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Correlations and bounds for stochastic volatility models

P.-L. Lions [a,b,*], M. Musiela [c]

[a] *Collège de France, 3, rue d'Ulm, 75005 Paris, France*
[b] *CEREMADE – UMR C.N.R.S. 7534, Université Paris-Dauphine, place du Maréchal de Lattre de Tassigny, 75775 Paris cedex 16, France*
[c] *FIRST, BNP-Paribas, 10 Harewood Avenue, London NW1 6AA, United Kingdom*

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Abstract

We investigate here, systematically and rigorously, various stochastic volatility models used in Mathematical Finance. Mathematically, such models involve coupled stochastic differential equations with coefficients that do not obey the natural and classical conditions required to make these models "well-posed". And we obtain necessary and sufficient conditions on the parameters, such as correlation, of these models in order to have integrable or L^p solutions (for $1 < p < \infty$).

1. Introduction

This paper is the first of a series devoted to the rigorous mathematical analysis of (some of) the most popular and classical stochastic volatility models in Finance (see [1–4] and [7–9]), some of which are indeed used by financial engineers for practical purposes.

In this paper, we present sharp conditions (most of the time, necessary and sufficient conditions) under which these models are meaningful – in a sense that will be made precise later on. And we shall show that the correlation parameter is *the* crucial parameter that makes the model meaningful or not. Let us immediately point out that we shall investigate, in the following articles of this series, various issues such as the long time behaviour of option prices, the well-posedness for fractional power models and semi-explicit formulas, or short time asymptotics... .

At this stage, let us detail the type of mathematical information we shall derive. And we begin with the following particular case – that we treat in Section 2 below together with some more general volatility equations...

$$\mathrm{d}F_t = \sigma_t F_t \,\mathrm{d}W_t, \quad F_0 = F > 0, \tag{1}$$

$$\mathrm{d}\sigma_t = \alpha\sigma_t \,\mathrm{d}Z_t, \quad \sigma_0 = \sigma > 0 \tag{2}$$

where $Z_t = \rho W_t + \sqrt{1-\rho^2} B_t$; (W_t, B_t) is a standard two-dimensional Brownian motion, $\alpha > 0$ and $\rho \in [-1, +1]$ is the correlation parameter.

Although (1), (2) is a very classical stochastic volatility model "used" in Mathematical Finance, we are not aware of much rigorous mathematical analysis of the stochastic system (1), (2). And, surprisingly enough, we show here that

* Corresponding author.
E-mail address: lions@ceremade.dauphine.fr (P.-L. Lions).

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stringent assumptions on the correlation ρ are necessary to make it a meaningful system! More precisely, we show below the following facts (after defining in a straightforward way $F_t \ldots$):

- F_t is, for all ρ, a continuous positive, integrable supermartingale;
- F_t is a martingale if and only if $\rho \leqslant 0$ (if $\rho > 0$, $E[F_t] < F$ for all $t > 0$!);
- If $\rho = 0$, F_t is a continuous *integrable* martingale such that $\sup_{t\in[0,T]} E[F_t|\mathrm{Log}\, F_t|] < 0$, $\sup_{t\in[0,T]} E[F_t^m] = +\infty$ for any $m > 1$, for all $T \in (0, +\infty)$;
- Let $m > 1$, then $\sup_{t\in[0,T]} E[F_t^m] < \infty$ for all $t \in (0, \infty)$ if and only if $\rho \leqslant -\sqrt{(m-1)/m}$;
- If $\rho = -1$, then F_t is bounded for all $t \geqslant 0$.

Additional informations are provided below. Let us only mention here that the difficulty with the system (1), (2) stems from the fact that the mapping $[(\sigma, F) \mapsto (\alpha\sigma, \sigma F)]$ does not satisfy the usual condition of linear growth at infinity. A slightly more precise interpretation consists in recalling the Novikov criterion (which is sufficient to guarantee the fact that "F_t" is a martingale, see for instance [5]...) namely

$$E\left[\exp\left\{\frac{1}{2}\int_0^T \sigma_t^2\,\mathrm{d}t\right\}\right] < \infty, \quad \text{for all } T > 0.$$

However, one can check that this never holds (for any ρ) if σ solves (2)!

The conclusions for our study in Section 2 are thus:

(i) the model (1), (2) is not, in general, well-posed and
(ii) in order to use a well-posed model, and if we wish to be able to manipulate standard objects such as the variance of F for instance, one needs to assume (at least) that $-1 \leqslant \rho \leqslant -1/\sqrt{2}$ (take $m = 2$ in the above results...).

The proof of the above results is given in Section 2 (2.1–2.4). The final Section 2.5 is devoted to the extension of these results to the case of a general stochastic equation for σ_t in face of (2).

Section 3 is devoted to the study of general models of the following type

$$\mathrm{d}F_t = \sigma_t^\delta F_t^\beta\,\mathrm{d}W_t, \tag{1'}$$

$$\mathrm{d}\sigma_t = \alpha\sigma_t^\gamma\,\mathrm{d}Z_t + b(\sigma_t)\,\mathrm{d}t \tag{2'}$$

where δ, β, γ are positive parameters (with some natural restrictions that we do not detail here) and b represents a trend (= drift) that could be for instance: $b(\sigma) = -b(\sigma - \sigma_*)$ for some given $b, \sigma_* > 0$.

We shall then give and prove necessary and sufficient conditions for the well-posedness of (1′), (2′) in terms of δ, β, γ and ρ (the correlation between W_t and Z_t). Once more, we shall also briefly mention extensions to more general expressions than power laws in (1′), (2′).

2. Log-normal like models

2.1. Preliminaries

We first wish to derive explicit formulae for σ and F. As is well-known, the solution σ of (2) is a continuous martingale (integrable to all powers...) given by

$$\sigma_t = \sigma\exp\left\{\alpha Z_t - \frac{\alpha^2}{2}t\right\}, \quad \text{for } t \geqslant 0. \tag{3}$$

Then, at least formally, we expect "the" solution of (1) to be given by

$$F_t = F\exp\left\{\int_0^t \sigma_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma_s^2\,\mathrm{d}s\right\}, \quad \text{for } t \geqslant 0, \tag{4}$$

which is a continuous positive process. This, however, needs to justified. And we do so by a classical localisation argument. We introduce, for instance, for all $n \geqslant 1$

$$\tau_n = \inf\{t \geqslant 0 \mid \sigma_t > n\}. \tag{5}$$

Of course, $\tau_n \underset{n}{\to} +\infty$ a.s. and solving (1) "up to time τ_n" yields

$$F_{t\wedge\tau_n} = F_t^n = F \exp\left\{\int_0^t \sigma_s 1_{(s<\tau_n)}\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma_s^2 1_{(s<\tau_n)}\,\mathrm{d}s\right\}, \quad \forall t \geqslant 0. \tag{6}$$

And F_t^n converges a.s. to F_t given by (4) (for all $t \geqslant 0$) as n goes to $+\infty$.

This allows to show the following easy (and general) fact

Proposition 2.1. *F_t is a continuous, positive, integrable supermartingale.*

Proof. The integrability is a trivial consequence of Fatou's lemma and of the fact that F_t^n is a martingale. Indeed, we have for all $t \geqslant 0$

$$E[F_t] \leqslant \underline{\lim}_n\, E\big[F_t^n\big] = F.$$

Similarly, F_t is a supermartingale since, for all $M > 0$, $F_t^n \wedge M$ is a supermartingale and $F_t^n \wedge M$ converges a.s. and thus in L^1 to $F_t \wedge M$. □

2.2. The independent case

Proposition 2.2. *If $\rho = 0$, then we have for all $T > 0$*

$$\sup_{t\in[0,T]} E\big[F_t|\mathrm{Log}\,F_t|\big] < \infty \tag{7}$$

and F_t is a continuous integrable martingale.

Proof. (i) Since the function $(x \mapsto x\mathrm{Log}\,x)$ is bounded when negative, it is enough to prove (7) replacing $|\mathrm{Log}\,F_t|$ by $\mathrm{Log}\,F_t$ and the above proof of Proposition 2.1 will yield (7) as soon as we show for all $t > 0$

$$\sup_n E\big[F_t^n \log F_t^n\big] < \infty \tag{8}$$

(observe indeed that $(x \mapsto x\mathrm{Log}\,x)$ is convex and thus $E[F_t^n \mathrm{Log}\,F_t^n]$ is nondecreasing in t since F_t^n is a martingale...).

Then, we observe that we have

$$\begin{aligned} E\big[F_t^n \mathrm{Log}\,F_t^n\big] &= E\left\{\left[\mathrm{Log}\,F + \int_0^t \sigma_s 1_{(s<\tau_n)}\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma_s^2 1_{(s<\tau_n)}\,\mathrm{d}s\right] F_t^n\right\} \\ &= F\mathrm{Log}\,F + F\widehat{E}\left[\int_0^t \sigma_s 1_{(s<\tau_n)}\big(\mathrm{d}W_s + \sigma_s 1_{(s<\tau_n)}\,\mathrm{d}s\big) - \frac{1}{2}\int_0^t \sigma_s^2 1_{(s<\tau_n)}\,\mathrm{d}s\right] \end{aligned}$$

where we used a Girsanov transform ("changing W_t into $W_t + \int_0^t \sigma_s 1_{(s<\tau_n)}\,\mathrm{d}s$").

Hence, we have for all $n \geqslant 1$, $t \geqslant 0$

$$\begin{aligned} E\big[F_t^n \mathrm{Log}\,F_t^n\big] &= F\mathrm{Log}\,F + \frac{F}{2}\widehat{E}\left(\int_0^t \sigma_s^2 1_{(s<\tau_n)}\,\mathrm{d}s\right) \leqslant F\mathrm{Log}\,F + \frac{F}{2}\widehat{E}\left(\int_0^t \sigma_s^2\,\mathrm{d}s\right) \\ &= F\mathrm{Log}\,F + \frac{F\sigma^2}{2}\left(\frac{\mathrm{e}^{\alpha^2 t} - 1}{\alpha^2}\right). \end{aligned}$$

(ii) The bound (7) combined with the proof of Proposition 1 now immediately yields the fact F_t^n converges in L^1 to F_t and, thus, F_t is a martingale. □

Remarks.

(i) As is well known, (7) yields the integrability of $\sup_{t\in[0,T]}|F_t|$ for all $T>0$;
(ii) A similar argument to the one used in the proof of Proposition 2.2 yields the following bounds for all $T>0$

$$\sup_{t\in[0,T]} E\big[F_t|\mathrm{Log}\,F_t|^m\big]<\infty, \quad \text{for all } m\geqslant 1; \tag{9}$$

$$\sup_{t\in[0,T]} E\big[\sigma_t^p F_t|\mathrm{Log}\,F_t|^m\big]<\infty, \quad \text{for all } m,p\geqslant 1; \tag{10}$$

(iii) As we shall see below, when $\rho=0$, F_t is *not* integrable to any power larger than 1.

2.3. The positive correlation case

We want to show now that $E[F_t]<F$ if $\rho>0$, $t>0$. In fact, the argument shown below yields the strict monotonicity of $E[F_t]$ with respect to t.

Indeed, we go back to the proof of Proposition 2.1 and write

$$F=E[F_{t\wedge\tau_n}]=E[F_t 1_{(t\leqslant\tau_n)}]+E[F_{\tau_n}1_{(\tau_n<t)}].$$

Since $E[F_t 1_{(t\leqslant\tau_n)}]\uparrow_n E[F_t]$, our claim will be shown if we prove that we have for all $t>0$

$$\varliminf_n E[F_{\tau_n}1_{(\tau_n<t)}]>0. \tag{11}$$

In order to do so, we write

$$E[F_{\tau_n}1_{(\tau_n<t)}]=FE\left\{1_{(\tau_n<t)}\exp\left\{\int_0^{\tau_n}\sigma_s\,\mathrm{d}W_s-\frac{1}{2}\int_0^{\tau_n}\sigma_s^2\,\mathrm{d}s\right\}\right\}$$

and we use Girsanov formula to obtain a new measure $\widehat{P}$ under which σ solves (up to τ_n)

$$\mathrm{d}\sigma_t=\alpha\sigma_t\,\mathrm{d}Z_t+\alpha\rho\sigma_t^2\,\mathrm{d}t. \tag{12}$$

And we have

$$E[F_{\tau_n}1_{(\tau_n<t)}]=F\widehat{P}(\tau_n<t).$$

Of course, it is well-known that the local solution of (12) blows up and thus $\widehat{P}(\tau_\infty<t)>0$ for $t>0$ large enough, where $\tau_\infty=\lim_n\uparrow\tau_n$ is the maximal existence time. This would, of course, be sufficient to yield the fact that $E[F_t]<F$ for t large enough. We are not aware, however, of more precise blow-up results which show that $\widehat{P}(\tau_\infty<t)>0$ for all $t>0$. And we do so by the following relatively easy argument where we fix $t=t_0>0$. First of all, all the differential equations we write below are understood to hold up to τ_∞. Then, we write $x_t=\mathrm{Log}\,\sigma_t$ and we have

$$\mathrm{d}x_t=\alpha\,\mathrm{d}Z_t+\left(\alpha\rho\,\mathrm{e}^{x_t}-\frac{\alpha^2}{2}\right)\mathrm{d}t, \quad x_0=x=\mathrm{Log}\,\sigma. \tag{13}$$

Next, we recall that for each $\lambda>0$, there exists a positive constant $\nu(=\nu(\lambda))$ such that $P(\sup_{t\in[0,t_0]}|Z_t-\lambda t|<1)\geqslant\nu$.

Hence, we have, on the set $A=\{\sup_{t\in[0,t_0]}|Z_t-\lambda t|<1\}$, the following inequalities

$$x_t\geqslant\alpha(\lambda t-1)+\alpha\rho\int_0^t \mathrm{e}^{x_s}\,\mathrm{d}s-\frac{\alpha^2}{2}t,$$

$$(x_t+\alpha) \geqslant (\alpha\rho\,\mathrm{e}^{-\alpha}) \int_0^t \left[\mathrm{e}^{x_s+\alpha} + \frac{\mathrm{e}^{\alpha}}{\rho}\left(\lambda - \frac{\alpha}{2}\right)\right] \mathrm{d}s,$$

$$x_t + \alpha \geqslant y_t$$

where y_t solves:

$$\dot{y}_t = (\alpha\rho\,\mathrm{e}^{-\alpha})\left(\mathrm{e}^{y} + \frac{\mathrm{e}^{\alpha}}{\rho}\left(\lambda - \frac{\alpha}{2}\right)\right), \quad y_0 = \operatorname{Log}\sigma + \alpha.$$

We then choose λ large enough such that the solution y_t of this ODE blows up for $t < t_0$. Thus, $\tau_\infty < t_0$ on A and $P(\tau_\infty < t_0) \geqslant \nu(\lambda)$.

2.4. The negative correlation case

Theorem 2.3. *If $\rho < 0$, F_t is a continuous integrable martingale and, for any $m > 1$, $\sup_{t\in[0,T]} E[F_t^m] < \infty$ for all $T > 0$ if and only if $\rho \leqslant -\sqrt{(m-1)/m}$.*

Remarks.

(i) The proof below may be refined to yield the following fact: if $\rho > -\sqrt{(m-1)/m}$, then $E[F_t^m] = +\infty$ for all $t > 0$.

(ii) Also, the proof below allows to show the following bounds for all $T \in (0, +\infty)$, $p \geqslant 1$: if $-1 \leqslant \rho \leqslant -\sqrt{(m-1)/m}$, then

$$\sup_{t\in[0,T]} E\left[F_t^m\left(1+\sigma_t^p\right)\right] < \infty.$$

In particular, if $m = 2$ and $-1 \leqslant \rho \leqslant -1/\sqrt{2}$, then, $\sup_{t\in[0,T]} E[F_t^2(1+\sigma_t^2)] < \infty$ and Eq. (1) holds!

(iii) Of course, if $\rho = -1$, Theorem 2.3 yields the finiteness of all moments of F_t. In that case, however, F_t is even bounded. Indeed, we have

$$F_t = F \exp\left\{\int_0^t \sigma_s\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \sigma_s^2\,\mathrm{d}s\right\}$$

and $\sigma_t = \sigma - \alpha\int_0^t \sigma_s\,\mathrm{d}W_s$. Hence,

$$F_t = F \exp\left\{\frac{1}{\alpha}(\sigma - \sigma_t) - \frac{1}{2}\int_0^t \sigma_s^2\,\mathrm{d}s\right\} \leqslant F \exp\left(\frac{\sigma}{\alpha}\right).$$

(iv) The proof below also shows the following monotonicity in ρ property of certain expectations. More precisely, if we consider for any $m \geqslant 1$ $\varphi(\rho) = E[F_t^m \Phi(\sigma)]$ for some functional Φ over $C([0,t])$ such that $\Phi(\sigma_1) \leqslant \Phi(\sigma_2)$ if $\sigma_1 \leqslant \sigma_2$ on $[0,t]$, then φ is nondecreasing with respect to ρ.

Proof of Theorem 2.3. (i) We begin with the case when $\rho \leqslant -\sqrt{(m-1)/m}$. In that case, the arguments introduced above show that it is enough to prove the following bound for all $t > 0$

$$\sup_n E\left[F_{t\wedge\tau_n}^m\right] < \infty.$$

Next, we write

$$\left[F_{t\wedge\tau_n}^m\right] = F^m E\left[\exp\left\{\frac{m^2-m}{2}\int_0^{t\wedge\tau_n} \sigma_s^2\,\mathrm{d}s\right\}\exp\left\{m\int_0^{t\wedge\tau_n} \sigma_s\,\mathrm{d}W_s - \frac{m^2}{2}\int_0^t \sigma_s^2\,\mathrm{d}s\right\}\right].$$

Hence, using once more Girsanov formula, we have

$$E\big[F^m_{t\wedge\tau_n}\big] = F^m \widehat{E}\left(\exp\left\{\frac{m^2-m}{2}\int_0^{t\wedge\tau_n}\sigma_s^2\,\mathrm{d}s\right\}\right)$$

where $\widehat{P}$ is a new measure under which σ solves

$$\mathrm{d}\sigma_t = \alpha\sigma_t\,\mathrm{d}Z_t + \alpha\rho m\sigma_t^2\,\mathrm{d}t. \tag{14}$$

We then introduce the function $w_n(\sigma,t)$ defined by

$$\widehat{E}\left(\exp\left\{\frac{m^2-m}{2}\int_0^{t\wedge\tau_n}\sigma_s^2\,\mathrm{d}s\right\}\right)$$

and, as is well known, w_n is the unique smooth solution of

$$\begin{cases} \dfrac{\partial w_n}{\partial t} - \dfrac{\alpha^2}{2}\sigma^2\dfrac{\partial^2 w_n}{\partial\sigma^2} - \alpha\rho m\sigma^2\dfrac{\partial w}{\partial\sigma} - \dfrac{m^2-m}{2}\sigma^2 w_n = 0 \quad \text{for } 0\leqslant\sigma\leqslant n,\ t\geqslant 0,\\ w_n|_{t=0} = 1 \quad \text{for } 0\leqslant\sigma\leqslant n, \quad w_n(n,t)=1 \quad \text{for } t\geqslant 0. \end{cases} \tag{15}$$

We finally obtain a bound on w_n by building an explicit supersolution of (15) and using the maximum principle (or if one prefers to avoid the use of differential equations, one may just apply Itô's formula to that supersolution ...). Indeed, let $\overline{w} = \exp(\mu\sigma)$ for $\mu\geqslant 0$. Obviously, $\overline{w}\geqslant 1$ if $\sigma\geqslant 0$ and

$$-\frac{\alpha^2}{2}\sigma^2\frac{\partial^2\overline{w}}{\partial\sigma^2} - \alpha\rho m\sigma^2\frac{\partial\overline{w}}{\partial\sigma} - \frac{m^2-m}{2}\sigma^2\overline{w} = \sigma^2\overline{w}\left\{-\frac{\alpha^2}{2}\mu^2 - \alpha\rho m\mu - \frac{m^2-m}{2}\right\}.$$

This quantity is nonnegative as soon as we can find $\mu\geqslant 0$ such that $-\alpha^2/2\mu^2 - \alpha\rho m\mu - (m^2-m)/2 \geqslant 0$ and this is possible if and only if $\rho^2 m^2 \geqslant m^2 - m$ i.e. $\rho^2 \geqslant (m-1)/m$.

(ii) Next, if $\rho\in(-\sqrt{(m-1)/m}, 0)$, we want to show that $E[F_t^m] = +\infty$ (at least for t large enough, see the above remarks). In order to do so, we observe that the proof made in step (i) yields the following inequality

$$E\big[F_t^m\big] \geqslant E\big[F_t^m 1_{(t<\tau_n)}\big] = F^m\widehat{E}\left(\exp\left\{\frac{m^2-m}{2}\int_0^t\sigma_s^2\,\mathrm{d}s\right\}1_{(t<\tau_n)}\right) = F^m z_n(\sigma,t)$$

and, z_n is the (smooth) solution of

$$\begin{cases} \dfrac{\partial z_n}{\partial t} - \dfrac{\alpha^2}{2}\sigma^2\dfrac{\partial^2 z_n}{\partial\sigma^2} - \alpha\rho m\sigma^2\dfrac{\partial z_n}{\partial\sigma} - \dfrac{m^2-m}{2}\sigma^2 z_n = 0 \quad \text{for } 0\leqslant\sigma\leqslant n,\ t\geqslant 0,\\ z_n|_{t=0} = 1 \quad \text{for } 0\leqslant\sigma\leqslant n, \quad z_n(n,t)=0 \quad \text{for } t>0. \end{cases} \tag{16}$$

It is then possible to show, by a rather technical argument detailed in Appendix A, that $z_n(\sigma,t)\uparrow_n +\infty$ for each $\sigma>0$ and for all $t>0$. □

Remarks. The monotonicity property stated in Remark (iv) after Theorem 2.3 is easily deduced from the preceding proof once we observe that the solution σ of (14) is nondecreasing with respect to ρ by classical results on stochastic differential equations.

2.5. More general volatility equations

In this section, we consider the case of a general equation for the volatility σ in place of (2) namely

$$\mathrm{d}\sigma_t = \mu(\sigma_t)\,\mathrm{d}Z_t + b(\sigma_t)\,\mathrm{d}t, \quad \sigma_0 = \sigma\geqslant 0 \tag{17}$$

where we assume that μ, b are smooth (for instance) functions on $[0,\infty)$ such that

$$\mu(0) = 0, \qquad b(0) \geqslant 0, \tag{18}$$

$$\mu(\xi) > 0 \quad \text{for } \xi > 0, \qquad \mu \text{ is Lipschitz on } [0, \infty), \tag{19}$$

$$b(\xi) \leqslant C(1 + \xi) \quad \text{on } [0, \infty), \text{ for some } C \geqslant 0. \tag{20}$$

These conditions, that we assume throughout this section and do not recall, insure the existence and uniqueness of a nonnegative solution of (17) (such that $E[\sup_{t\in[0,T]} |\sigma_t|^p] < \infty$ for all $1 \leqslant p < \infty$, $T \in (0, \infty)$). Let us immediately mention that it is straightforward to adapt the arguments and thus the results below to situations where (19) or (20) do not hold anymore (assuming, for example, that (19), (20) are replaced by a local Lipschitz on $[0, \infty)$ condition for $\mu \ldots$). We shall not detail here such easy adaptations.

First of all, Proposition 2.1 and its proof are still valid. Next, the analogues of Proposition 2.2 and of the facts stated and shown in Section 2.2 are given by the following

Theorem 2.4.

(i) *If the following condition holds*

$$\limsup_{\xi\to+\infty} \bigl(\rho\mu(\xi)\xi + b(\xi)\bigr)\xi^{-1} < \infty, \tag{21}$$

then $E[F_t |\mathrm{Log}\, F_t|] < \infty$, $E[\sup_{0\leqslant s\leqslant t} |F_s|] < \infty$ *for all* $t \geqslant 0$ *and* F_t *is an integrable nonnegative martingale.*

(ii) *If the following condition holds*

$$\liminf_{\xi\to+\infty} \bigl(\rho\mu(\xi)\xi + b(\xi)\bigr)\varphi(\xi)^{-1} > 0 \tag{22}$$

for some smooth, positive, increasing function φ *such that* $\int^\infty \frac{1}{\varphi}\,\mathrm{d}\xi < \infty$, *then* F_t *is not a martingale and we have*:

$$E[F_t] < F_0 \quad \textit{for all } t > 0.$$

Remarks.

(i) Notice that, if $b = 0$ and $\mu(\xi) = \alpha\xi\,(\alpha > 0)$, in which case (17) reduces to (2) and (21) is equivalent to $\rho \leqslant 0$, while (22) is equivalent to $\rho > 0$ (take $\varphi(\xi) = \xi^2$). And we recover, as a very particular case, the results contained in the preceding sections that concern the integrability of F_t.

(ii) If we assume that μ and b satisfy

$$\lim_{\xi\to+\infty} \frac{\mu(\xi)}{\xi} = \mu_\infty \geqslant 0, \tag{23}$$

$$\lim_{\xi\to+\infty} \frac{b(\xi)}{\xi^2} = b_\infty \in [-\infty, 0], \tag{24}$$

then (21) holds if $\rho\mu_\infty + b_\infty < 0$, while (22) holds if $\rho\mu_\infty + b_\infty > 0$ (again, take $\varphi(\xi) = \xi^2$).

(iii) In the case when $\rho = 0$ (the independent case) or $\rho < 0$ (the negative correlation case), then (20) implies obviously that (21) holds.

Proof of Theorem 2.4. *Proof of part* (i): We only sketch it since it is almost the same as the one of Proposition 2.1. It suffices to observe that, if (21) holds, then we have for some $C \geqslant 0$

$$\rho\mu(\xi)\xi + b(\xi) \leqslant C(1 + \xi) \quad \text{on } [0, \infty)$$

and this allows as to obtain some bounds on $\widehat{E}(\sigma_t^2)$ for all $t > 0$ by a simple application of Itô's formula. The rest of the proof is then exactly the same as the one of Proposition 2.1.

Proof of part (ii): Once more, we only sketch it since it is very similar to the arguments made in Section 2.3: indeed, we only need to show that the blowup time of σ_t solution of

$$\mathrm{d}\sigma_t = \mu(\sigma_t)\,\mathrm{d}Z_t + \bigl(\rho\mu(\sigma_t)\sigma_t + b(\sigma_t)\bigr)\,\mathrm{d}t, \quad \sigma_0 = \sigma > 0 \tag{25}$$

may be made, with positive probability, as small as we wish. In order to do so, we introduce $\psi(\xi)=\int_1^\xi \frac{1}{\mu(\eta)}\,\mathrm{d}\eta$ and consider $\eta_t=\psi(\sigma_t)$ which solves (up to the first blowup time)

$$\mathrm{d}\eta_t=\mathrm{d}Z_t+\left\{\frac{\beta(\sigma_t)}{\mu(\sigma_t)}-\frac{1}{2}\frac{\mu'(\sigma_t)}{\mu^2(\sigma_t)}\right\}\mathrm{d}t$$

where $\beta(\xi)=\rho\sigma(\xi)\xi+b(\xi)$.

We may then conclude as in Section 2.3: observe indeed that we have for some $\nu>0$, $C\geqslant 0$ and for all $\lambda\geqslant 0$

$$\beta(\xi)\geqslant \nu\varphi(\xi)-C,$$
$$\infty>\int_\eta^{+\infty}\left(\frac{\varphi(\psi^{-1}(\eta))}{\mu(\psi^{-1}(\eta))}+\lambda\right)^{-1}\mathrm{d}\eta=\int_\sigma^{+\infty}\big(\varphi(\xi)+\lambda\mu(\xi)\big)^{-1}\,\mathrm{d}\xi\to 0\quad\text{as }\lambda\to+\infty.$$

We next investigate conditions under which $F_t\in L^m$ for some $m\in(1,\infty)$. And we begin with sufficient conditions for the finiteness of $E[F_t^m]$ (or equivalently of $E[\sup_{0\leqslant s\leqslant t}F_s^m]$) for all $t>0$. We thus follow the approach introduced in Section 2.4 above and write, at least formally, using once more a Girsanov transform

$$E\big[F_t^m\big]=F_0^m E\left[\exp\left\{m\int_0^t\sigma_s\,\mathrm{d}W_s-\frac{m}{2}\int_0^t\sigma_s^2\,\mathrm{d}s\right\}\right]=F_0^m\widehat{E}\left[\exp\left(\frac{m^2-m}{2}\int_0^t\sigma_s^2\,\mathrm{d}s\right)\right]$$

where $\widehat{P}$ is a probability under which σ_t solves

$$\mathrm{d}\sigma_t=\mu(\sigma_t)\,\mathrm{d}Z_t+\beta(\sigma_t)\,\mathrm{d}t \tag{26}$$

and $\beta(\xi)=m\mu(\xi)\xi+b(\xi)$.

Still arguing formally, we need to obtain some upper bounds on the solution w of

$$\begin{cases}\dfrac{\partial w}{\partial t}-\dfrac{1}{2}\mu^2(\xi)\dfrac{\partial^2 w}{\partial\xi^2}-\beta(\xi)\dfrac{\partial w}{\partial\xi}-\dfrac{m^2-m}{2}\xi^2 w=0\quad\text{for }\xi\geqslant 0,\ t\geqslant 0,\\ w|_{t=0}=1\quad\text{for }\xi\geqslant 0.\end{cases}\tag{27}$$

If we follow the argument introduced in Section 2.4 above and look for an exponential supersolution of the form

$$\overline{w}(\xi,t)=\exp(A\xi+Bt)$$

where $A\geqslant 0$, $B\geqslant 0$, we are led to the following condition

$$\liminf_{\xi\to+\infty}\left[-\frac{1}{2}A^2\mu^2(\xi)-A\beta(\xi)-\frac{m^2-\mu}{2}\xi^2\right]>-\infty\quad\text{for some }A\geqslant 0.\tag{28}$$

Example. If we assume (23), (24), (28) holds if there exists $A\geqslant 0$ such that

$$-\frac{1}{2}A^2\mu_\infty^2-A(m\rho\mu_\infty+b_\infty)-\frac{m^2-m}{2}>0$$

or equivalently if

$$\rho<-\sqrt{\frac{m-1}{m}}-\frac{b_\infty}{m\mu_\infty}.\tag{29}$$

And, if $\rho m\mu_\infty+b_\infty=-\mu_\infty\sqrt{m^2-m}$, (28) holds if we have

$$\liminf_{\xi\to+\infty}\left\{-\frac{1}{2}A_0^2\big(\mu^2-\mu_\infty^2\xi^2\big)-A_0\big(\beta-(\rho m\mu_\infty+b_\infty)\xi^2\big)-\frac{m^2-m}{2}\xi^2\right\}>-\infty$$

where $A_0=\sqrt{m^2-m}/\mu_\infty$.

In particular, if we have $\mu(\xi) \equiv \mu_\infty \xi$, $b(\xi) = b_\infty \xi^2$ then the condition (28) is equivalent to

$$\rho \leqslant -\sqrt{\frac{m-1}{m}} - \frac{b_\infty}{m\mu_\infty} \tag{29$'$}$$

(and in the case when $b_\infty = 0$, we recover the condition obtained in Section 2.4 above). □

If (28) holds, we can find $A \geqslant 0$, $B \geqslant 0$ such that we have on $[0, \infty)$

$$-\frac{1}{2}A^2\mu^2(\xi) - A\beta(\xi) - \frac{m^2-m}{2}\xi^2 \geqslant -B.$$

Then, $\overline{w}(\xi, t) = \exp(A\xi + Bt)$ is indeed a supersolution of (26). From there on, one can justify as in Section 2.4 that we have

$$E\left[F_t^m\right] \leqslant \overline{w}(\sigma_0, t) \quad \text{for all } t \geqslant 0.$$

and we obtain the following

Theorem 2.5. *If* (28) *holds, and thus in particular if* (23), (24) *and* (29) *hold, then we have for all* $T > 0$

$$E\Big[\sup_{0\leqslant t\leqslant T} F_t^m\Big] < \infty.$$

Remark. Instead of assuming (23), (24) and (29), we may simply assume that (23) and (29) hold, denoting by

$$b_\infty = \limsup_{\xi\to+\infty}\left\{b(\xi)\xi^{-2}\right\} \in [-\infty, 0].$$

We conclude this section with a brief study of necessary conditions. More precisely, we are going to give conditions on μ, b that insure that $E[F_t^m] = +\infty$ for all $t > 0$, $F_0 > 0$, $\sigma_0 > 0$. In order to simplify the presentation, we assume that (μ, b) satisfies (23), (24) (we could, in fact ignore (24) and simply denote by $b_\infty = \liminf_{\xi\to+\infty}\{b(\xi)\xi^{-2}r\}$ where $b_\infty \in [-\infty, 0]$ in view of (20)). Next, we introduce the following condition (to be compared with (29) or (29$'$))

$$\rho > -\sqrt{\frac{m-1}{m}} - \frac{b_\infty}{m\mu_\infty}. \tag{30}$$

And we have the

Theorem 2.6. *If* (30) *holds, then we have for all* $F_0 > 0$, $\sigma_0 > 0$

$$E\left[F_t^m\right] = +\infty.$$

We skip the proof of Theorem 2.6 since it follows step by step the one made in Section 2.4 (and in Appendix A).

3. General models

3.1. Preliminaries

We consider throughout Section 3 the extension of the results and arguments introduced in the previous sections to a (more) general stochastic volatility model of the following type

$$\mathrm{d}F_t = \sigma_t^\delta F_t^\beta\,\mathrm{d}W_t, \quad F_0 = F \geqslant 0, \tag{1$'$}$$

$$\mathrm{d}\sigma_t = \alpha\sigma_t^\gamma\,\mathrm{d}Z_t + b(\sigma_t)\,\mathrm{d}t, \quad \sigma_0 = \sigma \geqslant 0 \tag{2$'$}$$

where $\alpha, \beta, \gamma, \delta > 0$, $b(0) \geqslant 0$, b is locally Lipschitz on $[0, \infty)$, b satisfies (20) and $\rho \in [-1, +1]$ stands for the correlation between the Brownian motions W_t and Z_t as before.

We next discuss some natural restrictions upon the parameters β and γ. We first assume that $\beta \leqslant 1$ and $\gamma \leqslant 1$. Indeed, if $\beta > 1$, or $\gamma > 1$, it is easy to check that (1$'$) or (2$'$) becomes in general (see for instance [6]) an explosive stochastic differential equation. Then, as is well-known, in order to have a unique local solution of (1′), (2′) one needs to assume that $\gamma \geqslant 1/2$ and $\beta \geqslant 1/2$. However, as we shall prove in another article of this series, one can solve (1′), (2′) for all $\beta, \gamma \in (0, 1]$ provided one restricts F_t, if $\beta < 1/2$, and σ_t, if $\gamma < 1/2$, to be nonnegative. This is why, with this further restriction, all the results we prove below are in fact valid for all $\beta, \gamma \in (0, 1]$ although the arguments may need to be adapted if β or $\gamma \in (0, \frac{1}{2})$ (through the use of techniques that we shall introduce for such fractional powers in a future work)... .

With these restrictions, there exists a unique solution σ_t of (2′) which remains nonnegative and we have for all $T > 0$

$$E\Big[\sup_{0\leqslant t\leqslant T} \sigma_t^p\Big] < \infty, \quad \text{for all } p \in (1, \infty). \tag{31}$$

At this point, we observe that the case when $\beta < 1$ is easily handled since, in that case, (1′) yields a unique solution whose moments (to an arbitrary high order) remains finite. Indeed, we have for any $m \in (1, \infty)$ and for any $n \geqslant 1$

$$E\big[F^m_{t\wedge\tau_n}\big] = F^m + E\int_0^t m\frac{m-1}{2}\sigma_s^{2\delta} F_{s\wedge\tau_n}^{2\beta+m-2} 1_{(\tau_n>s)}\,\mathrm{d}s$$

where $\tau_n = \inf(s \geqslant 0/F_s \geqslant n)$. Therefore, if $T \in (0, \infty)$ is fixed, we have for all $t \in [0, T]$

$$E\big[F^m_{t\wedge\tau_n}\big] \leqslant F^m + m\frac{m-1}{2p}E\bigg[\int_0^t \sigma_s^p\,\mathrm{d}s\bigg] + \frac{m(m-1)}{2q}\cdot E\bigg[\int_0^t F^m_{s\wedge\tau_n}\,\mathrm{d}s\bigg]$$

where $q = m/(2\beta + m - 2) \in (1, \infty)$ and $1/p + 1/q = 1$. We then deduce from (31) a bound independent of $n \geqslant 1$ on $\sup_{t\in[0,T]} E[F^m_{t\wedge\tau_n}]$ and we conclude (letting n go to $+\infty$) that, if $\beta < 1$, then (1′) defines uniquely a martingale F_t which belongs to L^m for all $1 < m < \infty$.

This is why we consider from now on the case $\beta = 1$ i.e.

$$\begin{cases} \mathrm{d}F_t = \sigma_t^\delta F_t\,\mathrm{d}W_t, \quad F_0 = F > 0, \\ \mathrm{d}\sigma_t = \alpha\sigma_t^\gamma\,\mathrm{d}Z_t + b(\sigma_t)\,\mathrm{d}t, \quad \sigma_0 = \sigma > 0 \end{cases} \tag{32}$$

(if $b(0) > 0$, we may consider as well the case when $\sigma_0 = 0\ldots$), with the above conditions on α, γ, δ, b and ρ.

3.2. When is F_t a martingale?

We begin with the study of the integrability of F_t and the related issue of the existence of a martingale F_t solving the first equation of (32).

Theorem 3.1.

(i) *If $\rho > 0$ and if $\gamma + \delta > 1$, we assume that b satisfies*

$$\limsup_{\xi\to+\infty} \frac{b(\xi) + \rho\alpha\xi^{\gamma+\delta}}{\xi} < \infty \tag{33}$$

(*and we make no restriction on b, ρ in the other parameter cases*). *Then, F_t is an integrable nonnegative martingale and*

$$E\big[F_t|\mathrm{Log}\,F_t|\big] < \infty, \quad E\Big[\sup_{0\leqslant s\leqslant t} |F_s|\Big] < \infty \quad \textit{for all } t \geqslant 0.$$

(ii) *If $\rho > 0$, $\gamma + \delta > 1$ and b satisfies*

$$\liminf_{\xi\to+\infty} \frac{b(\xi) + \rho\alpha\xi^{\gamma+\delta}}{\varphi(\xi)} > 0 \tag{34}$$

for some smooth, positive, increasing φ such that $\int^\infty \frac{1}{\varphi}\,\mathrm{d}\xi < \infty$, then F_t is not a martingale and we have:

$$E[F_t] < F_0 \quad \textit{for all } t > 0.$$

Remarks.

(i) It is worth comparing (33), (34) with (21), (22): those are obviously the "same" conditions with $\mu(\xi)$ replaced by $\alpha\xi^\gamma$ and ξ by ξ^δ.
(ii) If $\rho \leqslant 0$ or if $\gamma + \delta \leqslant 1$, (33) obviously holds.

Sketch of proof. The proof of the above result is once more entirely similar to the one made in Sections 2.2 or 2.5. And we shall not repeat it. Let us only mention that we have

$$E[F_t \operatorname{Log} F_t] = \frac{1}{2}\widehat{E}\int_0^t \sigma_s^{2\delta}\,\mathrm{d}s$$

where σ_t, under $\widehat{P}$, solves

$$\mathrm{d}\sigma_t = \alpha\sigma_t^\gamma\,\mathrm{d}Z_t + \left(\rho\alpha\sigma_t^{\gamma+\delta} + b(\sigma_t)\right)\mathrm{d}t, \quad \sigma_0 = \sigma < 0. \tag{35}$$

And this equation then leads naturally to conditions (33) and (34). □

3.3. Sufficient conditions

Let $1 < m < \infty$. We state in the following result conditions that insure the L^m integrability of F_t or equivalently

$$E\Big[\sup_{0\leqslant s\leqslant t} F_s^m\Big] < \infty \tag{36}$$

for some (or any...) $t > 0$.

Theorem 3.2.

(i) *If* $\delta < \gamma \leqslant 1/2$ *or* $\gamma < \delta \leqslant 1/2$ *or* $\gamma = \delta < 1/2$, *then* (36) *holds for all* $t \geqslant 0$.
(ii) *If* $\gamma < \delta$, $\delta > 1/2$ (*resp.* $\delta < \gamma$, $\gamma > 1/2$) *and* $\gamma + \delta < 1$, *then* (36) *holds for all* $t \geqslant 0$.
(iii) *If* $\gamma < \delta$, $\delta > 1/2$ (*resp.* $\delta < \gamma$, $\gamma > 1/2$) *and* $\gamma + \delta = 1$, *we set* $b_\infty = \limsup_{\xi\to+\infty}(b(\xi)/\xi)$ *and* $q = 1 + \delta - \gamma = 2\delta$. *Then, if* $\rho \leqslant -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, (36) *holds for all* $t \geqslant 0$ *and, if* $\rho > -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, *we consider the solution* $A(t)$ *of*

$$\dot{A} = \frac{\alpha^2}{2}q^2A^2 + q(\rho\alpha m + b_\infty)A + \frac{m^2 - m}{2}, \quad A(0) = 0, \tag{37}$$

then (36) *holds for all* $t < T_0$ *where* $T_0 \in (0, \infty)$ *is the blow-up time of* A.
(iv) *If* $\gamma < \delta$, $\delta > 1/2$ (*resp.* $\delta < \gamma$, $\gamma > 1/2$) *and* $\gamma + \delta > 1$, *then* (36) *holds for all* $t > 0$ *if* $\rho < -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$ *where* $b_\infty = \limsup_{\xi\to+\infty}(b(\xi)/\xi^{\gamma+\delta})$.
(v) *If* $\gamma = \delta > 1/2$, *we set* $b_\infty = \limsup_{\xi\to+\infty}(b(\xi)/\xi^{2\gamma})$. *Then, if* $\rho < -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, (36) *holds for all* $t > 0$.
(vi) *If* $\gamma = \delta = 1/2$, *we set* $b_\infty = \limsup_{\xi\to+\infty}(b(\xi)/\xi)$ *and we denote by* A *the solution of* (37) *with* $q = 1$. *Then,* (36) *holds for all* $t \geqslant 0$ *if* $\rho \leqslant -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$ *while* (36) *holds for all* $0 \leqslant t < T_0$ *if* $\rho > -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$.

Remarks.

(i) We shall see in the next section that this result is essentially optimal.
(ii) The case studied in Section 2 above corresponds to part (v) of the above result ($\gamma = \delta = 1\ldots$).

Sketch of proof of Theorem 3.2. Exactly as in the proof of Theorem 2.3 (see Section 2.4), we need to obtain upper bounds (independent of $n > 1$) on $z_n(t,\sigma) = E(\exp(\frac{m^2-m}{2}\int_0^{t\wedge\tau_n}\sigma_s^{2\delta}\,\mathrm{d}s))$ where σ solves

$$\mathrm{d}\sigma_t = \alpha\sigma_t^\gamma\,\mathrm{d}Z_t + \left(\rho\alpha m\sigma_t^{\gamma+\delta} + b(\sigma_t)\right)\mathrm{d}t, \quad \sigma_0 = \sigma > 0, \tag{38}$$

where $\tau_n = \inf(t \geqslant 0,\ \sigma_t > n)$. Obviously, z_n solves uniquely (in viscosity sense for instance)

$$\begin{cases} \dfrac{\partial z^n}{\partial t} - \dfrac{\alpha^2}{2}\xi^{2\gamma}\dfrac{\partial^2 z_n}{\partial \xi^2} - \big(\rho\alpha m \xi^{\gamma+\delta} + b(\xi)\big)\dfrac{\partial z_n}{\partial \xi} - \dfrac{m^2-m}{2}\xi^{2\delta} z_n = 0 \\ \quad \text{for } 0 < \xi < n,\ t > 0, \\ z_n|_{t=0} \equiv 1, \quad z_n(n,t) \equiv 1. \end{cases} \tag{39}$$

And we shall obtain those upperbounds by considering $\overline{w} = \exp(A(t)\xi^q + B(t))$ where $A(0) = 0$, $B(0) = 0$, $A \geqslant 0$, $B \geqslant 0$ and $q > 0$ are to be determined in such a way that $\overline{w}$ is a supersolution of (39) that is

$$0 \leqslant \frac{\partial \overline{w}}{\partial t} - \frac{\alpha^2}{2}\xi^{2\delta}\frac{\partial^2 \overline{w}}{\partial \xi^2} - \big(\rho\alpha m \xi^{\gamma+\delta} + b(\xi)\big)\frac{\partial \overline{w}}{\partial \xi} - \frac{m^2-m}{2}\xi^{2\delta}\overline{w}$$

or

$$\begin{cases} 0 \leqslant \overline{w}\Big\{\dot{A}\xi^q + \dot{B} - \dfrac{\alpha^2}{2}Aq(q-1)\xi^{2\gamma+q-2} - \dfrac{\alpha^2}{2}q^2A^2\xi^{2(\gamma+q-1)} \\ \qquad - \big(\rho\alpha m\xi^{\gamma+\delta} + b\big)qA\xi^{q-1} - \dfrac{m^2-m}{2}\xi^{2\delta}\Big\}. \end{cases} \tag{40}$$

We first choose $q = 1$. We begin with case (i) (expect for the subcase $\delta < \gamma = 1/2$ that we shall study separately below). We use the fact that $b(\xi) \leqslant C_0(1+\xi)$ on $[0,\infty)$ for some $C_0 \geqslant 0$. And we check easily that (40) holds provided we choose $A = \mathrm{e}^{Mt} - 1$ for some $M > 0$ large enough and then $B = K(\mathrm{e}^{2MT} - 1)$ for some $K > 0$ large enough.

We next turn to case (ii). We choose $q = 1 + \delta - \gamma$. Notice that, since $\gamma + \delta < 1$, $q > 2\delta$ while $\gamma + \delta + q - 1 = 2(\gamma + q - 1) = 2\delta$. Hence, (40) holds for ξ large provided we choose $A = \mathrm{e}^{Mt} - 1$ for some $M > qC_0$. It is then easy to choose B in such a way that (40) holds for all $\xi \geqslant 0$.

We complete the study of case (i) by considering the subcase when $1/2 = \gamma > \delta$. In that case, we choose $q < 1$, close to 1 so that $q > 2\delta$, $q > \gamma + \delta + q - 1$, $q > 2(1/2 + q - 1)$. And we conclude choosing $A = \mathrm{e}^{Mt} - 1$ for some $M > qC_0\ldots$.

We now consider the case (iii) and we choose $q = 1 + \delta - \gamma$. In that case, the leading order terms (as ξ goes to $+\infty$) in (40) are given (or can be estimated from below) by

$$\xi^{2\delta}\left\{\dot{A} - \frac{\alpha^2}{2}q^2A^2 - (\rho\alpha m + b')qA - \frac{m^2-m}{2}\right\}$$

for any $b' > b_\infty$. We next observe that, if $\rho < -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, there exists for b' close to b_∞ a positive root $\bar{x}$ of

$$\frac{\alpha^2}{2}x^2 + (\rho\alpha m + b')x + \frac{m^2-m}{2} = 0.$$

And, we choose, for such a b', A to be the solution of

$$\dot{A} = \frac{\alpha^2}{2}q^2A^2 + (\rho\alpha m + b')qA + \frac{m^2-m}{2}, \quad A(0) = 0, \tag{41}$$

we deduce that $A(t) \leqslant \bar{x}/q$ for all $t \geqslant 0$. It is then easy to complete the construction of $\overline{w}$. Finally, if $\rho \geqslant -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, we solve (37) with b_∞ replaced by $b' > b_\infty$ i.e. we solve (41), and we build a supersolution for all $t < T'$ where T' is the blow-up time of A' (solution of (41)). And we conclude since $T' \to T_0$ as $b' \to b_\infty$ (and $T_0 = +\infty$ if $\rho = -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$).

The argument for case (vi) is the same than the one we just made provided we choose $q = 1$ (and solve (41) with $q = 1\ldots$).

We next study case (iv). And we choose $q = 1 + \delta - \gamma$. Since $q < 2\delta$, the leading order terms (as ξ goes to $+\infty$) in (40) can be estimated from below by

$$\xi^{2\delta}\left\{-\frac{\alpha^2}{2}q^2A^2 - (\rho\alpha m + b')qA - \frac{m^2-m}{2}\right\}.$$

And, if $\rho < -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, we may find $A > 0$ such that the quantity between brackets is strictly positive. We then choose $A(t) \equiv A$ (and $B(t) \equiv B > 0$ large enough) to complete the construction of $\overline{w}$.

There only remains to treat case (v). We may then simply take $q = 1$ and follow the argument made in the preceding case.... □

Remark. In cases (iv) and (v), the borderline value $\rho = -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$ is slightly more delicate since the result depends on the behaviour of "$b(\xi) - b_\infty \xi^{\gamma+\delta}$". Indeed, in case (v), one may prove that, for $\rho = -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, (36) holds for all $t \geqslant 0$ if $b - b^\infty \xi^{2\gamma}$ is bounded from above on $[0, m)$.

Similarly, in case (iv), one may check that, still for $\rho = -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, (36) holds for all $t \geqslant 0$ if $\{(b - b^\infty \xi^{\gamma+\delta})\xi^{\delta-\gamma} + \alpha^2(\delta-\gamma)\xi^{\gamma+\delta-1}/2\}$ is bounded from above on $[0, \infty)$.

3.4. Necessary conditions

We consider in this section some necessary conditions for the boundedness of $E[F_t^m]$. As we shall see, these conditions are very close to the ones derived in the preceding section. They are summarised in the following

Theorem 3.3.

(i) *If $\gamma < \delta$, $\delta > 1/2$ (resp. $\delta < \gamma$, $\gamma > 1/2$) and $\gamma + \delta = 1$, we let $b_\infty = \liminf_{\xi \to +\infty}(b(\xi)/\xi)$ and $q = 1 + \delta - \gamma = 2\delta$. Then, if $\rho > -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, $E[F_t^m] = +\infty$ for all $t > T_0$ where $T_0 \in (0, \infty)$ is the blow-up time of A solution of* (37).

(ii) *If $\gamma < \delta$, $\delta > 1/2$ (resp. $\delta < \gamma$, $\gamma > 1/2$) and $\gamma + \delta > 1$, we set $b_\infty = \liminf_{\xi \to +\infty}(b(\xi)/\xi^{\gamma+\delta})$. Then, if $\rho > -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, $E[F_t^m] = +\infty$ for all $t > 0$.*

(iii) *If $\gamma = \delta > 1/2$, we set $b_\infty = \liminf_{\xi \to +\infty}(b(\xi)/\xi^{2\gamma})$. Then, if $\rho > -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, $E[F_t^m] = +\infty$ for all $t > 0$.*

(iv) *If $\gamma = \delta = 1/2$, we set $b_\infty = \liminf_{\xi \to +\infty}(b(\xi)/\xi)$. Then, if $\rho > -\sqrt{(m-1)/m} - b_\infty/(\alpha m)$, $E[F_t^m] = +\infty$ for all $t > T_0$ where $T_0 \in (0, \infty)$ is the blow-up time of A solution of* (37) *with* $q = 1$.

Remarks.

(i) Comparing Theorems 3.3 and 3.2 (and the remarks made afterwards) shows that the conditions introduced in Theorem 3.2 are sufficient and "almost" necessary.

(ii) As an example, we mention the so-called Heston model which corresponds to the choice $\gamma = \delta = 1/2$ and $b(\xi) = b(\bar{\sigma} - \xi)$ for some $b, \bar{\sigma} > 0$. Obviously, we have $b_\infty = -b$. And Theorems 3.2 and 3.3 imply the following facts:
   - if $\rho \leqslant -\sqrt{(m-1)/m} + b/(\alpha m)$, then $E[F_t^m] < \infty$ for all $t \geqslant 0$;
   - if $\rho > -\sqrt{(m-1)/m} + b/(\alpha m)$, then $E[F_t^m] < \infty$ for all $t < T_0$ and $E[F_t^m] = +\infty$ for all $t \geqslant T_0$, where $T_0 \in (0, \infty)$ is the blow-up time of the solution A of (37) with $q = 1$.

Sketch of proof of Theorem 3.3. We begin with the proof of case (iii). In that case, the argument is almost the same as the one presented in Appendix A, replacing in (A.1) by $(\nu \exp(\nu t \sigma^{2\gamma}))$.

We now turn to the proof of case (ii). Again, one can follow the argument introduced in Appendix A, replacing σ by σ^q (observe indeed that, at least formally, $\eta_t = \sigma_t^q$ satisfies

$$
\begin{aligned}
\mathrm{d}\eta_t &= q\sigma_t^{q-1}\,\mathrm{d}\sigma_t + \frac{q(q-1)}{2}\sigma_t^{q-2}\big(\alpha\sigma_t^\gamma\big)^2\,\mathrm{d}t \\
&= \alpha q \sigma_t^\delta\,\mathrm{d}Z_t + \big(\rho\alpha m \sigma_t^{2\delta} + b(\sigma_t)\sigma_t^{\delta-\gamma}\big)\,\mathrm{d}t + \frac{q(q-1)}{2}\alpha^2\sigma_t^{\delta+\gamma-1}\,\mathrm{d}t \\
&= \alpha q \eta_t^{\delta/q}\,\mathrm{d}Z_t + \big(\rho\alpha m \eta_t^{2\delta/q} + b\big(\eta_t^{1/q}\big)\eta_t^{((\delta-\gamma)/q)}\big)\,\mathrm{d}t + \frac{q(q-1)}{2}\alpha^2\eta_t^{((\delta+\gamma-1)/q)}\,\mathrm{d}t
\end{aligned}
$$

and $\delta + \gamma - 1 < 2\delta$).

Cases (i) and (iv) are of a slightly different nature since the moment $E[F_t^m]$ becomes infinite only after time T_0. In order to keep the presentation as simple as possible, we simply consider the case when $b(\xi) = b_\infty \xi$ (the general case being an easy technical modification of that one, exactly as we did in Section 2.5...). We begin with case (v) and we wish to prove that, with the rotation of Appendix A, $z(\sigma,t)\equiv+\infty$ for $t>T_0$. In order to do so, we observe that $z(\sigma,t)\geqslant 1$ for all $\sigma\in[0,\infty)$ and for all $t\geqslant 0$. Next, we remark that

$$\underline{w}(\sigma,t)=\exp\bigl(A(t)\sigma\bigr)$$

is a solution of

$$\frac{\partial\underline{w}}{\partial t}-\frac{\alpha^2}{2}\sigma\frac{\partial^2\underline{w}}{\partial\sigma^2}-(\rho\alpha m+b_\infty)\sigma\frac{\partial\underline{w}}{\partial\sigma}-\frac{m^2-m}{2}\sigma\underline{w}=0 \tag{42}$$

if A solves (37).

We then claim that we have for all $\sigma\geqslant 0$, $T_0-h>t\geqslant 0$, $s\geqslant 0$, $h>0$, $\delta>0$

$$z(\sigma,t+s)\geqslant\underline{w}(\sigma,t)-\delta\underline{w}(\sigma,t+h). \tag{43}$$

Indeed, the right-hand side of (43) solves (42) and, since $A(t)$ is increasing with respect to t, goes to $-\infty$ as σ goes to $+\infty$. Then, (43) follows from the maximum principle. And we conclude easily letting h and δ go to 0_+ since (43) yields

$$z(\sigma,t+s)\geqslant\underline{w}(\sigma,t)\quad\text{for all }\sigma\geqslant 0,\ s\geqslant 0,\ 0\leqslant t<T_0,$$

hence $z(\sigma,T_0+s)\equiv+\infty$ for all $\sigma>0$, $s\geqslant 0$.

We conclude with case (i). And we observe that $\underline{w}(\sigma,t)=\exp(A(t)\sigma^q+B(t))$ satisfies

$$\begin{aligned}&\frac{\partial\underline{w}}{\partial t}-\frac{\alpha^2}{2}\sigma^{2\gamma}\frac{\partial^2\underline{w}}{\partial\sigma^2}-(\rho\alpha m+b_\infty)\sigma\frac{\partial\underline{w}}{\partial\sigma}-\frac{m^2-m}{2}\sigma^{2\delta}\underline{w}\\&\quad=\sigma^q\underline{w}\left(\dot A-\frac{\alpha^2}{2}q^2A^2-(\rho\alpha m+b_\infty)qA-\frac{m^2-m}{2}\right)-\frac{\alpha^2}{2}q(q-1)A\underline{w}+\dot B\underline{w}.\end{aligned}$$

We may then choose (for example) A to be solution of (37) and B to satisfy: $\dot B=\frac{\alpha^2}{2}q(q-1)A$, $B(0)=0$. At this point, we can follow the argument above (made in case (iv)) since $A(t)\sigma^q+B(t)$ converges to $+\infty$ as t goes to T_0- for all $\sigma>0$. □

We conclude by briefly mentioning that all the above results can be extended to situations of the following type

$$\mathrm{d}F_t=g(\sigma_t)F_t^\beta\,\mathrm{d}W_t,\quad F_0=F\geqslant 0, \tag{44}$$

$$\mathrm{d}\sigma_t=\mu(\sigma_t)\,\mathrm{d}Z_t+b(\sigma_t)\,\mathrm{d}t,\quad \sigma_0=\sigma\geqslant 0 \tag{45}$$

where $0<\beta\leqslant 1$, μ and b satisfy (18) and (20), μ "behaves" like σ^γ near 0 for some $0<\gamma\leqslant 1$ and μ is Lipschitz on $(0,+\infty)$. The function g is assumed to be locally Lipschitz on $[0,+\infty)$ with a polynomial growth at infinity that is

$$\bigl|g(\xi)\bigr|\leqslant C\bigl(1+|\xi|^\delta\bigr)\quad\text{for all }\xi>0 \tag{46}$$

for some $C\geqslant 0$, $\delta>0$.

The case when $\beta<1$ is easily handled as we did above. When $\beta=1$, we may then easily adapt the arguments introduced in the preceding sections as we adapted in Section 2.4 the approach developed in Sections 2.1–2.3. In this way, one derives general results under natural assumptions on g, μ, b (involving polynomial behaviours as σ goes to $+\infty$) that we leave to the reader, since the adaptations are straightforward.

Appendix A. Blow-up of z_n if $\rho\in(-\sqrt{(m-1)/m},0)$

We show here that

$$\underset{F^m}{\perp}E\bigl[F_t^m\bigr]=\widehat{E}\left(\exp\left\{\frac{m^2-m}{2}\int_0^t\sigma_s^2\,\mathrm{d}s\right\}\right)=z(\sigma,t)=\lim_n\uparrow z_n(\sigma,t)=+\infty$$

for all $\sigma>0$, $t>0$. This will be done in several steps.

Step 1: A lower bound for z. We first show the existence of some $\vartheta \in (0,1)$ such that we have for all $\sigma, t \geqslant 0$

$$z(\sigma,t) \geqslant \vartheta \exp\{\vartheta t\sigma^2\}. \tag{A.1}$$

In order to do so, we first observe that, since $\rho \in (-\sqrt{(m-1)/m}, 0)$, there exists $\mu > 0$, $k > 0$ such that, denoting by $\varphi(x) = \mathrm{e}^{\mu x}\,\mathrm{e}^{\mathrm{i}kx}$

$$-\frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial\sigma^2} - \alpha\rho m\frac{\partial\varphi}{\partial\sigma} - \frac{m^2-m}{2}\varphi = 0 \quad \text{on } \mathbb{R}.$$

Hence, the first eigenvalue of the preceding differential operator on $[0, \frac{2\pi}{k}]$ with Dirichlet boundary conditions is strictly positive. Simple comparison properties for the first eigenvalue of elliptic differential operators then yield the existence of a positive constant $C_0 > 1$ such that for each $a \geqslant 1$, the first eigenvalue of the operator

$$-\frac{\alpha^2}{2}\sigma^2\frac{\partial^2}{\partial\sigma^2} - \alpha\rho m\sigma^2\frac{\partial}{\partial\sigma} - \frac{m^2-m}{2}\sigma^2$$

on $[a, \frac{2\pi}{k} + a]$ with Dirichlet boundary conditions satisfies

$$\frac{1}{C_0} \leqslant \frac{\lambda_a}{a^2} \leqslant C_0. \tag{A.2}$$

We then introduce φ_a the corresponding first eigenfunction (normalised by $\max\varphi_a = 1$) and we extend φ_a to $\mathbb{R}$ by 0. Since λ_a/σ^2 is bounded (from above and from below) on $[a, \frac{2\pi}{k} + a]$, we deduce easily that φ_a is, uniformly in a, bounded from below on $[a+\delta, \frac{2\pi}{k} + a - \delta]$ (for any $\delta \in (0, \frac{\pi}{k})$).

We may now apply the maximum principle to deduce that

$$z_n(\sigma,t) \geqslant \mathrm{e}^{\lambda_a t}\varphi_a(\sigma) \quad \text{if } \frac{2\pi}{k} + a \leqslant n. \tag{A.3}$$

Indeed, z_n and $\mathrm{e}^{\lambda_a t}\varphi_a(\sigma)$ both solve the same equation on $[a, \frac{2\pi}{k} + a] \times [0,\infty)$; $z_n(a)$, $z_n(\frac{2\pi}{k} + a) \geqslant 0$ while $\varphi_a(a) = \varphi_a(\frac{2\pi}{k} + a) = 0$; and $z_n(\sigma, 0) = 1 \geqslant \varphi_a$ on $[a, \frac{2\pi}{k} + a]$.

Letting n go to $+\infty$ in (A.3), we deduce (A.1) from (A.2) or at least we obtain (A.1) on $[1+\delta, \infty)$. Since z is easily shown to be nondecreasing with respect to t and with respect to σ, and $z \geqslant 1$ (A.1) follows easily for all $\sigma, t \geqslant 0$.

Step 2: Blow-up for some time. Let $\sigma_0 > 0$. We show here that $z \equiv +\infty$ for $\sigma \geqslant \sigma_0$, $t \geqslant T(\sigma_0)$ where $T(\sigma_0)$ goes to 0_+ as σ_0 goes to $+\infty$. In order to do so, we build a convenient subsolution that blows up in finite time. Before doing so, we introduce $z_*(\sigma,t) = \liminf(z_n(\sigma_n, t_n) \mid n \to \infty,\ \sigma_n \underset{n}{\to} \sigma,\ t_n \to t)$ and observe that by the monotonicity of z mentioned above, we have in fact: $z \geqslant z_* = \sup\{z(\sigma', t') \mid 0 \leqslant \sigma' < \sigma,\ 0 \leqslant t' < t\}$. And, as is well-known in viscosity solutions theory, z_* is a (viscosity) supersolution (with values in $[1, +\infty]$) of

$$\frac{\partial z}{\partial t} - \frac{\alpha^2}{2}\sigma^2\frac{\partial^2 z}{\partial\sigma^2} - \alpha\rho m\sigma^2\frac{\partial z}{\partial\sigma} - \frac{m^2-m}{2}\sigma^2 z = 0 \quad \text{on } [0,\infty)\times[0,\infty), \tag{A.4}$$

$$z|_{t=0} \equiv 1 \quad \text{on } [0,\infty). \tag{A.5}$$

We now look for a subsolution of (A.4), (A.5) of the following form

$$\underline{z}(t) = \exp\{b(t)(\sigma-\sigma_0)_+\} \tag{A.6}$$

where $b(0) = 1$ and b is determined below. We only have to check that $\underline{z}$ is a subsolution of (A.4) for $\sigma > \sigma_0$ and we find on that set

$$\frac{\partial \underline{z}}{\partial t} - \frac{\alpha^2}{2}\sigma^2\frac{\partial^2 \underline{z}}{\partial\sigma^2} - \alpha\rho m\sigma^2\frac{\partial \underline{z}}{\partial\sigma} - \frac{m^2-m}{2}\sigma^2\underline{z} = \underline{z}\left\{\dot{b}(\sigma-\sigma_0) - \frac{\sigma^2}{2}\left[\alpha^2 b^2 + 2\alpha\rho m b + (m^2-m)\right]\right\}.$$

Then, since $\rho \in (-\sqrt{(m-1)/m}, 0)$, there exists $\gamma > 0$ such that for all $b \in \mathbb{R}$

$$\alpha^2 b^2 + 2\alpha\rho m b + m^2 - m \geqslant 2\gamma b^2.$$

Hence, $\underline{z}$ is the desired subsolution provided b solves

$$\dot{b} = \gamma\sigma_0 b^2$$

i.e. $b = (1 - \gamma\sigma_0 t)^{-1}$ for $t < T(\sigma_0) = (\gamma\sigma_0)^{-1}$.

Next, we apply the maximum principle on $[0, \infty) \times [0, T(\sigma_0))$ and deduce easily that we have

$$z_*(\sigma, t) \geqslant \exp\{(1 - \gamma\sigma_0 t)^{-1}(\sigma - \sigma_0)_+\} \quad \text{for } \sigma \geqslant 0,\ 0 \leqslant t < T(\sigma_0). \tag{A.7}$$

Indeed, the lower bound (A.8) allows to work on a finite interval in σ and then we just have to use standard comparison arguments.

Of course, the lower bound (A.7) yields the following information valid for any $\sigma_0 > 0$

$$z = z_*(\sigma, t) = +\infty \quad \text{for all } \sigma > \sigma_0,\ t \geqslant (\gamma\sigma_0)^{-1}. \tag{A.8}$$

Step 3: $z \equiv +\infty$ for all $\sigma > 0$, $t > 0$. This follows in fact easily from (A.8) and from the fact that z_* is a (viscosity) supersolution of (A.4). Indeed, we first deduce that, for each $\sigma_0 > 0$, $z = z_* = +\infty$ for all $\sigma > 0$, $t \geqslant (\gamma\sigma_0)^{-1}$.

Indeed, let $\sigma_1 > \sigma_0$, we have for all $M > 0$, by the maximum principle

$$z \geqslant z_* \geqslant M\tilde{z} \quad \text{on } [0, \sigma_1] \times [0, \infty) \tag{A.9}$$

where $\tilde{z}$ solves (A.4), (A.5) on $[0, \sigma_1] \times [0, \infty)$ with $\tilde{z}(\sigma_1, t) = 0$ if $0 \leqslant t < (\gamma\sigma_0)^{-1}$ and $\tilde{z}(\sigma_1, t) = 1$ if $t \geqslant (\gamma\sigma_0)^{-1}$.

Then, the strong maximum principle ensures that $\tilde{z}(\sigma, t)$ is strictly positive. Hence, we deduce our claim from (A.9) upon letting M go to $+\infty$.

Having shown that $z(\sigma, t) = +\infty$ for all $\sigma > 0$, $t \geqslant (\gamma\sigma_0)^{-1}$ and for any $\sigma_0 > 0$, we only have to let σ_0 go to $+\infty$ in order to conclude that $z(\sigma, t) \equiv +\infty$ for all $\sigma, t > 0$.

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