

On the relaxation of some classes of pointwise gradient constrained energies

Sur la relaxation de quelques classes d'énergies avec contraintes ponctuelles sur le gradient

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Abstract

The integral representation problem on $BV(\Omega)$ for the $L^1(\Omega)$ -lower semicontinuous envelope \bar{F} of the functional $F : u \in W^{1,\infty}(\Omega) \mapsto \int_{\Omega} f(\nabla u) dx$ is approached when f is a Borel function, not necessarily convex, with values in $[0, +\infty]$. The presence of the value $+\infty$ in the image of f involves a pointwise gradient constraint on the admissible configurations, since those generating the relaxation process must satisfy the condition $\nabla u(x) \in \text{dom } f$ for a.e. $x \in \Omega$. The main novelty relies in the absence of any convexity assumption on the domain of f . For every convex bounded open set Ω , \bar{F} is represented on the whole $BV(\Omega)$ as an integral of the calculus of variations by means of the convex lower semicontinuous envelope of f . Due to the lack of the convexity properties of $\text{dom } f$, the classical integral representation techniques, based on measure theoretic arguments, seem not to work properly, thus an alternative approach is proposed. Applications are given to the relaxation of Dirichlet variational problems and to first order differential inclusions.

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Résumé

Le problème de représentation intégrale sur $BV(\Omega)$ pour l'enveloppe $L^1(\Omega)$ -semi-continue inférieurement \bar{F} de la fonctionnelle $F : u \in W^{1,\infty}(\Omega) \mapsto \int_{\Omega} f(\nabla u) dx$ est considéré dans le cas où f est Borelienne non nécessairement convexe, à valeurs dans $[0, +\infty]$. La présence de la valeur $+\infty$ dans l'image de f implique une contrainte ponctuelle sur le gradient des configurations admissibles, puisque celles qui jouent un rôle dans le processus de relaxation doivent satisfaire la condition $\nabla u(x) \in \text{dom } f$ p.p. dans Ω . La nouveauté principale consiste en l'absence d'hypothèses de convexité sur le domaine de f . Pour tout ensemble ouvert borné et convexe Ω , \bar{F} admet une représentation sur $BV(\Omega)$ tout entier comme une intégrale du calcul des variations au moyen de l'enveloppe convexe et semi-continue inférieurement de f . En raison du manque des propriétés de convexité de $\text{dom } f$, les techniques classiques de représentation intégrale, basées sur des arguments de théorie de la mesure, semblent ne pas fonctionner convenablement, donc une approche alternative est proposée. Des applications sont données à la relaxation des problèmes variationnels de type Dirichlet et aux inclusions différentielles du premier ordre.

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0. Introduction

Let U be a set, and let $F : U \rightarrow [0, +\infty]$. Then the approach to the minimization problem for F on U by means of the direct methods of the calculus of variations naturally leads to the introduction of a topology τ on U enjoying good compactness properties, and to the study of the τ -lower semicontinuity of F . If F is not τ -lower semicontinuous, one is naturally led to introduce the τ -relaxed functional \bar{F} of F , defined on the τ -closure \bar{U} of U as the greatest τ -lower semicontinuous functional on \bar{U} less than or equal to F on U . Indeed, \bar{F} turns out to be τ -lower semicontinuous, the minimum of \bar{F} on \bar{U} exists provided F satisfies suitable coerciveness conditions, and

$$\min_{u \in \bar{U}} \bar{F}(u) = \inf_{u \in U} F(u).$$

When $f : \mathbb{R}^n \rightarrow [0, +\infty[$ is Borel, Ω is a smooth bounded open subset of \mathbb{R}^n , $U \subseteq W^{1,1}(\Omega)$, and F is the integral energy defined by (here and in the following \mathcal{L}^n denotes the Lebesgue measure on \mathbb{R}^n)

$$F : u \in U \mapsto \int_{\Omega} f(\nabla u) \, d\mathcal{L}^n, \quad (0.1)$$

the above described relaxation process has been widely developed in the last decades for various choices of U , each one determining a particular variational problem (Neumann, Dirichlet, mixed, etc.), and under different assumptions on f . We refer to the books [1,2,8,11,14,20,31] for complete references on the subject, also in more general frameworks.

By choosing τ equal to the $L^1(\Omega)$ -topology, and under convexity assumptions on f , in [23] and in [10] the case of the Neumann problem has been treated when U is a Sobolev space, or a space of smooth functions. In these papers, integral representation results for \bar{F} have been proved on the space $BV(\Omega)$ of the functions with bounded variation in Ω . In the same framework, in [22] and [16] the case of the Dirichlet problem has been treated by imposing a boundary trace condition on the elements of U , and again proving integral representation results for \bar{F} on $BV(\Omega)$.

When f is not convex, relaxation processes have been carried out in both the cases of Neumann and Dirichlet problems when U is a Sobolev space, or a space of smooth functions, and τ is either the sequential weak- $W^{1,p}(\Omega)$ topology, with p depending on the choice of U , or the $L^1(\Omega)$ one (cf. for example [2,9,14,20,30] and the references quoted therein). In these papers integral representation results for \bar{F} have been proved in Sobolev spaces also in more general settings, for example by allowing a dependence of f also on the space variable x , under additional coerciveness and growth assumptions. It has been shown that the relaxation process produces a density convexification. For example, when $U = W^{1,\infty}(\Omega)$ and τ agrees with the sequential weak*- $W^{1,\infty}(\Omega)$ -topology, it turns out that

$$\bar{F}(u) = \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n \quad \text{for every } u \in W^{1,\infty}(\Omega), \quad (0.2)$$

where f^{**} is the convex envelope of f , i.e. the greatest convex function less than or equal to f .

We point out that results in the same spirit hold also in different settings. For example, when f is defined on the set of the $n \times m$ matrices and the elements of U are \mathbb{R}^m -valued, (0.2) still holds provided f^{**} is replaced by the quasiconvex envelope of f (cf. for example [2,11,31]).

In the above results the gradients of the elements of U are allowed to lie in the whole of \mathbb{R}^n without any restriction. When this does not occur, namely when a condition like (unless differently specified, a.e. means \mathcal{L}^n -a.e.)

$$\nabla u(x) \in E \quad \text{for a.e. } x \in \Omega,$$

must be fulfilled by the elements of U for some given subset E of \mathbb{R}^n , the corresponding relaxation processes become pointwise gradient constrained. The treatment of this case can be handled by allowing the value $+\infty$ in the target

space of f . Indeed, in this case the only elements of U that play a role are those that satisfy the following pointwise gradient constraint

$$\nabla u(x) \in \text{dom } f \quad \text{for a.e. } x \in \Omega, \tag{0.3}$$

where $\text{dom } f = \{z \in \mathbb{R}^n : f(z) < +\infty\}$.

Several situations in applications, for example in elastic-plastic torsion theory, in nonlinear elastomers modeling, and in optimal control problems, lead to classes of variational inequalities and of relaxation problems on sets of admissible configurations subject to pointwise gradient constraints of the above type (cf. for example [8,19,27,28,34,35] and the references quoted therein).

It is clear that, in general, (0.3) can be a very restrictive condition, entailing serious technical difficulties and hindering the development of a wide range of results like those described in the unconstrained case. Indeed, few results exist in literature on pointwise gradient constrained relaxation. We quote [20,30,5–7], and the monograph [8] in which additional gradient constrained variational problems are considered as well. In particular, we quote [4,18], and [3] for the treatment of the corresponding homogenization processes. In these papers, under various sets of assumptions on f , and with different choices of U and τ , again formulas similar to (0.2) have been proved, where now, since f takes its values in $[0, +\infty]$, f^{**} is the convex lower semicontinuous envelope of f . In particular, in [5] and [6] these formulas have been extended to BV spaces as well, and some cases in which $\text{dom } f$ has empty interior have been treated. In spite of this, it must be emphasized that all these papers assume the structure condition

$$\text{dom } f \text{ is convex,} \tag{0.4}$$

that however forestalls the approach in this context to the cases in which the gradients of the admissible configurations lie in disconnected or finite sets.

Finally, we point out that recently, in [15] and [17], gradient constrained relaxation processes for Neumann problems have been investigated by allowing a true dependence on the space variable x in f , either under continuity or just measurability assumptions on the multifunction

$$x \mapsto \text{dom } f(x, \cdot), \tag{0.5}$$

but assuming that for a.e. x , $\text{dom } f(x, \cdot)$ is convex, uniformly bounded, and with nonempty interior. In these papers a formula like

$$\overline{F}(u) = \int_{\Omega} f^{**}(x, \nabla u) \, d\mathcal{L}^n \quad \text{for every } u \in W^{1,\infty}(\Omega), \tag{0.6}$$

where for a.e. x , $f^{**}(x, \cdot)$ is the convex lower semicontinuous envelope of $f(x, \cdot)$, has been proved under continuity type assumptions on the multifunction in (0.5), but has been shown to fail under just measurability ones. Nevertheless, in this last case, it must be pointed out that \overline{F} still has an integral form as in (0.6), but with f^{**} replaced by a suitable integrand $\tilde{f} = \tilde{f}(x, z)$ convex and lower semicontinuous in the z variable.

In the present paper we study pointwise gradient constrained relaxation processes for functionals as in (0.1) when assumption (0.4) is dropped.

Actually, very little is known on this problem, and the measure theoretic techniques developed in the above mentioned papers seem not to be well suited for this case. Consequently, we propose an approach based on a new technique, that allows us to treat both the cases of Neumann and Dirichlet problems. More precisely, if $f : \mathbb{R}^n \rightarrow [0, +\infty]$ is Borel and Ω is a convex bounded open subset of \mathbb{R}^n , we prove, in the case of Neumann problems with $U = W^{1,\infty}(\Omega)$ and τ equal to the $L^1(\Omega)$ -topology, that (cf. Theorem 3.10)

$$\overline{F}(u) = \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + \int_{\Omega} (f^{**})^{\infty} \left(\frac{dD^s u}{d|D^s u|} \right) \, d|D^s u| \quad \text{for every } u \in BV(\Omega),$$

where $(f^{**})^{\infty}$ is the recession function of f^{**} (cf. Section 1 for the definition), and, for every $u \in BV(\Omega)$, ∇u is the density of the \mathcal{L}^n -absolutely continuous part of the \mathbb{R}^n -valued measure gradient of u , $D^s u$ is the \mathcal{L}^n -singular part of the gradient of u , $|D^s u|$ is its total variation, and $\frac{dD^s u}{d|D^s u|}$ is the Radon–Nikodym derivative of $D^s u$ with respect to $|D^s u|$.

Of course, the above formula agrees with the above recalled one established under (0.4), but we emphasize that here we do not need to assume any topological or geometrical condition on $\text{dom } f$.

We also observe explicitly that the constraint condition involved in the relaxed problem, at least on Sobolev functions, is given by

$$\nabla u(x) \in \overline{\text{co}(\text{dom } f)} \quad \text{for a.e. } x \in \Omega,$$

where $\text{co}(\text{dom } f)$ is the convex envelope of $\text{dom } f$.

In the case of Dirichlet problems, we first remark that the only nontrivial results occur when $\text{co}(\text{dom } f)$ has nonempty interior (cf. Proposition 4.1). Then we take $z_0 \in \text{int}(\text{co}(\text{dom } f))$, and consider the case in which $U = u_{z_0} + W_0^{1,\infty}(\Omega)$ and τ is again the $L^1(\Omega)$ -topology, where u_{z_0} is the linear function whose gradient is z_0 and $W_0^{1,\infty}(\Omega)$ the set of the Lipschitz functions on Ω whose (unique) extension on $\overline{\Omega}$ is equal to 0 on $\partial\Omega$. We prove that (cf. Theorem 4.5)

$$\overline{F}(u) = \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + \int_{\Omega} (f^{**})^{\infty} \left(\frac{dD^s u}{|dD^s u|} \right) \, d|D^s u| + \int_{\partial\Omega} (f^{**})^{\infty} ((u_{z_0} - u)\mathbf{n}_{\Omega}) \, d\mathcal{H}^{n-1}$$

for every $u \in BV(\Omega)$,

where \mathbf{n}_{Ω} is the unit outward normal to $\partial\Omega$ and \mathcal{H}^{n-1} the $(n-1)$ -dimensional Hausdorff measure.

To prove the above results, in both the cases for U , we proceed by means of successive representations of \overline{F} on wider and wider function classes, starting from the one of affine configurations. The main novelty of the paper is just in the techniques introduced to represent \overline{F} on such space and on the one of piecewise affine functions. By improving an idea from [24], we are able to approximate every linear function u_z by means of a sequence of functions $\{u_h\}$ whose gradients take only a finite number of values in $\text{dom } f$ and such that $\lim_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h) \, d\mathcal{L}^n \leq f^{**}(z) \mathcal{L}^n(\Omega)$ (cf. Proposition 3.5). A refinement of this result also provides a tool to replace the boundary datum of a function with a prescribed linear one without perturbing too much the corresponding energy, thus allowing the treatment of Dirichlet problems. It also generalizes the so called zig–zag lemma (cf. [14]) to the case of convex combinations of more than two vectors. The representation of \overline{F} on piecewise affine functions is then deduced by means of a structure result establishing that every piecewise affine function $u = \sum_j (u_{z_j} + s_j) \chi_{P_j}$ can be expressed at each point of a convex open set Ω as a maximum of minima of some of its components $u_{z_j} + s_j$ whose corresponding P_j have a nonempty intersection with Ω (cf. Theorem 2.1). Finally, the representation on BV spaces is achieved by means of suitable approximation processes, and of a general inner regularity result for abstract functionals (cf. Proposition 1.7).

We point out that pointwise gradient constrained relaxation problems are related to first order differential inclusions and Hamilton–Jacobi equations (cf. [29,13], and also [12] where existence results of a.e. solutions of differential inclusions are proved without assuming convexity hypotheses on the inclusion sets). In fact (for simplicity we discuss only the particular case of Sobolev functions), when applied to $f = I_E$, where E is a Borel subset of \mathbb{R}^n and I_E is its indicator function defined as $I_E(z) = 0$ if $z \in E$ and $I_E(z) = +\infty$ if $z \in \mathbb{R}^n \setminus E$, our results imply that for every $u \in W^{1,1}(\Omega)$ satisfying $\nabla u(x) \in \overline{\text{co}(E)}$ for a.e. $x \in \Omega$, there exists $\{u_h\}$ in $W^{1,\infty}(\Omega)$ such that $u_h \rightarrow u$ in $L^1(\Omega)$, and $\nabla u_h(x) \in E$ for every $h \in \mathbb{N}$ and a.e. $x \in \Omega$ (cf. Corollary 3.12). In addition, if $\text{int}(\text{co}(E)) \neq \emptyset$, $z_0 \in \text{int}(\text{co}(E))$, and the u above is in $u_{z_0} + W_0^{1,1}(\Omega)$, then $\{u_h\}$ can be taken in $u_{z_0} + W_0^{1,\infty}(\Omega)$ (cf. Corollary 4.7).

In both these results $\{u_h\}$ can be any sequence of solutions of the differential inclusion $\nabla v \in E$ a.e. in Ω , possibly satisfying a boundary condition. Conversely, we emphasize that, when f is not merely an indicator function, if $u \in W^{1,1}(\Omega)$ satisfies $\int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n < +\infty$, an additional difficulty occurs in the selection of the optimal sequences $\{u_h\}$, converging to u in $L^1(\Omega)$, provided by Theorem 3.10. Indeed, beside the differential inclusion $\nabla u_h(x) \in \text{dom } f$ for every $h \in \mathbb{N}$ and a.e. $x \in \Omega$, they must satisfy also the additional minimality condition expressed by the convergence of $\{\int_{\Omega} f(\nabla u_h) \, d\mathcal{L}^n\}$ to $\int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n$. Analogous remarks hold in the case of Dirichlet problems and Theorem 4.5 as well.

Eventually, we observe that our results are connected to those of the recent papers [25] and [26], where a relaxation phenomenon for Hamilton–Jacobi equations is pointed out by showing that the pointwise supremum of certain a.e. subsolutions of a Hamilton–Jacobi equation yields a viscosity solution of the corresponding convexified equation.

1. Recalls and preliminary results

In the present section we recall some notions, and prove some preliminary result, needed in the paper. Eventually, we establish some notations on the relaxed functionals we will be concerned with.

1.1. Recalls of convex analysis

We recall here some basics of convex analysis. We refer to [32] and [33] for a more complete exposition of the matter.

For a given $S \subseteq \mathbb{R}^n$ we denote by $\text{aff}(S)$ the affine hull of S , defined as the intersection of all the affine sets containing S . It is clear that $\text{aff}(S)$ is the smallest affine set containing S .

For every $S \subseteq \mathbb{R}^n$ we denote by $\text{co}(S)$ the convex hull of S , i.e. the intersection of all the convex subsets of \mathbb{R}^n containing S . It is clear that $\text{co}(S)$ is the smallest convex set containing S .

If $C \subseteq \mathbb{R}^n$ is convex, we denote by $\text{ri}(C)$ the relative interior of C , i.e. the set of the interior points of C , in the topology of $\text{aff}(C)$, once we regard it as a subspace of $\text{aff}(C)$. We recall that $\text{ri}(C) \neq \emptyset$ provided $C \neq \emptyset$. When $\text{aff}(C) = \mathbb{R}^n$ we write as usual $\text{ri}(C) = \text{int}(C)$. Moreover, we also recall that

$$tz + (1 - t)z_0 \in \text{ri}(C) \quad \text{whenever } z_0 \in \text{ri}(C), z \in \overline{C}, \text{ and } t \in]0, 1[. \tag{1.1}$$

For $\nu \in \{0, \dots, n\}$ and $z_0, \dots, z_\nu \in \mathbb{R}^n$, we say that z_0, \dots, z_ν are affinely independent if the dimension of $\text{aff}(\{z_0, \dots, z_\nu\})$ is ν . We recall that if z_0, \dots, z_ν are affinely independent, then the expression of each element of $\text{co}(\{z_0, \dots, z_\nu\})$ as a convex combination of z_0, \dots, z_ν is unique. In addition, if $\nu = n$, then $\text{int}(\text{co}(\{z_0, \dots, z_n\})) \neq \emptyset$ and

$$\text{int}(\text{co}(\{z_0, \dots, z_n\})) = \left\{ \sum_{j=0}^n t_j z_j : t_j \in]0, 1[\text{ for every } j \in \{0, \dots, n\}, \sum_{j=0}^n t_j = 1 \right\}. \tag{1.2}$$

A subset P of \mathbb{R}^n is said to be a polyhedral set if it is the intersection of a finite family of closed half-spaces. Clearly, a polyhedral set is closed and convex. Moreover, a bounded polyhedral set turns out to be the convex envelope of finitely many of its points.

For every $C \subseteq \mathbb{R}^n$ with $0 \in C$, the polar C° of C is defined by

$$C^\circ = \{x \in \mathbb{R}^n : z \cdot x \leq 1 \text{ for every } z \in C\}.$$

It is clear that polar sets are closed and convex. The result below describes some of their additional properties (cf. 11.20 Exercise in [33]).

Proposition 1.1. *Let $C \subseteq \mathbb{R}^n$ be closed, convex, and with $0 \in C$. Then*

C is bounded if and only if $0 \in \text{int}(C^\circ)$,

C° is bounded if and only if $0 \in \text{int}(C)$,

and

C is a polyhedral set if and only if so is C° .

Eventually, we recall that for every $C \subseteq \mathbb{R}^n$, the support function σ_C of C is defined by

$$\sigma_C : x \in \mathbb{R}^n \mapsto \sup\{z \cdot x : z \in C\}.$$

It is clear that support functions are convex, lower semicontinuous, and positively 1-homogeneous. Moreover, if $C \subseteq \mathbb{R}^n$ satisfies $0 \in C$, it is easy to verify that

$$0 \leq \sigma_C(x) \leq 1 \quad \text{for every } x \in C^\circ, \tag{1.3}$$

and, by using Proposition 1.1, that

$$\sigma_C(x) = 1 \quad \text{for every } x \in \partial C^\circ, \text{ provided } C \text{ is compact and convex.} \tag{1.4}$$

Let $f: \mathbb{R}^n \rightarrow [0, +\infty]$ be convex. Then it is well known that $\text{dom } f$ is convex, that f is lower semicontinuous in $\text{ri}(\text{dom } f)$, and that the restriction of f to $\text{ri}(\text{dom } f)$ is continuous. In particular, if $\text{int}(\text{dom } f) \neq \emptyset$, then f is continuous in $\text{int}(\text{dom } f)$.

For every $f: \mathbb{R}^n \rightarrow [0, +\infty]$ we denote by $\text{co } f$ the convex envelope of f , i.e. the function

$$\text{co } f: z \in \mathbb{R}^n \mapsto \sup \{ \phi(z): \phi: \mathbb{R}^n \rightarrow [0, +\infty] \text{ convex, } \phi(\zeta) \leq f(\zeta) \text{ for every } \zeta \in \mathbb{R}^n \}.$$

Clearly, $\text{co } f$ is convex, and $\text{co } f(z) \leq f(z)$ for every $z \in \mathbb{R}^n$. Consequently, $\text{co } f$ turns out to be the greatest convex function on \mathbb{R}^n less than or equal to f . The representation result below comes from Carathéodory Theorem (cf. Corollary 17.1.3 in [32]).

Theorem 1.2. *Let $f: \mathbb{R}^n \rightarrow [0, +\infty]$. Then, for every $z \in \mathbb{R}^n$,*

$$\text{co } f(z) = \inf \sum_{j=0}^v t_j f(z_j),$$

where the infimum is taken over all the $v \in \{0, \dots, n\}$, $z_0, \dots, z_v \in \mathbb{R}^n$, and $t_0, \dots, t_v \in]0, 1]$ such that z_0, \dots, z_v are affinely independent, $\sum_{j=0}^v t_j = 1$, and $\sum_{j=0}^v t_j z_j = z$.

For every $f: \mathbb{R}^n \rightarrow [0, +\infty]$ we denote by f^{**} the convex lower semicontinuous envelope of f , i.e. the function defined by

$$f^{**}: z \in \mathbb{R}^n \mapsto \sup \{ \phi(z): \phi: \mathbb{R}^n \rightarrow [0, +\infty] \text{ convex and lower semicontinuous, } \phi(\zeta) \leq f(\zeta) \text{ for every } \zeta \in \mathbb{R}^n \}.$$

Clearly, f^{**} is convex and lower semicontinuous, and $f^{**}(z) \leq f(z)$ for every $z \in \mathbb{R}^n$. Consequently, f^{**} turns out to be the greatest convex lower semicontinuous function on \mathbb{R}^n less than or equal to f .

Proposition 1.3. *Let $f: \mathbb{R}^n \rightarrow [0, +\infty]$. Then $\text{ri}(\text{dom } f^{**}) = \text{ri}(\text{dom}(\text{co } f)) = \text{ri}(\text{co}(\text{dom } f))$, and*

$$f^{**}(z) = \text{co } f(z) \quad \text{for every } z \in \text{ri}(\text{co}(\text{dom } f)) \cup (\mathbb{R}^n \setminus \overline{\text{co}(\text{dom } f)}).$$

We now define recession functions. To do it properly, we recall that for a given $f: \mathbb{R}^n \rightarrow [0, +\infty]$ convex, and $z_0 \in \text{dom } f$, the limit $\lim_{t \rightarrow +\infty} (f(z_0 + tz) - f(z_0))/t$ exists for every $z \in \mathbb{R}^n$. Therefore we define the recession function of f by

$$f^\infty: z \in \mathbb{R}^n \mapsto \lim_{t \rightarrow +\infty} \frac{f(z_0 + tz) - f(z_0)}{t}.$$

It is well known that f^∞ is positively 1-homogeneous, and that, if in addition f is also lower semicontinuous, then the definition of f^∞ does not depend on z_0 when it varies in $\text{dom } f$.

1.2. Lower semicontinuity results in BV spaces

Let Ω be an open subset of \mathbb{R}^n . By $BV(\Omega)$ we denote the set of the functions in $L^1(\Omega)$ having distributional partial derivatives that are Borel measures with bounded total variation in Ω . $BV(\Omega)$ is a Banach space with the norm $u \in BV(\Omega) \mapsto \|u\|_{L^1(\Omega)} + |Du|(\Omega)$, where, for every $u \in BV(\Omega)$, Du denotes the \mathbb{R}^n -valued measure gradient of u , and $|Du|$ its total variation. We refer, for example, to [36] for a complete treatment of BV spaces.

If Ω has Lipschitz boundary, then functions in $BV(\Omega)$ have traces on $\partial\Omega$ in the sense that for every $u \in BV(\Omega)$ there exists an element in $L^1(\partial\Omega)$, still denoted by u , such that

$$\int_{\Omega} u \operatorname{div} \varphi \, d\mathcal{L}^n = - \int_{\Omega} \varphi \cdot dDu + \int_{\partial\Omega} \varphi \cdot \mathbf{n}_{\Omega} u \, d\mathcal{H}^{n-1} \quad \text{for every } \varphi \in (C^1(\mathbb{R}^n))^n.$$

We also recall that, if Ω' is another open subset of \mathbb{R}^n such that $\overline{\Omega} \subseteq \Omega'$, and $v \in BV(\Omega' \setminus \overline{\Omega})$, then the function w , defined a.e. in Ω' by $w = u$ in Ω and $w = v$ in $\Omega' \setminus \overline{\Omega}$, is in $BV(\Omega')$, and

$$Dw(E) = \int_E (v - u) \mathbf{n}_\Omega \, d\mathcal{H}^{n-1} \quad \text{for every Borel set } E \subseteq \partial\Omega. \tag{1.5}$$

By $BV_{\text{loc}}(\mathbb{R}^n)$ we denote the set of the functions in $L^1_{\text{loc}}(\mathbb{R}^n)$ that are in $BV(\Omega)$ for every bounded open set Ω . We recall that $BV_{\text{loc}}(\mathbb{R}^n)$ is a Fréchet space.

By $\mathcal{A}_0(\mathbb{R}^n)$ we denote the set of the bounded open subsets of \mathbb{R}^n .

If $f : \mathbb{R}^n \rightarrow [0, +\infty]$ is convex and lower semicontinuous, we define the functional F_f as

$$F_f : (\Omega, u) \in \mathcal{A}_0(\mathbb{R}^n) \times BV_{\text{loc}}(\mathbb{R}^n) \mapsto \int_\Omega f(\nabla u) \, d\mathcal{L}^n + \int_\Omega f^\infty \left(\frac{dD^s u}{|D^s u|} \right) d|D^s u|. \tag{1.6}$$

It is clear that

$$F_f(\Omega, u) = \int_\Omega f(\nabla u) \, d\mathcal{L}^n \quad \text{for every } (\Omega, u) \in \mathcal{A}_0(\mathbb{R}^n) \times W^{1,1}_{\text{loc}}(\mathbb{R}^n). \tag{1.7}$$

The following lower semicontinuity property holds (cf. for example Theorem 7.4.6 in [8]).

Proposition 1.4. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be convex and lower semicontinuous, and let F_f be defined in (1.6). Then, for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$, $F_f(\Omega, \cdot)$ is $L^1_{\text{loc}}(\Omega)$ -lower semicontinuous.*

Analogously, if f is as above, $z_0 \in \text{dom } f$, and $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ has Lipschitz boundary, we define $F_f(u_{z_0}, \Omega, \cdot)$ as

$$F_f(u_{z_0}, \Omega, \cdot) : u \in BV_{\text{loc}}(\mathbb{R}^n) \mapsto \int_\Omega f(\nabla u) \, d\mathcal{L}^n + \int_\Omega f^\infty \left(\frac{dD^s u}{|D^s u|} \right) d|D^s u| + \int_{\partial\Omega} f^\infty((u_{z_0} - u) \mathbf{n}_\Omega) \, d\mathcal{H}^{n-1}. \tag{1.8}$$

It is clear that

$$F_f(u_{z_0}, \Omega, u) = \int_\Omega f(\nabla u) \, d\mathcal{L}^n \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n) \text{ with Lipschitz boundary, } u \in u_{z_0} + W^{1,1}_0(\Omega). \tag{1.9}$$

Since (1.5), and the 1-homogeneity of f^∞ imply

$$F_f(u_{z_0}, \Omega, u) = F_f(\Omega', u) - f(z_0) \mathcal{L}^n(\Omega' \setminus \Omega) \quad \text{whenever } z_0 \in \text{dom } f, \Omega \in \mathcal{A}_0(\mathbb{R}^n) \text{ has Lipschitz boundary, } \Omega' \in \mathcal{A}_0(\mathbb{R}^n) \text{ satisfies } \overline{\Omega} \subseteq \Omega', \text{ and } u \in BV(\Omega') \text{ is such that } u = u_{z_0} \text{ a.e. in } \Omega' \setminus \Omega,$$

the lower semicontinuity result below follows from Proposition 1.4.

Proposition 1.5. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be convex and lower semicontinuous, and let $z_0 \in \text{dom } f$. Let $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ have Lipschitz boundary, and let $F_f(u_{z_0}, \Omega, \cdot)$ be defined in (1.8). Then $F_f(u_{z_0}, \Omega, \cdot)$ is $L^1(\Omega)$ -lower semicontinuous.*

Let B_1 be the unit open ball of \mathbb{R}^n centred in 0, and let α be a symmetric mollifier, namely a nonnegative function in $C^\infty_0(B_1)$, symmetric with respect to 0, and such that $\int_{B_1} \alpha \, d\mathcal{L}^n = 1$. For every $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\eta > 0$, and $x \in \mathbb{R}^n$, let us denote by $u_\eta(x)$ the regularization of u at x defined by

$$u_\eta(x) = \frac{1}{\eta^n} \int_{\mathbb{R}^n} \alpha \left(\frac{x - y}{\eta} \right) u(y) \, dy.$$

It is well known that for every $\eta > 0$, $u_\eta \in C^\infty(\mathbb{R}^n)$, and that $u_\eta \rightarrow u$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ as $\eta \rightarrow 0$.

The following approximation in energy result for F_f holds (cf. Lemma 7.4.4 in [8]). In it and in the following, for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ and $\eta > 0$, we set $\Omega^-_\eta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \eta\}$.

Proposition 1.6. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be convex and lower semicontinuous, and let F_f be defined in (1.6). Then*

$$F_f(\Omega^-_\eta, u_\eta) \leq F_f(\Omega, u) \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n), u \in BV_{\text{loc}}(\mathbb{R}^n), \text{ and } \eta > 0.$$

1.3. Measure theoretic preliminaries

Let $\vartheta : \mathcal{A}_0(\mathbb{R}^n) \rightarrow [0, +\infty]$. We say that ϑ is increasing if

$$\vartheta(\Omega_1) \leq \vartheta(\Omega_2) \quad \text{for every } \Omega_1, \Omega_2 \in \mathcal{A}_0(\mathbb{R}^n) \text{ such that } \Omega_1 \subseteq \Omega_2.$$

We denote by ϑ_- the inner regular envelope of ϑ defined by

$$\vartheta_- : \Omega \in \mathcal{A}_0(\mathbb{R}^n) \mapsto \sup\{\vartheta(A) : A \in \mathcal{A}_0(\mathbb{R}^n), \bar{A} \subseteq \Omega\},$$

and say that ϑ is inner regular in $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ if

$$\vartheta(\Omega) = \vartheta_-(\Omega).$$

It is clear that ϑ_- is increasing, and that it is inner regular in Ω for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$.

Hereafter, if U is a set, $\Phi : \mathcal{A}_0(\mathbb{R}^n) \times U \rightarrow [0, +\infty]$, and $u \in U$, we denote by $\Phi_-(\cdot, u)$ the inner regular envelope of $\Phi(\cdot, u)$, namely the function defined, for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$, by $\Phi_-(\Omega, u) = \Phi_-(\cdot, u)_-(\Omega)$.

For every $S \subseteq \mathbb{R}^n$, every function u on S , $x_0 \in \mathbb{R}^n$, and $t > 0$, we define $u_{x_0,t}$ as

$$u_{x_0,t} : x \in x_0 + \frac{1}{t}(S - x_0) \mapsto \frac{1}{t}u(x_0 + t(x - x_0)). \tag{1.10}$$

The following inner regularity criterion is proved, also in a more general setting, in Proposition 2.7.4 of [8].

Proposition 1.7. *Let $\Phi : \mathcal{A}_0(\mathbb{R}^n) \times BV_{\text{loc}}(\mathbb{R}^n) \rightarrow [0, +\infty]$ be such that for every $u \in BV_{\text{loc}}(\mathbb{R}^n)$, $\Phi(\cdot, u)$ is increasing. Let $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ be convex, $u \in BV_{\text{loc}}(\mathbb{R}^n)$, $x_0 \in \Omega$, and assume that*

$$\begin{aligned} \liminf_{t \rightarrow 1^-} \Phi(\Omega, u_{x_0,t}) &\geq \Phi(\Omega, u), \\ \limsup_{t \rightarrow 1^+} \Phi_-(x_0 + t(\Omega - x_0), u_{x_0,1/t}) &\leq \Phi_-(\Omega, u). \end{aligned}$$

Then

$$\Phi(\Omega, u) = \Phi_-(\Omega, u).$$

The following paving result comes from the Vitali Covering Theorem.

Proposition 1.8. *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let K be a compact subset of \mathbb{R}^n such that $0 \in K$ and $\mathcal{L}^n(K) > 0$. Then there exist $\{x_h\} \subseteq \Omega$ and $\{t_h\} \subseteq]0, 1]$ such that the sets $\{x_h + t_h K : h \in \mathbb{N}\}$ are contained in Ω , pairwise disjoint, and*

$$\mathcal{L}^n\left(\Omega \setminus \bigcup_{h \in \mathbb{N}} (x_h + t_h K)\right) = 0.$$

Proof. The Vitali Covering Theorem (cf. for example Corollary 2 at page 28 of [21]) provides a sequence $\{C_h\}$ of pairwise disjoint closed balls in Ω with radius less than or equal to $\text{diam}(K)$, and $\mathcal{L}^n(\Omega \setminus \bigcup_{h \in \mathbb{N}} C_h) = 0$. Let $\{B_h\}$ the sequence of the corresponding open balls, then it is easy to verify that also $\mathcal{L}^n(\Omega \setminus \bigcup_{h \in \mathbb{N}} B_h) = 0$. Let B be an open ball centred in 0 and with radius strictly greater than $\text{diam}(K)$. Then, if for every $h \in \mathbb{N}$, x_h^1 is the centre of B_h and r_h^1 is its radius divided by the radius of B , we deduce that $\{r_h^1\} \subseteq]0, 1]$, and, since $0 \in K$, that $x_h^1 + r_h^1 K \subseteq x_h^1 + r_h^1 B = B_h$ for every $h \in \mathbb{N}$. Because of this, the sets $\{x_h^1 + r_h^1 K : h \in \mathbb{N}\}$ too are pairwise disjoint, and

$$\begin{aligned} \mathcal{L}^n\left(\Omega \setminus \bigcup_{h \in \mathbb{N}} (x_h^1 + r_h^1 K)\right) &= \mathcal{L}^n\left(\left(\bigcup_{h \in \mathbb{N}} B_h\right) \setminus \left(\bigcup_{h \in \mathbb{N}} (x_h^1 + r_h^1 K)\right)\right) = \sum_{h \in \mathbb{N}} \mathcal{L}^n((x_h^1 + r_h^1 B) \setminus (x_h^1 + r_h^1 K)) \\ &= \frac{\mathcal{L}^n(B \setminus K)}{\mathcal{L}^n(B)} \sum_{h \in \mathbb{N}} (r_h^1)^n \mathcal{L}^n(B) = \frac{\mathcal{L}^n(B \setminus K)}{\mathcal{L}^n(B)} \mathcal{L}^n(\Omega). \end{aligned}$$

Now, let us set $\Omega_1 = (\bigcup_{h \in \mathbb{N}} B_h) \setminus \bigcup_{h \in \mathbb{N}} (x_h^1 + r_h^1 K)$. Then, since $\Omega_1 = \bigcup_{h \in \mathbb{N}} (B_h \setminus (x_h^1 + r_h^1 K))$, Ω_1 turns out to be open. Consequently, we can repeat the above argument starting from Ω_1 in place of Ω , thus getting $\{x_h^2\} \subseteq \Omega_1$ and $\{r_h^2\} \subseteq]0, 1]$ such that the sets $\{x_h^i + r_h^i K : i \in \{1, 2\}, h \in \mathbb{N}\}$ are pairwise disjoint,

$$\begin{aligned} \mathcal{L}^n\left(\Omega_1 \setminus \bigcup_{h \in \mathbb{N}} (x_h^2 + r_h^2 K)\right) &= \frac{\mathcal{L}^n(B \setminus K)}{\mathcal{L}^n(B)} \mathcal{L}^n(\Omega_1) = \frac{\mathcal{L}^n(B \setminus K)}{\mathcal{L}^n(B)} \mathcal{L}^n\left(\left(\bigcup_{h \in \mathbb{N}} B_h\right) \setminus \bigcup_{h \in \mathbb{N}} (x_h^1 + r_h^1 K)\right) \\ &= \frac{\mathcal{L}^n(B \setminus K)}{\mathcal{L}^n(B)} \mathcal{L}^n\left(\Omega \setminus \bigcup_{h \in \mathbb{N}} (x_h^1 + r_h^1 K)\right) = \left(\frac{\mathcal{L}^n(B \setminus K)}{\mathcal{L}^n(B)}\right)^2 \mathcal{L}^n(\Omega), \end{aligned}$$

and

$$\mathcal{L}^n\left(\Omega \setminus \bigcup_{i \in \{1,2\}} \bigcup_{h \in \mathbb{N}} (x_h^i + r_h^i K)\right) = \mathcal{L}^n\left(\Omega_1 \setminus \bigcup_{h \in \mathbb{N}} (x_h^2 + r_h^2 K)\right) = \left(\frac{\mathcal{L}^n(B \setminus K)}{\mathcal{L}^n(B)}\right)^2 \mathcal{L}^n(\Omega).$$

By iterating the above argument, we obtain that for every $m \in \mathbb{N}$ there exist $\{x_h^m\} \subseteq \Omega$ and $\{r_h^m\} \subseteq]0, 1]$ such that the sets $\{x_h^i + r_h^i K : i \in \{1, \dots, m\}, h \in \mathbb{N}\}$ are pairwise disjoint, and

$$\mathcal{L}^n\left(\Omega \setminus \bigcup_{i \in \{1, \dots, m\}} \bigcup_{h \in \mathbb{N}} (x_h^i + r_h^i K)\right) = \left(\frac{\mathcal{L}^n(B \setminus K)}{\mathcal{L}^n(B)}\right)^m \mathcal{L}^n(\Omega).$$

Because of this, we conclude that the sets $\{x_h^i + r_h^i K : i \in \mathbb{N}, h \in \mathbb{N}\}$ satisfy the properties required in the proposition. \square

1.4. Relaxed functionals

Here we define the relaxed functionals that we will consider in this the paper.

For every $f : \mathbb{R}^n \rightarrow [0, +\infty]$ Borel, $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$, and $u_0 \in W^{1,\infty}(\Omega)$, we define

$$\bar{F}(\Omega, \cdot) : u \in L^1(\Omega) \mapsto \min \left\{ \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h) \, d\mathcal{L}^n : \{u_h\} \subseteq W^{1,\infty}(\Omega), u_h \rightarrow u \text{ in } L^1(\Omega) \right\} \quad (1.11)$$

and, in order to analyze Dirichlet type problems,

$$\bar{F}(u_0, \Omega, \cdot) : u \in L^1(\Omega) \mapsto \min \left\{ \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h) \, d\mathcal{L}^n : \{u_h\} \subseteq u_0 + W_0^{1,\infty}(\Omega), u_h \rightarrow u \text{ in } L^1(\Omega) \right\}, \quad (1.12)$$

where the minima are trivially attained, since the $L^1(\Omega)$ -topology is metric.

For technical reasons, and to deduce sharper estimates as well, we need to introduce the two additional relaxed functionals below, analogous to the above ones, but in the uniform convergence topology. To this aim, when no confusion occurs, we denote, for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$, by $C^0(\bar{\Omega})$ both the space of the continuous functions on $\bar{\Omega}$ and the usual topology of the uniform convergence on $\bar{\Omega}$. We define

$$\bar{G}(\Omega, \cdot) : u \in C^0(\bar{\Omega}) \mapsto \min \left\{ \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h) \, d\mathcal{L}^n : \{u_h\} \subseteq W^{1,\infty}(\Omega), u_h \rightarrow u \text{ in } C^0(\bar{\Omega}) \right\} \quad (1.13)$$

and

$$\bar{G}(u_0, \Omega, \cdot) : u \in C^0(\bar{\Omega}) \mapsto \min \left\{ \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h) \, d\mathcal{L}^n : \{u_h\} \subseteq u_0 + W_0^{1,\infty}(\Omega), u_h \rightarrow u \text{ in } C^0(\bar{\Omega}) \right\}. \quad (1.14)$$

We recall that, for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ and $u_0 \in W^{1,\infty}(\Omega)$, $\bar{F}(\Omega, \cdot)$ and $\bar{F}(u_0, \Omega, \cdot)$ are $L^1(\Omega)$ -lower semicontinuous, and that $\bar{G}(\Omega, \cdot)$ and $\bar{G}(u_0, \Omega, \cdot)$ are $C^0(\bar{\Omega})$ -lower semicontinuous. Obviously,

$$\bar{F}(\Omega, u) \leq \bar{G}(\Omega, u), \quad \bar{F}(u_0, \Omega, u) \leq \bar{G}(u_0, \Omega, u) \quad \text{for every } u \in C^0(\bar{\Omega}). \quad (1.15)$$

Finally, we observe that (1.7) and Proposition 1.4 provide

$$\int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + \int_{\Omega} (f^{**})^{\infty} \left(\frac{dD^s u}{|dD^s u|} \right) |dD^s u| \leq \bar{F}(\Omega, u) \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n), u \in BV(\Omega), \quad (1.16)$$

whilst (1.9) and Proposition 1.5 imply

$$\int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + \int_{\Omega} (f^{**})^{\infty} \left(\frac{dD^s u}{|dD^s u|} \right) d|D^s u| + \int_{\partial\Omega} (f^{**})^{\infty}((u_{z_0} - u)\mathbf{n}_{\Omega}) \, d\mathcal{H}^{n-1} \leq \bar{F}(u_{z_0}, \Omega, u)$$

for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ with Lipschitz boundary, $u \in BV(\Omega)$. (1.17)

2. A representation result for piecewise affine functions

For every $E \subseteq \mathbb{R}^n$, we denote by χ_E the characteristic function of E defined as $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(z) = 0$ if $x \in \mathbb{R}^n \setminus E$.

Let u be a continuous function on \mathbb{R}^n . We say that u is piecewise affine if

$$u(x) = \sum_{j=1}^m (u_{z_j}(x) + s_j) \chi_{P_j}(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{2.1}$$

for some $m \in \mathbb{N}$, $z_1, \dots, z_m \in \mathbb{R}^n$, $s_1, \dots, s_m \in \mathbb{R}$, and some polyhedral sets P_1, \dots, P_m with nonempty pairwise disjoint open interiors such that $\bigcup_{j=1}^m P_j = \mathbb{R}^n$. We observe explicitly that the presence of the a.e. requirement in (2.1) is simply due to the closedness of polyhedral sets. Actually (2.1) holds for every x in $\bigcup_{j=1}^m \text{int}(P_j)$.

We denote by $PA(\mathbb{R}^n)$ the set of the piecewise affine functions.

In the theorem below we prove that every piecewise affine function u as in (2.1) can be represented on a convex open set Ω as a maximum of minima of some of its components $u_{z_j} + s_j$ for which $\text{int}(P_j) \cap \Omega \neq \emptyset$. The result has been already proved in [4] in the framework of homogenization problems. Here we propose a more direct and elementary proof.

Given $x_1, x_2 \in \mathbb{R}^n$, we set $[x_1, x_2] = \{tx_1 + (1 - t)x_2 : t \in [0, 1]\}$, and so on for $]x_1, x_2[$, $[x_1, x_2[$, $]x_1, x_2]$.

Theorem 2.1. *Let $u = \sum_{j=1}^m (u_{z_j} + s_j) \chi_{P_j}$ be in $PA(\mathbb{R}^n)$, and let Ω be a convex open subset of \mathbb{R}^n . Then there exist $k \in \mathbb{N}$ and $N_1, \dots, N_k \subseteq \{j \in \{1, \dots, m\} : \text{int}(P_j) \cap \Omega \neq \emptyset\}$ such that*

$$u(x) = \max_{i \in \{1, \dots, k\}} \min_{j \in N_i} (u_{z_j}(x) + s_j) \quad \text{for every } x \in \Omega.$$

Proof. For simplicity, let us assume that for some $m_{\Omega} \in \{1, \dots, m\}$, $\text{int}(P_j) \cap \Omega \neq \emptyset$ if and only if $j \in \{1, \dots, m_{\Omega}\}$. Let us denote by I the set of the indexes in $\{1, \dots, m_{\Omega}\}$ corresponding to the truly different components of u living in Ω , namely $I = \{1\} \cup \{j \in \{2, \dots, m_{\Omega}\} : z_j \neq z_i \text{ or } s_j \neq s_i \text{ for every } i \in \{1, \dots, j - 1\}\}$, and let A be the subset of Ω where such components are different, i.e. $A = \Omega \setminus \{x \in \Omega : \text{there exist } i, j \in I \text{ with } i \neq j \text{ and } u_{z_i}(x) + s_i = u_{z_j}(x) + s_j\}$. It is clear that A is open, dense in Ω , and that it possesses a finite number of connected components, say A_1, \dots, A_k . We observe that for every $i \in \{1, \dots, k\}$, A_i is open, that u turns out to be affine in A_i , and that

$$\begin{aligned} &\text{for every } p, q \in I \text{ with } p \neq q \text{ it results that } u_{z_p}(x) + s_p > u_{z_q}(x) + s_q \text{ for every } x \in A_i, \\ &\text{or that } u_{z_p}(x) + s_p < u_{z_q}(x) + s_q \text{ for every } x \in A_i. \end{aligned} \tag{2.2}$$

Let us prove that for every $i \in \{1, \dots, k\}$ there exists $N_i \subseteq I$ such that, if

$$v_i : x \in \mathbb{R}^n \mapsto \min_{j \in N_i} (u_{z_j}(x) + s_j),$$

then

$$v_i(x) = u(x) \quad \text{for every } x \in \overline{A_i}, \tag{2.3}$$

and

$$v_i(x) \leq u(x) \quad \text{for every } x \in \Omega. \tag{2.4}$$

To do this, let $i \in \{1, \dots, k\}$, and let us define

$$N_i = \{j \in I : u_{z_j}(x) + s_j \geq u(x) \text{ for every } x \in A_i\}.$$

Then, since u is affine in A_i , it is clear that (2.3) holds.

To prove (2.4), we take for the moment $x \in A \setminus \overline{A_i}$, and fix $x_0 \in A_i$. Then (2.3) implies that $\{y \in [x_0, x]: v_i \leq u \text{ in } [x_0, y]\}$ is nonempty, so that it possesses a point that is the farthest from x_0 . Call x_1 such point, and observe that, by the continuity of v_i and of u ,

$$v_i(x_1) = u(x_1).$$

Moreover, by (2.3), $x_1 \notin A_i$ and, in particular, $x_1 \neq x_0$. If

$$v_i(x) > u(x), \tag{2.5}$$

x_1 turns out to be different from x , hence there exists $x_2 \in]x_1, x[$ such that

$$v_i(x_2) > u(x_2).$$

Now, we observe that the convexity of Ω implies $[x_0, x] \subseteq \Omega$, so that $x_2 \in \Omega$. Moreover, since $x \in A$ and $\Omega \setminus A$ is made up by the intersection of a finite numbers of hyperplanes with Ω , x_2 too can be taken in A and so close to x_1 to belong to the same connected component of A whose closure contains x_1 . In such connected component, $u = u_{z_l} + s_l$ for some $l \in I$, thus we have that $v_i(x_1) = u_{z_l}(x_1) + s_l$ and $v_i(x_2) > u_{z_l}(x_2) + s_l$. Consequently, the concavity of v_i implies that $v_i < u_{z_l} + s_l$ in $[x_0, x_1[$, and, since $x_1 \neq x_0$, that

$$v_i(x_0) < u_{z_l}(x_0) + s_l.$$

Once we recall that u is affine in A_i , (2.2), the previous inequality, and (2.3) imply that $u < u_{z_l} + s_l$ in A_i , and hence, by the definition of v_i , that

$$v_i(x_2) \leq u_{z_l}(x_2) + s_l = u(x_2).$$

This yields a contradiction, since $v_i(x_2) > u(x_2)$. Consequently, (2.5) cannot hold, and, by using also (2.3), we conclude that $v_i(x) \leq u(x)$ for every $x \in A$. Because of this, and of the continuity of v_i and of u , we deduce (2.4).

By (2.3) and (2.4), the theorem follows. \square

We conclude this section by recalling that Example 2.2 in [4] shows that, in general, in Theorem 2.1 the convexity assumption on Ω cannot be dropped, and not even replaced by a connectedness one.

3. Relaxation of gradient constrained Neumann problems

Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel. In this section we prove the representation result for the functional \overline{F} in (1.11).

We start with some preparatory convex analysis lemmas. Since the first two need to be established in an \mathbb{R}^v -space, with v not necessarily equal to n , the symbol ∇ and the expression a.e. used there will be referred to the \mathbb{R}^v -environment and to \mathcal{L}^v respectively.

Lemma 3.1. *Let $v \in \mathbb{N}$, let $\zeta_0, \dots, \zeta_v \in \mathbb{R}^v$ be affinely independent, let $t_0, \dots, t_v \in]0, 1[$ be such that $\sum_{j=0}^v t_j = 1$ and $\sum_{j=0}^v t_j \zeta_j = 0$, and let $\omega \in \mathcal{A}_0(\mathbb{R}^v)$. Then there exists $v \in W_0^{1,\infty}(\omega)$ such that $-1 \leq v(x) < 0$ and $\nabla v(x) \in \{\zeta_0, \dots, \zeta_v\}$ for a.e. $x \in \omega$, and $\mathcal{L}^v(\{x \in \omega: \nabla v(x) = \zeta_j\}) = t_j \mathcal{L}^v(\omega)$ for every $j \in \{0, \dots, v\}$.*

Proof. Let us set $C = \text{co}(\{\zeta_0, \dots, \zeta_v\})$. Then C is a bounded polyhedral set. Moreover, by (1.2), $0 \in \text{int}(C)$, and, by Proposition 1.1, C° turns out to be polyhedral, compact, and $0 \in \text{int}(C^\circ)$.

By Proposition 1.8 applied to $n = v$, $\Omega = \omega$, and $K = C^\circ$, there exist $\{x_k\} \subseteq \omega$ and $\{t_k\} \subseteq]0, 1]$ such that the sets $\{x_k + t_k C^\circ: k \in \mathbb{N}\}$ are contained in ω , pairwise disjoint, and $\mathcal{L}^v(\omega \setminus \bigcup_{k \in \mathbb{N}} (x_k + t_k C^\circ)) = 0$.

Let σ_C be the support function of C . Then it is easy to verify that

$$\sigma_C(x) = \max\{u_{\zeta_j}(x): j \in \{0, \dots, v\}\} \quad \text{for every } x \in \mathbb{R}^v,$$

from which it follows that $\sigma_C \in PA(\mathbb{R}^v)$.

For every $k \in \mathbb{N}$ and $x \in x_k + t_k C^\circ$, we now define

$$w_k(x) = t_k \sigma_C\left(\frac{x - x_k}{t_k}\right) - t_k.$$

Then, by (1.3) and (1.4), we deduce that $-t_k \leq w_k(x) < 0$ for every $k \in \mathbb{N}$ and $x \in \text{int}(x_k + t_k C^\circ)$, and that $w_k(x) = 0$ for every $k \in \mathbb{N}$ and $x \in \partial(x_k + t_k C^\circ)$. Consequently, if we define for every $x \in \omega$,

$$v(x) = \begin{cases} w_k(x) & \text{if } k \text{ is the only integer such that } x \in \text{int}(x_k + t_k C^\circ), \\ 0 & \text{if } x \notin \bigcup_{k \in \mathbb{N}} \text{int}(x_k + t_k C^\circ), \end{cases}$$

it turns out that $v \in W_0^{1,\infty}(\omega)$, and that $-1 \leq v(x) < 0$ and $\nabla v(x) \in \{\zeta_0, \dots, \zeta_k\}$ for a.e. $x \in \omega$.

Eventually, by the divergence theorem, we have that

$$0 = \int_{\omega} \nabla v(x) \, d\mathcal{L}^v = \sum_{j=0}^v \mathcal{L}^v(\{x \in \omega : \nabla v(x) = \zeta_j\}) \zeta_j.$$

Because of this, since ζ_0, \dots, ζ_v are affinely independent and hence 0 can be uniquely expressed as a convex combination of ζ_0, \dots, ζ_v , we also obtain that

$$\mathcal{L}^v(\{x \in \omega : \nabla v(x) = \zeta_j\}) = t_j \mathcal{L}^v(\omega) \quad \text{for every } j \in \{0, \dots, v\}.$$

This completes the proof of the lemma. \square

Hereafter, given $\{u_h\} \subseteq L_{\text{loc}}^\infty(\mathbb{R}^n)$ and $u \in L_{\text{loc}}^\infty(\mathbb{R}^n)$, we say that $u_h \rightarrow u$ in weak*- $L_{\text{loc}}^\infty(\mathbb{R}^n)$ if $u_h \rightarrow u$ in weak*- $L^\infty(A)$ for every $A \in \mathcal{A}_0(\mathbb{R}^n)$. Analogously, if $\{u_h\} \subseteq W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$ and $u \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$, we say that $u_h \rightarrow u$ in weak*- $W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$ if $u_h \rightarrow u$ in weak*- $W^{1,\infty}(A)$ for every $A \in \mathcal{A}_0(\mathbb{R}^n)$.

Lemma 3.2. *Let $v \in \mathbb{N}$, let $\zeta_0, \dots, \zeta_v \in \mathbb{R}^v$ be affinely independent, and let $t_0, \dots, t_v \in]0, 1[$ be such that $\sum_{j=0}^v t_j = 1$ and $\sum_{j=0}^v t_j \zeta_j = 0$. Then there exists $\{v_h\} \subseteq W_{\text{loc}}^{1,\infty}(\mathbb{R}^v)$, with $v_h \rightarrow 0$ in weak*- $W_{\text{loc}}^{1,\infty}(\mathbb{R}^v)$, such that $-\frac{1}{h} \leq v_h(x) < 0$ and $\nabla v_h(x) \in \{\zeta_0, \dots, \zeta_v\}$ for every $h \in \mathbb{N}$ and a.e. $x \in \mathbb{R}^v$. In particular, $\chi_{\{x \in \mathbb{R}^v : \nabla v_h(x) = \zeta_j\}} \rightarrow t_j$ in weak*- $L_{\text{loc}}^\infty(\mathbb{R}^v)$ for every $j \in \{0, \dots, v\}$.*

Proof. An application of Lemma 3.1 with $\omega =]0, 1[^v$ provides $v \in W_0^{1,\infty}(]0, 1[^v)$ such that $-1 \leq v(x) < 0$ and $\nabla v(x) \in \{\zeta_0, \dots, \zeta_v\}$ for a.e. $x \in]0, 1[^v$, and

$$\mathcal{L}^v(\{x \in]0, 1[^v : \nabla v(x) = \zeta_j\}) = t_j \quad \text{for every } j \in \{0, \dots, v\}. \tag{3.1}$$

Since v is equal to 0 on $\partial]0, 1[^v$, we can extend it by periodicity to the whole \mathbb{R}^v . Call again v such extension. For every $h \in \mathbb{N}$ and every $x \in \mathbb{R}^v$, now we set $v_h(x) = \frac{1}{h}v(hx)$. Then $\{v_h\}$ satisfies the required properties. In particular, if $A \in \mathcal{A}_0(\mathbb{R}^v)$, $j \in \{0, \dots, v\}$ and $\psi \in L^1(A)$, by the $]0, 1[^v$ -periodicity of $\chi_{\{x \in \mathbb{R}^v : \nabla v(x) = \zeta_j\}}$ and (3.1), we have that

$$\lim_{h \rightarrow +\infty} \int_A \chi_{\{x \in \mathbb{R}^v : \nabla v_h(x) = \zeta_j\}}(y) \psi(y) \, d\mathcal{L}^v = \lim_{h \rightarrow +\infty} \int_A \chi_{\{x \in \mathbb{R}^v : \nabla v(x) = \zeta_j\}}(hy) \psi(y) \, d\mathcal{L}^v = t_j \int_A \psi(y) \, d\mathcal{L}^v,$$

from which also the last part of the lemma follows. \square

Lemma 3.3. *Let $n \in \mathbb{N}$ and $v \in \{0, \dots, n\}$, let $z_0, \dots, z_v \in \mathbb{R}^n$ be affinely independent, and let $t_0, \dots, t_v \in]0, 1[$ be such that $\sum_{j=0}^v t_j = 1$ and $\sum_{j=0}^v t_j z_j = 0$. Then there exists $\{v_h\} \subseteq W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$, with $v_h \rightarrow 0$ in weak*- $W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$, such that $-\frac{1}{h} \leq v_h(x) < 0$ and $\nabla v_h(x) \in \{z_0, \dots, z_v\}$ for every $h \in \mathbb{N}$ and a.e. $x \in \mathbb{R}^n$. In particular, $\chi_{\{x \in \mathbb{R}^n : \nabla v_h(x) = z_j\}} \rightarrow t_j$ in weak*- $L_{\text{loc}}^\infty(\mathbb{R}^n)$ for every $j \in \{0, \dots, v\}$.*

Proof. If $v = 0$, then $z_0 = 0$ and $t_0 = 1$. Then the lemma follows by considering $\{v_h\}$ with $v_h(x) = -\frac{1}{h}$ for every $h \in \mathbb{N}$ and every $x \in \mathbb{R}^n$.

If $v = n$ the lemma follows from Lemma 3.2.

If $v \in \{1, \dots, n - 1\}$, we observe that $0 \in \text{aff}(\{z_0, \dots, z_v\})$, and consider an orthogonal matrix $R : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$R(\mathbb{R}^v \times \{0_{n-v}\}) = \text{aff}(\{z_0, \dots, z_v\}),$$

where 0_{n-v} is the origin of \mathbb{R}^{n-v} . Let us set $\zeta_0 = \text{Pr}_v(R^{-1}z_0), \dots, \zeta_v = \text{Pr}_v(R^{-1}z_v)$, where Pr_v is the projection operator from \mathbb{R}^n to \mathbb{R}^v . Then ζ_0, \dots, ζ_v turn out to be affinely independent, and $\sum_{j=0}^v t_j \zeta_j = \text{Pr}_v(R^{-1}(\sum_{j=0}^v t_j z_j)) = 0_v$.

Let $\{w_h\} \subseteq W_{\text{loc}}^{1,\infty}(\mathbb{R}^v)$ be given by Lemma 3.2 applied to the above ζ_0, \dots, ζ_v . For every $h \in \mathbb{N}$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, and $x \in \mathbb{R}^n$, we set $\tilde{v}_h(y) = w_h(y_1, \dots, y_v)$ and $v_h(x) = \tilde{v}_h(R^{-1}x)$. Then $v_h \rightarrow 0$ in weak*- $W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$, and $-\frac{1}{h} \leq v_h(x) < 0$ for every $h \in \mathbb{N}$ and a.e. $x \in \mathbb{R}^n$. Moreover, since $R^{-1} = R^T$ (the transpose of R), we have

$$(\nabla_x v_h(x))^T = \nabla_y \tilde{v}_h(R^{-1}x)R^{-1} = \nabla_y \tilde{v}_h(R^{-1}x)R^T = (R \nabla_y \tilde{v}_h(R^{-1}x))^T \quad \text{for every } h \in \mathbb{N} \text{ and a.e. } x \in \mathbb{R}^n,$$

from which, once we recall that $\nabla_y \tilde{v}_h(y) \in \{(\zeta_0, 0_{n-v}), \dots, (\zeta_v, 0_{n-v})\} = \{R^{-1}z_0, \dots, R^{-1}z_v\}$ for every $h \in \mathbb{N}$ and a.e. $y \in \mathbb{R}^n$, we conclude that

$$\nabla_x v_h(x) \in \{z_0, \dots, z_v\} \quad \text{for every } h \in \mathbb{N} \text{ and a.e. } x \in \mathbb{R}^n.$$

Now, we fix $j \in \{0, \dots, v\}$, and set for every $h \in \mathbb{N}$,

$$A_{j,h} = \{x \in \mathbb{R}^n : \nabla v_h(x) = z_j\}, \quad B_{j,h} = \{y \in \mathbb{R}^v : \nabla w_h(y) = \zeta_j\}.$$

Then

$$A_{j,h} = R(B_{j,h} \times \mathbb{R}^{n-v}) \quad \text{for every } h \in \mathbb{N}. \tag{3.2}$$

By (3.2), for every $h \in \mathbb{N}$, $A \in \mathcal{A}_0(\mathbb{R}^n)$, and $\psi \in L^1(A)$, we have that

$$\begin{aligned} \int_A \chi_{A_{j,h}}(x) \psi(x) \, d\mathcal{L}^n &= \int_{A_{j,h}} \chi_A(x) \psi(x) \, d\mathcal{L}^n = \int_{R(B_{j,h} \times \mathbb{R}^{n-v})} \chi_A(x) \psi(x) \, d\mathcal{L}^n = \int_{B_{j,h} \times \mathbb{R}^{n-v}} \chi_A(Ry) \psi(Ry) \, d\mathcal{L}^n \\ &= \int_{\mathbb{R}^{n-v}} \left(\int_{B_{j,h}} \chi_A(Ry) \psi(Ry) \, dy_1 \cdots dy_v \right) dy_{v+1} \cdots dy_n \\ &= \int_{\mathbb{R}^{n-v}} \left(\int_{\mathbb{R}^v} \chi_{B_{j,h}}(y) \chi_A(Ry) \psi(Ry) \, dy_1 \cdots dy_v \right) dy_{v+1} \cdots dy_n, \end{aligned}$$

from which, again by Lemma 3.2 and Lebesgue Dominated Convergence Theorem, we deduce that

$$\begin{aligned} \lim_{h \rightarrow +\infty} \int_A \chi_{A_{j,h}}(x) \psi(x) \, d\mathcal{L}^n &= t_j \int_{\mathbb{R}^{n-v}} \left(\int_{\mathbb{R}^v} \chi_A(Ry) \psi(Ry) \, dy_1 \cdots dy_v \right) dy_{v+1} \cdots dy_n \\ &= t_j \int_{\mathbb{R}^n} \chi_A(Ry) \psi(Ry) \, d\mathcal{L}^n = t_j \int_A \psi(x) \, d\mathcal{L}^n \\ &\quad \text{for every } A \in \mathcal{A}_0(\mathbb{R}^n) \text{ and } \psi \in L^1(A). \end{aligned}$$

Because of this, we have that $\chi_{\{x \in \mathbb{R}^n : \nabla v_h(x) = z_j\}} \rightarrow t_j$ in weak*- $L_{\text{loc}}^\infty(\mathbb{R}^n)$, and the lemma follows also in this case. \square

The above lemmas allow us to prove the basic inequality below, that is the starting point for the proof of the representation result.

Lemma 3.4. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel. Then*

$$\inf \left\{ \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h) \, d\mathcal{L}^n : \{u_h\} \subseteq W^{1,\infty}(\Omega), u_h \rightarrow u_z \text{ in weak*-}W^{1,\infty}(\Omega) \right\} \leq \text{co } f(z) \mathcal{L}^n(\Omega)$$

for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$, $z \in \mathbb{R}^n$.

Proof. Let Ω, z be as above. Let us assume that $\text{co } f(z) < +\infty$, and take $\varepsilon > 0$. Then Theorem 1.2 provides $v \in \{0, \dots, n\}$, $z_0, \dots, z_v \in \mathbb{R}^n$, and $t_0, \dots, t_v \in]0, 1]$ such that z_0, \dots, z_v are affinely independent, $\sum_{j=0}^v t_j = 1$, $\sum_{j=0}^v t_j z_j = z$, and

$$\sum_{j=0}^{\nu} t_j f(z_j) \leq \text{co } f(z) + \varepsilon. \tag{3.3}$$

Because of this, the vectors $z_0 - z, \dots, z_\nu - z$ turn out to be affinely independent, and $\sum_{j=0}^{\nu} t_j(z_j - z) = 0$. Let $\{v_h\} \subseteq W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$ be given by Lemma 3.3 applied to $z_0 - z, \dots, z_\nu - z$. Then $u_z + v_h \rightarrow u_z$ in weak*- $W^{1,\infty}(\Omega)$, and

$$\chi_{\{x \in \mathbb{R}^n: z + \nabla v_h(x) = z_j\}} \rightarrow t_j \quad \text{in weak*-}L^\infty(\Omega) \text{ for every } j \in \{0, \dots, \nu\}. \tag{3.4}$$

Consequently, by (3.4) and (3.3), we obtain

$$\begin{aligned} & \inf \left\{ \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h) \, d\mathcal{L}^n : \{u_h\} \subseteq W^{1,\infty}(\Omega), u_h \rightarrow u_z \text{ in weak*-}W^{1,\infty}(\Omega) \right\} \\ & \leq \liminf_{h \rightarrow +\infty} \int_{\Omega} f(z + \nabla v_h) \, d\mathcal{L}^n = \liminf_{h \rightarrow +\infty} \sum_{j=0}^{\nu} \mathcal{L}^n(\{x \in \Omega: z + \nabla v_h(x) = z_j\}) f(z_j) \\ & = \sum_{j=0}^{\nu} t_j \mathcal{L}^n(\Omega) f(z_j) \leq (\text{co } f(z) + \varepsilon) \mathcal{L}^n(\Omega) \quad \text{for every } \varepsilon > 0. \end{aligned}$$

As $\varepsilon \rightarrow 0$, the lemma follows. \square

In the results below we establish some inequalities for \bar{G} in (1.13). These will provide the representation result for \bar{F} in (1.11).

Proposition 3.5. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel, and let \bar{G} be given by (1.13). Then*

$$\bar{G}(\Omega, u_z) \leq f^{**}(z) \mathcal{L}^n(\Omega) \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n), z \in \mathbb{R}^n.$$

Proof. Let Ω, z be as above, and take $t \in [0, 1[$. Let us assume that $f^{**}(z) < +\infty$, and take $z_0 \in \text{ri}(\text{dom } f^{**})$. Then (1.1) and Proposition 1.3 yield $tz + (1 - t)z_0 \in \text{ri}(\text{dom } f^{**})$ and $f^{**}(tz + (1 - t)z_0) = \text{co } f(tz + (1 - t)z_0)$. Consequently, Lemma 3.4 and the convexity of f^{**} provide

$$\begin{aligned} \bar{G}(\Omega, u_{tz+(1-t)z_0}) & \leq \inf \left\{ \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h) \, d\mathcal{L}^n : \{u_h\} \subseteq W^{1,\infty}(\Omega), u_h \rightarrow u_{tz+(1-t)z_0} \text{ in weak*-}W^{1,\infty}(\Omega) \right\} \\ & \leq \text{co } f(tz + (1 - t)z_0) \mathcal{L}^n(\Omega) = f^{**}(tz + (1 - t)z_0) \mathcal{L}^n(\Omega) \leq (tf^{**}(z) + (1 - t)f^{**}(z_0)) \mathcal{L}^n(\Omega). \end{aligned}$$

Because of this, and of the $C^0(\bar{\Omega})$ -lower semicontinuity of $\bar{G}(\Omega, \cdot)$, we conclude as $t \rightarrow 1^-$ that

$$\bar{G}(\Omega, u_z) \leq \liminf_{t \rightarrow 1^-} \bar{G}(\Omega, u_{tz+(1-t)z_0}) \leq f^{**}(z) \mathcal{L}^n(\Omega),$$

from which the proposition follows. \square

In order to extend Proposition 3.5 to piecewise affine functions, we need the preparatory lemma below.

Lemma 3.6. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel, and let \bar{G} be given by (1.13). Let $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$, and let $U \subseteq W^{1,1}(\Omega) \cap C^0(\bar{\Omega})$ be such that*

$$\bar{G}(\Omega, u) \leq \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n < +\infty \quad \text{for every } u \in U. \tag{3.5}$$

Then, for every $m \in \mathbb{N}$ and every $u_1, \dots, u_m \in U$, it results that

$$\bar{G}(\Omega, \min\{u_1, \dots, u_m\}) \leq \int_{\Omega} f^{**}(\nabla \min\{u_1, \dots, u_m\}) \, d\mathcal{L}^n < +\infty \tag{3.6}_m$$

and

$$\overline{G}(\Omega, \max\{u_1, \dots, u_m\}) \leq \int_{\Omega} f^{**}(\nabla \max\{u_1, \dots, u_m\}) \, d\mathcal{L}^n < +\infty. \tag{3.7}_m$$

Proof. Let us prove the inequalities in (3.6), the proof for those in (3.7) being similar.

We argue by induction on m .

If $m = 1$, (3.5) clearly implies (3.6)₁.

Let now $m \in \mathbb{N}$, and prove that (3.6)_m implies (3.6)_{m+1}. To do this, let $u_1, \dots, u_{m+1} \in U$, and set $u = \min\{u_1, \dots, u_{m+1}\}$ and $v = \min\{u_1, \dots, u_m\}$. Then, by (3.5) and (3.6)_m, there exist $\{u_h^{m+1}\}$ and $\{v_h\}$ in $W^{1,\infty}(\Omega)$ such that $u_h^{m+1} \rightarrow u_{m+1}$ in $C^0(\overline{\Omega})$, $v_h \rightarrow v$ in $C^0(\overline{\Omega})$, and

$$\begin{cases} \limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h^{m+1}) \, d\mathcal{L}^n \leq \int_{\Omega} f^{**}(\nabla u_{m+1}) \, d\mathcal{L}^n < +\infty, \\ \limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla v_h) \, d\mathcal{L}^n \leq \int_{\Omega} f^{**}(\nabla v) \, d\mathcal{L}^n < +\infty. \end{cases} \tag{3.8}$$

Now, we observe that $\min\{v_h, u_h^{m+1}\} \rightarrow u$, and $\max\{v_h, u_h^{m+1}\} \rightarrow \max\{v, u_{m+1}\}$ in $C^0(\overline{\Omega})$, and that

$$\begin{aligned} f(\nabla \min\{v_h, u_h^{m+1}\})(x) &= f(\nabla v_h(x)) + f(\nabla u_h^{m+1}(x)) - f(\nabla \max\{v_h, u_h^{m+1}\})(x) \\ &\text{for every } h \in \mathbb{N} \text{ and a.e. } x \in \Omega. \end{aligned} \tag{3.9}$$

Therefore, since clearly (3.8) implies

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla \max\{v_h, u_h^{m+1}\}) \, d\mathcal{L}^n \leq \limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla v_h) \, d\mathcal{L}^n + \limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h^{m+1}) \, d\mathcal{L}^n < +\infty,$$

by (3.8), (3.9), (1.15), and (1.16), we have

$$\begin{aligned} \overline{G}(\Omega, u) &\leq \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla \min\{v_h, u_h^{m+1}\}) \, d\mathcal{L}^n \\ &\leq \limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla v_h) \, d\mathcal{L}^n + \limsup_{h \rightarrow +\infty} \int_{\Omega} f(\nabla u_h^{m+1}) \, d\mathcal{L}^n - \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla \max\{v_h, u_h^{m+1}\}) \, d\mathcal{L}^n \\ &\leq \int_{\Omega} f^{**}(\nabla u_{m+1}) \, d\mathcal{L}^n + \int_{\Omega} f^{**}(\nabla v) \, d\mathcal{L}^n - \overline{G}(\Omega, \max\{v, u_{m+1}\}) \\ &\leq \int_{\Omega} f^{**}(\nabla u_{m+1}) \, d\mathcal{L}^n + \int_{\Omega} f^{**}(\nabla v) \, d\mathcal{L}^n - \int_{\Omega} f^{**}(\nabla \max\{v, u_{m+1}\}) \, d\mathcal{L}^n < +\infty. \end{aligned} \tag{3.10}$$

By (3.10), inequality (3.6)_{m+1} and the lemma follow once we observe that, since

$$f^{**}(\nabla u_{m+1}(x)) + f^{**}(\nabla v(x)) - f^{**}(\nabla \max\{v, u_{m+1}\})(x) = f^{**}(\nabla u(x)) \quad \text{for a.e. } x \in \Omega$$

and

$$\int_{\Omega} f^{**}(\nabla \max\{v, u_{m+1}\}) \, d\mathcal{L}^n \leq \int_{\Omega} f^{**}(\nabla v) \, d\mathcal{L}^n + \int_{\Omega} f^{**}(\nabla u_{m+1}) \, d\mathcal{L}^n < +\infty,$$

the last line of (3.10) is equal to $\int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n$. \square

Proposition 3.7. Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel, and let \overline{G} be given by (1.13). Then

$$\overline{G}(\Omega, u) \leq \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n) \text{ convex, } u \in PA(\mathbb{R}^n).$$

Proof. Let $\Omega, u = \sum_{j=1}^m (u_{z_j} + s_j)\chi_{P_j}$ be as above. Then, by Theorem 2.1, we obtain the existence of $k \in \mathbb{N}$ and of $N_1, \dots, N_k \subseteq \{j \in \{1, \dots, m\}: \text{int}(P_j) \cap \Omega \neq \emptyset\}$ such that

$$u(x) = \max_{i \in \{1, \dots, k\}} \min_{j \in N_i} (u_{z_j}(x) + s_j) \quad \text{for every } x \in \Omega.$$

Let us observe that it is not restrictive to assume that $\int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n < +\infty$, that is

$$f^{**}(z_j) < +\infty \quad \text{for every } j \in \{1, \dots, m\} \text{ such that } \text{int}(P_j) \cap \Omega \neq \emptyset. \tag{3.11}$$

Let $i \in \{1, \dots, k\}$, m_i be the cardinality of N_i , and $v_i = \min_{j \in N_i} (u_{z_j} + s_j)$. Then, by Proposition 3.5, (3.11), and (3.6) _{m_i} of Lemma 3.6 applied with $U = \{u_{z_j} + s_j: j \in N_i\}$, we obtain

$$\overline{G}(\Omega, v_i) \leq \int_{\Omega} f^{**}(\nabla v_i) \, d\mathcal{L}^n < +\infty \quad \text{for every } i \in \{1, \dots, k\}. \tag{3.12}$$

At this point, by (3.12) and (3.7) _{k} of Lemma 3.6 applied with $U = \{v_i: i \in \{1, \dots, k\}\}$, we deduce the proposition. \square

Lemma 3.8. Let $f: \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel, and let \overline{G} be given by (1.13). Then

$$\overline{G}_-(\Omega, u) \leq \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n) \text{ convex, } u \in C^1(\mathbb{R}^n).$$

Proof. Let Ω, u be as above. Obviously, we can assume that $\nabla u(x) \in \text{dom } f^{**}$ for every $x \in \Omega$. If $\text{dom } f^{**}$ contains only a single point then the thesis follows by Proposition 3.5, therefore it is not restrictive to assume that the dimension ν of $\text{aff}(\text{dom } f^{**})$ is bigger than zero.

We first consider the case in which

$$0 \in \text{ri}(\text{dom } f^{**}). \tag{3.13}$$

If $\nu = n$, let R be the identity matrix on \mathbb{R}^n . If $\nu < n$, let R be an orthogonal matrix such that

$$R(\text{aff}(\text{dom } f^{**})) = \mathbb{R}^{\nu} \times \{0_{n-\nu}\}. \tag{3.14}$$

In both cases, let us define the function \tilde{u} by

$$\tilde{u}: y \in \mathbb{R}^n \mapsto u(R^{-1}y),$$

then, as in the proof of Lemma 3.3, we have that

$$\nabla_y \tilde{u}(y) = R \nabla_x u(R^{-1}y) \quad \text{for every } y \in \mathbb{R}^n. \tag{3.15}$$

Since $\nabla u(x) \in \text{dom } f^{**}$ for every $x \in \Omega$, by (3.15) and (3.14) we infer that, when $\nu < n$, $\nabla \tilde{u}(y)$ has the last $n - \nu$ entries equal to zero for every $y \in R\Omega$. Hence, by taking into account the convexity of $R\Omega$, it turns out that \tilde{u} depends only on (y_1, \dots, y_{ν}) when $y = (y_1, \dots, y_n)$ varies in $R\Omega$. Because of this, we can define \hat{u} by

$$\hat{u}: (y_1, \dots, y_{\nu}) \in \mathbb{R}^{\nu} \mapsto \begin{cases} \tilde{u}(y_1, \dots, y_n) & \text{if } \nu = n, \\ \tilde{u}(y_1, \dots, y_{\nu}, \beta(y_1, \dots, y_{\nu})) & \text{if } \nu < n, \end{cases}$$

where, if $\nu < n$, $\beta: \mathbb{R}^{\nu} \rightarrow \mathbb{R}^{n-\nu}$ is any function such that $(y_1, \dots, y_{\nu}, \beta(y_1, \dots, y_{\nu})) \in R\Omega$ for every $(y_1, \dots, y_{\nu}) \in \text{Pr}_{\nu}(R\Omega)$. Then $\hat{u} \in C^1(\text{Pr}_{\nu}(R\Omega))$ and, since $\nabla u(x) \in \text{dom } f^{**}$ for every $x \in \Omega$, we obtain that $\nabla \hat{u}(y) \in \text{Pr}_{\nu}(R(\text{dom } f^{**}))$ for every $y \in \text{Pr}_{\nu}(R\Omega)$.

Let us fix $s \in [0, 1[$. Then, by (3.13), there exists a compact subset H of $\text{ri}(\text{Pr}_{\nu}(R(\text{dom } f^{**})))$ such that

$$s \nabla \hat{u}(y) \in H \quad \text{for every } y \in \text{Pr}_{\nu}(R\Omega). \tag{3.16}$$

Let $A \in \mathcal{A}_0(\mathbb{R}^n)$ be convex and such that $\overline{A} \subseteq \Omega$. Then obviously $\overline{\text{Pr}_{\nu}(RA)} \subseteq \text{Pr}_{\nu}(R\Omega)$. Let $\{\hat{u}_h\} \subseteq PA(\mathbb{R}^{\nu})$ be such that $\hat{u}_h \rightarrow s\hat{u}$ in $W^{1,\infty}(\text{Pr}_{\nu}(RA))$. Then, by (3.16), we obtain that

$$\nabla \hat{u}_h(y) \in \widehat{K} \quad \text{for every } h \in \mathbb{N} \text{ large enough and a.e } y \in \text{Pr}_{\nu}(RA), \tag{3.17}$$

\widehat{K} being a suitable compact subset of $\text{ri}(\text{Pr}_{\nu}(R(\text{dom } f^{**})))$.

For every $h \in \mathbb{N}$ let us now define the functions \tilde{u}_h and u_h by

$$\tilde{u}_h : (y_1, \dots, y_n) \in \mathbb{R}^n \mapsto \hat{u}_h(y_1, \dots, y_n)$$

and

$$u_h : x \in \mathbb{R}^n \mapsto \tilde{u}_h(Rx).$$

Then obviously $\{u_h\} \subseteq PA(\mathbb{R}^n)$, and

$$u_h \rightarrow su \quad \text{in } W^{1,\infty}(A). \tag{3.18}$$

Moreover, by (3.17) we deduce the existence of a compact subset K of $\text{ri}(\text{dom } f^{**})$ such that

$$\nabla u_h(x) \in K \quad \text{for every } h \in \mathbb{N} \text{ large enough and a.e. } x \in A. \tag{3.19}$$

At this point, by the convexity of A and Proposition 3.7, we obtain

$$\overline{G}(A, u_h) \leq \int_A f^{**}(\nabla u_h) \, d\mathcal{L}^n \quad \text{for every } h \in \mathbb{N}, \tag{3.20}$$

whilst, by (3.18), (3.19), and the local Lipschitz continuity of f^{**} in $\text{ri}(\text{dom } f^{**})$, we have

$$\lim_{h \rightarrow +\infty} \int_A f^{**}(\nabla u_h) \, d\mathcal{L}^n = \int_A f^{**}(s\nabla u) \, d\mathcal{L}^n. \tag{3.21}$$

Therefore, by (3.18), the $C^0(\overline{A})$ -lower semicontinuity of $\overline{G}(A, \cdot)$, (3.20), (3.21), and the convexity of f^{**} , we obtain

$$\begin{aligned} \overline{G}(A, su) &\leq \liminf_{h \rightarrow +\infty} \overline{G}(A, u_h) \leq \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + (1-s)\mathcal{L}^n(\Omega) f^{**}(0) \\ &\quad \text{for every } s \in [0, 1[\text{ and every } A \in \mathcal{A}_0(\mathbb{R}^n) \text{ convex with } \overline{A} \subseteq \Omega. \end{aligned} \tag{3.22}$$

Now, we observe that the convexity of Ω yields

$$\sup\{\overline{G}(A, u) : A \in \mathcal{A}_0(\mathbb{R}^n), A \text{ convex}, \overline{A} \subseteq \Omega\} = \overline{G}_-(\Omega, u). \tag{3.23}$$

Therefore, taking the limits in (3.22) first as s tends to 1 and then as A increases to Ω , by (3.13), (3.22), again the $C^0(\overline{A})$ -lower semicontinuity of $\overline{G}(A, \cdot)$, and (3.23), the lemma follows if (3.13) holds.

In the general case, if (3.13) does not hold, we only have to take $z_0 \in \text{ri}(\text{dom } f^{**})$ and consider the function f_0 defined by $f_0 : z \in \mathbb{R}^n \mapsto f(z_0 + z)$. We have, with the obvious meaning for the symbols adopted,

$$f_0^{**}(z) = f^{**}(z_0 + z) \quad \text{for every } z \in \mathbb{R}^n, \tag{3.24}$$

$$\overline{G}_0(A, v) = \overline{G}(A, u_{z_0} + v) \quad \text{for every } A \in \mathcal{A}_0(\mathbb{R}^n) \text{ and } v \in C^0(\overline{A}), \tag{3.25}$$

and

$$0 \in \text{ri}(\text{dom } f_0^{**}). \tag{3.26}$$

Therefore, by (3.25), (3.26), the previously treated case applied to f_0 , and (3.24), we obtain that

$$\overline{G}_-(\Omega, u) \leq \int_{\Omega} f_0^{**}(\nabla u - z_0) \, d\mathcal{L}^n = \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n,$$

from which the lemma follows also in the general case. \square

Lemma 3.9. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel, and let \overline{F} be given by (1.11). Then*

$$\overline{F}_-(\Omega, u) \leq \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + \int_{\Omega} (f^{**})^{\infty} \left(\frac{dD^s u}{|dD^s u|} \right) |dD^s u| \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n) \text{ convex, } u \in BV(\Omega).$$

Proof. Let Ω , u be as above. Since convex open sets have Lipschitz boundary, the zero extension of u from Ω to \mathbb{R}^n is in $BV(\mathbb{R}^n)$. Call again u such extension.

Let $A \in \mathcal{A}_0(\mathbb{R}^n)$ with $\bar{A} \subseteq \Omega$, A being also convex, and take $\eta > 0$ so small that $\bar{A} \subseteq \Omega_\eta^-$. Then, by (1.15), the convexity of A , Lemma 3.8, and Proposition 1.6, we infer

$$\begin{aligned} \bar{F}_-(A, u_\eta) &\leq \int_A f^{**}(\nabla u_\eta) d\mathcal{L}^n \leq \int_{\Omega_\eta^-} f^{**}(\nabla u_\eta) d\mathcal{L}^n \\ &\leq \int_\Omega f^{**}(\nabla u) d\mathcal{L}^n + \int_\Omega (f^{**})^\infty \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u| \quad \text{for every } \eta > 0 \text{ small enough.} \end{aligned}$$

Taking the limits first as η tends to 0 and then as A increases to Ω in the above inequalities, by the $L^1(A)$ -lower semicontinuity of $\bar{F}_-(A, \cdot)$, and the inner regularity of \bar{F}_- , we deduce the lemma. \square

We are now able to prove the representation result.

Theorem 3.10. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel, and let \bar{F} be given by (1.11). Then*

$$\bar{F}(\Omega, u) = \int_\Omega f^{**}(\nabla u) d\mathcal{L}^n + \int_\Omega (f^{**})^\infty \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u| \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n) \text{ convex, } u \in BV(\Omega).$$

Proof. By (1.16) and Lemma 3.9, the theorem follows if we prove that

$$\bar{F}(\Omega, u) = \bar{F}_-(\Omega, u) \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n) \text{ convex, } u \in BV(\Omega).$$

To do this, we exploit Proposition 1.7 with Φ equal to the restriction of \bar{F} to $\mathcal{A}_0(\mathbb{R}^n) \times BV_{\text{loc}}(\mathbb{R}^n)$.

It is clear that for every $u \in BV_{\text{loc}}(\mathbb{R}^n)$, $\bar{F}(\cdot, u)$ is increasing.

Let $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ be convex, and let $u \in BV(\Omega)$. As in the proof of Lemma 3.9, it is not restrictive to assume that $u \in BV(\mathbb{R}^n)$. Let $x_0 \in \Omega$, $t \in]0, 1]$, and let $u_{x_0, t}$ be defined in (1.10). Then the $L^1(\Omega)$ -lower semicontinuity of $\bar{F}(\Omega, \cdot)$ implies that

$$\liminf_{t \rightarrow 1^-} \bar{F}(\Omega, u_{x_0, t}) \geq \bar{F}(\Omega, u).$$

Moreover, since by means of a change of variables it is easy to verify that

$$\limsup_{t \rightarrow 1^+} \bar{F}_-(x_0 + t(\Omega - x_0), u_{x_0, 1/t}) \leq \bar{F}_-(\Omega, u),$$

also the last requirement of Proposition 1.7 is fulfilled by Φ . Consequently, Proposition 1.7 applies, and the theorem follows. \square

From Theorem 3.10 we deduce the following corollary, in which the constraint condition is emphasized.

Corollary 3.11. *Let $g : \mathbb{R}^n \rightarrow [0, +\infty[$ be Borel, and let E be a Borel subset of \mathbb{R}^n . Then*

$$\begin{aligned} &\inf \left\{ \liminf_{h \rightarrow +\infty} \int_\Omega g(\nabla u_h) d\mathcal{L}^n : \{u_h\} \subseteq W^{1, \infty}(\Omega), \nabla u_h(x) \in E \text{ for every } h \in \mathbb{N} \text{ and a.e. } x \in \Omega, \right. \\ &\quad \left. u_h \rightarrow u \text{ in } L^1(\Omega) \right\} \\ &= \int_\Omega (g + I_E)^{**}(\nabla u) d\mathcal{L}^n + \int_\Omega ((g + I_E)^{**})^\infty \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u| \\ &\quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n) \text{ convex, } u \in BV(\Omega). \end{aligned}$$

Proof. Follows from Theorem 3.10 applied to $f = g + I_E$. \square

Eventually, Theorem 3.10 provides information on the structure of the set of the solutions of first order differential inclusions.

Corollary 3.12. *Let $E \subseteq \mathbb{R}^n$ be Borel. Then, for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ convex, and $u \in W^{1,1}(\Omega)$ such that $\nabla u(x) \in \overline{\text{co}(E)}$ for a.e. $x \in \Omega$, there exists $\{u_h\}$ in $W^{1,\infty}(\Omega)$ such that $u_h \rightarrow u$ in $L^1(\Omega)$, and $\nabla u_h(x) \in E$ for every $h \in \mathbb{N}$ and a.e. $x \in \Omega$.*

Proof. Follows from Corollary 3.11 applied with $g = 0$, once we recall that $(I_E)^{**} = I_{\overline{\text{co}(E)}}$. \square

4. Relaxation of gradient constrained Dirichlet problems

Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel. In the present section we exploit the previous results to represent the functional defined in (1.12).

First of all, we point out that, when dealing with gradient constrained Dirichlet problems, a condition like

$$\text{int}(\text{co}(\text{dom } f)) \neq \emptyset \tag{4.1}$$

turns out to be necessary in order to avoid trivial cases, as shown in the result below (cf. also Lemma 3.6 in [6]).

Proposition 4.1. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel with $\text{int}(\text{co}(\text{dom } f)) = \emptyset$. Let $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$, $u_0 \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$ satisfy $\nabla u_0(x) \in \text{aff}(\text{dom } f)$ for a.e. $x \in \Omega$, and let $\overline{F}(u_0, \Omega, \cdot)$ be given by (1.12). Then*

$$\overline{F}(u_0, \Omega, u) = \begin{cases} \int_{\Omega} f(\nabla u) \, d\mathcal{L}^n & \text{if } u = u_0 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise} \end{cases} \quad \text{for every } u \in L^1(\Omega).$$

Proof. It is clear that

$$\overline{F}(u_0, \Omega, u) \leq \begin{cases} \int_{\Omega} f(\nabla u) \, d\mathcal{L}^n & \text{if } u = u_0 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise} \end{cases} \quad \text{for every } u \in L^1(\Omega). \tag{4.2}$$

To prove the reverse inequality, we show that if $v \in u_0 + W_0^{1,\infty}(\Omega)$ is such that $\int_{\Omega} f(\nabla v) \, d\mathcal{L}^n < +\infty$, then $v = u_0$. Clearly, this holds if $\text{dom } f = \{0\}$.

If this is not the case, let v be as above, and let us assume for the moment that

$$\text{aff}(\text{dom } f) = \mathbb{R}^v \times \{0_{n-v}\} \quad \text{for some } v \in \{1, \dots, n-1\}. \tag{4.3}$$

Then, since $\nabla v(x) \in \text{dom } f$ for a.e. $x \in \Omega$ and $\nabla u_0 \in \text{aff}(\text{dom } f)$ for a.e. $x \in \Omega$, by (4.3) we infer that $v - u_0 \in W_0^{1,\infty}(\Omega)$ and that $\nabla_{v+1}(v - u_0) = \dots = \nabla_n(v - u_0) = 0$ a.e. in Ω . Because of this, we get that $v = u_0$.

When (4.3) is dropped, let us observe that $\text{int}(\text{co}(\text{dom } f)) = \emptyset$ implies $\text{aff}(\text{dom } f) \neq \mathbb{R}^n$. Let $v \in \{1, \dots, n-1\}$ be the dimension of $\text{aff}(\text{dom } f)$, and let

$$A : y \in \mathbb{R}^n \mapsto M_A y + b \in \mathbb{R}^n$$

be an affine transformation such that $\det M_A = 1$, and $A(\text{aff}(\text{dom } f)) = \mathbb{R}^v \times \{0_{n-v}\}$. Let us set

$$f_A : z \in \mathbb{R}^n \mapsto f(A^{-1}(z)), \quad u_0^A : y \in \mathbb{R}^n \mapsto u_0(A(y)) + b \cdot y, \quad v^A : y \in \mathbb{R}^n \mapsto v(A(y)) + b \cdot y.$$

Then f_A is Borel and satisfies (4.3) with f_A in place of f , $\nabla u_0^A(y) \in \text{aff}(\text{dom } f_A)$ for a.e. $y \in A^{-1}(\Omega)$, $v^A \in u_0^A + W_0^{1,\infty}(A^{-1}(\Omega))$, and $\int_{A^{-1}(\Omega)} f_A(\nabla v^A) \, d\mathcal{L}^n = \int_{\Omega} f(\nabla v) \, d\mathcal{L}^n < +\infty$. Therefore, by the particular case above considered, we conclude that $v^A = u_0^A$, that is $v = u_0$.

The above considerations imply the reverse inequality of (4.2), that completes the proof of the proposition. \square

The lemma below allows us to compare the functionals in (1.13) and (1.14), and is the key lemma to recover the results of Section 3 for the treatment of the relaxation of Dirichlet problems.

Lemma 4.2. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel and satisfy (4.1). Assume that $0 \in \text{int}(\text{co}(\text{dom } f))$, and let \overline{G} and $\overline{G}(0, \cdot, \cdot)$ be given by (1.13) and (1.14) with $u_0 = 0$. Then*

$$\overline{G}(0, \Omega, u) \leq \overline{G}(\Omega, u) \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n), \quad u \in PA(\mathbb{R}^n) \text{ with compact support in } \Omega.$$

Proof. Let Ω, u be as above.

Clearly, we can assume that $\overline{G}(\Omega, u)$ is finite, so that, if Ω_0 is the open set defined by $\Omega_0 = \{x \in \Omega: u(x) \neq 0\}$, so is also $\overline{G}(\Omega_0, u)$. Consequently, there exists $\{u_h\} \subseteq W^{1,\infty}(\Omega_0)$ such that $u_h \rightarrow u$ in $C^0(\overline{\Omega_0})$, and

$$\overline{G}(\Omega_0, u) = \lim_{h \rightarrow +\infty} \int_{\Omega_0} f(\nabla u_h) \, d\mathcal{L}^n \tag{4.4}$$

(note that this holds also if $\Omega_0 = \emptyset$).

Since $0 \in \text{int}(\text{co}(\text{dom } f))$, by Proposition 1.3 it follows that $f^{**}(0) < +\infty$. Let $\varepsilon > 0$. Then, again by Proposition 1.3 and by Theorem 1.2, there exist $v \in \{0, \dots, n\}$, $\xi_0, \dots, \xi_v \in \text{dom } f$ affinely independent, and $t_0, \dots, t_v \in]0, 1[$ with $\sum_{j=0}^v t_j = 1$, such that $\sum_{j=0}^v t_j \xi_j = 0$, and

$$\sum_{j=0}^v t_j f(\xi_j) \leq f^{**}(0) + \varepsilon. \tag{4.5}$$

Let $\{v_k\} \subseteq W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$ be given by Lemma 3.3 applied to the above ξ_0, \dots, ξ_v . Then $v_k \rightarrow 0$ in $\text{weak}^*-W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$, and $-\frac{1}{k} \leq v_k(x) < 0$ and $\nabla v_k(x) \in \{\xi_0, \dots, \xi_v\}$ for every $k \in \mathbb{N}$ and a.e. $x \in \mathbb{R}^n$. In particular, $\chi_{\{x \in \mathbb{R}^n: \nabla v_k(x) = \xi_j\}} \rightarrow t_j$ in $\text{weak}^*-L_{\text{loc}}^\infty(\mathbb{R}^n)$.

Let $\{\varepsilon_h\}$ be a sequence of positive numbers converging to 0, and observe that, for every $h \in \mathbb{N}$, $C_h = \{x \in \Omega_0: |u_h(x)| \geq \sup_{\Omega_0} |u_h - u| + \varepsilon_h\}$ is a closed subset of \mathbb{R}^n . In fact, if $h \in \mathbb{N}$ and $\{x_k\} \subseteq C_h$ converges to $x \in \mathbb{R}^n$, the continuity of u and of u_h implies that

$$|u(x)| = \lim_{k \rightarrow +\infty} |u(x_k)| \geq \lim_{k \rightarrow +\infty} |u_h(x_k)| - \sup_{\Omega_0} |u_h - u| \geq \varepsilon_h > 0.$$

In addition, since u has compact support in Ω , we also have that $x \in \overline{C_h} \subseteq \overline{\Omega_0} \subseteq \Omega$, so that $x \in \Omega_0$. Because of this, and of the continuity of u_h , we conclude that $x \in C_h$, and hence that C_h is closed in \mathbb{R}^n .

Let $z_0, \dots, z_n \in \text{dom } f$ be affinely independent and such that $0 \in \text{int}(\text{co}(\{z_0, \dots, z_n\}))$. Then (1.2) provides $s_0, \dots, s_n \in]0, 1[$, with $\sum_{j=0}^n s_j = 1$, such that $\sum_{j=0}^n s_j z_j = 0$. For every $h \in \mathbb{N}$, let $v_h^0 \in W_0^{1,\infty}(\Omega \setminus C_h)$ be given by Lemma 3.1 applied to the z_j and s_j above with $v = n$ and $\omega = \Omega \setminus C_h$. Then $-1 \leq v_h^0(x) < 0$ and $\nabla v_h^0(x) \in \{z_0, \dots, z_n\}$ for a.e. $x \in \Omega \setminus C_h$. Moreover, since $v_k \rightarrow 0$ in $C^0(\overline{\Omega})$, we can find $k_h \in \mathbb{N}$ such that

$$\lim_{h \rightarrow +\infty} \mathcal{L}^n(\{x \in \Omega \setminus C_h: v_{k_h}(x) < v_h^0(x)\}) = 0. \tag{4.6}$$

For every $h \in \mathbb{N}$, we now set

$$\chi_h : t \in \mathbb{R} \mapsto \begin{cases} t + \sup_{\Omega_0} |u_h - u| + \varepsilon_h & \text{if } t \leq -\sup_{\Omega_0} |u_h - u| - \varepsilon_h, \\ 0 & \text{if } -\sup_{\Omega_0} |u_h - u| - \varepsilon_h < t < \sup_{\Omega_0} |u_h - u| + \varepsilon_h, \\ t - \sup_{\Omega_0} |u_h - u| - \varepsilon_h & \text{if } t \geq \sup_{\Omega_0} |u_h - u| + \varepsilon_h, \end{cases}$$

and

$$w_h(x) = \begin{cases} \chi_h(u_h(x)) & \text{if } x \in C_h, \\ \max\{v_{k_h}(x), v_h^0(x)\} & \text{if } x \in \Omega \setminus C_h. \end{cases}$$

Then, since $\chi_h(u_h(x)) = 0$ for every $x \in \partial C_h$ and $\partial C_h \subseteq \Omega$, we have that $\{w_h\} \subseteq W_0^{1,\infty}(\Omega)$. Moreover, it is not difficult to verify that $w_h \rightarrow u$ in $C^0(\overline{\Omega})$.

Let now $m \in \mathbb{N}$. We have

$$\begin{aligned} & \int_{\Omega} f(\nabla w_h) \, d\mathcal{L}^n \\ &= \int_{C_h} f(\nabla u_h) \, d\mathcal{L}^n + \int_{\{x \in \Omega \setminus C_h: v_{k_h}(x) \geq v_h^0(x)\}} f(\nabla v_{k_h}) \, d\mathcal{L}^n + \int_{\{x \in \Omega \setminus C_h: v_{k_h}(x) < v_h^0(x)\}} f(\nabla v_h^0) \, d\mathcal{L}^n \end{aligned}$$

$$\begin{aligned} &\leq \int_{\Omega_0} f(\nabla u_h) d\mathcal{L}^n + \int_{\Omega \setminus C_h} f(\nabla v_{k_h}) d\mathcal{L}^n + \max_{j \in \{0, \dots, n\}} f(z_j) \mathcal{L}^n(\{x \in \Omega \setminus C_h : v_{k_h}(x) < v_h^0(x)\}) \\ &\leq \int_{\Omega_0} f(\nabla u_h) d\mathcal{L}^n + \int_{\bigcup_{k \geq m} (\Omega \setminus C_k)} f(\nabla v_{k_h}) d\mathcal{L}^n + \max_{j \in \{0, \dots, n\}} f(z_j) \mathcal{L}^n(\{x \in \Omega \setminus C_h : v_{k_h}(x) < v_h^0(x)\}) \\ &\text{for every } h \in \mathbb{N} \text{ with } h \geq m, \end{aligned}$$

from which, together with (4.4), the properties of $\{v_{k_h}\}$, (4.5), and (4.6), we deduce that

$$\overline{G}(0, \Omega, u) \leq \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla w_h) d\mathcal{L}^n \leq \overline{G}(\Omega_0, u) + (f^{**}(0) + \varepsilon) \mathcal{L}^n\left(\Omega \setminus \bigcap_{k \geq m} C_k\right).$$

Now, we observe that

$$\bigcup_{m \in \mathbb{N}} \bigcap_{k \geq m} C_k = \Omega_0,$$

therefore, as m increases and ε goes to 0, we obtain

$$\overline{G}(0, \Omega, u) \leq \overline{G}(\Omega_0, u) + f^{**}(0) \mathcal{L}^n(\Omega \setminus \Omega_0). \tag{4.7}$$

Finally we observe that, since $u \in PA(\mathbb{R}^n)$, then $\mathcal{L}^n(\partial\Omega_0) = 0$. Consequently, since trivially

$$\overline{G}(\Omega_0, u) + \overline{G}(\Omega \setminus \overline{\Omega_0}, u) \leq \overline{G}(\Omega, u),$$

by (4.7), (1.16), and (1.15), we obtain

$$\overline{G}(0, \Omega, u) \leq \overline{G}(\Omega_0, u) + f^{**}(0) \mathcal{L}^n(\Omega \setminus \overline{\Omega_0}) \leq \overline{G}(\Omega_0, u) + \overline{G}(\Omega \setminus \overline{\Omega_0}, u) \leq \overline{G}(\Omega, u)$$

from which the lemma follows. \square

Lemma 4.3. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel and satisfy (4.1). Assume that $z_0 \in \text{int}(\text{co}(\text{dom } f))$, and let $\overline{F}(u_{z_0}, \cdot, \cdot)$ be given by (1.12). Then*

$$\overline{F}(u_{z_0}, \Omega, u) \leq \int_{\Omega} f^{**}(\nabla u) d\mathcal{L}^n + \int_{\Omega} (f^{**})^\infty\left(\frac{dD^s u}{|D^s u|}\right) d|D^s u|$$

for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ convex, $u \in BV(\Omega)$ such that $u - u_{z_0}$ has compact support in Ω .

Proof. Let Ω be as above, and assume for the moment that $z_0 = 0$, so that, by Proposition 1.3, $f^{**}(0) < +\infty$. Moreover, let us first consider the case in which $u \in C_0^1(\Omega)$ and $\int_{\Omega} f^{**}(\nabla u) d\mathcal{L}^n < +\infty$.

Let $t \in [0, 1[$, and let $\{u_h\} \subseteq PA(\mathbb{R}^n)$ be a sequence of functions with compact support in Ω such that $u_h \rightarrow tu$ in $W^{1,\infty}(\Omega)$. Then, since $\nabla u(x) \in \text{dom } f^{**}$ for every $x \in \Omega$, and since $0 \in \text{int}(\text{co}(\text{dom } f)) = \text{int}(\text{dom } f^{**})$ by Proposition 1.3, it turns out that $\nabla u_h(x)$ belongs to a fixed compact subset of $\text{int}(\text{dom } f^{**})$ for every h sufficiently large and a.e. $x \in \Omega$. Because of this, of the continuity of f^{**} in $\text{int}(\text{dom } f^{**})$, and of the convexity of f^{**} , it then follows that

$$\lim_{h \rightarrow +\infty} \int_{\Omega} f^{**}(\nabla u_h) d\mathcal{L}^n = \int_{\Omega} f^{**}(t\nabla u) d\mathcal{L}^n \leq t \int_{\Omega} f^{**}(\nabla u) d\mathcal{L}^n + (1-t) f^{**}(0) \mathcal{L}^n(\Omega). \tag{4.8}$$

Let now $\overline{G}(0, \cdot, \cdot)$ be given by (1.14) with $u_0 = 0$. Then, by the $C^0(\overline{\Omega})$ -lower semicontinuity of $\overline{G}(0, \Omega, \cdot)$, Lemma 4.2, Proposition 3.7, and (4.8) we obtain that

$$\begin{aligned} \overline{G}(0, \Omega, tu) &\leq \liminf_{h \rightarrow +\infty} \overline{G}(0, \Omega, u_h) \leq \liminf_{h \rightarrow +\infty} \overline{G}(\Omega, u_h) \leq \liminf_{h \rightarrow +\infty} \int_{\Omega} f^{**}(\nabla u_h) d\mathcal{L}^n \\ &\leq \int_{\Omega} f^{**}(\nabla u) d\mathcal{L}^n + (1-t) f^{**}(0) \mathcal{L}^n(\Omega), \end{aligned}$$

from which, together again the $C^0(\overline{\Omega})$ -lower semicontinuity of $\overline{G}(0, \Omega, \cdot)$ and the finiteness of $f^{**}(0)$, we conclude that

$$\overline{G}(0, \Omega, u) \leq \liminf_{t \rightarrow 1^-} \overline{G}(0, \Omega, tu) \leq \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n \quad \text{for every } u \in C_0^1(\Omega). \tag{4.9}$$

If u is in $BV(\Omega)$ has compact support in Ω , then clearly $u \in BV(\mathbb{R}^n)$ and, by the $L^1(\Omega)$ -lower semicontinuity of $\overline{F}(0, \Omega, \cdot)$, (1.15), (4.9), the finiteness of $f^{**}(0)$, and Proposition 1.6, we obtain

$$\begin{aligned} \overline{F}(0, \Omega, u) &\leq \liminf_{\eta \rightarrow 0^+} \overline{F}(0, \Omega, u_\eta) \leq \liminf_{\eta \rightarrow 0^+} \overline{G}(0, \Omega, u_\eta) \leq \liminf_{\eta \rightarrow 0^+} \int_{\Omega} f^{**}(\nabla u_\eta) \, d\mathcal{L}^n \\ &\leq \limsup_{\eta \rightarrow 0^+} \left\{ \int_{\Omega_\eta^-} f^{**}(\nabla u_\eta) \, d\mathcal{L}^n + f^{**}(0) \mathcal{L}^n(\Omega \setminus \Omega_\eta^-) \right\} \\ &\leq \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + \int_{\Omega} (f^{**})^\infty \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u|, \end{aligned}$$

that completes the proof of the lemma when $z_0 = 0$.

Finally, the case when $z_0 \neq 0$ follows from the previous considered one applied to $f_0 : z \in \mathbb{R}^n \mapsto f(z_0 + z)$, as in the proof of Lemma 3.8. \square

Lemma 4.4. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel and satisfy (4.1). Assume that $z_0 \in \text{int}(\text{co}(\text{dom } f))$, and let $\overline{F}(u_{z_0}, \cdot, \cdot)$ be given by (1.12). Then*

$$\overline{F}(u_{z_0}, \Omega, u) \leq \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + \int_{\Omega} (f^{**})^\infty \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u| + \int_{\partial\Omega} (f^{**})^\infty (u_{z_0} - u) \mathbf{n}_\Omega \, d\mathcal{H}^{n-1}$$

for every $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ convex, $u \in BV(\Omega)$.

Proof. First of all, we observe that

$$\overline{F}(u_{z_0}, \Omega, u) \leq \liminf_{t \rightarrow 1^+} \overline{F}(u_{z_0}, x_0 + t(\Omega - x_0), u) \quad \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n), u \in L^1_{\text{loc}}(\mathbb{R}^n). \tag{4.10}$$

Indeed, let Ω, u be as in (4.10), and let us take $t > 1$ such that $\overline{F}(u_{z_0}, x_0 + t(\Omega - x_0), u) < +\infty$, so that there exists $\{u_h\} \subseteq u_{z_0} + W_0^{1,\infty}(x_0 + t(\Omega - x_0))$, with $u_h \rightarrow u$ in $L^1(x_0 + t(\Omega - x_0))$, and

$$\liminf_{h \rightarrow +\infty} \int_{x_0 + t(\Omega - x_0)} f(\nabla u_h) \, d\mathcal{L}^n \leq \overline{F}(u_{z_0}, x_0 + t(\Omega - x_0), u). \tag{4.11}$$

For every $h \in \mathbb{N}$ we set, as in (1.10),

$$v_h = u_{z_0}(x_0) + (u_h - u_{z_0}(x_0))_{x_0,t} = u_{z_0}(x_0) + \frac{1}{t}(u_h(x_0 + t(\cdot - x_0)) - u(x_0)).$$

Then, since

$$u_{z_0}(x_0) + (u_{z_0} - u_{z_0}(x_0))_{x_0,t} = u_{z_0},$$

it follows that

$$v_h \in u_{z_0}(x_0) + (u_{z_0} - u_{z_0}(x_0))_{x_0,t} + W_0^{1,\infty}(\Omega) = u_{z_0} + W_0^{1,\infty}(\Omega) \quad \text{for every } h \in \mathbb{N}.$$

Moreover

$$v_h \rightarrow u_{z_0}(x_0) + (u - u_{z_0}(x_0))_{x_0,t} \quad \text{in } L^1(\Omega),$$

and

$$\int_{\Omega} f(\nabla v_h) \, d\mathcal{L}^n = \frac{1}{t^n} \int_{x_0+t(\Omega-x_0)} f(\nabla u_h) \, d\mathcal{L}^n \quad \text{for every } h \in \mathbb{N}.$$

This, together with (4.11), yields

$$t^n \overline{F}(u_{z_0}, \Omega, u_{z_0}(x_0) + (u - u_{z_0}(x_0))_{x_0,t}) \leq t^n \liminf_{h \rightarrow +\infty} \int_{\Omega} f(\nabla v_h) \, d\mathcal{L}^n \leq \overline{F}(u_{z_0}, x_0 + t(\Omega - x_0), u). \quad (4.12)$$

In conclusion, (4.10) follows by (4.12) and the $L^1(\Omega)$ -lower semicontinuity of $\overline{F}(u_{z_0}, \Omega, \cdot)$, since

$$u_{z_0}(x_0) + (u - u_{z_0}(x_0))_{x_0,t} \rightarrow u$$

in $L^1(\Omega)$ as $t \rightarrow 1^+$.

Let now $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ be convex, $u \in BV(\Omega)$, and let us define \hat{u} as the extension of u to the whole \mathbb{R}^n obtained by defining $\hat{u} = u_{z_0}$ in $\mathbb{R}^n \setminus \Omega$. Then $u \in BV_{\text{loc}}(\mathbb{R}^n)$. Let us fix $x_0 \in \Omega$, and take $t > 1$. Then the convexity of Ω yields $\overline{\Omega} \subseteq x_0 + t(\Omega - x_0)$ and $\text{spt}(\hat{u} - u_{z_0}) \subseteq x_0 + t(\Omega - x_0)$.

By Lemma 4.3, we have that

$$\begin{aligned} \overline{F}(u_{z_0}, x_0 + t(\Omega - x_0), \hat{u}) &\leq \int_{x_0+t(\Omega-x_0)} f^{**}(\nabla \hat{u}) \, d\mathcal{L}^n + \int_{x_0+t(\Omega-x_0)} (f^{**})^\infty \left(\frac{dD^s \hat{u}}{d|D^s \hat{u}|} \right) d|D^s \hat{u}| \\ &= \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + f^{**}(z_0) \mathcal{L}^n((x_0 + t(\Omega - x_0)) \setminus \Omega) \\ &\quad + \int_{\Omega} (f^{**})^\infty \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u| + \int_{\partial\Omega} (f^{**})^\infty \left(\frac{dD^s \hat{u}}{d|D^s \hat{u}|} \right) d|D^s \hat{u}|. \end{aligned} \quad (4.13)$$

At this point, once we recall that, by (1.5), $D\hat{u} = (u_{z_0} - u)\mathbf{n}_\Omega \mathcal{H}^{n-1}$ on $\partial\Omega$, by (4.13), the finiteness of $f^{**}(z_0)$, and the 1-homogeneity of $(f^{**})^\infty$, we infer that

$$\begin{aligned} \limsup_{t \rightarrow 1^+} \overline{F}(u_{z_0}, x_0 + t(\Omega - x_0), \hat{u}) \\ \leq \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + \int_{\Omega} (f^{**})^\infty \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u| + \int_{\partial\Omega} (f^{**})^\infty ((u_{z_0} - u)\mathbf{n}_\Omega) \, d\mathcal{H}^{n-1}. \end{aligned} \quad (4.14)$$

Therefore, by (4.14) and (4.10), since clearly $\overline{F}(u_{z_0}, \Omega, \hat{u}) = \overline{F}(u_{z_0}, \Omega, u)$, the lemma follows. \square

The previous lemmas allow us to prove the representation result for the functional in (1.12).

Theorem 4.5. *Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be Borel and satisfy (4.1). Assume that $z_0 \in \text{int}(\text{co}(\text{dom } f))$, and let $\overline{F}(u_{z_0}, \cdot, \cdot)$ be given by (1.12). Then*

$$\begin{aligned} \overline{F}(u_{z_0}, \Omega, u) &= \int_{\Omega} f^{**}(\nabla u) \, d\mathcal{L}^n + \int_{\Omega} (f^{**})^\infty \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u| + \int_{\partial\Omega} (f^{**})^\infty ((u_{z_0} - u)\mathbf{n}_\Omega) \, d\mathcal{H}^{n-1} \\ &\text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n) \text{ convex, } u \in BV(\Omega). \end{aligned}$$

Proof. Follows from (1.17) and Lemma 4.4. \square

As in the case of Neumann problems, from Theorem 4.5 we deduce the corollaries below. In the first one the constraint condition is emphasized. The second one provides information on the structure of the set of the solutions Dirichlet problems with linear boundary data for first order differential inclusions.

Corollary 4.6. *Let $g : \mathbb{R}^n \rightarrow [0, +\infty[$ be Borel, and let E be a Borel subset of \mathbb{R}^n with $\text{int}(\text{co}(E)) \neq \emptyset$. Let $z_0 \in \text{int}(\text{co}(E))$. Then*

$$\inf \left\{ \liminf_{h \rightarrow +\infty} \int_{\Omega} g(\nabla u_h) \, d\mathcal{L}^n : \{u_h\} \subseteq u_{z_0} + W_0^{1,\infty}(\Omega), \nabla u_h(x) \in E \text{ for every } h \in \mathbb{N} \text{ and a.e. } x \in \Omega, \right. \\ \left. u_h \rightarrow u \text{ in } L^1(\Omega) \right\} \\ = \int_{\Omega} (g + I_E)^{**}(\nabla u) \, d\mathcal{L}^n + \int_{\Omega} ((g + I_E)^{**})^{\infty} \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u| + \int_{\partial\Omega} ((g + I_E)^{**})^{\infty} ((u_{z_0} - u)\mathbf{n}_{\Omega}) \, d\mathcal{H}^{n-1} \\ \text{for every } \Omega \in \mathcal{A}_0(\mathbb{R}^n) \text{ convex, } u \in BV(\Omega).$$

Proof. Follows from Theorem 4.5 applied to $f = g + I_E$. \square

Corollary 4.7. *Let $E \subseteq \mathbb{R}^n$ be Borel with $\text{int}(\text{co}(E)) \neq \emptyset$. Then, for every $z_0 \in \text{int}(\text{co}(E))$, $\Omega \in \mathcal{A}_0(\mathbb{R}^n)$ convex, and $u \in u_{z_0} + W_0^{1,1}(\Omega)$ such that $\nabla u(x) \in \overline{\text{co}(E)}$ for a.e. $x \in \Omega$, there exists $\{u_h\}$ in $u_{z_0} + W_0^{1,\infty}(\Omega)$ such that $u_h \rightarrow u$ in $L^1(\Omega)$, and $\nabla u_h(x) \in E$ for every $h \in \mathbb{N}$ and a.e. $x \in \Omega$.*

Proof. Follows from Corollary 4.6 applied with $g = 0$, once we recall that $(I_E)^{**} = I_{\overline{\text{co}(E)}}$. \square

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