Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 24 (2007) 167–169
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Corrigendum

Corrigendum for the comparison theorems in: "A new definition of viscosity solutions for a class of second-order degenerate elliptic integro-differential equations" [Ann. I. H. Poincaré – AN 23 (5) (2006) 695–711]

Mariko Arisawa

GSIS, Tohoku University, Aramaki 09, Aoba-ku, Sendai 980-8579, Japan

Available online 13 November 2006

In this note, we shall present the correction of the proofs of the comparison results in the paper [1]. In order to show clearly the correct way of the demonstration, we shall simplify the problem to the following.

(Problem (I)):

$$F\big(x,u,\nabla u,\nabla^2 u\big)-\int\limits_{\mathbf{R}^\mathbf{N}} u(x+z)-u(x)-\mathbf{1}_{|z|\leqslant 1}\big\langle z,\nabla u(x)\big\rangle q(\mathrm{d}z)=0 \quad \text{in } \Omega, \tag{1}$$

(Problem (II)):

$$F\big(x,u,\nabla u,\nabla^2 u\big)-\int\limits_{\{z\in\mathbf{R}^\mathbf{N}\mid x+z\in\overline{\Omega}\}} u(x+z)-u(x)-\mathbf{1}_{|z|\leqslant 1}\big\langle z,\nabla u(x)\big\rangle q(\mathrm{d}z)=0 \quad \text{in } \Omega, \tag{2}$$

where $\Omega\subset\mathbf{R}^\mathbf{N}$ is open, and $q(\mathrm{d}z)$ is a positive Radon measure such that $\int_{|z|\leqslant 1}|z|^2q(\mathrm{d}z)+\int_{|z|>1}1q(\mathrm{d}z)<\infty$. Although in [1] only (II) was studied, in order to avoid the non-essential technical complexity, here, let us give the explanation mainly for (I). For (I), we consider the Dirichlet B.C.:

$$u(x)=g(x)\quad \forall x\in\Omega^c, \tag{3}$$

where g is a given continuous function in Ω^c. For (II), we assume that Ω is a precompact convex open subset in $\mathbf{R}^\mathbf{N}$ with C^1 boundary satisfying the uniform exterior sphere condition, and consider either the Dirichlet B.C.:

$$u(x)=h(x)\quad \forall x\in\partial\Omega, \tag{4}$$

where h is a given continuous function on $\partial\Omega$, or the Neumann B.C.:

$$\big\langle\nabla u(x),\mathbf{n}(x)\big\rangle=0\quad \forall x\in\partial\Omega, \tag{5}$$

where $\mathbf{n}(x)\in\mathbf{R}^\mathbf{N}$ the outward unit normal vector field defined on $\partial\Omega$. The above problems are studied in the framework of the viscosity solutions introduced in [1]. Under all the assumptions in [1], for (I) the following comparison result holds, and for (II), although the proofs therein are incomplete, the comparison results stated in [1] hold, and we shall show in a future article.

DOI of original article: 10.1016/j.anihpc.2005.09.002.

E-mail address: arisawa@math.is.tohoku.ac.jp (M. Arisawa).

doi:10.1016/j.anihpc.2006.10.001

Theorem 1.1 *(Problem (I) with Dirichlet B.C.). Assume that Ω is bounded, and the conditions for F in* [1] *hold. Let $u \in \mathrm{USC}(\mathbf{R^N})$ and $v \in \mathrm{LSC}(\mathbf{R^N})$ be respectively a viscosity subsolution and a supersolution of* (1) *in Ω, which satisfy $u \leqslant v$ on Ω^c. Then, $u \leqslant v$ in Ω.*

To prove Theorem 1.1, we approximate the solutions u and v by the supconvolution: $u^r(x) = \sup_{y\in\mathbf{R^N}}\{u(y) - \frac{1}{2r^2}|x-y|^2\}$ and the infconvolution: $v_r(x) = \inf_{y\in\mathbf{R^N}}\{v(y) + \frac{1}{2r^2}|x-y|^2\}$ ($x \in \mathbf{R^N}$), where $r > 0$.

Lemma 1.2 *(Approximation for Problem (I)). Let u and v be respectively a viscosity subsolution and a supersolution of* (1). *For any $\nu > 0$ there exists $r > 0$ such that u^r and v_r are respectively a subsolution and a supersolution of the following problems.*

$$F\big(x, u, \nabla u, \nabla^2 u\big) - \int_{\mathbf{R^N}} u(x+z) - u(x) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla u(x)\big\rangle q(\mathrm{d}z) \leqslant \nu, \tag{6}$$

$$F\big(x, v, \nabla v, \nabla^2 v\big) - \int_{\mathbf{R^N}} v(x+z) - v(x) - \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla v(x)\big\rangle q(dz) \geqslant -\nu, \tag{7}$$

in $\Omega_r = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \sqrt{2Mr}\}$, where $M = \max\{\sup_{\overline{\Omega}} |u|, \sup_{\overline{\Omega}} |v|\}$.

Remark that u^r is semiconvex, v_r is semiconcave, and both are Lipschitz continuous in $\mathbf{R^N}$. We then deduce from the Jensen's maximum principle and the Alexandrov's theorem (deep results in the convex analysis, see [2] and [3]), the following lemma, the last claim of which is quite important in the limit procedure in the nonlocal term.

Lemma 1.3. *Let U be semiconvex and V be semiconcave in Ω. For $\phi(x,y) = \alpha|x-y|^2$ ($\alpha > 0$) consider $\Phi(x,y) = U(x) - V(y) - \phi(x,y)$, and assume that $(\bar{x}, \bar{y})$ is an interior maximum of Φ in $\overline{\Omega} \times \overline{\Omega}$. Assume also that there is an open precompact subset O of $\Omega \times \Omega$ containing $(\bar{x}, \bar{y})$, and that $\mu = \sup_O \Phi(x,y) - \sup_{\partial O} \Phi(x,y) > 0$. Then, the following holds.*

(i) *There exists a sequence of points $(x_m, y_m) \in O$ ($m \in \mathbf{N}$) such that $\lim_{m\to\infty}(x_m, y_m) = (\bar{x}, \bar{y})$, and $(p_m, X_m) \in J_{\Omega}^{2,+}U(x_m)$, $(p'_m, Y_m) \in J_{\Omega}^{2,-}V(y_m)$ such that*

$$\lim_{m\to\infty} p_m = \lim_{m\to\infty} p'_m = 2\alpha(x_m - y_m) = p,$$

and $X_m \leqslant Y_m$ $\forall m$.

(ii) *For $P_m = (p_m - p, -(p'_m - p))$, $\Phi_m(x,y) = \Phi(x,y) - \langle P_m, (x,y)\rangle$ takes a maximum at (x_m, y_m) in O.*

(iii) *The following holds for any $z \in \mathbf{R^N}$ such that $(x_m + z, y_m + z) \in O$.*

$$U(x_m+z) - U(x_m) - \langle p_m, z\rangle \leqslant V(y_m+z) - V(y_m) - \langle p'_m, z\rangle. \tag{8}$$

By admitting these lemmas here, let us show how Theorem 1.1 is proved.

Proof of Theorem 1.1. We use the argument by contradiction, and assume that $\max_{\overline{\Omega}}(u-v) = (u-v)(x_0) = M_0 > 0$ for $x_0 \in \Omega$. Then, we approximate u by u^r (supconvolution) and v by v_r (infconvolution), which are a subsolution and a supersolution of (6) and (7), respectively. Clearly, $\max_{\overline{\Omega}}(u^r - v_r) \geqslant M_0 > 0$. Let $\bar{x} \in \Omega$ be the maximizer of $u^r - v_r$. In the following, we abbreviate the index and write $u = u^r$, $v = v_r$ without any confusion. As in the PDE theory, consider $\Phi(x,y) = u(x) - v(y) - \alpha|x-y|^2$, and let $(\hat{x}, \hat{y})$ be the maximizer of Φ. Then, from Lemma 1.3 there exists $(x_m, y_m) \in \Omega$ ($m \in \mathbf{N}$) such that $\lim_{m\to\infty}(x_m, y_m) = (\hat{x}, \hat{y})$, and we can take $(\varepsilon_m, \delta_m)$ a pair of positive numbers such that

$$u(x_m+z) \leqslant u(x_m) + \langle p_m, z\rangle + \frac{1}{2}\langle X_m z, z\rangle + \delta_m|z|^2, \qquad v(y_m+z) \geqslant v(y_m) + \langle p'_m, z\rangle + \frac{1}{2}\langle Y_m z, z\rangle - \delta_m|z|^2,$$

for $\forall|z| \leqslant \varepsilon_m$. From the definition of the viscosity solutions, we have

$$F\big(x_m, u(x_m), p_m, X_m\big) - \int_{|z|\leqslant\varepsilon_m} \frac{1}{2}\big\langle (X_m + 2\delta_m I)z, z\big\rangle \mathrm{d}q(z)$$

$$- \int_{|z|\geqslant\varepsilon_m} u(x_m+z) - u(x_m) - \mathbf{1}_{|z|\leqslant 1}\langle z, p_m\rangle q(\mathrm{d}z) \leqslant \nu,$$

$$F\big(y_m, v(y_m), p'_m, Y_m\big) - \int_{|z|\leqslant\varepsilon_m} \frac{1}{2}\big\langle (Y_m - 2\delta_m I)z, z\big\rangle \mathrm{d}q(z)$$

$$- \int_{|z|\geqslant\varepsilon_m} v(y_m+z) - v(y_m) - \mathbf{1}_{|z|\leqslant 1}\langle z, p'_m\rangle q(\mathrm{d}z) \geqslant -\nu.$$

By taking the difference of the above two inequalities, by using (8), and by passing $m \to \infty$ (thanking to (8), it is now available), we can obtain the desired contradiction. The claim $u \leqslant v$ is proved. □

Remark 1.1. To prove the comparison results for (II) (in [1]), we do the approximation by the supconvolution: $u^r(x) = \sup_{y\in\overline{\Omega}}\{u(y) - \frac{1}{2r^2}|x-y|^2\}$, and the infconvolution: $v_r(x) = \inf_{y\in\overline{\Omega}}\{v(y) + \frac{1}{2r^2}|x-y|^2\}$ as in Lemma 1.2. Because of the restriction of the domain of the integral of the nonlocal term and the Neumann B.C., a slight technical complexity is added. The approximating problem for (2)–(5) in $\overline{\Omega}$ is as follows.

$$\min\Bigg[F\big(x, u(x), \nabla u(x), \nabla^2 u(x)\big) + \min_{y\in\overline{\Omega},\, |x-y|\leqslant\sqrt{2Mr}} \Bigg\{- \int_{\{z\in\mathbf{R^N} \mid y+z\in\overline{\Omega}\}} u(x+z) - u(x)$$

$$- \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla u(x)\big\rangle q(\mathrm{d}z), \min_{y\in\partial\Omega,\, |x-y|\leqslant\sqrt{2Mr}} \big\{\big\langle \mathbf{n}(y), \nabla u(x)\big\rangle + \rho\big\}\Bigg\}\Bigg] \leqslant \nu,$$

$$\max\Bigg[F\big(x, v(x), \nabla v(x), \nabla^2 v(x)\big) + \max_{y\in\overline{\Omega},\, |x-y|\leqslant\sqrt{2Mr}} \Bigg\{- \int_{\{z\in\mathbf{R^N} \mid y+z\in\overline{\Omega}\}} v(x+z) - v(x)$$

$$- \mathbf{1}_{|z|\leqslant 1}\big\langle z, \nabla v(x)\big\rangle q(\mathrm{d}z), \max_{y\in\partial\Omega,\, |x-y|\leqslant\sqrt{2Mr}} \big\{\big\langle \mathbf{n}(y), \nabla v(x)\big\rangle - \rho\big\}\Bigg\}\Bigg] \geqslant -\nu.$$

We deduce the comparison result from this approximation and Lemma 1.3, by using the similar argument as in the proof of Theorem 1.1.

References

[1] M. Arisawa, A new definition of viscosity solutions for a class of second-order degenerate elliptic integro-differential equations, Ann. I. H. Poincaré – AN 23 (5) (2006) 695–711.
[2] M.G. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1) (1992).
[3] W.H. Fleming, H.M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, 1992.