A_e} \left( \int_{\mathbb{R}} |(w_\delta^n)_e^y(t+h) - (w_\delta^n)_e^y(t)| dt \right) dy \\ &\leq 2\delta + \int_{A_e} \omega_\delta^y(|h|) dy. \end{aligned} \tag{3.26}$$

Now we set  $\omega_\delta(s) = \int_{A_e} \omega_\delta^y(s) dy$ . Then  $\omega_\delta \leq 2C(e, \delta)|A_e| \leq 2C(e, \delta)$ . Also,  $\omega_\delta(s) \rightarrow 0$  as  $s \rightarrow 0$  by the corresponding convergence for every  $\omega_\delta^y$  and the dominated convergence theorem. We now define  $\omega(s) := \inf_{\delta > 0} (2\delta + \omega_\delta(s))$ , which is a bounded function with  $\omega(s) \rightarrow 0$  as  $s \rightarrow 0$ . By (3.26), we have

$$\int_{\mathbb{R}^N} |v^n(x+he) - v^n(x)| dx \leq \omega(|h|) \tag{3.27}$$

for all  $n \in \mathbb{N}$  and  $h \in \mathbb{R}$ .

Repeating this construction for  $N$  linearly independent vectors  $e_1, \dots, e_N$  shows the analog of (3.27) for all of these vectors, and now the Fréchet–Kolmogorov theorem shows that  $(v^n)$  is precompact in  $L^1(A)$ .  $\square$

#### 4. A product estimate

In this section, we prove a product estimate similar to that of Sandier and Serfaty [24] that allows us to use the lower bounds of the last section just for specific directions. This will later be useful to separate time- and space-derivatives. As in the previous section,  $\Omega$  is a three-dimensional domain. The following theorem is our main result in this chapter. We will prove it later in this section after establishing some preliminary results.

We will need the notion of a defect measure: Let  $(f_\varepsilon)$  be a sequence of functions and assume  $f_\varepsilon \rightharpoonup f$  in  $L^2$ . Then  $|f_\varepsilon|^2 dx$  converges weakly in the sense of measures to  $|f|^2 dx + \nu$ , where  $\nu$  is called the defect measure of  $L^2$  convergence of this sequence.

**Theorem 4.1.** Let  $X, Y \in C^0(\overline{\Omega}; \mathbb{R}^3)$  be vector fields. Let  $(u_\varepsilon)$  be a sequence of functions such that  $F_\varepsilon(u_\varepsilon; \Omega; \partial\Omega) \leq M \log \frac{1}{\varepsilon}$  and  $u_\varepsilon \rightarrow u$  in  $L^1(\partial\Omega)$ . Let  $\nu_X$  and  $\nu_Y$  denote the defect measures of  $L^2$  convergence of  $\frac{1}{\sqrt{\log(1/\varepsilon)}} X \cdot \nabla u_\varepsilon$  and  $\frac{1}{\sqrt{\log(1/\varepsilon)}} Y \cdot \nabla u_\varepsilon$ . Then  $u \in BV(\partial\Omega; \pi\mathbb{Z})$ , and we have the inequality

$$\|\nu_X\|^{1/2} \|\nu_Y\|^{1/2} \geq \frac{1}{2} \int_{\partial\Omega} |D^\perp u \cdot (X \times Y)|, \tag{4.1}$$

where  $D^\perp u$  denotes the vector-valued measure obtained by rotating  $Du$  in the tangent space to  $\partial\Omega$  by  $\frac{\pi}{2}$ . It follows that we have

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \left( \int_{\Omega} |X \cdot \nabla u_\varepsilon|^2 \right)^{1/2} \left( \int_{\Omega} |Y \cdot \nabla u_\varepsilon|^2 \right)^{1/2} \geq \frac{1}{2} \int_{\partial\Omega} |D^\perp u \cdot (X \times Y)|. \tag{4.2}$$

**Corollary 4.2.** Let  $G \subset \mathbb{R}^2$  open, with  $\partial G \in C^1$ , and  $I \subset \mathbb{R}$ . Then for every  $X, Y \in C^0(\overline{G \times I})$  and  $(u_\varepsilon)$  with  $F_\varepsilon(u_\varepsilon; G \times I; \partial G \times I) \leq M \log \frac{1}{\varepsilon}$  and  $\int_{G \times \partial I} |\nabla u_\varepsilon|^2 \leq M \log \frac{1}{\varepsilon}$  there holds

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \left( \int_{G \times I} |X \cdot \nabla u_\varepsilon|^2 \right)^{1/2} \left( \int_{G \times I} |Y \cdot \nabla u_\varepsilon|^2 \right)^{1/2} \geq \frac{1}{2} \int_{\partial G \times I} |D^\perp u \cdot (X \times Y)|. \tag{4.3}$$

**Proof.** This follows from Theorem 4.1 as follows: Since  $G \times I$  is a cylinder and not  $C^1$ , we cannot apply Theorem 4.1 directly. However, we can extend  $u_\varepsilon$  to  $G \times I_{2\delta}$ , where  $I_\delta$  is the interval  $I$  extended by  $\delta$  on both ends, by setting  $u_\varepsilon = u_\varepsilon|_{\partial I}$  on the two components on  $G_{2\delta} \setminus G$ . We also extend  $X$  and  $Y$  to continuous  $X_\delta$  and  $Y_\delta$  with  $X_\delta = Y_\delta = 0$  on  $G \times (I_{2\delta} \setminus I_\delta)$ . The theorem now applies on any  $C^1$  domain  $\Omega_\delta$  with  $G \times I_\delta \subset \Omega_\delta \subset G \times I_{2\delta}$ . Letting  $\delta \rightarrow 0$ , we obtain the claim.  $\square$

**Remark 4.3.** The results of Theorems 3.1 and 4.1 and Corollary 4.2 also hold *mutatis mutandis* for the functional  $\mathcal{F}^\varepsilon$  defined in (1.3): The sequence  $(u_\varepsilon)$  still satisfies the same compactness properties, with the limit  $u$  now satisfying the condition  $v := u - g \in BV(\partial\Omega, \pi\mathbb{Z})$ . Furthermore, the lower bounds (4.2) and (4.3) hold with  $u$  on the right-hand sides replaced by  $v$ . This can be proved similarly to the argument in Section 3 of [17].

**Lemma 4.4.** Let  $X$  and  $Y$  be linearly independent vectors in  $\mathbb{R}^3$ . Then for every  $q \in \mathbb{R}^3$  and  $\lambda > 0$  and defining the map  $\Phi = \Phi_{\lambda,q,X,Y} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  by

$$\Phi(a, b) = q + \frac{1}{\sqrt{\lambda}} aX + \sqrt{\lambda} bY, \tag{4.4}$$

we have for any domain  $B \subset \mathbb{R}^2$ ,  $v \in H^1_{\text{loc}}(B)$  and  $u = v \circ \Phi^{-1}$  the relation

$$|X \times Y| \int_B |\nabla v|^2 = \int_{\Phi(B)} \frac{1}{\lambda} |X \cdot \nabla u|^2 + \lambda |Y \cdot \nabla u|^2. \tag{4.5}$$

**Proof.** This is easily seen by change of variables.  $\square$

**Proposition 4.5.** Let  $X, Y \in \mathbb{R}^3$ . Let  $D = D_r$  be a half-ball and  $E = E_r$  the flat part of its boundary as in Proposition 3.8. Then for any  $\lambda > 0$  there exist sets  $A_\varepsilon \subset D$  with  $|A_\varepsilon| \rightarrow 0$  such that

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{A_\varepsilon} \frac{1}{\lambda} |X \cdot \nabla u_\varepsilon|^2 + \lambda |Y \cdot \nabla u_\varepsilon|^2 \geq \int_E |D^\perp u \cdot (X \times Y)|. \tag{4.6}$$

If  $\eta \in C^1(D_r)$  is zero on the curved part of  $\partial D_r$ , we also have

$$\liminf_{\eta \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{A_\varepsilon} \frac{1}{\lambda} |\eta X \cdot \nabla u_\varepsilon|^2 + \lambda |\eta Y \cdot \nabla u_\varepsilon|^2 \geq \int_E \eta^2 |D^\perp u \cdot (X \times Y)|. \tag{4.7}$$

**Proof.** If  $X$  and  $Y$  are linearly dependent or both lie in the plane  $P := \{x_3 = 0\}$ , this is trivial. Otherwise, let  $P_{XY}$  be the plane spanned by  $X$  and  $Y$ . Let  $e \in P$  be a unit vector orthogonal to  $P_{XY} \cap P$ . Let  $p: \mathbb{R}^3 \rightarrow P_{XY}$  denote the projection parallel to  $P$ . We set  $E_\varepsilon := p(E) = P \cap P_{XY} \cap E$  and  $E^y := p^{-1}(y) \cap E$  as well as  $D^y := p^{-1}(y) \cap D$  for every  $y \in E_\varepsilon$  as before. Using Fubini’s theorem and writing  $u^y = u_\varepsilon^y$  for the slices of  $u_\varepsilon$  on  $D^y$ , we obtain

$$\begin{aligned} & \int_D \frac{1}{\lambda} |X \cdot \nabla u^y|^2 + \lambda |Y \cdot \nabla u^y|^2 + \frac{1}{\varepsilon} \int_E V(u^y) \\ &= \int_{E_\varepsilon} \left( \int_{D^y} \frac{1}{\lambda} |X \cdot \nabla u^y|^2 + \lambda |Y \cdot \nabla u^y|^2 + \frac{1}{\varepsilon} \int_{E^y} V(u^y) \right) dy. \end{aligned} \tag{4.8}$$

We can find a map  $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  as in Lemma 4.4 such that  $v^y = u^y \circ \Phi$  satisfies

$$\int_{D^y} \frac{1}{\lambda} |X \cdot \nabla u^y|^2 + \lambda |Y \cdot \nabla u^y|^2 + \frac{1}{\varepsilon} \int_{E^y} V(u^y) = |X \times Y| \int_{\Phi^{-1}(D^y)} |\nabla v^y|^2 + \frac{1}{\varepsilon} \int_{\Phi^{-1}(E^y)} \alpha_{XY}^\lambda V(v^y(x)) dx, \tag{4.9}$$

where  $\alpha = \alpha_{XY}^\lambda > 0$  is obtained from the change of variables on  $E^y$ . For almost every  $y$ , we have as in the proof of Proposition 3.8 that  $u^y \rightarrow u$  in  $L^1(E^y)$ , and  $u \in BV(E^y)$ , which translates for  $v$  to  $v_\varepsilon^y \rightarrow v = u \circ \Phi$  and  $v \in BV(\Phi^{-1}(E^y))$ . Using Proposition 3.5, we find two-dimensional half-balls  $B_i^{\varepsilon;y}$  inside  $\Phi^{-1}(D^y)$  that cover  $S_v$ , which is a finite set for a.e.  $y$ . On these balls, we can reduce the functional to  $G_\varepsilon$  on the boundary as before, just changing  $V$  to  $\alpha V$ , and use Corollary 3.6 to obtain

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{B_i^{\varepsilon;y}} |\nabla v_\varepsilon^y|^2 \geq |X \times Y| \int_{S_v} |Dv|. \tag{4.10}$$

Changing back to the original variables, we obtain the estimate

$$\frac{1}{\log(1/\varepsilon)} \int_{\Phi(B_i^{\varepsilon;y})} \frac{1}{\lambda} |X \cdot \nabla u_\varepsilon|^2 + \lambda |Y \cdot \nabla u_\varepsilon|^2 \geq |X \times Y| \int_{S_u} |Du| - o(1). \tag{4.11}$$

Multiplying with  $\eta^2$ , we see that

$$\frac{1}{\log(1/\varepsilon)} \int_{\Phi(B_i^{\varepsilon;y})} \frac{1}{\lambda} |\eta X \cdot \nabla u_\varepsilon|^2 + \lambda |\eta Y \cdot \nabla u_\varepsilon|^2 \geq \min_{\Phi(B_i^{\varepsilon;y})} \eta^2 \left( |X \times Y| \int_{S_u} |Du| - o(1) \right). \tag{4.12}$$

Since  $\eta$  is  $C^1$  and  $\text{diam } \Phi(B_i^{\varepsilon;y}) \leq C(\lambda)o(1) = o(1)$ , we can let  $\varepsilon \rightarrow 0$  in (4.12), whence the minimum converges to the value at the center of the ball, and we obtain

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{\Phi(B_i^{\varepsilon;y})} \frac{1}{\lambda} |\eta X \cdot \nabla u_\varepsilon|^2 + \lambda |\eta Y \cdot \nabla u_\varepsilon|^2 \geq |X \times Y| \int_{S_u} \eta^2 |Du|. \tag{4.13}$$

Integrating over  $E_\varepsilon$ , that shows

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{A_\varepsilon} \frac{1}{\lambda} |\eta X \cdot \nabla u_\varepsilon|^2 + \lambda |\eta Y \cdot \nabla u_\varepsilon|^2 \geq |X \times Y| \int_E \eta^2 |e \cdot Du|, \tag{4.14}$$

where  $A_\varepsilon = \bigcup_{i,y} \Phi(B_i^{\varepsilon;y})$  satisfies  $|A_\varepsilon| \leq C(\lambda) \int_{E_\varepsilon} \sum_i |B_i^{\varepsilon;y}| dy \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Since  $e \cdot Du = e^\perp \cdot D^\perp u$ ,  $e^\perp$  is parallel to the projection of  $X \times Y$  onto  $P$ , and  $e_3 \cdot Du = 0$ , (4.14) now implies (4.7). The equation without  $\eta$  follows similarly.  $\square$

**Proof of Theorem 4.1.** We follow the proof of Theorem 1 in [24]. By the energy bound, for every bounded vector field  $Z$ , e.g.  $X$  and  $Y$ , we have that the functions  $f_Z^\varepsilon = (Z \cdot \nabla u_\varepsilon) / \sqrt{\log(1/\varepsilon)}$  are bounded in  $L^2$ , hence  $f_Z^\varepsilon \rightharpoonup f_Z$  for

a subsequence. Since  $|f_Z^\varepsilon|^2 dx$  is bounded in the sense of measures, we have  $|f_Z^\varepsilon|^2 dx \rightharpoonup |f_Z|^2 dx + \nu_Z$ , where  $\nu_Z$  is the defect measure.

For every  $r > 0$ , cover  $\partial\Omega$  by  $B_i = B_r(m_i) \cap \Omega$  with  $m_i \in \partial\Omega$ . Let  $\eta_i \in C^1(\overline{\Omega})$  such that  $\eta_i$  is supported in  $B_i$  and  $\sum_i \eta_i^2 = 1$  on  $\partial\Omega$ . Setting  $X_i = X(m_i)$  and  $Y_i = Y(m_i)$ , we have by continuity that, as  $r \rightarrow 0$ ,

$$\delta(r) = \sup_i \sup_{B_i} (|X - X_i| \vee |Y - Y_i|) \rightarrow 0.$$

We now map  $B_i$  into a half-ball  $B_i^r$  by a diffeomorphism  $\Psi_i^r$ . We can assume that the supremum over  $i$  of the isometry defects of the  $\Psi_i^r$ , denoted  $\beta(r)$ , vanishes as  $r \rightarrow 0$ . Using the results on the isometry defect in [3] and by Proposition 4.5, we find sets  $A'_{\varepsilon;i} \subset B_i^r$  of measure tending to 0 with  $\varepsilon$  such that

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{A'_{\varepsilon;i}} \frac{1}{\lambda} |\eta_i X_i \cdot \nabla u_\varepsilon|^2 + \lambda |\eta_i Y_i \cdot \nabla u_\varepsilon|^2 \geq (1 - \beta(r))^5 \int_{\partial\Omega} \eta_i^2 |D^\perp u \cdot (X \times Y)|. \tag{4.15}$$

Summing over  $i$  and setting  $A_\varepsilon = \bigcup_i (\Psi_i^r)^{-1}(A'_{\varepsilon;i})$ , we still have  $|A_\varepsilon| \rightarrow 0$  and using that  $X_i$  is close to  $X$ , we continue as in [24] to obtain

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{A_\varepsilon} \frac{1}{\lambda} |X \cdot \nabla u_\varepsilon|^2 + \lambda |Y \cdot \nabla u_\varepsilon|^2 \geq (1 - \beta(r))^5 \left( \int_{\partial\Omega} |D^\perp u \cdot (X \times Y)| - C\delta(r) \right). \tag{4.16}$$

We can now let  $r \rightarrow 0$  and finish as in [24] to obtain that the defect measures  $\nu_X$  and  $\nu_Y$  satisfy

$$\frac{1}{\lambda} \|\nu_X\| + \lambda \|\nu_Y\| \geq \int_{\partial\Omega} |D^\perp u \cdot (X \times Y)|, \tag{4.17}$$

from which the claim follows by optimizing over  $\lambda$ .  $\square$

Just as in [24], we can derive the following corollaries:

**Corollary 4.6.** *If  $F_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ , then  $u_\varepsilon \xrightarrow{S} \vec{a} \in (\partial\Omega)^N$  with  $d \in \mathbb{Z}^N$ , and there holds*

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{2 \log(1/\varepsilon)} \int_{\Omega} |\nabla u_\varepsilon|^2 \geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \left( \int_{\Omega} |\partial_1 u_\varepsilon|^2 \int_{\Omega} |\partial_2 u_\varepsilon|^2 \right) \geq \frac{\pi}{2} \sum_{i=1}^N |d_i|. \tag{4.18}$$

**Corollary 4.7.** *If in addition to the assumptions of the previous theorem we have  $u_\varepsilon \xrightarrow{S} \vec{a} \in (\partial\Omega)_*^N$  with  $\vec{d} \in \{\pm 1\}^N$ , and  $\frac{1}{2} \int_{\Omega} |\nabla u_\varepsilon|^2 \leq \frac{\pi N}{2} \log \frac{1}{\varepsilon} (1 + o(1))$  as  $\varepsilon \rightarrow 0$ , there holds for any  $X, Y \in C^0(\overline{\Omega})$*

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \frac{1}{\log(1/\varepsilon)} (X \cdot \nabla u_\varepsilon)(Y \cdot \nabla u_\varepsilon) = \frac{\pi}{2} \sum_{i=1}^N X(a_i) \cdot Y(a_i). \tag{4.19}$$

### 5. The lower bound in time

In this section we use the approach of Sandier and Serfaty [24,23] to show how the product estimate leads to  $H^1$  in time motion of the vortices, and the lower bound part required for the application of the theory of  $\Gamma$ -convergence of gradient flows.

We will need the following norm on the space  $\mathcal{M}(\partial\Omega)$  of measures on  $\partial\Omega$ :

$$\|\mu\|_1 := \sup \left\{ \left| \int_{\partial\Omega} \zeta \mu \right| : \int_{\partial\Omega} \zeta = 0, \left| \frac{\partial}{\partial \tau} \zeta \right| \leq 1 \right\}, \tag{5.1}$$

where  $\frac{\partial}{\partial \tau}$  denotes tangential differentiation.



**Theorem 5.1.** Let  $(u_\varepsilon)$  with  $u_\varepsilon = u_\varepsilon(x, t) : \Omega \times [0, T] \rightarrow \mathbb{R}$  be a sequence of functions such that for some  $M > 0$ ,

$$\int_{\Omega \times (0, T)} |\partial_t u_\varepsilon|^2 \leq M \log \frac{1}{\varepsilon} \tag{5.2}$$

and

$$\max_{t \in [0, T]} \mathcal{F}^\varepsilon(u_\varepsilon(\cdot, t)) \leq M \log \frac{1}{\varepsilon}. \tag{5.3}$$

Then  $(u_\varepsilon)$  converges in  $L^1(\partial\Omega \times (0, T))$  to a function  $u$  with  $v = u - g \in BV(\partial\Omega \times (0, T))$ . The measures  $\mu = \partial_\tau v$  and  $\sigma = \partial_t v$  satisfy  $\mu \in L^\infty((0, T), \mathcal{M}(\partial\Omega))$  and  $\sigma \in L^2((0, T), \mathcal{M}(\partial\Omega))$ . Furthermore, we have  $\mu \in C^{0,1/2}([0, T], (\mathcal{M}(\partial\Omega), \|\cdot\|_1))$ , and for every space-vector field  $X$  and every continuous function  $f$  there holds

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \left( \int_{\Omega \times (0, T)} |X \cdot \nabla u_\varepsilon|^2 - \int_{\Omega \times (0, T)} f^2 |\partial_t u_\varepsilon|^2 \right)^{1/2} \geq \frac{1}{2} \left| \int_{\partial\Omega \times [0, T]} (X \cdot \nu) f \sigma \right|. \tag{5.4}$$

**Proof.** We follow the proof of Theorem 3 in [24]. We use a coordinate system on  $\partial\Omega \times [0, T]$  given by  $\hat{e}_\nu, \hat{e}_\tau, \hat{e}_t$ , where  $\hat{e}_t$  is the unit vector pointing in time-direction,  $\hat{e}_\nu = \nu$  is the outer normal to  $\partial\Omega$ , and  $\hat{e}_\tau$  is a tangent vector to  $\partial\Omega$  with  $\hat{e}_t \cdot (\hat{e}_\nu \times \hat{e}_\tau) = 1$ . We split the measure  $Du$  on  $\partial\Omega$  as  $Du = \partial_t u \hat{e}_t + \partial_\tau u \hat{e}_\tau$ . So  $D^\perp u = -\partial_\tau u \hat{e}_t + \partial_t u \hat{e}_\tau$ . Eq. (5.4) is now a direct consequence of (4.3) and Remark 4.3. Setting  $f = 1$  and using the estimates (5.2), (5.3), we see that

$$\left| \int_{\partial\Omega \times [0, T]} \sigma x_\nu \right|^2 \leq C \int_0^T \|X\|_{L^\infty(\Omega)}^2. \tag{5.5}$$

Choosing  $X$  as  $X = x_\nu \hat{e}_\nu$  on  $\partial\Omega \times [0, T]$ , and extending to  $\Omega$  with  $\|X\|_{L^\infty(\Omega)} = \|x_\nu\|_{L^\infty(\partial\Omega)}$ , we see that

$$\left| \int_{\partial\Omega \times [0, T]} \sigma x_\nu \right|^2 \leq C \int_0^T \|x_\nu\|_{L^\infty(\partial\Omega)}^2, \tag{5.6}$$

which shows by duality that  $\sigma \in L^2([0, T], \mathcal{M}(\partial\Omega))$ , so for every  $X$  with  $|X| \leq 1$  there holds

$$\left| \int_{\partial\Omega \times [t_1, t_2]} (X \cdot \nu) \sigma \right| \leq C \sqrt{t_2 - t_1}, \tag{5.7}$$

where  $C = \|\sigma\|_{L^2([0, T], \mathcal{M}(\partial\Omega))}$ .

Now we choose a vector field  $\zeta \in C^1(\bar{\Omega}; \mathbb{R}^3)$  with  $|\frac{\partial}{\partial \tau} \zeta| \leq 1$  on  $\partial\Omega$  and  $\int_{\partial\Omega} \zeta = 0$ . Integrating by parts we have, using the fact that distributional derivatives commute and  $\partial_t \zeta = 0$

$$\begin{aligned} \int_{\partial\Omega \times [t_1, t_2]} \sigma \partial_\tau \zeta &= \int_{\partial\Omega \times [t_1, t_2]} \partial_t v \partial_\tau \zeta \\ &= \int_{\partial\Omega} v(t_2) \partial_\tau \zeta - \int_{\partial\Omega} v(t_1) \partial_\tau \zeta - \int_{\partial\Omega \times [t_1, t_2]} v \partial_t \partial_\tau \zeta \\ &= - \int_{\partial\Omega} \partial_\tau v(t_2) \zeta + \int_{\partial\Omega} \partial_\tau v(t_1) \zeta - \int_{\partial\Omega \times [t_1, t_2]} v \partial_\tau \partial_t \zeta \\ &= - \int_{\partial\Omega} \mu(t_2) \zeta + \int_{\partial\Omega} \mu(t_1) \zeta. \end{aligned}$$

Hence

$$\left| \int_{\partial\Omega} (\mu(t_2) - \mu(t_1)) \zeta \right| \leq \|\sigma\|_{L^2([0,T], \mathcal{M}(\partial\Omega))} \sqrt{t_2 - t_1}, \tag{5.8}$$

which shows that  $t \mapsto \mu(t)$  is Hölder continuous with respect to the  $\|\cdot\|_1$ -norm, and

$$[\mu]_{C^{0,1/2}([0,T], (\mathcal{M}(\partial\Omega), \|\cdot\|_1))} \leq \|\sigma\|_{L^2([0,T], \mathcal{M}(\partial\Omega))}. \quad \square \tag{5.9}$$

**Proposition 5.2.** *If  $(u_\varepsilon)$  satisfy  $\mathcal{F}^\varepsilon(u_\varepsilon(t)) \leq M \log \frac{1}{\varepsilon}$  for all  $t \in [0, T]$ ,  $\int_{\Omega \times [0,T]} |\partial_t u_\varepsilon|^2 \leq C \log \frac{1}{\varepsilon}$ , and  $u_\varepsilon \rightarrow u$  in  $L^1(\partial\Omega)$  with  $u - g \in BV(\partial\Omega, \pi\mathbb{Z})$ , then  $\sigma = \partial_\tau(u - g)$  is of the form  $\sigma(t) = \pi \sum_{i=1}^{n(t)} d_i(t) \delta_{a_i(t)}$  for some  $a_i \in \partial\Omega$ ,  $d_i \in \mathbb{Z}$ .*

*In addition, for any  $\zeta \in C^1(\overline{\Omega})$ , the map  $t \mapsto \int_{\partial\Omega} \zeta \mu(t)$  is of class  $H^1((0, T))$ .*

*If in addition there holds  $\sum_i |d_i(t)| \leq \sum_i |d_i(0)|$  for all  $t \in [0, T]$ ,  $d_i(0) \in \{\pm 1\}$ , and  $a_i(0)$  are distinct, then there exists a time  $T^* \in (0, T]$  and  $n = n(0)$  maps  $a_i(t) \in H^1((0, T^*), \partial\Omega)$  such that the  $a_i(t)$  are distinct for  $0 \leq t < T^*$  and  $\mu(t) = \pi \sum_i d_i(0) \delta_{a_i(t)}$ .*

*If  $T^* < T$ , then there exists  $i \neq j$  such that  $\lim_{t \rightarrow T^*} a_i(t) = \lim_{t \rightarrow T^*} a_j(t)$ .*

**Proof.** This follows by using Theorem 5.1 just as Propositions 3.2 and 3.3 of [23].  $\square$

**Proposition 5.3.** *If in addition to the conditions of the last proposition there holds*

$$\int_{\Omega} |\nabla u_\varepsilon|^2 \leq \frac{\pi}{2} \sum_i \log \frac{1}{\varepsilon} (1 + o(1)),$$

*then for all intervals  $[t_1, t_2] \subset [0, T]$  on which  $a_i(t)$  remain distinct,*

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{t_1}^{t_2} \int_{\Omega} |\partial_t u_\varepsilon|^2 \geq \frac{\pi}{2} \sum_i \int_{t_1}^{t_2} |\partial_t a_i|^2 \tag{5.10}$$

**Proof.** This follows as Corollary 7 in [24] from the proof of Theorem 5.1 and Corollary 4.7.  $\square$

### 6. Construction of a “recovery sequence”

Here we perform the construction necessary for the application of the gradient flow  $\Gamma$ -convergence theorem. Our construction follows the same general idea as that of [23] (pushing the vortices in the direction of the limit flow). However, we need to refine the construction since isometric pushing as in [23] only works for constant curvature.

**Theorem 6.1.** *Let  $(u_\varepsilon)$  be a sequence with  $u_\varepsilon \xrightarrow{S} (\vec{a}, \vec{d})$  with  $\vec{a} \in (\partial\Omega)_*^N$  and  $\vec{d} \in \{\pm 1\}^N$ . Assume that*

$$\mathcal{F}^\varepsilon(u_\varepsilon) - \frac{\pi N}{2} \log \frac{1}{\varepsilon} \leq C$$

and

$$\|\Delta u_\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{C}{\log(1/\varepsilon)}.$$

*Let  $\vec{V} = (V_1, \dots, V_N)$  be a collection of tangent vectors to  $\partial\Omega$  at  $a_i$ , and let  $\vec{b}(t) \in (\partial\Omega)_*^N$  be such that  $\vec{b}(0) = \vec{a}$  and  $\frac{d\vec{b}}{dt}(0) = \vec{V}$ .*

*Then there exist  $v_\varepsilon = v_\varepsilon(x, t)$  such that  $v_\varepsilon(0) = u_\varepsilon(0)$  and a locally bounded function  $G$  on  $(\partial\Omega)_*^N$  such that*

$$\frac{1}{\log(1/\varepsilon)} \int_{\Omega} |\partial_t v_\varepsilon(0)|^2 = \frac{\pi}{2} |\vec{V}|^2 + o(1) \tag{6.1}$$

and

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{E}^\varepsilon(v_\varepsilon(t)) \leq \left. \frac{d}{dt} \right|_{t=0} \mathcal{E}(\bar{b}(t)) + G(\bar{a})D_\varepsilon + o(1), \tag{6.2}$$

where  $D_\varepsilon = \mathcal{E}^\varepsilon(u_\varepsilon) - \mathcal{E}(\bar{a})$  is the energy excess of  $u_\varepsilon$ .

**Proof.** We want to “push” the vortices along  $\partial\Omega$ . The “pushing” will need to be nearly isometric on the boundary and infinitesimally conformal in the interior near the vortices in order not to change the energy of the vortex cores.

We define the family of diffeomorphisms  $\chi_t : \bar{\Omega} \rightarrow \bar{\Omega}$  as the solutions of the flow given by the vector field  $\lambda$  of Proposition 6.2, i.e.  $\frac{d}{dt}\chi_t(x) = \lambda(\chi_t(x))$  and  $\chi_0(x) = x$ .

Let  $u_*^t = u_*(\chi_t(a_1), \dots, \chi_t(a_N))$  denote the singular harmonic function jumping by  $-\pi d_i$  at  $\chi_t(a_i)$  and  $u_* = u_*^0 = u_*(\bar{a})$ , and set  $\psi_t = u_*^t \circ \chi_t - u_*$ .

Now we define  $v_\varepsilon(x, t)$  via  $v_\varepsilon(\chi_t(x), t) = u_\varepsilon(x) + \psi_t(x)$ . Then we calculate  $\mathcal{F}^\varepsilon(v_\varepsilon)$  by changing variables as

$$\begin{aligned} \mathcal{F}^\varepsilon(v_\varepsilon) &= \frac{1}{2} \int_{\Omega} |(\nabla v_\varepsilon) \circ \chi_t|^2 \det D\chi_t + \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin^2(v_\varepsilon \circ \chi_t - g \circ \chi_t) \tau \cdot \frac{\partial \chi_t}{\partial \tau} \\ &= \frac{1}{2} \int_{\Omega} |D\chi_t^{-1} \nabla(u_\varepsilon + \psi_t)|^2 \det D\chi_t + \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin^2(u_\varepsilon + \psi_t - g \circ \chi_t) \tau \cdot \frac{\partial \chi_t}{\partial \tau}. \end{aligned} \tag{6.3}$$

Differentiating and using the definition of  $\chi_t$  and  $\psi_0 = 0$ , we obtain

$$\left. \frac{d}{dt} \right|_{t=0} \frac{1}{2} \int_{\Omega} |D\chi_t^{-1} \nabla(u_\varepsilon + \psi_t)|^2 \det D\chi_t = \int_{\Omega} (-D\lambda \nabla u_\varepsilon) \cdot \nabla u_\varepsilon + \frac{1}{2} |\nabla u_\varepsilon|^2 \operatorname{div} \lambda + \left. \frac{d}{dt} \right|_{t=0} \nabla \psi_t \cdot \nabla u_\varepsilon. \tag{6.4}$$

In the balls  $B_\rho(a_i)$  where  $\lambda$  is holomorphic, we have  $-D\lambda \nabla u_\varepsilon \cdot \nabla u_\varepsilon + \frac{1}{2} \operatorname{div} \lambda |\nabla u_\varepsilon|^2 = 0$  so

$$\left. \frac{d}{dt} \right|_{t=0} \frac{1}{2} \int_{\Omega} |\nabla v_\varepsilon|^2 = \int_{\Omega_\rho} -D\lambda \nabla u_\varepsilon \cdot \nabla u_\varepsilon + \frac{1}{2} \operatorname{div} \lambda |\nabla u_\varepsilon|^2 + \int_{\Omega} \left. \frac{d}{dt} \right|_{t=0} \nabla \psi_t \cdot \nabla u_\varepsilon. \tag{6.5}$$

On the other hand,

$$\int_{\chi_t(\Omega_\rho)} |\nabla u_*^t|^2 = \int_{\Omega_\rho} |(\nabla u_*^t) \circ \chi_t| \det D\chi_t, \tag{6.6}$$

and since  $\nabla \psi_t = D\chi_t \nabla u_*^t \circ \chi_t - \nabla u_*$ , this can be rewritten as

$$\int_{\Omega_\rho} |D\chi_t^{-1} (\nabla \psi_t + \nabla u_*)|^2 \det D\chi_t.$$

Differentiating, we obtain

$$\left. \frac{d}{dt} \right|_{t=0} \frac{1}{2} \int_{\chi_t(\Omega_\rho)} |\nabla u_*^t|^2 = \int_{\Omega_\rho} (-D\lambda \nabla u_*) \cdot \nabla u_* + \frac{1}{2} |\nabla u_*|^2 \operatorname{div} \lambda + \left. \frac{d}{dt} \right|_{t=0} \nabla \psi_t \cdot \nabla u_*. \tag{6.7}$$

Since  $\Delta u_\varepsilon \rightarrow 0$  in  $L^2$  and  $u_\varepsilon \rightarrow u_*$  in  $L^2(\partial\Omega)$ , it is not hard to see that  $u_\varepsilon \rightarrow u_*$  in  $L^2(\Omega)$ . If  $\limsup_{\varepsilon \rightarrow 0} D_\varepsilon < \infty$  (otherwise (6.2) is trivial), then we can use (2.15) and see that  $\nabla u_\varepsilon - \nabla u_* \rightharpoonup A$  in  $L^2(\Omega_\rho; \mathbb{R}^2)$  for every  $\rho > 0$ . Testing this with  $\nabla \varphi$  and  $\nabla^\perp \varphi$  for smooth  $\varphi$  with  $\varphi = 0$  in  $B_{\rho/2}(a_i)$ , we obtain after integration by parts and using the  $L^2(\Omega)$  and  $L^2(\partial\Omega)$  convergences that  $A = 0$ .

Comparing (6.5) and (6.7), we calculate the difference of corresponding terms. Note

$$D\lambda \nabla u_\varepsilon \cdot \nabla u_\varepsilon - D\lambda \nabla u_* \cdot \nabla u_* = D\lambda (\nabla u_\varepsilon - \nabla u_*) \cdot (\nabla u_\varepsilon - \nabla u_*) + (D\lambda + {}^t(D\lambda)) \nabla u_* \cdot (\nabla u_\varepsilon - \nabla u_*). \tag{6.8}$$

Hence the weak convergence and (2.15) imply that this difference is bounded by  $G(\bar{a})D_\varepsilon + o(1)$ . The divergence term can be treated similarly. For the term with  $\left. \frac{d}{dt} \right|_{t=0} \nabla \psi_t$ , we use that Lemma 2.10 and the method of proof of Theorem 4.2 in [16] actually imply that  $\nabla u_\varepsilon - \nabla u_* \rightarrow 0$  in all  $L^p(\Omega)$ ,  $p < 2$ , so this term is also  $o(1)$  by the regularity of  $\psi_t$ .

For the boundary term, we have

$$\begin{aligned} & \left. \frac{d}{dt} \right|_{t=0} \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin^2(u_\varepsilon + \psi_t - g \circ \chi_t) \tau \cdot \frac{\partial \chi_t}{\partial \tau} \chi_t \\ &= \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin^2(u_\varepsilon - g) \tau \cdot \frac{\partial \lambda}{\partial \tau} + \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin 2(u_\varepsilon - g) \left. \frac{d}{dt} \right|_{t=0} (u_*^t \circ \chi_t - g \circ \chi_t). \end{aligned} \tag{6.9}$$

Since the last term is 0 and  $|\tau \cdot \frac{\partial \lambda}{\partial \tau}| \leq C\rho$  in  $B_\rho \cap \Omega$ , we can use (2.14) and (2.16) and obtain that the boundary contribution is bounded by  $G(\bar{a})D_\varepsilon + O(\sigma \log \frac{1}{\sigma})$  for every  $\sigma < \rho$ , and letting  $\sigma \rightarrow 0$  we obtain, taking into account Lemma 6.3 below, (6.2).

Eq. (6.1) follows from Corollary 4.7.  $\square$

**Proposition 6.2.** *Let  $\Omega \in C^{2,\alpha}$  for some  $\alpha > 0$ , and let  $\rho > 0$  be such that  $B_\rho(a_i)$  are disjoint. Then there exists a vector field  $\lambda \in C^1(\bar{\Omega}; \mathbb{R}^2)$  with  $\lambda(a_i) = (V_i \cdot \tau)\tau$ ,  $\lambda$  tangential to  $\partial\Omega$  everywhere,  $\lambda$  holomorphic in  $B_\rho(a_i) \cap \Omega$  and  $\frac{\partial}{\partial \tau}(\lambda \cdot \tau) = 0$  at the points  $a_i$ . The  $C^1(\bar{\Omega})$ -norm of  $\lambda$  can here be bounded by a function  $G(\bar{a})$  that is locally bounded on  $(\partial\Omega)_*^N$ .*

**Proof.** Let  $h: \mathbb{R}_+^2 \rightarrow \Omega$  be a conformal map. By the Kellogg–Warschawski theorem, it is  $C^{2,\alpha}$  up to the boundary. For  $a \in \mathbb{R}$ ,  $z \mapsto h(z + a)$  is also such a conformal map, hence the derivative  $h'(z)$  is tangent to  $\partial\Omega$ . If  $g = h^{-1}$ , this means that  $\lambda = 1/g'$  is a tangent holomorphic function. With a suitable Möbius transformation, we can achieve the tangential derivative condition at any given point, and patching together gives the desired vector field.  $\square$

**Lemma 6.3.** *With  $\Omega_\rho(t) = \Omega \setminus \bigcup B_\rho(\chi_t(a_i))$  there holds*

$$\lim_{\rho \rightarrow 0} \left( \left. \frac{d}{dt} \right|_{t=0} \frac{1}{2} \int_{\Omega_\rho(t)} |\nabla u_*^t|^2 - \frac{\pi N}{2} \log \frac{1}{\rho} \right) = \lim_{\rho \rightarrow 0} \left( \left. \frac{d}{dt} \right|_{t=0} \frac{1}{2} \int_{\chi_t(\Omega_\rho)} |\nabla u_*^t|^2 - \frac{\pi N}{2} \log \frac{1}{\rho} \right). \tag{6.10}$$

**Proof.** It suffices to show that

$$\lim_{\rho \rightarrow 0} \lim_{t \rightarrow 0} \frac{1}{t} \int_{\Delta_\rho(t)} |\nabla u_*^t|^2 = 0, \tag{6.11}$$

where  $\Delta_\rho(t) = (\Omega_\rho(t) \setminus \chi_t(\Omega_\rho)) \cup (\chi_t(\Omega_\rho) \setminus \Omega_\rho(t))$ .

From the construction of  $\chi_t$ , we can infer the existence of  $C > 0$  such that  $\Delta_\rho(t) \subset \bigcup_i B_{\rho(1+Ct\rho)}(a_i(t)) \setminus B_{\rho(1-Ct\rho)}(a_i(t))$ . Since  $|\nabla u_*^t(z)| \leq \frac{c}{|z-a_i(t)|}$  near  $a_i$ , we can estimate

$$\frac{1}{t} \int_{\Delta_\rho(t)} |\nabla u_*^t|^2 \leq \frac{c}{t} \int_{\rho(1-Ct\rho)}^{\rho(1+Ct\rho)} \frac{1}{r} dr = \frac{c}{t} \log \frac{1+Ct\rho}{1-Ct\rho}, \tag{6.12}$$

and letting first  $t \rightarrow 0$  and then  $\rho \rightarrow 0$  we arrive at the claim.  $\square$

### 7. Proof of Theorem 1.4

In this section, we show how we can combine all the previous results to prove Theorem 1.4. We closely follow the corresponding proofs in [23].

As a first step, we prove the case of the natural time scaling for small times:

**Proposition 7.1.** *Under the assumptions of Theorem 1.4 and with  $\lambda_\varepsilon = \frac{1}{\log(1/\varepsilon)}$ , there exists some time  $T_0 > 0$  such that for  $t \leq T_0$ , we have  $u_\varepsilon(t) \xrightarrow{S} (\bar{a}(t), \bar{d}(0))$  with  $\bar{a}(t)$  satisfying the gradient flow law (1.8).*

**Proof.** Note that for  $t < s$

$$\mathcal{F}^\varepsilon(u_\varepsilon(t)) - \mathcal{F}^\varepsilon(u_\varepsilon(s)) = \frac{1}{\log(1/\varepsilon)} \int_t^s \int_\Omega |\partial_t u_\varepsilon|^2, \tag{7.1}$$

hence  $t \mapsto \mathcal{F}^\varepsilon(u_\varepsilon(t))$  is decreasing, and so  $\mathcal{F}^\varepsilon(u_\varepsilon(t)) \leq C \log \frac{1}{\varepsilon}$  for all  $t \in [0, T]$ . To use Proposition 5.2, we will need to find some  $T_0 > 0$  with

$$\int_0^{T_0} \int_\Omega |\partial_t u_\varepsilon|^2 \leq C \log \frac{1}{\varepsilon}. \tag{7.2}$$

We proceed to prove (7.2) with  $C = 1$  by contradiction, almost verbatim as in [23]:

Rescale time via  $v_\varepsilon(x, (\log \frac{1}{\varepsilon})t) = u_\varepsilon(x, t)$ . If (7.2) is not true, then we can find  $s_\varepsilon \ll \log \frac{1}{\varepsilon}$  such that

$$\mathcal{F}^\varepsilon(v_\varepsilon(0)) - \mathcal{F}^\varepsilon(v_\varepsilon(s_\varepsilon)) = \int_0^{s_\varepsilon} \int_\Omega |\partial_t v_\varepsilon|^2 = 1. \tag{7.3}$$

Rescaling time again and setting  $w_\varepsilon(x, t) = v_\varepsilon(x, s_\varepsilon t)$ , we obtain

$$\int_0^1 \int_\Omega |\partial_t w_\varepsilon|^2 = s_\varepsilon \int_0^{s_\varepsilon} \int_\Omega |\partial_t v_\varepsilon|^2 = s_\varepsilon. \tag{7.4}$$

Now we can use Theorem 5.1 on  $w_\varepsilon$ . Since due to (7.4), the left-hand side of (5.4) is 0, it follows that the  $L^1(\partial\Omega \times [0, T])$  limit  $w$  of  $w_\varepsilon$  satisfies  $\partial_t w = 0$ . Hence the vortices do not move, so  $v_\varepsilon(s_\varepsilon) = w_\varepsilon(1) \xrightarrow{S} (\vec{a}(0), \vec{d}(0))$ . The  $\Gamma$ -convergence relation (2.1) and the well-preparedness assumption (1.7) thus show

$$\liminf_{\varepsilon \rightarrow 0} (\mathcal{F}^\varepsilon(v_\varepsilon(s_\varepsilon)) - \mathcal{F}^\varepsilon(v_\varepsilon(0))) \geq 0. \tag{7.5}$$

However, this contradicts (7.3), so (7.2) is true. Using the fact that the energy is decreasing and by the first-order  $\Gamma$ -convergence theorem of [17], it follows that  $\sum |d_i(t)| \leq \sum |d_i(0)|$ .

Now we can apply Proposition 5.2 to  $u_\varepsilon$ . This shows that for  $t < T_1 = \min(T_0, T^*)$ , with  $T^*$  defined in Proposition 5.2, we have  $u_\varepsilon(t) \xrightarrow{S} (\vec{a}(t), \vec{d})$  with  $\vec{a} \in (\partial\Omega)_*^N$  and  $\vec{d} = \vec{d}(0) \in \{\pm 1\}^N$ .

To further prepare our use of Sandier–Serfaty’s abstract result Theorem 1.6, we set  $\mathcal{E}^\varepsilon(v) = \mathcal{F}^\varepsilon(v) - \frac{\pi N}{2} \log \frac{1}{\varepsilon}$  for  $v \in H^1(\Omega) =: \mathcal{M}$  and  $\mathcal{E}(\vec{a}) = W(\vec{a}, \vec{d})$  for  $\vec{a} \in (\partial\Omega)_*^N =: \mathcal{N}$ . As metric structures  $X_\varepsilon$  we use  $\|v\|_{X_\varepsilon}^2 = \frac{1}{\log(1/\varepsilon)} \int_\Omega v^2$ , and as metric for  $T\mathcal{N} \subset \mathbb{R}^{2N}$  we use  $\|X\|_{T\mathcal{N}}^2 = \frac{\pi}{2} \|X\|_{\mathbb{R}^{2N}}^2$ . Hypothesis (LB) is provided by Proposition 5.3, while (CON) follows from Theorem 6.1 (the assumption on  $\limsup_{\varepsilon \rightarrow 0} \int_\Omega |\Delta u_\varepsilon|^2$  can be made as in Lemma 2.1 of [23]).

We can therefore apply Theorem 1.6, which proves the claim for  $t < T_1$ .  $\square$

**Proof of Theorem 1.4.** Part (ii) and (iii) of Theorem 1.4 follow from the use of  $\Gamma$ -convergence of gradient flows in Proposition 7.1 and Proposition 1.5 of [23].

We still have to show that for part (i), the trajectories of the vortices follow the gradient flow not only for a short time, but indeed until collision time. To do so, we assume that  $s < T$  is smaller than the collision time and is the maximal such time for which  $u_\varepsilon(t) \xrightarrow{S} (\vec{a}(t), \vec{d})$  for  $t \in [0, s)$ , with  $a_i$  that satisfy (1.8).

For every  $t < s$  we have

$$\frac{1}{\log(1/\varepsilon)} \int_0^t \int_\Omega |\partial_t u_\varepsilon|^2 = \mathcal{F}^\varepsilon(u_\varepsilon(t)) - \mathcal{F}^\varepsilon(u_\varepsilon(0)) \longrightarrow W(\vec{a}(t), \vec{d}) - W(\vec{a}(0), \vec{d}). \tag{7.6}$$

Passing to the limit  $t \rightarrow s$  and using that  $s$  is less than collision time, we obtain

$$\int_0^s \int_\Omega |\partial_t u_\varepsilon|^2 \leq C \log \frac{1}{\varepsilon}. \tag{7.7}$$

Using Theorem 5.1 and using the continuity of the measures there, it follows that  $u_\varepsilon(s) \xrightarrow{S} (\vec{a}(s), \vec{d})$ . To see that (1.7) holds at  $s$ , we use  $\mathcal{F}^\varepsilon(u_\varepsilon(s)) \leq \mathcal{F}^\varepsilon(u_\varepsilon(t))$  for all  $t$  and let  $t \rightarrow s$  in (1.9).

Hence, we can use Proposition 7.1 at time  $s$ , contradicting the maximality of  $s$ .  $\square$

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