

Minimizing L^∞ -norm functional on divergence-free fields

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Abstract

In this paper, we study the minimization problem on the L^∞ -norm functional over the divergence-free fields with given boundary normal component. We focus on the computation of the minimum value and the classification of certain special minimizers including the so-called absolute minimizers. In particular, several alternative approaches for computing the minimum value are given using L^q -approximations and the sets of finite perimeter. For problems in two dimensions, we establish the existence of absolute minimizers using a similar technique for the absolute minimizers of L^∞ -functionals of gradient fields. In some special cases, precise characterizations of all minimizers and the absolute minimizers are also given based on equivalent descriptions of the absolutely minimizing Lipschitz extensions of boundary functions.

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1. Introduction and main results

Let Ω be a bounded domain in \mathbf{R}^n ($n \geq 2$) with Lipschitz continuous boundary $\partial\Omega$ and σ be a positive continuous function on $\bar{\Omega}$. Given $H : \Omega \rightarrow \mathbf{R}^n$ and $\beta : \partial\Omega \rightarrow \mathbf{R}$, we study the value of the following minimization problem:

$$\rho(\beta, H) = \min_{G \in \mathcal{S}_\beta(\Omega)} \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(\Omega)}, \quad (1.1)$$

where $\mathcal{S}_\beta(\Omega)$ is the set of all divergence-free fields G in Ω of fixed boundary normal-component $G \cdot \nu|_{\partial\Omega} = \beta$.

The motivation for studying such a problem is two-folds. First, in many variational problems, it is typical that finding the best constant for some inequalities to hold or the certain threshold condition for a problem to have some special solutions will eventually lead to computing optimal values involving the L^∞ -norm of divergence-controlled quantities [4,8,12,13,22]. It is certainly desirable to find alternative ways to compute such values. Second, the study of the L^∞ -norm of divergence-free fields is a special case of the study of the L^∞ -functionals of general functions with certain \mathcal{A} -quasiconvexity [9,15]. In working with the special problem (1.1) for divergence-free fields, we are hoping to further explore the similar ideas from the study of gradient fields, as in [6,7,19,23]. In particular, for our problem (1.1), we would like to study whether some special (hopefully unique) minimizers can be obtained through

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certain underlying selection principles similar to those for the viscosity solutions and the absolute minimizers in the gradient case.

We first address the issues concerning the alternative ways to compute the value $\rho(\beta, H)$. Motivated by some results in [10], we assume the following natural conditions for H and β :

$$H \in L^\infty(\Omega; \mathbf{R}^n), \quad \operatorname{div} H \in L^n(\Omega); \quad \beta \in L^\infty(\partial\Omega), \quad \int_{\partial\Omega} \beta \, d\mathcal{H}^{n-1} = 0. \tag{1.2}$$

The space of functions H satisfying the conditions in (1.2) will be denoted by $\mathcal{X}_n(\Omega)$ and has been studied by many authors [5,8,11–13], even with $\operatorname{div} H$ being a Radon measure. In particular, if $H \in \mathcal{X}_n(\Omega)$, then a *normal-component* $H \cdot \nu \in L^\infty(\partial\Omega)$ can be defined \mathcal{H}^{n-1} -a.e. on $\partial\Omega$ in such a way that the generalized divergence formula

$$\int_{\Omega} \zeta \operatorname{div} H \, dx = \int_{\partial\Omega} \zeta (H \cdot \nu) \, d\mathcal{H}^{n-1} - \int_{\Omega} H \cdot \nabla \zeta \, dx \tag{1.3}$$

holds for all $H \in \mathcal{X}_n(\Omega)$ and $\zeta \in W^{1,1}(\Omega)$; this formula can be extended to $\zeta \in BV(\Omega)$, the space of functions of bounded variation in Ω . The admissible class $\mathcal{S}_\beta(\Omega)$ is defined by

$$\mathcal{S}_\beta(\Omega) = \{G \in L^\infty(\Omega; \mathbf{R}^n) \mid \operatorname{div} G = 0, \, G \cdot \nu = \beta\} \tag{1.4}$$

and is nonempty under the assumptions on β in (1.2). For $G \in \mathcal{S}_\beta(\Omega)$ and $\zeta \in W^{1,1}(\Omega)$, by (1.3), it follows that

$$\int_{\partial\Omega} \zeta (\beta + H \cdot \nu) - \int_{\Omega} \zeta \operatorname{div} H = \int_{\Omega} (G + H) \cdot \nabla \zeta \leq \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(\Omega)} \int_{\Omega} \sigma |\nabla \zeta|$$

and hence we have that

$$\rho(\beta, H) \geq \sup_{\substack{\zeta \in W^{1,1}(\Omega) \\ \zeta \neq \text{const}}} \frac{\int_{\partial\Omega} \zeta (\beta + H \cdot \nu) \, d\mathcal{H}^{n-1} - \int_{\Omega} \zeta \operatorname{div} H \, dx}{\int_{\Omega} \sigma |\nabla \zeta| \, dx}. \tag{1.5}$$

One of the main motivations of the paper is that the equality holds in this relation:

Theorem 1.1. *Let H, β satisfy (1.2). Then $\rho(\beta, H) = \mu(\beta + H \cdot \nu, \operatorname{div} H)$, where the quantity $\mu(g, h)$ is defined by*

$$\mu(g, h) = \sup_{\substack{\zeta \in W^{1,1}(\Omega) \\ \zeta \neq \text{const}}} \frac{\int_{\partial\Omega} \zeta g \, d\mathcal{H}^{n-1} - \int_{\Omega} \zeta h \, dx}{\int_{\Omega} \sigma(x) |\nabla \zeta| \, dx} \tag{1.6}$$

for functions g and h satisfying the condition

$$g \in L^\infty(\partial\Omega), \quad h \in L^n(\Omega), \quad \int_{\partial\Omega} g \, d\mathcal{H}^{n-1} = \int_{\Omega} h \, dx. \tag{1.7}$$

The optimization problem (1.6) is similar to the problems appearing in several important studies, such as the dual variational principle for plasticity [12], the best constant for the Sobolev trace-embedding of $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$ [4,24], the eigenvalue problem for 1-Laplacian operator [8,13], and the generalized Cheeger problems [2,18,20]. One readily verifies that

$$\mu(g, h) = \sup_{\substack{\zeta \in BV(\Omega) \\ \zeta \neq \text{const}}} \frac{\int_{\partial\Omega} \gamma(\zeta) g \, d\mathcal{H}^{n-1} - \int_{\Omega} \zeta h \, dx}{\int_{\Omega} \sigma \, d|D\zeta|}, \tag{1.8}$$

where $\gamma : BV(\Omega) \rightarrow L^1(\partial\Omega)$ is the trace operator and $|D\zeta|$ is the total variation measure of the vector Radon measure $D\zeta$. The formula (1.8) is different from the one used in the generalized Cheeger problem studied in [18] because we do not have any boundary condition on $\zeta \in BV(\Omega)$. However, as in [18], we will see that the number $\mu(g, h)$ can also be characterized in terms of sets of finite perimeter instead of functions of bounded variation (see Theorem 3.3 below).

We discuss another approach for $\mu(g, h)$ based on approximation by power-law functionals. Let $B : BV(\Omega) \rightarrow \mathbf{R}$ be defined by

$$B(\zeta) = \int_{\partial\Omega} \gamma(\zeta)g \, d\mathcal{H}^{n-1} - \int_{\Omega} \zeta h \, dx. \tag{1.9}$$

Note that $B \equiv 0$ if and only if $g = h = 0$. Assume $B \not\equiv 0$. We define the following constrained minimization problems:

$$\lambda(g, h) = \inf_{\substack{\zeta \in BV(\Omega) \\ B(\zeta)=1}} \int_{\Omega} \sigma \, d|D\zeta|, \quad \lambda_p(g, h) = \inf_{\substack{\zeta \in W^{1,p}(\Omega) \\ B(\zeta)=1}} \int_{\Omega} \sigma^p |\nabla \zeta|^p \quad (1 \leq p < \infty). \tag{1.10}$$

If $B \equiv 0$ (i.e., $g = h = 0$), define $\lambda(0, 0) = \lambda_p(0, 0) = +\infty$.

The following result provides another way to compute the quantity $\mu(g, h)$.

Theorem 1.2. *It follows that $\lambda_1(g, h) = \lambda(g, h) = \frac{1}{\mu(g,h)} > 0$ and*

$$\lim_{p \rightarrow 1^+} \lambda_p(g, h) = \lambda_1(g, h) = \lambda(g, h). \tag{1.11}$$

Assume $B \not\equiv 0$. For each $1 < p < \infty$, a standard direct method in the calculus of variations shows that there exists a unique function $u_p \in W^{1,p}(\Omega)$ with $B(u_p) = 1$ satisfying $\int_{\Omega} u_p \, dx = 0$ that minimizes the problem (1.10); that is, $\int_{\Omega} \sigma^p |\nabla u_p|^p = \lambda_p(g, h)$. We have the following result.

Theorem 1.3. *There exist subsequence $p_j \rightarrow 1^+$ as $j \rightarrow \infty$, functions $\bar{u} \in BV(\Omega)$ and $\bar{F} \in L^\infty(\Omega; \mathbf{R}^n)$ with $\|\bar{F}\|_{L^\infty(\Omega)} \leq 1$ such that, as $j \rightarrow \infty$,*

$$u_{p_j} \rightarrow \bar{u} \quad \text{in } L^{\frac{n}{n-1}}(\Omega), \quad \nabla u_{p_j} \overset{*}{\rightharpoonup} D\bar{u} \quad \text{as measures on } \Omega, \tag{1.12}$$

$$|\nabla u_{p_j}|^{p_j-2} \nabla u_{p_j} \rightharpoonup \bar{F} \quad \text{in } L^r(\Omega) \text{ for any } r > 1, \tag{1.13}$$

$$\operatorname{div}(\sigma \bar{F}) = \lambda h, \quad \sigma \bar{F} \cdot \nu = \lambda g, \quad \text{where } \lambda = \lambda(g, h). \tag{1.14}$$

The function \bar{u} so determined is a minimizer for $\lambda(g, h)$ in $BV(\Omega)$ if and only if $B(\bar{u}) = 1$.

Since the trace operator $\gamma : BV(\Omega) \rightarrow L^1(\partial\Omega)$ is not continuous under the weak-star convergence of $BV(\Omega)$, one may not have $B(\bar{u}) = 1$ for the function \bar{u} determined in the theorem; so \bar{u} may not be a minimizer for $\mu(g, h)$ in $BV(\Omega)$. But any such limit \bar{u} is a weak solution to a Neumann problem for a 1-Laplacian-type equation (see Remark 3.1). In general, the existence of a minimizer for $\lambda(g, h)$ in $BV(\Omega)$ is unknown. However, under certain conditions (see Theorem 5.2 below), any such function \bar{u} will satisfy $B(\bar{u}) = 1$ and hence is a minimizer for $\lambda(g, h)$.

We now address the issues concerning the special (hopefully unique) minimizers for $\rho(\beta, H)$ in problem (1.1). Note that, with $g = \beta + H \cdot \nu$, $h = \operatorname{div} H$ and \bar{F} so determined in Theorem 1.3, the relationship

$$\bar{G} = \frac{\sigma \bar{F}}{\lambda(g, h)} - H \tag{1.15}$$

defines a minimizer \bar{G} for $\rho(\beta, H)$; all such minimizers \bar{G} can also be characterized by minimizing the L^q -norm as $q \rightarrow \infty$ (see Proposition 4.2 below) in much similar way as for the L^∞ -functionals of gradients of scalar functions [6,7,19]. The Γ -convergence of the general power-law functionals of divergence-free fields as power tends to infinity has been studied in [9]. However, unlike the gradient case, viscosity and comparison principles seem intractable for our problem (1.1) with divergence-free vector-fields. Instead, we focus on the principle of absolute minimizers. In a natural analogy to the absolute minimizers for L^∞ -functionals of gradients, we make the following definition.

Definition 1.1. A minimizer $\bar{G} \in \mathcal{S}_\beta(\Omega)$ for $\rho(\beta, H)$ is called an *absolute minimizer* provided that $\|\frac{\bar{G}+H}{\sigma}\|_{L^\infty(E)} \leq \|\frac{G+H}{\sigma}\|_{L^\infty(E)}$ holds for all open sets $E \Subset \Omega$ with *connected* $\Omega \setminus E$ and all fields $G \in \mathcal{S}_\beta(\Omega)$ satisfying

$$\int_E G \cdot \nabla \zeta \, dx = \int_E \bar{G} \cdot \nabla \zeta \, dx \quad \forall \zeta \in C_0^\infty(\mathbf{R}^n). \tag{1.16}$$

If E is a set of finite perimeter in Ω , then there is a local characterization of the condition (1.16) in terms of the interior normal-components relative to Ω on ∂^*E ; see Remark 2.2(b). The requirement of connectedness on $\Omega \setminus E$ seems necessary as seen in the two dimensional case.

The existence of an absolute minimizer is unknown in general. However, in dimension $n = 2$, using the fact that a divergence-free field is a rotated gradient and some results about absolute minimizers in the gradient case, we are able to show that any minimizer \bar{G} obtained through (1.15) from a vector-field \bar{F} determined in Theorem 1.3 is an absolute minimizer. With given Dirichlet boundary conditions, the similar result in the gradient case would follow from the Γ -convergence of the power-law energies to the L^∞ -energy [7,9]. However, some care need to be taken here because the Dirichlet boundary conditions are not uniquely determined from the normal trace of the divergence-free fields, especially when $\partial\Omega$ consists of disjoint closed curves.

We summarize the results for the two-dimensional problem in the following theorem, which provides a concrete procedure of finding the absolute minimizers for $\rho(\beta, H)$ in the special case and also indicates that the absolute minimizers may not be unique; the convex set Σ and the Lipschitz continuous functions $\alpha_{\mathbf{c}}$ given in the theorem will be specified later.

Theorem 1.4. *Let $n = 2$. Then any minimizer \bar{G} determined by a function \bar{F} in Theorem 1.3 through (1.15) is an absolute minimizer for $\rho(\beta, H)$.*

In the special case when $\sigma = 1$, $H = 0$ and $\partial\Omega$ consists of $k + 1$ disjoint Lipschitz Jordan curves, there exist nonempty compact convex set $\Sigma \subset \mathbf{R}^k$ and certain given Lipschitz continuous functions $\alpha_{\mathbf{c}}$ on $\partial\Omega$, distinct for different $\mathbf{c} \in \Sigma$, such that any absolute minimizer $\bar{G} \in \mathcal{S}_\beta(\Omega)$ is representable as $\bar{G} = (\bar{\varphi}_{x_2}, -\bar{\varphi}_{x_1})$, where $\bar{\varphi}$ is the absolute minimizing Lipschitz extension of $\alpha_{\mathbf{c}}$ for some $\mathbf{c} \in \Sigma$.

The paper is organized as follows. In Section 2, we collect some notation and preliminary results on functions of bounded variation and sets of finite perimeter, mostly from [3,17], and on the normal-components for functions in $\mathcal{X}_n(\Omega)$ and we define the measures (F, Dv) for functions $F \in \mathcal{X}_n(\Omega)$ and $v \in BV(\Omega)$ in a slightly different way from those used in [4,5,8,11–13]. In Section 3, we prove Theorems 1.2 and 1.3 by giving several characterizations of $\mu(g, h)$. In Section 4, we present two proofs of Theorem 1.1, one based on Theorem 1.3 and the other on a natural direct approach analogous to the approach for L^∞ -functionals of gradient fields given in [7,9] using the limit of p -power functionals as $p \rightarrow \infty$. In Section 5, we provide a sufficient condition for the existence of minimizers for $\lambda(g, h)$ in $BV(\Omega)$ and maximizing sets of $\mu(g, h)$. A highly non-trivial interesting example is also given (see Example 5.1). In Section 6, we study two-dimensional problems and we prove Theorem 1.4 as two separate theorems (Theorems 6.2 and 6.5). The proof of Theorem 6.5 relies on several equivalent descriptions of the absolutely minimizing Lipschitz extension as the viscosity solution to the infinity Laplacian equation as given in [6,19,23].

2. Notation and preliminaries

Let U be an open set in \mathbf{R}^n . Let $L^p(U)$ and $W^{1,p}(U)$ be the usual Lebesgue and Sobolev spaces [1]. A function $u \in L^1(U)$ is said to have *bounded variation* in U if $|Du|(U) = \int_U |Du| < \infty$, where

$$\int_U |Du| = \sup \left\{ \int_U u \operatorname{div} \varphi \, dx \mid \varphi \in C_0^1(U; \mathbf{R}^n), \|\varphi\|_{L^\infty(U)} \leq 1 \right\}. \quad (2.1)$$

We denote by $BV(U)$ the space of all functions in $L^1(U)$ having bounded variation in U ; this is a Banach space with norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$. It is well-known that $u \in BV(U)$ if and only if $u \in L^1(U)$ and, for each $i = 1, 2, \dots, n$, the distributional derivative u_{x_i} is a measure μ_i of finite total variation in the space $\mathcal{M}(U)$ of all Radon measures on U . Hence, the distributional gradient of u is a vector measure $Du = (\mu_1, \mu_2, \dots, \mu_n)$.

Let E be a Borel set in \mathbf{R}^n . The *perimeter* $P(E, U)$ of E in U is defined to be $P(E, U) = \int_U |D\chi_E|$; write $P(E) = P(E, \mathbf{R}^n)$. We say that E is a set of *finite perimeter* in U if $P(E, U) < \infty$. A set E is called a *Caccioppoli set* if $P(E, U) < \infty$ for every bounded open set U in \mathbf{R}^n . For a Caccioppoli set E , a point $x \in \mathbf{R}^n$ is said to be in the *reduced boundary* ∂^*E of E if $\int_{B_\epsilon(x)} |D\chi_E| > 0$ for all $\epsilon > 0$ and the vector limit $\nu_E(x) = \lim_{\epsilon \rightarrow 0} \frac{\int_{B_\epsilon(x)} D\chi_E}{\int_{B_\epsilon(x)} |D\chi_E|}$ exists and satisfies $|\nu_E(x)| = 1$. (Note that we use the same notation as [17] for the reduced boundary, which is different

from the notation used in [3], where $\mathcal{F}E$ is used.) This unit vector $\nu_E(x)$ is called the *generalized inner normal* to E at $x \in \partial^*E$. By Theorem 3.59 in [3] or Theorem 4.4 in [17], we know that, as Radon measures in $\mathcal{M}(\mathbf{R}^n)$,

$$D\chi_E = \nu_E |D\chi_E|, \quad |D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^*E. \tag{2.2}$$

Given a measurable set E in \mathbf{R}^n and a number $t \in [0, 1]$, the set E^t of all points where E has density t is defined by

$$E^t = \left\{ x \in \mathbf{R}^n \mid \lim_{\epsilon \rightarrow 0} \frac{|E \cap B_\epsilon(x)|}{|B_\epsilon(x)|} = t \right\}.$$

The sets E^0 and E^1 can be considered as the measure-theoretic interior and exterior of E . So the set $\partial^m E = \mathbf{R}^n \setminus (E^0 \cup E^1)$ is defined to be the *measure-theoretic boundary* of E (or the *essential boundary* of E); clearly $\partial^m E \subset \partial E$. (Note again that our notation for the essential boundary is different from that used in [3].) A well-known theorem (cf., [3, Theorem 3.61]) states that if E has finite perimeter in \mathbf{R}^n , then

$$\partial^*E \subset E^{\frac{1}{2}} \subset \partial^m E, \quad \mathcal{H}^{n-1}(\partial^m E \setminus \partial^*E) = 0. \tag{2.3}$$

In particular, E has density either 0, $\frac{1}{2}$ or 1 at \mathcal{H}^{n-1} -a.e. $x \in \mathbf{R}^n$.

In what follows, we assume Ω is a bounded domain with Lipschitz boundary $\partial\Omega$ in \mathbf{R}^n and define the family

$$\mathcal{P}(\Omega) = \{ E \subset \Omega \mid 0 < P(E, \Omega) < \infty \}. \tag{2.4}$$

The trace operator $u|_{\partial\Omega}$ can be extended as a linear bounded operator $\gamma = \gamma_\Omega : BV(\Omega) \rightarrow L^1(\partial\Omega)$ (see [3, Theorem 3.87] and [17, Theorem 2.10]) so that, for each $u \in BV(\Omega)$,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^n} \int_{B_\epsilon(a) \cap \Omega} |u(x) - \gamma(u)(a)| dx = 0 \tag{2.5}$$

for \mathcal{H}^{n-1} -almost every $a \in \partial\Omega$; moreover, for all $\zeta \in C^1(\mathbf{R}^n; \mathbf{R}^n)$,

$$\int_\Omega u \operatorname{div} \zeta dx = - \int_\Omega \zeta \cdot d(Du) + \int_{\partial\Omega} \gamma(u) \zeta \cdot \nu d\mathcal{H}^{n-1}. \tag{2.6}$$

The trace operator γ_Ω is onto from $W^{1,1}(\Omega)$ to $L^1(\partial\Omega)$ (see, e.g., [5, Lemma 5.5]).

It is well-known that (see [5, Lemma 5.2] and [17, Remark 2.12]), for each $u \in BV(\Omega)$, there exists a sequence $u_j \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that

$$(a) \quad u_j \rightarrow u \quad \text{in } L^{\frac{n}{n-1}}(\Omega) \quad (b) \quad \int_\Omega |\nabla u_j| dx \rightarrow \int_\Omega |Du| \quad (c) \quad \gamma(u_j) = \gamma(u). \tag{2.7}$$

By [3, Corollary 3.49 and Remark 3.50], we have the following Poincaré inequality: there exists a constant C such that, for all $u \in BV(\Omega)$,

$$\|u - (u)_\Omega\|_{L^{\frac{n}{n-1}}(\Omega)} \leq C \int_\Omega |Du|, \tag{2.8}$$

where $(u)_\Omega = \frac{1}{|\Omega|} \int_\Omega u dx$ is the average of u on Ω .

Let $u \in BV(\Omega)$. By [3, Theorem 3.40], for almost every $t \in \mathbf{R}$, the set $\{u > t\} = \{x \in \Omega \mid u(x) > t\}$ has finite perimeter in Ω and the *coarea formula*

$$|Du|(B) = \int_{-\infty}^{\infty} |D\chi_{\{u>t\}}|(B) dt, \quad Du(B) = \int_{-\infty}^{\infty} D\chi_{\{u>t\}}(B) dt \tag{2.9}$$

holds for any Borel set $B \subset \Omega$. If $u \in W^{1,1}(\Omega)$ and is *precisely* represented in Ω , say, $u \in C(\Omega) \cap W^{1,1}(\Omega)$ (see [21]), then for all Borel functions $\psi : \Omega \rightarrow \mathbf{R}$

$$\int_\Omega \psi |\nabla u| = \int_{-\infty}^{\infty} \left(\int_{u^{-1}(t)} \psi d\mathcal{H}^{n-1} \right) dt = \int_{-\infty}^{\infty} \left(\int_\Omega \psi d|D\chi_{\{u>t\}}| \right) dt. \tag{2.10}$$

We gather some useful convergence results in $BV(\Omega)$ in the following proposition; see, e.g., [3, Propositions 3.6, 3.13 and Theorem 3.88] and [17, Theorem 2.11].

Proposition 2.1. *Let $u, u_j \in BV(\Omega)$, $|Du_j|(\Omega) \leq M$ and $u_j \rightarrow u$ in $L^1(\Omega)$ as $j \rightarrow \infty$. Then*

$$u_j \rightharpoonup u \quad \text{in } L^{\frac{n}{n-1}}(\Omega), \tag{2.11}$$

$$u_j \rightarrow u \quad \text{in } L^q(\Omega) \text{ for each } 1 \leq q < \frac{n}{n-1}, \tag{2.12}$$

$$|Du|(A) \leq \liminf_{j \rightarrow \infty} |Du_j|(A) \quad \forall A \subset \Omega \text{ open}, \tag{2.13}$$

$$\int_{\Omega} \phi d|Du| \leq \liminf_{j \rightarrow \infty} \int_{\Omega} \phi d|Du_j|, \tag{2.14}$$

where ϕ is any nonnegative lower semi-continuous function in Ω . If, in addition, we assume

$$\lim_{j \rightarrow \infty} \int_{\Omega} |Du_j| = \int_{\Omega} |Du|, \tag{2.15}$$

then we have

$$\gamma(u_j) \rightarrow \gamma(u) \quad \text{in } L^1(\partial\Omega), \tag{2.16}$$

$$\lim_{j \rightarrow \infty} \int_{\Omega} \psi d|Du_j| = \int_{\Omega} \psi d|Du| \tag{2.17}$$

for all bounded continuous functions $\psi \in C(\Omega)$. In particular, $|Du_j| \xrightarrow{*} |Du|$ in $\mathcal{M}(\Omega)$.

We prove the following result providing a formula for the traces of certain functions.

Proposition 2.2. *For each $u \in BV(\Omega)$,*

$$\gamma(u)(a) = \lim_{\epsilon \rightarrow 0} \frac{2}{|B_{\epsilon}(a)|} \int_{B_{\epsilon}(a)} \chi_{\Omega}(x)u(x) dx$$

for \mathcal{H}^{n-1} -a.e. $a \in \partial\Omega$; for each $E \in \mathcal{P}(\Omega)$, $\gamma(\chi_E) = \chi_{\partial\Omega \cap \partial^*E}$ in $L^1(\partial\Omega)$.

Proof. Note that Ω has finite perimeter in \mathbf{R}^n (see [3, Proposition 3.62]) and that $\partial^*\Omega \subset \partial\Omega = \partial^m\Omega$. Hence $\mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^{\frac{1}{2}}) = 0$. By (2.5), for \mathcal{H}^{n-1} -a.e. $a \in \partial\Omega$,

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^n} \left| \int_{B_{\epsilon}(a) \cap \Omega} (u(x) - \gamma(u)(a)) \right| \leq \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^n} \int_{B_{\epsilon}(a) \cap \Omega} |u(x) - \gamma(u)(a)| = 0,$$

from which we have

$$\gamma(u)(a) \lim_{\epsilon \rightarrow 0} \frac{|B_{\epsilon}(a) \cap \Omega|}{|B_{\epsilon}(a)|} = \lim_{\epsilon \rightarrow 0} \frac{1}{|B_{\epsilon}(a)|} \int_{B_{\epsilon}(a)} \chi_{\Omega}(x)u(x) dx.$$

Since $\mathcal{H}^{n-1}(\partial\Omega \setminus \Omega^{\frac{1}{2}}) = 0$, for \mathcal{H}^{n-1} -a.e. $a \in \partial\Omega$, it follows that $\lim_{\epsilon \rightarrow 0} \frac{|B_{\epsilon}(a) \cap \Omega|}{|B_{\epsilon}(a)|} = \frac{1}{2}$; the first statement is proved.

To prove the second statement, let $\phi = \gamma(\chi_E) \in L^1(\partial\Omega)$ be the trace function. Then

$$\phi(a) = 2 \lim_{\epsilon \rightarrow 0} \frac{|B_{\epsilon}(a) \cap E|}{|B_{\epsilon}(a)|}. \tag{2.18}$$

We write $\partial\Omega = (\partial\Omega \cap E^0) \cup (\partial\Omega \cap E^1) \cup (\partial\Omega \cap \partial^*E) \cup [\partial\Omega \cap (\partial^m E \setminus \partial^*E)]$. First, since $E \subset \Omega$, $\Omega^{\frac{1}{2}} \cap E^1 = \emptyset$; thus $\mathcal{H}^{n-1}(\partial\Omega \cap E^1) = 0$. If $a \in \partial\Omega \cap E^0$, then $\phi(a) = 0$. If $a \in \partial\Omega \cap \partial^*E$ then, by (2.3), $a \in E^{\frac{1}{2}}$ and hence $\phi(a) = 1$.

Finally, by Lemma 2.3, $\mathcal{H}^{n-1}(\partial^m E \setminus \partial^* E) = 0$, and hence $\phi(a) = \chi_{\partial\Omega \cap \partial^* E}(a)$ for \mathcal{H}^{n-1} -a.e. $a \in \partial\Omega$, which proves $\gamma(\chi_E) = \chi_{\partial\Omega \cap \partial^* E}$ in $L^1(\partial\Omega)$. \square

In the rest of this section, we review the space $\mathcal{X}_n(\Omega)$ and normal-components on $\partial\Omega$. The normal-component operator (called *interior normal trace*) can be defined on the reduced boundary of a set of finite perimeter for the so-called *divergence-measure* vector fields $F \in L^\infty(\Omega; \mathbf{R}^n)$ with $\operatorname{div} F$ being a Radon measure [11]. However, for our purpose, we only consider the vector fields of $\mathcal{X}_n(\Omega)$ with L^n integrable divergence. The following result is proved in [5].

Theorem 2.3. (See Theorem 1.2 in [5].) *There exists a linear (outward) normal-component operator $\delta = \delta_\Omega : \mathcal{X}_n(\Omega) \rightarrow L^\infty(\partial\Omega)$ such that, for all $F \in \mathcal{X}_n(\Omega)$,*

$$\|\delta(F)\|_{L^\infty(\partial\Omega)} \leq \|F\|_{L^\infty(\Omega)}, \tag{2.19}$$

$$\delta(\zeta|_\Omega) = \zeta(x) \cdot \nu(x) \quad \forall \zeta \in C^1(\mathbf{R}^n; \mathbf{R}^n), \tag{2.20}$$

$$\int_\Omega v \operatorname{div} F + \int_\Omega F \cdot \nabla v = \int_{\partial\Omega} \gamma(v) \delta(F) d\mathcal{H}^{n-1} \quad \forall v \in W^{1,1}(\Omega). \tag{2.21}$$

Moreover, if $F_j, F \in \mathcal{X}_n(\Omega)$ satisfy $F_j \xrightarrow{*} F$ in $L^\infty(\Omega; \mathbf{R}^n)$ and $\operatorname{div} F_j \rightharpoonup \operatorname{div} F$ in $L^n(\Omega)$, then $\delta(F_j) \xrightarrow{*} \delta(F)$ in $L^\infty(\partial\Omega)$.

We extend the function $F \cdot \nabla v$ in (2.21) from $v \in W^{1,1}(\Omega)$ to $v \in BV(\Omega)$ by defining the pairing (F, Dv) as a measure for $F \in \mathcal{X}_n(\Omega)$ and $v \in BV(\Omega)$. We do this in a slightly different way from [4,5,8,11–13] by making (F, Dv) a Radon measure on whole \mathbf{R}^n . To do so, given $v \in BV(\Omega)$, $F \in \mathcal{X}_n(\Omega)$, define a distribution $L : C_0^\infty(\mathbf{R}^n) \rightarrow \mathbf{R}$ by

$$L(\varphi) = \int_{\partial\Omega} \varphi \gamma(v) \delta(F) d\mathcal{H}^{n-1} - \int_\Omega (\varphi v \operatorname{div} F + (\nabla \varphi \cdot F)v) dx \quad \forall \varphi \in C_0^\infty(\mathbf{R}^n). \tag{2.22}$$

Theorem 2.4. *There exists a unique Radon measure ω in $\mathcal{M}(\mathbf{R}^n)$, denoted by $\omega = (F, Dv)$, such that $L(\varphi) = \int_{\mathbf{R}^n} \varphi d\omega$ for all $\varphi \in C_0^\infty(\mathbf{R}^n)$. Moreover, for any positive bounded continuous function σ on Ω and any Borel set B in \mathbf{R}^n ,*

$$\left| \int_B d\omega \right| \leq \int_B d|\omega| \leq \left\| \frac{F}{\sigma} \right\|_{L^\infty(U \cap \Omega)} \int_{B \cap \Omega} \sigma d|Dv| \tag{2.23}$$

where U is any open set containing B . In particular, the measure $\omega = (F, Dv)$ is concentrated on Ω and absolutely continuous with respect to $|Dv| \llcorner \Omega$.

Proof. Let v_j be an approximation sequence as determined in (2.7) for $v \in BV(\Omega)$. Using (2.21) with $v = \varphi v_j$, we have

$$L(\varphi) = \lim_{j \rightarrow \infty} \int_\Omega (F \cdot \nabla v_j) \varphi dx \quad \forall \varphi \in C_0^\infty(\mathbf{R}^n). \tag{2.24}$$

Hence

$$|L(\varphi)| \leq \|F\|_{L^\infty} \|\varphi\|_{L^\infty} \lim_{j \rightarrow \infty} \int_\Omega |\nabla v_j| dx = \|F\|_{L^\infty} \|\varphi\|_{L^\infty} \int_\Omega |Dv|$$

for all $\varphi \in C_0^\infty(\mathbf{R}^n)$. Since $C_0^\infty(\mathbf{R}^n)$ is dense in $C_0(\mathbf{R}^n)$, L can be uniquely extended as a linear functional \tilde{L} on $C_0(\mathbf{R}^n)$ that still satisfies

$$|\tilde{L}(\varphi)| \leq \|F\|_{L^\infty(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \int_\Omega |Dv| \quad \forall \varphi \in C_0(\mathbf{R}^n).$$

By Riesz’s theorem, there exists a unique finite Radon measure ω on \mathbf{R}^n such that $\tilde{L}(\varphi) = \int_{\mathbf{R}^n} \varphi d\omega$ for all $\varphi \in C_0(\mathbf{R}^n)$. We now prove (2.23). Let B be any Borel set in \mathbf{R}^n and let $K \subset B \subset U$, where K is compact and U is open. For any $\epsilon > 0$, by the outer regularity of measure $\sigma d|Dv|$, there exists an open set $V \subset U$ containing K such that

$$\int_{V \cap \Omega} \sigma d|Dv| < \int_{K \cap \Omega} \sigma d|Dv| + \epsilon.$$

Given any $\varphi \in C_c(V)$ with $\|\varphi\|_{L^\infty(V)} \leq 1$, let $\varphi_k \in C_c^\infty(V)$ be such that $\|\varphi_k - \varphi\|_{L^\infty(V)} \rightarrow 0$ as $k \rightarrow \infty$. Let $U_k = \text{supp } \varphi_k$ be the compact support of φ_k in V . Let us consider the measures $\mu_j = \sigma |\nabla v_j| dx$ and $\lambda = \sigma d|Dv|$ in $\mathcal{M}(\Omega)$. Since $\int_\Omega |\nabla v_j| dx \rightarrow \int_\Omega |Dv|$, by Proposition 2.1, we have $\mu_j(\Omega) \rightarrow \lambda(\Omega)$ and $\lambda(A) \leq \liminf_{j \rightarrow \infty} \mu_j(A)$ for all open sets $A \subset \Omega$. This implies $\lambda(\Omega \setminus A) \geq \limsup_{j \rightarrow \infty} \mu_j(\Omega \setminus A)$ for all open sets $A \subset \Omega$. Taking $A = U_k \cap \Omega$, we have, for each $k = 1, 2, \dots$,

$$\limsup_{j \rightarrow \infty} \int_{U_k \cap \Omega} \sigma |\nabla v_j| dx \leq \int_{U_k \cap \Omega} \sigma d|Dv|.$$

Therefore, by (2.24),

$$\begin{aligned} |\tilde{L}(\varphi)| &= \lim_{k \rightarrow \infty} |L(\varphi_k)| = \lim_{k \rightarrow \infty} \left| \lim_{j \rightarrow \infty} \int_{U_k \cap \Omega} (F \cdot \nabla v_j) \varphi_k dx \right| \\ &\leq \limsup_{k \rightarrow \infty} \left[\left\| \frac{F}{\sigma} \right\|_{L^\infty(U \cap \Omega)} \|\varphi_k\|_{L^\infty} \left(\limsup_{j \rightarrow \infty} \int_{U_k \cap \Omega} \sigma |\nabla v_j| dx \right) \right] \\ &\leq \left\| \frac{F}{\sigma} \right\|_{L^\infty(U \cap \Omega)} \left(\limsup_{k \rightarrow \infty} \int_{U_k \cap \Omega} \sigma d|Dv| \right) \leq \left\| \frac{F}{\sigma} \right\|_{L^\infty(U \cap \Omega)} \int_{V \cap \Omega} \sigma d|Dv|. \end{aligned}$$

Since $|\omega|(V) = \sup\{\tilde{L}(\varphi) \mid \varphi \in C_c(V), \|\varphi\|_{L^\infty(V)} \leq 1\}$, we have

$$|\omega|(V) \leq \left\| \frac{F}{\sigma} \right\|_{L^\infty(U \cap \Omega)} \int_{V \cap \Omega} \sigma d|Dv|.$$

Therefore

$$|\omega|(K) \leq |\omega|(V) \leq \left\| \frac{F}{\sigma} \right\|_{L^\infty(U \cap \Omega)} \left(\int_{K \cap \Omega} \sigma d|Dv| + \epsilon \right),$$

for all $\epsilon > 0$. This proves

$$|\omega|(K) \leq \left\| \frac{F}{\sigma} \right\|_{L^\infty(U \cap \Omega)} \int_{K \cap \Omega} \sigma d|Dv|$$

for all $K \subset B \subset U$, K compact and U open. Hence (2.23) follows by the inner regularity of $|\omega|$ and $\sigma d|Dv| \ll \Omega$. \square

Remark 2.1. Since $\omega = (F, Dv)$ is concentrated on Ω , we can write $\int_{\mathbf{R}^n} \varphi d\omega = \int_\Omega \varphi d\omega$. Therefore, by (2.22) we obtain a more general divergence formula

$$\int_\Omega \varphi d(F, Dv) = \int_{\partial\Omega} \varphi \gamma(v) \delta(F) d\mathcal{H}^{n-1} - \int_\Omega (\varphi v \operatorname{div} F + (\nabla \varphi \cdot F) v) dx \tag{2.25}$$

for all $F \in \mathcal{X}_n(\Omega)$, $v \in BV(\Omega)$ and all Lipschitz functions $\varphi \in W^{1,\infty}(\Omega)$.

We have the following compensated compactness result; see also [5, Theorems 4.1 and 4.2].

Proposition 2.5. Let $u_j, u \in BV(\Omega)$ satisfy, for some $1 \leq q \leq \frac{n}{n-1}$, $u_j \rightarrow u$ in $L^q(\Omega)$ and let $F_j, F \in \mathcal{X}_n(\Omega)$ satisfy $F_j \xrightarrow{*} F$ in $L^\infty(\Omega)$, $\operatorname{div} F_j \rightarrow \operatorname{div} F$ in $L^{q'}(\Omega)$, where $q' = \frac{q}{q-1}$. Then $(F_j, Du_j) \xrightarrow{*} (F, Du)$ as Radon measures in $\mathcal{M}(\Omega)$. Furthermore, if, in addition,

$$\lim_{j \rightarrow \infty} \int_{\Omega} |Du_j| = \int_{\Omega} |Du|, \tag{2.26}$$

then $(F_j, Du_j) \xrightarrow{*} (F, Du)$ in $\mathcal{M}(\mathbf{R}^n)$.

Proof. Given any $\varphi \in C_0^1(\mathbf{R}^n)$, by (2.25), we have, for each $j = 1, 2, \dots$,

$$\int_{\Omega} \varphi d(F_j, Du_j) = \int_{\partial\Omega} \varphi \gamma(u_j) \delta(F_j) d\mathcal{H}^{n-1} - \int_{\Omega} (\varphi u_j \operatorname{div} F_j + (\nabla\varphi \cdot F_j) u_j).$$

First, if $\varphi \in C_0^1(\Omega)$, then there vanishes the boundary term and hence we have

$$\lim_{j \rightarrow \infty} \int_{\Omega} \varphi d(F_j, Du_j) = \int_{\Omega} \varphi d(F, Du) \tag{2.27}$$

for all $\varphi \in C_0^1(\Omega)$. This proves the weak-star convergence in $\mathcal{M}(\Omega)$. Now, assume (2.26). Then by Theorem 2.3, $\delta(F_j) \xrightarrow{*} \delta(F)$ in $L^\infty(\partial\Omega)$, and by (2.16) in Proposition 2.1, $\gamma(u_j) \rightarrow \gamma(u)$ in $L^1(\partial\Omega)$. Hence, (2.27) holds for all $\varphi \in C_0^1(\mathbf{R}^n)$, which proves the weak-star convergence in $\mathcal{M}(\mathbf{R}^n)$. \square

The following result defines a boundary normal-component for functions in $\mathcal{X}_n(\Omega)$ on sets of finite perimeter; the similar definition has been given in [11, Theorem 5.2] for a broader class of functions.

Proposition 2.6. Given any $F \in \mathcal{X}_n(\Omega)$ and any set $E \subset \Omega$ of finite perimeter, there exists a function $\tilde{\theta}_E(F) \in L^\infty(\partial^*E; d\mathcal{H}^{n-1})$, called the interior normal-component of F on ∂^*E , such that

$$\int_E (\varphi \operatorname{div} F + \nabla\varphi \cdot F) dx = - \int_{\partial^*E} \varphi \tilde{\theta}_E(F) d\mathcal{H}^{n-1} \quad \forall \varphi \in C^1(\mathbf{R}^n). \tag{2.28}$$

Moreover, if E is an open set with Lipschitz boundary, then $\tilde{\theta}_E(F) = -\delta_E(F)$, where $\delta_E(F) \in L^\infty(\partial E)$ is the normal-component of $F \in \mathcal{X}_n(E)$ on ∂E defined above.

Proof. Since $\chi_E \in BV(\Omega)$, the measure $(F, D\chi_E)$ is well-defined above as a Radon measure in \mathbf{R}^n concentrated on Ω and absolutely continuous relative to $|D\chi_E| \llcorner \Omega = \mathcal{H}^{n-1} \llcorner (\Omega \cap \partial^*E)$. Hence, by the Radon–Nikodym theorem, there exists a function $\theta_E(F) \in L^\infty(\Omega \cap \partial^*E; d\mathcal{H}^{n-1})$ with $\|\theta_E(F)\|_{L^\infty} \leq \|F\|_{L^\infty(U)}$, $U \subset \Omega$ any open set containing E , such that

$$d(F, D\chi_E) = \theta_E(F) d\mathcal{H}^{n-1} \llcorner (\Omega \cap \partial^*E) \quad \text{on } \mathbf{R}^n. \tag{2.29}$$

(The function $\theta_E(F)$ has been called the interior normal-trace relative to E of F on ∂^*E in [11, Theorem 5.2] for a broader class of functions F .) Let us define

$$\tilde{\theta}_E(F)(x) = \begin{cases} \theta_E(F)(x) & x \in \Omega \cap \partial^*E, \\ -\delta(F)(x) & x \in \partial\Omega \cap \partial^*E, \end{cases} \tag{2.30}$$

where $\delta(F) = \delta_\Omega(F)$ is the normal-component of F on $\partial\Omega$ defined in Theorem 2.3. Combining formula (2.25) with (2.29) and Proposition 2.2, we have the following general Gauss–Green formula:

$$\int_E (\varphi \operatorname{div} F + \nabla\varphi \cdot F) dx = - \int_{\partial^*E} \varphi \tilde{\theta}_E(F) d\mathcal{H}^{n-1} \quad \forall \varphi \in C^1(\mathbf{R}^n).$$

This proves (2.28). If E is a Lipschitz domain itself, then by (2.21) we easily see that $\tilde{\theta}_E(F) = -\delta_E(F)$. \square

Remark 2.2. (a) The formula (2.28) generalizes a case of the Gauss–Green formula in [11, Theorem 5.3] since we allow $\varphi \in C^1(\bar{\Omega})$ and $\partial E \cap \partial\Omega \neq \emptyset$; in [11, Theorem 5.3], only sets $E \Subset \Omega$ are considered. Furthermore, we have $\tilde{\theta}_E(F) = F \cdot \nu_E$ on ∂^*E if $F \in C^1(\mathbf{R}^n; \mathbf{R}^n)$.

(b) Let $G, \bar{G} \in \mathcal{S}_\beta(\Omega)$ and $E \subset \Omega$ be a set of finite perimeter in Ω . Then the condition (1.16) in Definition 1.1 is equivalent to the local condition: $\tilde{\theta}_E(G) = \tilde{\theta}_E(\bar{G})$ in $L^\infty(\partial^*E; d\mathcal{H}^{n-1})$.

3. Characterization of $\mu(g, h)$ and proof of Theorems 1.2 and 1.3

Given any function u on Ω , denote by $E_t(u)$ the upper-level set $\{u > t\} = \{x \in \Omega \mid u(x) > t\}$ for each $t \in \mathbf{R}$. We first prove the following useful result.

Lemma 3.1. *Let $u \in BV(\Omega)$. Then*

$$u(x) = \int_0^\infty \chi_{E_t(u)}(x) dt - \int_{-\infty}^0 (1 - \chi_{E_t(u)}(x)) dt \quad (\mathcal{L}^n\text{-a.e. } x \in \Omega),$$

$$\gamma(u)(a) = \int_0^\infty \gamma(\chi_{E_t(u)})(a) dt - \int_{-\infty}^0 (1 - \gamma(\chi_{E_t(u)})(a)) dt \quad (\mathcal{H}^{n-1}\text{-a.e. } a \in \partial\Omega).$$

Proof. By writing $u = u^+ - u^-$, without loss of generality, we assume $u \geq 0$, so in the two identities there are only integral terms from 0 to ∞ . The first identity is easy; so we only prove the second identity. We proceed to prove

$$\gamma\left(\int_0^\infty \chi_{E_t(u)}(x) dt\right)(a) = \int_0^\infty \gamma(\chi_{E_t(u)})(a) dt \tag{3.1}$$

for \mathcal{H}^{n-1} -a.e. $a \in \partial\Omega$. Note that, for almost every $t \in \mathbf{R}$, the set $E_t(u)$ has finite perimeter in Ω and so $\chi_{E_t(u)} \in BV(\Omega)$. Let $\{t_j\}$, $j = 1, 2, \dots$, be a dense sequence of $(0, \infty)$ such that each $\chi_{E_{t_j}(u)} \in BV(\Omega)$. By Proposition 2.2, we have that, for \mathcal{H}^{n-1} -a.e. $a \in \partial\Omega$,

$$\gamma(\chi_{E_{t_j}(u)})(a) = \lim_{\epsilon \rightarrow 0} \frac{2}{|B_\epsilon(a)|} \int_{B_\epsilon(a)} \chi_{E_{t_j}(u)}(x) dx \quad \forall j = 1, 2, \dots, \tag{3.2}$$

and for each $N = 1, 2, \dots$,

$$\gamma\left(\int_0^N \chi_{E_t(u)}(x) dt\right)(a) = \lim_{\epsilon \rightarrow 0} \frac{2}{|B_\epsilon(a)|} \int_{B_\epsilon(a)} \left(\int_0^N \chi_{E_t(u)}(x) dt\right) dx.$$

For such an $a \in \partial\Omega$, the function $r(t) = \gamma(\chi_{E_t(u)})(a)$ is a non-increasing function in $t \in (0, \infty)$ and hence is continuous almost everywhere. At any continuity point t_0 of this function, by (3.2), it follows that

$$\gamma(\chi_{E_{t_0}(u)})(a) = \lim_{\epsilon \rightarrow 0} \frac{2}{|B_\epsilon(a)|} \int_{B_\epsilon(a)} \chi_{E_{t_0}(u)}(x) dx. \tag{3.3}$$

Hence, by Fubini’s theorem,

$$\begin{aligned} \gamma\left(\int_0^N \chi_{E_t(u)}(x) dt\right)(a) &= \lim_{\epsilon \rightarrow 0} \int_0^N \frac{2}{|B_\epsilon(a)|} \int_{B_\epsilon(a)} \chi_{E_t(u)}(x) dx dt \\ &= \int_0^N \lim_{\epsilon \rightarrow 0} \frac{2}{|B_\epsilon(a)|} \int_{B_\epsilon(a)} \chi_{E_t(u)}(x) dx dt = \int_0^N \gamma(\chi_{E_t(u)})(a) dt, \end{aligned}$$

where the change of order of the limit into the integral is justified by the dominated convergence theorem. Finally note that, by [16, Proposition 6, p. 340],

$$\int_0^N \chi_{E_t(u)}(x) dt = \min\{u(x), N\} \rightarrow u(x) = \int_0^\infty \chi_{E_t(u)}(x) dt$$

in the norm topology of $BV(\Omega)$ as $N \rightarrow \infty$. Hence, by the continuity of the trace-operator,

$$\gamma \left(\int_0^N \chi_{E_t(u)} dt \right) \rightarrow \gamma \left(\int_0^\infty \chi_{E_t(u)} dt \right)$$

in $L^1(\partial\Omega)$ as $N \rightarrow \infty$. This proves (3.1) for \mathcal{H}^{n-1} -a.e. $a \in \partial\Omega$. \square

In what follows, we assume σ is the function given in the introduction; that is, σ is continuous function in Ω and satisfies, for two positive constants σ_0 and M_0 , that

$$0 < \sigma_0 \leq \sigma(x) \leq M_0 < \infty \quad \forall x \in \Omega. \tag{3.4}$$

Let $g \in L^\infty(\partial\Omega)$ and $h \in L^n(\Omega)$ satisfy (1.7) above. Define the following functionals on $BV(\Omega)$:

$$N(u) = \int_\Omega \sigma d|Du|, \quad B(u) = \int_{\partial\Omega} \gamma(u)g d\mathcal{H}^{n-1} - \int_\Omega uh dx. \tag{3.5}$$

Proposition 3.2. *Given $0 < m < \infty$, the following statements are equivalent:*

- (a) $mN(\zeta) \geq B(\zeta) \quad \forall \zeta \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$.
- (b) $mN(u) \geq B(u) \quad \forall u \in BV(\Omega)$.
- (c) $mN(\chi_E) \geq B(\chi_E) \quad \forall E \in \mathcal{P}(\Omega)$.
- (d) $mN(\chi_E) \geq |B(\chi_E)| \quad \forall E \in \mathcal{P}(\Omega)$ open.

Proof. That (a) implies (b) follows by the approximation (2.7) and the convergence result Proposition 2.1. That (b) implies (c) is immediate as $\chi_E \in BV(\Omega)$ for all $E \in \mathcal{P}(\Omega)$. To prove that (c) implies (d), note that if $E \in \mathcal{P}(\Omega)$ then $\Omega \setminus E \in \mathcal{P}(\Omega)$ and $D\chi_{\Omega \setminus E} = -D\chi_E$ and $\partial^*(\Omega \setminus E) = \partial^*E$; hence, by condition (1.7), $N(\chi_{\Omega \setminus E}) = N(\chi_E)$ and $B(\chi_{\Omega \setminus E}) = -B(\chi_E)$. This proves that (c) implies (d). Finally we prove that (d) implies (a). Let $L^\pm(u) = mN(u) \pm B(u)$. Then, by (d),

$$L^\pm(\chi_E) \geq 0 \quad \forall E \in \mathcal{P}(\Omega) \text{ open.}$$

Given any $\zeta \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$, we write $\zeta = \zeta^+ - \zeta^-$, where $\zeta^\pm(x) = \max\{\pm\zeta(x), 0\}$. Then $\zeta^\pm \in C(\Omega) \cap W^{1,1}(\Omega)$ are nonnegative and

$$\gamma(\zeta) = \gamma(\zeta^+) - \gamma(\zeta^-), \quad |\nabla\zeta| = |\nabla\zeta^+| + |\nabla\zeta^-|.$$

Hence $mN(\zeta) - B(\zeta) = L^-(\zeta) = L^-(\zeta^+ - \zeta^-) = L^-(\zeta^+) + L^+(\zeta^-)$. We claim that $L^-(\zeta^+) \geq 0$ and $L^+(\zeta^-) \geq 0$ and hence $mN(\zeta) \geq B(\zeta)$ follows, as desired of (a). Since the argument is similar, we only prove $L^-(\zeta^+) \geq 0$. Note that, by Lemma 3.1 and Fubini’s theorem,

$$\int_\Omega h(x)\zeta^+(x) dx = \int_\Omega \int_0^\infty \chi_{\{\zeta^+ > t\}}(x)h(x) dt dx = \int_0^\infty \left(\int_\Omega \chi_{\{\zeta^+ > t\}}(x)h(x) dx \right) dt; \tag{3.6}$$

$$\int_{\partial\Omega} g\gamma(\zeta^+) d\mathcal{H}^{n-1} = \int_0^\infty \left(\int_{\partial\Omega} \gamma(\chi_{\{\zeta^+ > t\}})g d\mathcal{H}^{n-1} \right) dt. \tag{3.7}$$

Also, by the coarea formula (2.10),

$$\int_{\Omega} \sigma(x) |\nabla \zeta^+(x)| dx = \int_0^{\infty} \left(\int_{\Omega} \sigma d|\mathcal{D}\chi_{\{\zeta^+ > t\}}| \right) dt. \quad (3.8)$$

Note that, for almost every $t \in (0, \infty)$, the open set $E_t = \{\zeta^+ > t\}$ is either in $\mathcal{P}(\Omega)$ or is Ω or empty. By assumption (d), $L^-(\chi_{E_t}) \geq 0$ for a.e. $t \in (0, \infty)$. Finally, combining (3.8), (3.6) and (3.7), we obtain

$$L^-(\zeta^+) = \int_0^{\infty} L^-(\chi_{\{\zeta^+ > t\}}) dt = \int_0^{\infty} L^-(\chi_{E_t}) dt \geq 0,$$

as claimed. This completes the proof. \square

Theorem 3.3. *Let σ, g, h be the functions satisfying conditions (3.4) and (1.7). Then the following quantities are all finite and equal:*

$$\begin{aligned} \mu_1 &= \sup_{\substack{\zeta \in W^{1,1}(\Omega) \\ \zeta \neq \text{const}}} \frac{\int_{\partial\Omega} \gamma(\zeta) g d\mathcal{H}^{n-1} - \int_{\Omega} \zeta h dx}{\int_{\Omega} \sigma(x) |\nabla \zeta| dx}, \\ \mu_2 &= \sup_{\substack{\zeta \in BV(\Omega) \\ \zeta \neq \text{const}}} \frac{\int_{\partial\Omega} \gamma(\zeta) g d\mathcal{H}^{n-1} - \int_{\Omega} \zeta h dx}{\int_{\Omega} \sigma(x) d|D\zeta|}, \\ \mu_3 &= \sup_{E \in \mathcal{P}(\Omega)} \frac{\int_{\partial\Omega \cap \partial^* E} g d\mathcal{H}^{n-1} - \int_E h dx}{\int_{\Omega \cap \partial^* E} \sigma d\mathcal{H}^{n-1}}, \\ \mu_4 &= \sup_{E \in \mathcal{P}(\Omega) \text{ open}} \frac{|\int_{\partial\Omega \cap \partial^* E} g d\mathcal{H}^{n-1} - \int_E h dx|}{\int_{\Omega \cap \partial^* E} \sigma d\mathcal{H}^{n-1}}. \end{aligned}$$

We denote the value of these quantities by $\mu(g, h)$. Moreover, $\mu(g, h) = 0$ if and only if $g = h = 0$.

Proof. Let $N(u)$ and $B(u)$ be defined by (3.5) above. Define $Q(\zeta) = \frac{B(\zeta)}{N(\zeta)}$ if ζ is not a constant function and let $Q(\zeta) = 0$ if ζ is a constant function. Note that for each $E \in \mathcal{P}(\Omega)$, by (2.2) and Proposition 2.2(b),

$$B(\chi_E) = \int_{\partial\Omega \cap \partial^* E} g d\mathcal{H}^{n-1} - \int_E h dx, \quad N(\chi_E) = \int_{\Omega \cap \partial^* E} \sigma d\mathcal{H}^{n-1}.$$

Hence, it is obvious that $\mu_4 \leq \mu_3 \leq \mu_2$ and $\mu_1 \leq \mu_2$. If $\mu_4 = 0$, then, as in the proof of the previous result, using (3.6)–(3.7), we have $B(\zeta) = 0$ for all $\zeta \in C(\Omega) \cap W^{1,1}(\Omega)$ and hence $B(u) = 0$ for all $u \in BV(\Omega)$; this implies $g = 0$ and $h = 0$. So, in this case, all the numbers are zero. Now assume $B \neq 0$. We show $\mu_2 < \infty$; this proves all these numbers are positive. Once we have proved this, the equality of them follows again from the previous result. To show $\mu_2 < \infty$, note that $Q(\zeta + C) = Q(\zeta)$ for all constants C . Given any nonconstant $u \in BV(\Omega)$, let $c = (u)_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u$. Then

$$\begin{aligned} |B(u - c)| &\leq \|g\|_{L^\infty(\partial\Omega)} \|\gamma(u - c)\|_{L^1(\partial\Omega)} + \|h\|_{L^n(\Omega)} \|u - c\|_{L^{\frac{n}{n-1}}(\Omega)} \\ &\leq C(\|g\|_{L^\infty(\partial\Omega)} + \|h\|_{L^n(\Omega)}) \int_{\Omega} |Du| \end{aligned}$$

and, by (3.4) and u being nonconstant, $N(u - c) \geq \sigma_0 \int_{\Omega} |Du| > 0$. Hence

$$Q(u) = Q(u - c) \leq \frac{C}{\sigma_0} (\|g\|_{L^\infty(\partial\Omega)} + \|h\|_{L^n(\Omega)})$$

for all $u \in BV(\Omega)$. This proves $\mu_2 \leq \frac{C}{\sigma_0} (\|g\|_{L^\infty(\partial\Omega)} + \|h\|_{L^n(\Omega)}) < \infty$. \square

Let $\lambda_p(g, h)$ and $\lambda(g, h)$ be defined as above in (1.10). If $B \neq 0$, we easily see that $\mu_1 = \frac{1}{\lambda_1(g, h)}$ and $\mu_2 = \frac{1}{\lambda(g, h)}$, where μ_1 and μ_2 are the equal numbers defined in Theorem 3.3; therefore $\lambda_1(g, h) = \lambda(g, h) = \frac{1}{\mu(g, h)}$. If $B \equiv 0$; that is, if $g = h = 0$, since $\mu(0, 0) = 0$, these equalities remain valid since we have defined $\lambda_p(0, 0) = \lambda(0, 0) = \infty$ for $1 \leq p < \infty$.

Proof of Theorem 1.2. The first statement of the theorem follows easily as explained above. We now prove (1.11). Without loss of generality, we assume $B \neq 0$. Given any $p > 1$, let $\zeta_j \in W^{1,p}(\Omega)$ be such that

$$\lambda_p(g, h) = \lim_{j \rightarrow \infty} \int_{\Omega} \sigma^p(x) |\nabla \zeta_j|^p dx, \quad B(\zeta_j) = 1.$$

Young’s inequality easily implies that

$$\sigma(x) |\nabla \zeta_j| \leq \frac{1}{p} \sigma^p(x) |\nabla \zeta_j|^p + \frac{p-1}{p}.$$

Since $\zeta_j \in W^{1,1}(\Omega)$ and $B(\zeta_j) = 1$, it follows that

$$\lambda_1(g, h) \leq \int_{\Omega} \sigma(x) |\nabla \zeta_j| dx \leq \frac{1}{p} \int_{\Omega} \sigma^p(x) |\nabla \zeta_j|^p dx + \frac{p-1}{p} |\Omega|.$$

Letting $j \rightarrow \infty$, we have $\lambda_1(g, h) \leq \frac{1}{p} \lambda_p(g, h) + \frac{p-1}{p} |\Omega|$. Letting $p \downarrow 1$, this implies

$$\lambda_1(g, h) \leq \liminf_{p \rightarrow 1^+} \lambda_p(g, h). \tag{3.9}$$

On the other hand, given any $\zeta \in C^1(\bar{\Omega})$ with $B(\zeta) = 1$, we have $\lambda_p(g, h) \leq \int_{\Omega} \sigma^p(x) |\nabla \zeta|^p dx$. Let $1 \leq p \leq 2$. Then $\sigma^p(x) |\nabla \zeta|^p \leq (1 + M_0)^2 (1 + |\nabla \zeta|^2) \leq K < \infty$. Letting $p \downarrow 1$ in the above inequality and using the dominated convergence theorem, we have

$$\limsup_{p \rightarrow 1^+} \lambda_p(g, h) \leq \int_{\Omega} \sigma(x) |\nabla \zeta| dx,$$

for all $\zeta \in C^1(\bar{\Omega})$ with $B(\zeta) = 1$. By the standard approximation argument, this inequality also holds for all $\zeta \in W^{1,1}(\Omega)$ with $B(\zeta) = 1$. Hence it follows that

$$\limsup_{p \rightarrow 1^+} \lambda_p(g, h) \leq \inf_{\substack{\zeta \in W^{1,1}(\Omega) \\ B(\zeta)=1}} \int_{\Omega} \sigma(x) |\nabla \zeta| dx = \lambda_1(g, h).$$

This, combined with (3.9), completes the proof. \square

The following theorem contains some results of Theorem 1.3 and other results that are useful later.

Theorem 3.4. Assume $B \neq 0$. Then, for each $1 < p < \infty$, there exists a unique $u_p \in W^{1,p}(\Omega)$ with $\int_{\Omega} u_p dx = 0$ such that

$$\int_{\Omega} \sigma^p(x) |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi dx = \lambda_p(g, h) B(\varphi) \quad \forall \varphi \in W^{1,p}(\Omega). \tag{3.10}$$

Furthermore, there exist subsequence $p_j \rightarrow 1^+$ as $j \rightarrow \infty$, functions $\bar{u} \in BV(\Omega)$ and $\bar{F} \in L^\infty(\Omega; \mathbf{R}^n)$ with $\|\bar{F}\|_{L^\infty(\Omega)} \leq 1$ such that, as $j \rightarrow \infty$,

$$u_{p_j} \rightharpoonup \bar{u} \text{ in } L^{\frac{n}{n-1}}(\Omega), \quad \nabla u_{p_j} \overset{*}{\rightharpoonup} D\bar{u} \text{ in } \mathcal{M}(\Omega), \tag{3.11}$$

$$|\nabla u_{p_j}|^{p_j-2} \nabla u_{p_j} \rightharpoonup \bar{F} \text{ in } L^r(\Omega) \text{ for each } r > 1, \tag{3.12}$$

$$\operatorname{div}(\sigma \bar{F}) = \lambda_1 h, \quad \delta(\sigma \bar{F}) = \lambda_1 g, \quad \text{where } \lambda_1 = \lambda_1(g, h) > 0, \tag{3.13}$$

$$\sigma^{p_j} |\nabla u_{p_j}|^{p_j} \overset{*}{\rightharpoonup} (\sigma \bar{F}, D\bar{u}) \text{ in } \mathcal{M}(\Omega), \tag{3.14}$$

$$(\sigma \bar{F}, D\bar{u}) = \sigma |D\bar{u}| \text{ as Radon measures in } \mathcal{M}(\Omega). \tag{3.15}$$

Proof. Since $B(\zeta_0) = 1$ for some $\zeta_0 \in C^\infty(\mathbf{R}^n)$, the constraining set $\{u \in W^{1,p}(\Omega) \mid B(u) = 1\}$ in defining $\lambda_p(g, h)$ above is nonempty. We first show that, for each $1 < p < \infty$, the number $\lambda_p(g, h)$ is attained by a unique function u_p in $W^{1,p}(\Omega)$ satisfying $B(u_p) = 1$ and $\int_\Omega u_p dx = 0$. Let $v_j \in W^{1,p}(\Omega)$ be a minimizing sequence for $\lambda_p(g, h)$; that is,

$$\lim_{j \rightarrow \infty} \int_\Omega \sigma^p(x) |\nabla v_j|^p dx = \lambda_p(g, h), \quad B(v_j) = 1.$$

Since $B(v + C) = B(v)$ for all constants C , we can assume $\int_\Omega v_j dx = 0$. Therefore, since $\{\|\nabla v_j\|_{L^p(\Omega)}\}$ is bounded, by Poincaré’s inequality for $W^{1,p}(\Omega)$ -functions [1], $\{v_j\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Hence, by a subsequence, $v_j \rightharpoonup u_p$, where $u_p \in W^{1,p}(\Omega)$ satisfies $\int_\Omega u_p dx = 0$. By the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^1(\partial\Omega) \cap L^{\frac{n}{n-1}}(\Omega)$ (see, e.g., [1]), we have $B(u_p) = \lim_j B(v_j) = 1$ and hence, by the convexity of $\int_\Omega \sigma^p |\nabla v|^p dx$ and the definition of $\lambda_p(g, h)$, it follows that

$$B(u_p) = 1, \quad \int_\Omega \sigma^p |\nabla u_p|^p dx = \lambda_p(g, h); \tag{3.16}$$

that is, u_p is a minimizer for $\lambda_p(g, h)$. The strict convexity of $\int_\Omega \sigma^p |\nabla v|^p dx$ implies that u_p is the unique minimizer of $\lambda_p(g, h)$ satisfying $\int_\Omega u_p dx = 0$. By the Lagrange theorem for constrained minimizers, u_p is a critical point of the functional $L(u) = \int_\Omega \sigma^p(x) |\nabla u|^p dx - \lambda B(u)$ on $u \in W^{1,p}(\Omega)$, where λ is a real number (the Lagrange multiplier). Hence we have

$$p \int_\Omega \sigma^p(x) |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi dx = \lambda B(\varphi) \quad \forall \varphi \in W^{1,p}(\Omega).$$

Taking $\varphi = u_p$, we have $\lambda = p\lambda_p(g, h)$ and this proves (3.10).

To prove the second part of the theorem, consider the set of functions $\{F_p = |\nabla u_p|^{p-2} \nabla u_p\}$ with $1 < p < 2$. Since $|F_p| = |\nabla u_p|^{p-1}$, by Young’s inequality, for any $0 < r < \frac{p}{p-1}$, it follows that

$$|F_p|^r = |\nabla u_p|^{r(p-1)} \leq \frac{r(p-1)}{p} |\nabla u_p|^p + \left(1 - \frac{r(p-1)}{p}\right),$$

and hence

$$\int_\Omega |F_p|^r dx \leq \frac{r(p-1)}{p\sigma_0^p} \lambda_p(g, h) + \left(1 - \frac{r(p-1)}{p}\right) |\Omega|. \tag{3.17}$$

Using this inequality with $r = \frac{1}{p-1}$ and $r = 2$, it follows that $\{u_p\}_{1 < p < 2}$ is bounded in $BV(\Omega)$ and $\{F_p\}_{1 < p < 2}$ is bounded in $L^2(\Omega; \mathbf{R}^n)$. Hence there exists a decreasing sequence $p_j \rightarrow 1$ such that $u_{p_j} \rightharpoonup \bar{u}$ in $L^{\frac{n}{n-1}}(\Omega)$ and $\nabla u_{p_j} \overset{*}{\rightharpoonup} D\bar{u}$ in $\mathcal{M}(\Omega)$, where $\bar{u} \in BV(\Omega)$ satisfies $\int_\Omega \bar{u} = 0$, and $F_{p_j} \rightharpoonup \bar{F}$ in $L^2(\Omega; \mathbf{R}^n)$, where $\bar{F} \in L^2(\Omega; \mathbf{R}^n)$. Given any $r > 1$, let $r < \frac{p_j}{p_j-1}$ for all $j \geq N_r$. Then (3.17) implies that the sequence $\{F_{p_j}\}_{j \geq N_r}$ is bounded in $L^r(\Omega; \mathbf{R}^n)$ and hence any subsequence of it with $p_j \rightarrow 1$ has a sub-subsequence $\{F_{p_{j_k}}\}$ weakly converging to a function $\tilde{F} \in L^r(\Omega; \mathbf{R}^n)$ as $p_{j_k} \rightarrow 1$. Again, by (3.17), we have

$$\int_\Omega |\tilde{F}|^r dx \leq \liminf_{p_{j_k} \rightarrow 1} \int_\Omega |F_{p_{j_k}}|^r dx \leq |\Omega|.$$

On the other hand, since the whole sequence $\{F_{p_j}\}$ is weakly convergent to \bar{F} in $L^2(\Omega; \mathbf{R}^n)$ as $p_j \rightarrow 1$, we must have $\tilde{F} = \bar{F}$. This shows that $\bar{F} \in L^r(\Omega; \mathbf{R}^n)$ for each $r > 1$ and the whole sequence $F_{p_j} \rightharpoonup \bar{F}$ as $p_j \rightarrow 1$ weakly in $L^r(\Omega; \mathbf{R}^n)$ for each $r > 1$ in the sense that, given any $1 < q < \infty$ and any $\Phi \in L^q(\Omega; \mathbf{R}^n)$,

$$\lim_{j \rightarrow \infty} \int_\Omega |\nabla u_{p_j}|^{p_j-2} \nabla u_{p_j} \cdot \Phi dx = \int_\Omega \bar{F} \cdot \Phi dx. \tag{3.18}$$

We have thus proved the convergences (3.11)–(3.12). Note also that $\int_{\Omega} |\bar{F}|^r dx \leq |\Omega|$ and thus $\|\bar{F}\|_{L^r(\Omega)} \leq |\Omega|^{1/r}$ for all $r > 1$. Hence

$$\|\bar{F}\|_{L^\infty(\Omega)} = \lim_{r \rightarrow \infty} \|\bar{F}\|_{L^r(\Omega)} \leq 1. \tag{3.19}$$

Letting $\varphi \in C^\infty(\mathbf{R}^n)$ and $p = p_j \rightarrow 1$ in (3.10), by Theorem 1.2, we have

$$\int_{\Omega} \sigma(x) \bar{F}(x) \cdot \nabla \varphi(x) dx = \lambda_1(g, h) B(\varphi) \quad \forall \varphi \in C^\infty(\mathbf{R}^n), \tag{3.20}$$

which implies that $\operatorname{div}(\sigma \bar{F}) = \lambda_1 h$ and $\delta(\sigma \bar{F}) = \lambda_1 g$; this proves (3.13). To prove (3.14), let $\varphi \in C_0^\infty(\Omega)$, using φu_p as test function in (3.10), and we obtain

$$\int_{\Omega} \varphi \sigma^p |\nabla u_p|^p dx = \lambda_p(g, h) B(\varphi u_p) - \int_{\Omega} u_p \sigma^p F_p \cdot \nabla \varphi dx \tag{3.21}$$

for all $1 < p < \infty$, where $B(\varphi u_p) = -\int_{\Omega} \varphi u_p h dx$. Hence

$$\lim_{p_j \rightarrow 1} B(\varphi u_{p_j}) = -\int_{\Omega} \varphi \bar{u} h dx.$$

Since $\sigma^{p_j} u_{p_j} \rightarrow \sigma \bar{u}$ in $L^q(\Omega)$ for any fixed $1 < q < \frac{n}{n-1}$ and $F_{p_j} \rightharpoonup \bar{F}$ weakly in $L^{q'}(\Omega; \mathbf{R}^n)$ with $q' = \frac{q}{q-1}$, we have, by (3.21),

$$\lim_{p_j \rightarrow 1} \int_{\Omega} \varphi \sigma^{p_j} |\nabla u_{p_j}|^{p_j} dx = -\lambda_1 \int_{\Omega} \varphi \bar{u} h dx - \int_{\Omega} \bar{u} \sigma \bar{F} \cdot \nabla \varphi dx,$$

where $\lambda_1 = \lambda_1(g, h)$. Using $\operatorname{div}(\sigma \bar{F}) = \lambda_1 h$, the right-hand side of this identity exactly becomes

$$-\int_{\Omega} [\varphi \bar{u} \operatorname{div}(\sigma \bar{F}) + \bar{u} \sigma \bar{F} \cdot \nabla \varphi] dx = \int_{\Omega} \varphi d(\sigma \bar{F}, D\bar{u}),$$

by the definition of the measure $(\sigma \bar{F}, D\bar{u})$ (see, e.g., (2.25)). Hence we have

$$\lim_{p_j \rightarrow 1} \int_{\Omega} \varphi \sigma^{p_j} |\nabla u_{p_j}|^{p_j} = \int_{\Omega} \varphi d(\sigma \bar{F}, D\bar{u})$$

for all $\varphi \in C_0^\infty(\Omega)$; this proves (3.14). Finally, we show (3.15). It is easy to see, along a subsequence of $p_j \rightarrow 1$, that $\sigma |\nabla u_{p_j}| dx \xrightarrow{*} \mu \geq \sigma |D\bar{u}| \llcorner \Omega$ in $\mathcal{M}(\mathbf{R}^n)$. By Young’s inequality,

$$\sigma |\nabla u_{p_j}| \leq \frac{1}{p} \sigma^p |\nabla u_p|^p + \frac{p-1}{p},$$

and hence by (3.14) we have that $\sigma |D\bar{u}| \llcorner \Omega \leq \mu \leq (\sigma \bar{F}, D\bar{u})$ as Radon measures in $\mathcal{M}(\Omega)$. However, using $\|\bar{F}\|_{L^\infty(\Omega)} \leq 1$ and Theorem 2.4, we easily see that $(\sigma \bar{F}, D\bar{u}) \leq \sigma |D\bar{u}| \llcorner \Omega$ as measures in $\mathcal{M}(\mathbf{R}^n)$. Therefore, $\sigma |D\bar{u}| = (\sigma \bar{F}, D\bar{u})$ in $\mathcal{M}(\Omega)$, which proves (3.15). \square

The following result indicates the condition (1.7) is the right condition for the solvability of $\operatorname{div} Y = h, Y \cdot \nu = g$ in $L^\infty(\Omega; \mathbf{R}^n)$; see also [10, Theorem 3’]. It also shows that the admissible set $S_\beta(\Omega)$ defined above is nonempty.

Corollary 3.5. *Let g, h satisfy (1.7). Then there exists a function $Y \in \mathcal{X}_n(\Omega)$ such that*

$$\operatorname{div} Y = h, \quad \delta(Y) = g.$$

In particular, for any $\beta \in L^\infty(\partial\Omega)$ with $\int_{\partial\Omega} \beta d\mathcal{H}^{n-1} = 0$, the admissible set

$$S_\beta(\Omega) = \{G \in \mathcal{X}_n(\Omega) \mid \operatorname{div} G = 0, \delta(G) = \beta\} \neq \emptyset. \tag{3.22}$$

Proof. If $g = h = 0$ then $Y = 0$ is a solution. If at least one of g and h is not zero, then (3.13) of Theorem 3.4 implies that there exists $\sigma \bar{F} \in \mathcal{X}_n(\Omega)$ such that

$$\operatorname{div}(\sigma \bar{F}) = \lambda_1 h, \quad \delta(\sigma \bar{F}) = \lambda_1 g,$$

where $\lambda_1 = \lambda(g, h) > 0$. Hence the function $Y = \frac{\sigma \bar{F}}{\lambda_1}$ in $\mathcal{X}_n(\Omega)$ will satisfy the required condition. \square

Remark 3.1. Given $\tilde{h} \in \mathcal{X}_n(\Omega)$, $\tilde{g} \in L^\infty(\partial\Omega)$, a function $u \in BV(\Omega)$ is said to be a *BV solution* to the Neumann problem of the equation:

$$\begin{cases} \operatorname{div}\left(\sigma \frac{Du}{|Du|}\right) = \tilde{h}, \\ \sigma \frac{Du}{|Du|} \cdot \nu|_{\partial\Omega} = \tilde{g}, \end{cases} \tag{3.23}$$

provided that there exists a function F with $\sigma F \in \mathcal{X}_n(\Omega)$ such that

$$\operatorname{div}(\sigma F) = \tilde{h}, \quad \delta(\sigma F) = \tilde{g}, \tag{3.24}$$

$$(\sigma F, Du) = \sigma d|Du| \quad \text{in } \mathcal{M}(\Omega). \tag{3.25}$$

By (3.13) and (3.15), the limit function \bar{u} is a BV solution of (3.23) with $\tilde{h} = \lambda_1 h$, $\tilde{g} = \lambda_1 g$.

The following result, combined with Theorem 3.4, completes the proof of Theorem 1.3.

Proposition 3.6. Any function \bar{u} determined in Theorem 3.4 satisfies $0 \leq B(\bar{u}) \leq 1$. Moreover, if $\bar{u} \neq 0$, then $v = \frac{\bar{u}}{B(\bar{u})} \in BV(\Omega)$ is a minimizer for $\lambda(g, h)$.

Proof. From (3.13) it follows that

$$\int_{\Omega} d(\sigma \bar{F}, Dv) = \lambda_1 B(v) \quad \forall v \in BV(\Omega). \tag{3.26}$$

Hence, by (3.15), $\lambda_1 B(\bar{u}) = \int_{\Omega} \sigma d|D\bar{u}| \leq \lambda_1$. Since $\lambda_1 > 0$, it follows that $0 \leq B(\bar{u}) \leq 1$. Note that, since $\int_{\Omega} \bar{u} = 0$, $B(\bar{u}) = 0$ if and only if $\bar{u} = 0$. Therefore, if $\bar{u} \neq 0$, then the function $\bar{v} = \frac{\bar{u}}{B(\bar{u})} \in BV(\Omega)$ will be a minimizer for $\lambda(g, h)$. Therefore \bar{u} is a minimizer for $\lambda(g, h)$ if and only if $B(\bar{u}) = 1$. We finally note that the condition $B(\bar{u}) = 1$ holds if and only if

$$\lim_{p_j \rightarrow 1} \int_{\partial\Omega} \gamma(u_{p_j})g \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} \gamma(\bar{u})g \, d\mathcal{H}^{n-1}. \quad \square \tag{3.27}$$

4. Special minimizers for $\rho(\beta, H)$ and proof of Theorem 1.1

In this section, we present two approaches for minimizers of $\rho(\beta, H)$ defined by (1.1) and thus provide two proofs of Theorem 1.1. One approach is based on Theorem 3.4 and the other is based on a natural direct approach analogous to the method for L^∞ -functionals in [6,7] using the limits of p -power functionals as $p \rightarrow \infty$.

First proof of Theorem 1.1. Let $H \in \mathcal{X}_n(\Omega)$, $\beta \in L^\infty(\partial\Omega)$ and let $g = \beta + \delta(H)$ and $h = \operatorname{div} H$. By Corollary 3.5, $S_\beta(\Omega) \neq \emptyset$. By (1.5), $\mu(g, h) \leq \rho(\beta, H)$. Therefore, it suffices to show that there exists a $\bar{G} \in S_\beta(\Omega)$ such that

$$\left\| \frac{\bar{G} + H}{\sigma} \right\|_{L^\infty(\Omega)} \leq \mu(g, h). \tag{4.1}$$

Let B be the linear functional defined above through g and h . Note that $B \equiv 0$ if and only if $g = 0$ and $h = 0$; that is, $\operatorname{div} H = 0$ and $\beta = -\delta(H)$. Hence $B \equiv 0$ if and only if $-H \in S_\beta(\Omega)$. If $-H \in S_\beta(\Omega)$ then $\bar{G} = -H$ will satisfy

(4.1) as both quantities are zero. So, we assume $B \neq 0$; that is, $-H \notin \mathcal{S}_\beta(\Omega)$. Let \bar{F} be any vector-field determined in Theorem 3.4. Since $h = \operatorname{div} H$ and $g = \beta + \delta(H)$, condition (3.13) exactly means

$$\bar{G} = \frac{\sigma \bar{F}}{\lambda_1(g, h)} - H \in \mathcal{S}_\beta(\Omega), \tag{4.2}$$

and hence, by (3.19) and Theorem 1.2,

$$\left\| \frac{\bar{G} + H}{\sigma} \right\|_{L^\infty(\Omega)} = \frac{\|\bar{F}\|_{L^\infty(\Omega)}}{\lambda_1(g, h)} \leq \frac{1}{\lambda_1(g, h)} = \mu(g, h),$$

which proves (4.1). This completes the proof. \square

For $1 < q < \infty$, define

$$\rho_q(\beta, H) = \inf_{G \in \mathcal{S}_\beta^q(\Omega)} \left\| \frac{G + H}{\sigma} \right\|_{L^q(\Omega)}, \tag{4.3}$$

where the admissible set $\mathcal{S}_\beta^q(\Omega)$ is defined as follows: $G \in \mathcal{S}_\beta^q(\Omega)$ if and only if $G \in L^q(\Omega; \mathbf{R}^n)$ and

$$\int_\Omega G(x) \cdot \nabla \varphi(x) \, dx = \int_{\partial\Omega} \beta \varphi \, d\mathcal{H}^{n-1} \quad \forall \varphi \in C_0^\infty(\mathbf{R}^n). \tag{4.4}$$

Note that

$$\emptyset \neq \mathcal{S}_\beta(\Omega) = \mathcal{S}_\beta^\infty(\Omega) \subset \mathcal{S}_\beta^q(\Omega) \quad \forall 1 \leq q < \infty$$

and $\mathcal{S}_\beta^q(\Omega)$ is closed under the weak (or weak-star if $q = \infty$) convergence of $L^q(\Omega; \mathbf{R}^n)$.

Proposition 4.1. *For any $1 < q < \infty$, there exists a unique $G_q \in \mathcal{S}_\beta^q(\Omega)$ such that*

$$\left\| \frac{G_q + H}{\sigma} \right\|_{L^q(\Omega)} = \min_{G \in \mathcal{S}_\beta^q(\Omega)} \left\| \frac{G + H}{\sigma} \right\|_{L^q(\Omega)} = \rho_q(\beta, H). \tag{4.5}$$

Furthermore, there exists an increasing sequence $q_j \rightarrow \infty$ and $\bar{G} \in \mathcal{S}_\beta(\Omega)$ such that $G_{q_j} \rightharpoonup \bar{G}$ in $L^r(\Omega)$ for all $1 < r < \infty$ and,

$$\left\| \frac{\bar{G} + H}{\sigma} \right\|_{L^\infty(\Omega)} = \min_{G \in \mathcal{S}_\beta(\Omega)} \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(\Omega)} = \rho(\beta, H).$$

Moreover, for all measurable sets $E \subset \Omega$,

$$\left\| \frac{\bar{G} + H}{\sigma} \right\|_{L^\infty(E)} \leq \liminf_{q_j \rightarrow \infty} \left\| \frac{G_{q_j} + H}{\sigma} \right\|_{L^{q_j}(E)}. \tag{4.6}$$

In particular, it follows that

$$\lim_{q \rightarrow \infty} \rho_q(\beta, H) = \rho(\beta, H). \tag{4.7}$$

Proof. The existence of minimizer G_q of $\rho_q(\beta, H)$ follows from the standard direct method of calculus of variations since $\mathcal{S}_\beta^q(\Omega)$ is nonempty and weakly closed in $L^q(\Omega; \mathbf{R}^n)$. The uniqueness of G_q follows from the strict convexity of the L^q -norm. We now prove the rest of the proposition. Given any $G \in \mathcal{S}_\beta(\Omega)$, since $G \in \mathcal{S}_\beta^q(\Omega)$, by the minimality of G_q , we have

$$\left\| \frac{G_q + H}{\sigma} \right\|_{L^q(\Omega)} \leq \left\| \frac{G + H}{\sigma} \right\|_{L^q(\Omega)} \leq \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{q}}. \tag{4.8}$$

For any $1 < r < q$ and any measurable set $E \subset \Omega$, by Hölder’s inequality,

$$\left\| \frac{G_q + H}{\sigma} \right\|_{L^r(E)} \leq \left\| \frac{G_q + H}{\sigma} \right\|_{L^q(E)} |E|^{\frac{1}{r} - \frac{1}{q}}. \tag{4.9}$$

In particular, using (4.8) and (4.9) with $E = \Omega$, we have

$$\left\| \frac{G_q + H}{\sigma} \right\|_{L^r(\Omega)} \leq \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{r}} \quad \forall 1 < r < q, G \in \mathcal{S}_\beta(\Omega). \tag{4.10}$$

Using this estimate with $r = 2$, we have that the sequence $\{G_q\}_{q>2}$ is bounded in $L^2(\Omega; \mathbf{R}^n)$ and hence there exists an increasing subsequence $q_j \rightarrow \infty$ such that $G_{q_j} \rightharpoonup \bar{G}$ in $L^2(\Omega; \mathbf{R}^n)$, where $\bar{G} \in \mathcal{S}_\beta^2(\Omega)$. Given any $r > 1$, let $q_j > r$ for all $j \geq J_r$. Then, using (4.10), the sequence $\{G_{q_j}\}_{j \geq J_r}$ is bounded in $L^r(\Omega; \mathbf{R}^n)$; hence a subsequence of it (with $q_j \rightarrow \infty$) will converge weakly to some function \tilde{G} in $L^r(\Omega; \mathbf{R}^n)$ and the limit function \tilde{G} will satisfy

$$\left\| \frac{\tilde{G} + H}{\sigma} \right\|_{L^r(\Omega)} \leq \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{r}}.$$

However, the whole sequence $\{G_{q_j}\}$ converges weakly to \bar{G} in $L^2(\Omega; \mathbf{R}^n)$ and thus $\tilde{G} = \bar{G}$. This proves that $\bar{G} \in L^r(\Omega; \mathbf{R}^n)$ for all $r > 1$; moreover, by (4.10),

$$\left\| \frac{\bar{G} + H}{\sigma} \right\|_{L^r(\Omega)} \leq \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{r}} \quad \forall r > 1.$$

Letting $r \rightarrow \infty$, we have

$$\left\| \frac{\bar{G} + H}{\sigma} \right\|_{L^\infty(\Omega)} \leq \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(\Omega)} \quad \forall G \in \mathcal{S}_\beta(\Omega).$$

This proves that $\bar{G} \in \mathcal{S}_\beta(\Omega)$ and is a minimizer for $\rho(\beta, H)$. In (4.9), letting first $q = q_j \rightarrow \infty$ and then $r \rightarrow \infty$, we have

$$\left\| \frac{\bar{G} + H}{\sigma} \right\|_{L^\infty(E)} \leq \liminf_{q_j \rightarrow \infty} \left\| \frac{G_{q_j} + H}{\sigma} \right\|_{L^{q_j}(E)}$$

for all measurable sets $E \subset \Omega$. Finally, (4.7) follows by combining (4.6) for $E = \Omega$ and (4.8) for $G = \bar{G}$. \square

The following result establishes the one-to-one correspondence between G_q, \bar{G} determined in Proposition 4.1 and u_p, \bar{F} determined in Theorem 3.4.

Proposition 4.2. *Let $g = \beta + \delta(H)$ and $h = \operatorname{div} H$. Then, G_q satisfies (4.5) if and only if*

$$G_q = \frac{\sigma^p |\nabla u_p|^{p-2} \nabla u_p}{\lambda_p(g, h)} - H, \quad p = \frac{q}{q-1}, \tag{4.11}$$

where $\lambda_p(g, h)$ is defined as above and $u_p \in W^{1,p}(\Omega)$ is the unique function determined in Theorem 3.4. Hence $\rho_q(\beta, H) = (\lambda_p(g, h))^{-\frac{1}{p}}$. Furthermore, any function \bar{G} determined in Proposition 4.1 corresponds to a function \bar{F} determined in Theorem 3.4 through the relation: $\sigma \bar{F} = \lambda(g, h)(\bar{G} + H)$.

Proof. We only need to establish the relation (4.11). To show (4.11), it suffices to show that the function G_q defined by (4.11) is the minimizer of $\rho_q(\beta, H)$. First, note that, by (3.10), $G_q \in \mathcal{S}_\beta^q(\Omega)$. To show G_q is a minimizer for $\rho_q(\beta, H)$, given any $G \in \mathcal{S}_\beta^q(\Omega)$, let $F = G - G_q$. Then

$$\int_{\Omega} \nabla v \cdot F \, dx = 0 \quad \forall v \in W^{1,p}(\Omega).$$

Note that, by (4.11), one easily verifies that

$$\frac{|G_q + H|^{q-2} (G_q + H)}{\sigma^q} = \frac{\nabla u_p}{\lambda_p(g, h)^{q-1}}.$$

Hence

$$\int_{\Omega} \frac{|G_q + H|^{q-2}(G_q + H)}{\sigma^q} \cdot F \, dx = \frac{1}{\lambda_p(g, h)^{q-1}} \int_{\Omega} \nabla u_p \cdot F \, dx = 0.$$

Finally, by the convexity of function $h(X) = |X|^q$ for $q > 1$, we have

$$\left| \frac{G + H}{\sigma} \right|^q \geq \left| \frac{G_q + H}{\sigma} \right|^q + \frac{q|G_q + H|^{q-2}(G_q + H)}{\sigma^q} \cdot F,$$

and hence

$$\int_{\Omega} \left| \frac{G + H}{\sigma} \right|^q \, dx \geq \int_{\Omega} \left| \frac{G_q + H}{\sigma} \right|^q \, dx.$$

This proves that $G_q \in \mathcal{S}_{\beta}^q(\Omega)$ is the minimizer for $\rho_q(\beta, H)$. The proof is complete. \square

Second proof of Theorem 1.1. Combining the first part of Proposition 4.2 and Proposition 4.1 and using (4.7), we obtain another proof of Theorem 1.1. We remark that this proof does not rely on the second part of either Theorem 3.4 or Proposition 4.2. \square

Given any measurable set $E \subset \Omega$ and any function $G \in L^q(\Omega; \mathbf{R}^n)$ with $\operatorname{div} G = 0$ in the sense of distributions on Ω , we denote the distribution $\operatorname{div}(G\chi_E)$ on \mathbf{R}^n by $\tilde{\delta}_E(G)$; that is,

$$(\tilde{\delta}_E(G), \zeta) = - \int_E G \cdot \nabla \zeta \, dx \quad \forall \zeta \in C_0^\infty(\mathbf{R}^n). \tag{4.12}$$

If $q = \infty$ and $E \subset \Omega$ is a set of finite perimeter in Ω then, by the generalized Green formula (2.28), it follows that $\tilde{\delta}_E(G) = \tilde{\theta}_E(G) \mathcal{L} \partial^* E$.

Lemma 4.3. *Let $1 \leq q \leq \infty$ and $G_1, G_2 \in \mathcal{S}_{\beta}^q(\Omega)$. Assume $E \subset \Omega$ is measurable and $\tilde{\delta}_E(G_1) = \tilde{\delta}_E(G_2)$ as distributions on \mathbf{R}^n . Then the function $G = G_1\chi_E + G_2\chi_{\Omega \setminus E}$ belongs to $\mathcal{S}_{\beta}^q(\Omega)$.*

Proof. Clearly $G \in L^q(\Omega; \mathbf{R}^n)$. Now, for any $\varphi \in C_0^\infty(\mathbf{R}^n)$, it follows that

$$\begin{aligned} \int_{\Omega} G \cdot \nabla \varphi \, dx &= \int_E G_1 \cdot \nabla \varphi \, dx + \int_{\Omega \setminus E} G_2 \cdot \nabla \varphi \, dx \\ &= \int_E G_2 \cdot \nabla \varphi \, dx + \int_{\Omega \setminus E} G_2 \cdot \nabla \varphi \, dx = \int_{\Omega} G_2 \cdot \nabla \varphi \, dx = \int_{\partial \Omega} \beta \varphi \, d\mathcal{H}^{n-1}, \end{aligned}$$

with the second equality resulting from $\tilde{\delta}_E(G_1) = \tilde{\delta}_E(G_2)$ and the last from the definition of $G_2 \in \mathcal{S}_{\beta}^q(\Omega)$. Hence, by definition (4.4), $G \in \mathcal{S}_{\beta}^q(\Omega)$. \square

We streamline a possible approach for proving that \tilde{G} is an absolute minimizer for $\rho(\beta, H)$ in much a similar way to [7].

Proposition 4.4. *Let \tilde{G} be any function determined in Proposition 4.1 and let $E \subset \Omega$ be an open set and $G \in \mathcal{S}_{\beta}(\Omega)$ satisfy $\tilde{\delta}_E(G) = \tilde{\delta}_E(\tilde{G})$. Suppose that there exist two sequences $\{E^k\}_{k=1,2,\dots}$ and $\{E^{j,k}\}_{j,k=1,2,\dots}$ of measurable subsets of E such that*

$$E^k \text{ is increasing on } k \text{ and } E = \bigcup_{k=1}^{\infty} E^k, \tag{4.13}$$

$$E^k \subset E^{j,k} \text{ for all sufficiently large } j, \tag{4.14}$$

$$\tilde{\delta}_{E^{j,k}}(G_{q_j}) = \tilde{\delta}_{E^{j,k}}(G). \tag{4.15}$$

Then $\|\frac{\tilde{G}+H}{\sigma}\|_{L^\infty(E)} \leq \|\frac{G+H}{\sigma}\|_{L^\infty(E)}$.

Proof. By (4.15) and Lemma 4.3, $\tilde{G} = G_{q_j} \chi_{\Omega \setminus E^{j,k}} + G \chi_{E^{j,k}} \in \mathcal{S}_\beta^{q_j}(\Omega)$; hence, upon testing the minimality of G_{q_j} with \tilde{G} and canceling common terms, we have (the absolutely minimizing property of G_{q_j})

$$\left\| \frac{G_{q_j} + H}{\sigma} \right\|_{L^{q_j}(E^{j,k})} \leq \left\| \frac{G + H}{\sigma} \right\|_{L^{q_j}(E^{j,k})} \leq \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(E)} |E|^{1/q_j}$$

for all $j, k = 1, 2, \dots$. Therefore, combining with (4.6), it follows that

$$\left\| \frac{\tilde{G} + H}{\sigma} \right\|_{L^\infty(E^k)} \leq \liminf_{j \rightarrow \infty} \left\| \frac{G_{q_j} + H}{\sigma} \right\|_{L^{q_j}(E^{j,k})} \leq \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(E)}$$

for all $k = 1, 2, \dots$. Letting $k \rightarrow \infty$, the result follows. \square

5. An existence result on minimizers of $\lambda(g, h)$ in $BV(\Omega)$

In this section, we give a sufficient condition for the existence of minimizers for $\lambda(g, h)$ in $BV(\Omega)$; we follow closely some idea of [12, Theorem 2.3]. First, we have the following result relying on the special property of nonnegative Radon measures.

Proposition 5.1. *Let σ be the function as given above. Let $w_k, w \in BV(\Omega)$ satisfy $w_k \rightarrow w$ in $L^1(\Omega)$, $Dw_k \xrightarrow{*} Dw$ in $\mathcal{M}(\Omega)$ and $\sigma d|Dw_k| \xrightarrow{*} \mu$ in $\mathcal{M}(\Omega)$. Let F_k, F with $\sigma F_k, \sigma F \in \mathcal{X}_n(\Omega)$ satisfy $\|F_k\|_{L^\infty(\Omega)} \leq 1$, $F_k \xrightarrow{*} F$ in $L^\infty(\Omega; \mathbf{R}^n)$ and $\text{div}(\sigma F_k) \rightarrow \text{div}(\sigma F)$ in $L^n(\Omega)$. Then*

$$0 \leq \int_{\Omega} (\sigma|Dw| - (\sigma F, Dw)) \leq \int_{\Omega} (\mu - (\sigma F, Dw)) \tag{5.1}$$

$$\leq \liminf_{k \rightarrow \infty} \left[\int_{\Omega} \sigma d|Dw_k| - \int_{\Omega} d(\sigma F_k, Dw_k) \right]. \tag{5.2}$$

Proof. By Theorem 2.4, the measure $\sigma|Dv| - (\sigma F, Dv)$ is nonnegative for all $v \in BV(\Omega)$. Given any $\epsilon > 0$, let $K \Subset \Omega$ be a compact set such that

$$\int_{\Omega} d(\mu - (\sigma F, Dw)) \leq \int_K d(\mu - (\sigma F, Dw)) + \epsilon.$$

Let $\phi \in C_c(\Omega)$ be a cut-off function such that $0 \leq \phi(x) \leq 1$ on Ω and $\phi(x) = 1$ on K . By Proposition 2.5, we have $(\sigma F_k, Dw_k) \xrightarrow{*} (\sigma F, Dw)$ in $\mathcal{M}(\Omega)$ and thus

$$\begin{aligned} 0 &\leq \int_{\Omega} (\sigma|Dw| - (\sigma F, Dw)) \leq \int_{\Omega} (\mu - (\sigma F, Dw)) \\ &\leq \int_K d(\mu - (\sigma F, Dw)) + \epsilon \leq \int_{\Omega} \phi d(\mu - (\sigma F, Dw)) + \epsilon \\ &= \lim_{k \rightarrow \infty} \int_{\Omega} \phi (\sigma d|Dw_k| - d(\sigma F_k, Dw_k)) + \epsilon \\ &\leq \liminf_{k \rightarrow \infty} \int_{\Omega} (\sigma d|Dw_k| - d(\sigma F_k, Dw_k)) + \epsilon \end{aligned}$$

$$= \liminf_{k \rightarrow \infty} \left[\int_{\Omega} \sigma d|Dw_k| - \int_{\Omega} d(\sigma F_k, Dw_k) \right] + \epsilon$$

for arbitrary $\epsilon > 0$. Hence, we have proved (5.1) and (5.2). \square

Assume g, h are given functions satisfying (1.7). Let

$$\bar{\rho}(g) = \inf_{\substack{Y \in \mathcal{X}_n(\Omega) \\ \delta(Y)=g}} \left\| \frac{Y}{\sigma} \right\|_{L^\infty(\Omega)}, \quad \bar{\rho}(g, h) = \inf_{\substack{Y \in \mathcal{X}_n(\Omega) \\ \text{div } Y=h, \delta(Y)=g}} \left\| \frac{Y}{\sigma} \right\|_{L^\infty(\Omega)}. \tag{5.3}$$

By Corollary 3.5, the set of functions Y defining $\bar{\rho}(g, h)$ is nonempty and in fact $\bar{\rho}(g, h)$ is attained as a minimum. Let $H \in \mathcal{X}_n(\Omega)$ be any function such that $\text{div } H = h$ and $\delta(H) = g$, and let $\beta = g - \delta(H)$. Then, by Theorem 1.1, $\bar{\rho}(g, h) = \rho(\beta, H) = \mu(g, h)$; hence

$$\mu(g, h) = \bar{\rho}(g, h) \geq \bar{\rho}(g). \tag{5.4}$$

We have the following existence result similar to [12, Theorem 2.3].

Theorem 5.2. Assume $\mu(g, h) > \bar{\rho}(g)$. Suppose $\{v_j\} \subset BV(\Omega)$ with $\int_{\Omega} v_j dx = 0$ is a minimizing sequence for $\lambda(g, h)$; that is,

$$\lim_{j \rightarrow \infty} \int_{\Omega} \sigma d|Dv_j| = \lambda(g, h), \quad B(v_j) = 1.$$

Then there exists a subsequence $\{w_k\} = \{v_{j_k}\}$ and $\bar{v} \in BV(\Omega)$ such that

$$w_k \rightarrow \bar{v} \text{ in } L^1(\Omega), \quad \int_{\Omega} |Dw_k| \rightarrow \int_{\Omega} |D\bar{v}|. \tag{5.5}$$

Consequently $\gamma(w_k) \rightarrow \gamma(\bar{v})$ in $L^1(\partial\Omega)$ and

$$\int_{\Omega} \sigma d|D\bar{v}| = \lambda(g, h), \quad B(\bar{v}) = 1;$$

that is, $\bar{v} \in BV(\Omega)$ is a minimizer for $\lambda(g, h)$.

Proof. We write $\lambda_1 = \lambda_1(g, h) = \lambda(g, h)$ for simplicity. The condition $\frac{1}{\lambda_1} = \mu(g, h) > \bar{\rho}(g)$ implies that there exists a $Y \in \mathcal{X}_n(\Omega)$ with $\delta(Y) = g$ such that $\frac{1}{t} = \left\| \frac{Y}{\sigma} \right\|_{L^\infty(\Omega)} < \frac{1}{\lambda_1}$ (hence $t > \lambda_1$). Let $F_1 = \frac{tY}{\sigma} \in \mathcal{X}_n(\Omega)$. Then $\|F_1\|_{L^\infty(\Omega)} = 1$ and $\delta(\sigma F_1) = tg$. We assume the subsequence $\{w_k\}$ satisfies that $w_k \rightarrow \bar{v}$ in $L^1(\Omega)$, $Dw_k \overset{*}{\rightharpoonup} D\bar{v}$ in $\mathcal{M}(\Omega)$ and $\sigma d|Dw_k| \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$. First use (5.1)–(5.2) of Proposition 5.1 with constant sequence $F_k = \bar{F}$, where \bar{F} is determined in Theorem 3.4. Since $\int_{\Omega} (\sigma \bar{F}, Dw_k) = \lambda_1 B(w_k) = \lambda_1$ and $\{w_k\}$ is minimizing, we have

$$\lim_{k \rightarrow \infty} \left[\int_{\Omega} \sigma d|Dw_k| - \int_{\Omega} d(\sigma \bar{F}, Dw_k) \right] = 0 \tag{5.6}$$

and hence $\mu = \sigma |D\bar{v}| = (\sigma \bar{F}, D\bar{v})$ in $\mathcal{M}(\Omega)$. We then use (5.1)–(5.2) with $F_k = F_1$ to obtain

$$\int_{\Omega} (\sigma |D\bar{v}| - (\sigma F_1, D\bar{v})) \leq \liminf_{k \rightarrow \infty} \left[\int_{\Omega} \sigma d|Dw_k| - \int_{\Omega} d(\sigma F_1, Dw_k) \right].$$

Using (5.6), we have

$$0 \leq \int_{\Omega} (\sigma \bar{F} - \sigma F_1, D\bar{v}) \leq \liminf_{k \rightarrow \infty} \int_{\Omega} (\sigma \bar{F} - \sigma F_1, Dw_k). \tag{5.7}$$

This inequality and the divergence formula imply, by virtue of $\delta(\sigma \bar{F} - \sigma F_1) = (\lambda_1 - t)g$ and $w_k \rightharpoonup \bar{v}$ in $L^{\frac{n}{n-1}}(\Omega)$, that

$$(\lambda_1 - t) \int_{\partial\Omega} \gamma(\bar{v})g \, d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} (\lambda_1 - t) \int_{\partial\Omega} \gamma(w_k)g \, d\mathcal{H}^{n-1}$$

and hence, since $\lambda_1 - t < 0$,

$$\int_{\partial\Omega} \gamma(\bar{v})g \, d\mathcal{H}^{n-1} \geq \limsup_{k \rightarrow \infty} \int_{\partial\Omega} \gamma(w_k)g \, d\mathcal{H}^{n-1}.$$

From this, we easily have

$$\begin{aligned} B(\bar{v}) &= \int_{\partial\Omega} \gamma(\bar{v})g \, d\mathcal{H}^{n-1} - \int_{\Omega} \bar{v}h \, dx \\ &\geq \limsup_{k \rightarrow \infty} \left[\int_{\partial\Omega} \gamma(w_k)g \, d\mathcal{H}^{n-1} - \int_{\Omega} w_k h \, dx \right] = \limsup_{k \rightarrow \infty} B(w_k) = 1. \end{aligned}$$

On the other hand,

$$\lambda_1 B(\bar{v}) = \int_{\Omega} (\sigma \bar{F}, D\bar{v}) = \int_{\Omega} \sigma \, d|D\bar{v}| \leq \liminf_{k \rightarrow \infty} \int_{\Omega} \sigma \, d|Dw_k| = \lambda_1,$$

and hence $B(\bar{v}) \leq 1$ since $\lambda_1 > 0$. So we have proved $B(\bar{v}) = 1$ and hence

$$\lim_{k \rightarrow \infty} \int_{\Omega} \sigma \, d|Dw_k| = \lambda_1 = \lambda_1 B(\bar{v}) = \int_{\Omega} (\sigma \bar{F}, D\bar{v}) = \int_{\Omega} \sigma \, d|D\bar{v}|. \tag{5.8}$$

This proves $\bar{v} \in BV(\Omega)$ is a minimizer of $\lambda(g, h)$. Obviously,

$$\int_A \sigma \, d|D\bar{v}| \leq \liminf_{k \rightarrow \infty} \int_A \sigma \, d|Dw_k| \quad \forall A \subset \Omega \text{ open.}$$

Let $\mu_k = \sigma \, d|Dw_k|$, $\lambda = \sigma \, d|D\bar{v}|$ and $\phi = \frac{1}{\sigma}$. Then we have proved that

$$\lim_{k \rightarrow \infty} \mu_k(\Omega) = \lambda(\Omega) \quad \text{and} \quad \lambda(A) \leq \liminf_{k \rightarrow \infty} \mu_k(A)$$

for all open sets $A \subset \Omega$. Hence, by [3, Proposition 1.80],

$$\lim_{k \rightarrow \infty} \int_{\Omega} \phi \, d\mu_k = \int_{\Omega} \phi \, d\lambda;$$

that is,

$$\lim_{k \rightarrow \infty} \int_{\Omega} d|Dw_k| = \int_{\Omega} d|D\bar{v}|.$$

This completes the proof. \square

Corollary 5.3. Assume $\mu(g, h) > \bar{\rho}(g)$. Then any function \bar{u} determined in Theorem 3.4 is a minimizer for $\lambda(g, h)$.

Proof. Since $\mu(g, h) > \bar{\rho}(g)$, it follows that $B \neq 0$. Let $v_j = u_{p_j}$, where u_p is the minimizer of $\lambda_p(g, h)$ in Theorem 3.4. Since $\lambda_p(g, h) \rightarrow \lambda(g, h)$ as $p \rightarrow 1$, by the Young inequality, we easily see that $\{v_j\}$ is the minimizing sequence of $\lambda(g, h)$. Hence the result follows from Theorem 5.2. \square

Corollary 5.4. Assume $\mu(g, h) > \bar{\rho}(g)$. Then there exists a set $A \in \mathcal{P}(\Omega)$ such that

$$\frac{\int_{\partial\Omega \cap \partial^*A} g \, d\mathcal{H}^{n-1} - \int_A h \, dx}{\int_{\Omega \cap \partial^*A} \sigma \, d\mathcal{H}^{n-1}} = \max_{E \in \mathcal{P}(\Omega)} \frac{\int_{\partial\Omega \cap \partial^*E} g \, d\mathcal{H}^{n-1} - \int_E h \, dx}{\int_{\Omega \cap \partial^*E} \sigma \, d\mathcal{H}^{n-1}}.$$

Proof. We use the same notation as in Theorem 3.3. Assume $E_j \in \mathcal{P}(\Omega)$ is a maximizing sequence:

$$\lim_{j \rightarrow \infty} \frac{B(\chi_{E_j})}{N(\chi_{E_j})} = \mu_3 = \sup_{E \in \mathcal{P}(\Omega)} \frac{B(\chi_E)}{N(\chi_E)}.$$

Again since $B \not\equiv 0$, we have $\mu_3 > 0$. Let $v_j = b_j(\chi_{E_j} - c_j)$, where $b_j > 0$ and $c_j \in [0, 1]$ are constants such that $B(v_j) = 1$ and $\int_{\Omega} v_j \, dx = 0$. Therefore

$$\lim_{j \rightarrow \infty} N(v_j) = \lim_{j \rightarrow \infty} \int_{\Omega} \sigma \, d|Dv_j| = \frac{1}{\mu_3} = \lambda(g, h).$$

Hence $\{v_j\}$ is a minimizing sequence of $\lambda(g, h)$ in $BV(\Omega)$. By Theorem 5.2, there exists a subsequence $w_k = v_{j_k} \rightarrow \bar{v}$ in $L^1(\Omega)$, where $\bar{v} \in BV(\Omega)$ is a minimizer for $\lambda(g, h)$. We may assume $w_k(x) \rightarrow \bar{v}(x)$ for almost every $x \in \Omega$, $-b_{j_k}c_{j_k} \rightarrow a \in [-\infty, 0]$ and $b_{j_k}(1 - c_{j_k}) \rightarrow d \in [0, \infty]$, as $k \rightarrow \infty$. Since the essential range of $\bar{v}(x)$ can only belong to $\{a, d\}$ and \bar{v} is not constant (since $B(\bar{v}) = 1$), a and d must be finite and distinct. Hence there exists a set $A \subset \Omega$ such that $\bar{v} = r(\chi_A - s)$ for some constants $r, s \in \mathbf{R}$. This shows $A \in \mathcal{P}(\Omega)$ is a maximizing set for μ_3 . \square

Remark 5.1. Assume that $\sigma \equiv 1$, Ω is C^2 and $g \in C^1(\partial\Omega)$. The result [13, Proposition 5] shows that, for each $\epsilon > 0$, there exists a $Y_\epsilon \in \mathcal{X}_n(\Omega)$ satisfying $\delta(Y_\epsilon) = g$ and $\|Y_\epsilon\|_{L^\infty(\Omega)} \leq (1 + \epsilon)\|g\|_{L^\infty(\partial\Omega)}$. So, by (2.19),

$$\|g\|_{L^\infty(\partial\Omega)} \leq \inf_{\substack{Y \in \mathcal{X}_n(\Omega) \\ \delta(Y)=g}} \|Y\|_{L^\infty(\Omega)} \leq \|Y_\epsilon\|_{L^\infty(\Omega)} \leq (1 + \epsilon)\|g\|_{L^\infty(\partial\Omega)}$$

for all $\epsilon > 0$. Hence, in this case, $\bar{\rho}(g) = \|g\|_{L^\infty(\partial\Omega)}$.

Example 5.1. We consider an interesting example. Let $n \geq 2$ and $\Omega = B_R \setminus \bar{B}_r = \{x \in \mathbf{R}^n \mid r < |x| < R\}$ and let $h = 0$ and $g : \partial\Omega = \partial B_R \cup \partial B_r \rightarrow \mathbf{R}$ be defined by

$$g(x) = \begin{cases} 0 & |x| = r, \\ x_1 & |x| = R. \end{cases}$$

By Remark 5.1 above, we have $\bar{\rho}(g) = \|g\|_{L^\infty(\partial\Omega)} = R$. However,

$$\mu(g, 0) = \sup_{E \in \mathcal{P}(\Omega)} \frac{\int_{\partial^*E \cap \partial B_R} x_1 \, d\mathcal{H}^{n-1}}{P(E, \Omega)} \geq \frac{R^n}{R^{n-1} - r^{n-1}} > R, \tag{5.9}$$

if we choose $E = \Omega^+ = \{x \in \Omega \mid x_1 \geq 0\}$. Therefore the condition $\mu(g, 0) > \bar{\rho}(g)$ holds and so Corollary 5.3 applies. Let

$$B(u) = \int_{|x|=1} x_1 \gamma_\Omega(u) \, d\mathcal{H}^{n-1}, \quad N_p(u) = \int_{\Omega} |Du|^p \, dx.$$

Note that $B(u) = B(\tilde{u})$, $N_p(u) = N_p(\tilde{u})$ for all $u \in W^{1,p}(\Omega)$, where $\tilde{u}(x_1, x') = \epsilon u(\epsilon x_1, O'x')$; here $x' \in \mathbf{R}^{n-1}$, $\epsilon = \pm 1$ and O' is any rotation of \mathbf{R}^{n-1} . This invariance property of B , N_p and the uniqueness of minimizer u_p show that $u_p(x) = U_p(x_1, |x'|)$, where $U_p(s, t)$ is a function of $s \in \mathbf{R}$ and $t \geq 0$ which is odd in s . For any function $u \in W^{1,p}(\Omega)$ odd in x_1 , let $u^* = |u|_{\chi_{\Omega^+}} - |u|_{\chi_{\Omega^-}}$, where $\Omega^- = \{x \in \Omega \mid x_1 \leq 0\}$. It is easily seen that $B(u^*) \geq B(u)$ and $N_p(u^*) = N_p(u)$. This shows that $u_p = u_p^* = |u_p|_{\chi_{\Omega^+}} - |u_p|_{\chi_{\Omega^-}}$, and hence $u_p \geq 0$ on Ω^+ .

Let \bar{u} be any function determined in Theorem 3.4 as a limit of a subsequence of u_p as $p = p_j \rightarrow 1$ in $L^1(\Omega)$. The invariance properties of u_p above show that $\bar{u}(x) = \bar{U}(x_1, |x'|)$, where $\bar{U}(s, t)$ is a function of $s \in \mathbf{R}$ and $t \geq 0$, odd in s and nonnegative for $s \geq 0$; moreover, \bar{u} is a minimizer for $\lambda(g, 0)$. Consider now the upper-level sets of \bar{u} :

$E_t = \{x \in \Omega \mid \bar{u}(x) > t\}$, $F_t = \{x \in \Omega \mid \bar{u}(x) \geq t\}$. Since $\bar{u} \geq 0$ on Ω^+ , we have $E_t \subseteq F_t \subseteq \Omega^+$ for all $t > 0$. As in Lemma 3.1, we have

$$u(x) = \int_0^\infty \chi_{E_t}(x) dt = \int_0^\infty \chi_{F_t}(x) dt \quad \forall \text{a.e. } x \in \Omega^+$$

and, integrating over Ω^+ and using Fubini’s theorem, we have $\int_0^\infty \mathcal{L}^n(F_t \setminus E_t) dt = 0$. This implies that $\mathcal{L}^n(F_t \setminus E_t) = 0$ for almost every $t > 0$. Hence, by [3, Proposition 3.38(c)], we have

$$P(F_t, \Omega) = P(E_t, \Omega) \quad \forall \text{a.e. } t > 0. \tag{5.10}$$

Let $x = (x_1, x') \in \mathbf{R}^n$ and $\tilde{x} = (-x_1, x')$. For any set $E \subseteq \mathbf{R}^n$, denote $\tilde{E} = \{\tilde{x} \mid x \in E\}$. We easily see that $P(\tilde{E}, \Omega) = P(E, \Omega)$. By the invariance property of \bar{u} , we have $E_{-t} = \Omega \setminus \tilde{F}_t$ for all $t \in \mathbf{R}$. From these properties, by Lemma 3.1 and the coarea formula, one eventually obtains that

$$B(\bar{u}) = 2 \int_I B(\chi_{E_t}) dt = 1; \quad N(\bar{u}) = 2 \int_I N(\chi_{E_t}) dt = \lambda(g, 0), \tag{5.11}$$

where the set $I \subseteq (0, \infty)$ is the set of $t > 0$ such that $E_t \in \mathcal{P}(\Omega)$ and the condition (5.10) holds. Obviously $\mathcal{L}^1((0, \infty) \setminus I) = 0$. For each $t \in I$, we have $\lambda(g, 0)B(\chi_{E_t}) \leq N(\chi_{E_t})$ and hence (5.11) implies that for each t

$$\frac{B(\chi_{E_t})}{N(\chi_{E_t})} = \frac{1}{\lambda(g, 0)} = \mu(g, 0);$$

that is, for each $t \in I$, the set E_t is a maximizer for $\mu(g, 0)$.

Let $E \subseteq \Omega^+$ be any set invariant with respect to rotation about the x_1 -axis. Then we can write E as

$$E = \{(x_1, x') \in \Omega^+ \mid (x_1, |x'|) \in K\} \cup Z, \tag{5.12}$$

where $Z = E \cap \{x' = 0\}$ and hence $\mathcal{L}^{n-1}(Z) = 0$ and K is a subset of $\omega^+ = \{(t, s) \mid t \geq 0, s > 0, r^2 < s^2 + t^2 < R^2\}$ in \mathbf{R}^2 and we also have that E has finite perimeter in Ω if and only if K has finite perimeter in the annulus $\Delta = \{(t, s) \mid r^2 < t^2 + s^2 < R^2\}$. In this case one obtains that

$$\frac{B(\chi_E)}{N(\chi_E)} = \frac{\int_{\partial^* K \cap \Gamma} t s^{n-2} d\mathcal{H}^1}{\int_{\partial^* K \cap \omega^+} s^{n-2} d\mathcal{H}^1} := I(K) \equiv \frac{\tilde{B}(K)}{\tilde{N}(K)}, \tag{5.13}$$

where $\Gamma = \{(t, s) \mid t^2 + s^2 = R^2, t \geq 0, s > 0\}$. Furthermore, E defined by (5.12) is a maximizer for $\mu(g, 0)$ if and only if K is a maximizer for the functional $I(K)$ over $K \subseteq \omega^+$.

Proposition 5.5. *Let K be a maximizer of functional I above over the sets of finite perimeter in ω^+ . Then $\partial^* K = \partial^* \omega^+$ and hence $\mu(g, 0) = I(K) = I(\omega^+) = \frac{R^n}{R^{n-1} - r^{n-1}}$.*

Proof. It suffices to show that $\partial^* K \cap \text{int}(\omega^+) = \emptyset$. Suppose for the contrary that $\partial^* K \cap \text{int}(\omega^+) \neq \emptyset$. Since $\partial^* K$ is the union of rectifiable curves [3, Theorem 3.59], any piece, say β , of these curves inside $\text{int}(\omega^+)$ can be parameterized as $\beta = (t(\tau), s(\tau))$, where $\tau \in [a, b]$ is the arc-length parameter of β ; the rectifiability of curve β implies that both $t(\tau)$ and $s(\tau)$ are differentiable a.e. and hence $\dot{t}(\tau)^2 + \dot{s}(\tau)^2 = 1$ on $[a, b]$. We claim $\dot{s}(\tau) = 0$ on $[a, b]$ and hence β is parallel to the t -axis. If not, assuming $\dot{s}(\tau_0) \neq 0$ for some $\tau_0 \in (a, b)$, then, near the point $P_0 = (t(\tau_0), s(\tau_0))$, the set K lies either on the left-hand side or on the right-hand side of β . We assume K near P_0 lies on the left-side of β ; that is, for some interval (c, d) containing τ_0 in $[a, b]$, any point $(t, s(\tau))$ with $t > t(\tau)$ for some $\tau \in (c, d)$ is not in K . Now let $\zeta \in C_0^\infty(c, d)$, $\zeta \geq 0$, be any given test function. For small $\epsilon \geq 0$, consider the sets $R_\epsilon = \{(t, s(\tau)) \mid t(\tau) - \epsilon \zeta(\tau) \leq t \leq t(\tau), \tau \in (c, d)\}$ and $L_\epsilon = \{(t, s(\tau)) \mid t(\tau) \leq t \leq t(\tau) + \epsilon \zeta(\tau), \tau \in (c, d)\}$. Let $K_\epsilon^+ = K \cup L_\epsilon$, $K_\epsilon^- = K \setminus R_\epsilon$. Then $\partial^*(K_\epsilon^\pm) = (\partial^* K \setminus \beta|_{\tau \in (c, d)}) \cup \beta_\epsilon^\pm$, where $\beta_\epsilon^\pm = \{(t(\tau) \pm \epsilon \zeta(\tau), s(\tau)) \mid \tau \in (c, d)\} \subset \text{int}(\omega^+)$ if $\epsilon \geq 0$ is sufficiently small. Let $h_\pm(\epsilon) = \tilde{N}(K_\epsilon^\pm) - \tilde{N}(K)$. Since K is maximizer of I and $\tilde{B}(K_\epsilon^\pm) = \tilde{B}(K)$, we must have $h_\pm(0) = 0$ and $h_\pm(\epsilon) \geq 0$ for all small $\epsilon \geq 0$ and hence $h'_\pm(0^+) \geq 0$. But, using $\dot{t}(\tau)^2 + \dot{s}(\tau)^2 = 1$,

$$h_\pm(\epsilon) = \int_c^d s(\tau)^{n-2} [\sqrt{\dot{s}^2 + (\dot{t} \pm \epsilon \dot{\zeta})^2} - 1] d\tau,$$

and hence we have $h'_{\pm}(0^+) = \pm \int_c^d s(\tau)^{n-2} \dot{\zeta} \, d\tau \geq 0$ for all test functions $\zeta \geq 0$. This implies $\int_c^d s^{n-2} \dot{\zeta} \, d\tau = 0$ for all test functions ζ and thus $s(\tau)$ must be constant on $\tau \in (c, d)$, which contradicts with $\dot{s}(\tau_0) \neq 0$. Hence every rectifiable piece of $\partial^* K \cap \text{int}(\omega^+)$ is a line segment parallel to the t -axis, and as such a line segment cannot have the end-points still in $\text{int}(\omega^+)$ it must reach to the boundary of ω^+ . Therefore $\partial^* K$ consists of a family of closed line segments with endpoints on $\partial\omega^+$; this implies that

$$K = \bigcup_{s \in \Lambda} \{(t, s) \mid \sqrt{(r^2 - s^2)^+} < t < \sqrt{R^2 - s^2}\} \cup N,$$

where $\Lambda \subseteq (0, R)$ and $\mathcal{H}^1(N) = 0$. Note that Λ is a parameter set for $\partial^* K \cap \Gamma$ since $\partial^* K \cap \Gamma = \{(\sqrt{R^2 - s^2}, s) \mid s \in \Lambda\}$. Using this parametrization, we can write

$$I(K) = \frac{R \int_{\Lambda} s^{n-2} \, ds}{\int_{\Lambda \cap (r, R)} s^{n-2} \, ds + \sum_{s \in (0, R) \cap \partial \Lambda} s^{n-2} (\sqrt{R^2 - s^2} - \sqrt{(r^2 - s^2)^+})} := J(\Lambda).$$

From the assumption $\partial^* K \cap \text{int}(\omega^+) \neq \emptyset$, we have already shown that $(0, R) \cap \partial \Lambda \neq \emptyset$. We proceed to derive the desired contradiction. First suppose there is a point $s_0 \in (0, r) \cap \partial \Lambda$. Define a set $\Lambda_0 = (\Lambda \cap [s_0, R]) \cup [0, s_0]$. Then $\Lambda \subseteq \Lambda_0$ and $(0, R) \cap \partial \Lambda_0 = (s_0, R) \cap \partial \Lambda$ is strictly contained in $(0, R) \cap \partial \Lambda$. Hence $J(\Lambda) < J(\Lambda_0)$. Use Λ_0 to define K_0 as Λ for K ; then we have $I(K) < I(K_0)$, a contradiction. Therefore $(0, R) \cap \partial \Lambda \subseteq (r, R)$ is nonempty; so $\Lambda \subseteq [r, R)$ and we arrive at

$$\mu(g, 0) = I(K) = J(\Lambda) = \frac{R \int_{\Lambda \cap (r, R)} s^{n-2} \, ds}{\int_{\Lambda \cap (r, R)} s^{n-2} \, ds + \sum_{s \in (r, R) \cap \partial \Lambda} s^{n-2} \sqrt{R^2 - s^2}} < R,$$

which is again a contradiction with (5.9). This completes the proof. \square

From this result, we also see that each upper-level set E_t of \bar{u} is Ω^+ . Therefore the function \bar{u} is constant on Ω^+ and hence \bar{u} is uniquely given by $\bar{u} = c(2\chi_{\Omega^+} - 1)$, where c is the constant such that $B(\bar{u}) = 1$. This also proves that the whole family $\{u_p\}$ converges in $L^1(\Omega)$ to the same function \bar{u} as $p \rightarrow 1^+$.

6. Problems in two dimensions and proof of Theorem 1.4

The main goal of this section is to prove Theorem 1.4. Assume $n = 2$ and let $\Omega \subset \mathbf{R}^2$ be a bounded domain such that its boundary $\partial\Omega$ consists of $k + 1$ simple closed Lipschitz (thus Jordan) curves denoted by $\Gamma_0, \Gamma_1, \dots, \Gamma_k$ ($k \geq 0$), with Γ_0 being the boundary of the unbounded component of $\mathbf{R}^2 \setminus \Omega$. Hence

$$\Omega = D_0 \setminus \bigcup_{i=1}^k \bar{D}_i, \tag{6.1}$$

where D_j is the simply-connected domain enclosed by Γ_j ; that is, the inside of the curve Γ_j for each $j = 0, 1, \dots, k$.

For each $1 \leq q \leq \infty$, define $S_0^q(\Omega) = S_\beta^q(\Omega)$ as above with $\beta = 0$. In order to characterize this set $S_0^q(\Omega)$, we introduce the following space

$$W_*^{1,q}(\Omega) = \left\{ \varphi \in W^{1,q}(\Omega) \mid \gamma_\Omega(\varphi) = \sum_{i=0}^k c_i \chi_{\Gamma_i}, \, c_i \in \mathbf{R} \right\}, \tag{6.2}$$

where γ_Ω is the trace-operator on $\partial\Omega$. Obviously,

$$\mathbf{R} \oplus W_0^{1,q}(\Omega) := \{c + \varphi \mid c \in \mathbf{R}, \varphi \in W_0^{1,q}(\Omega)\} \subset W_*^{1,q}(\Omega);$$

however, if $k \geq 1$, the two spaces are not the same. Given any $\varphi \in W_*^{1,q}(\Omega)$, if we extend φ by constant c_i onto \bar{D}_i ($i = 1, 2, \dots, k$), then the extended function $\tilde{\varphi}$ belongs to $\mathbf{R} \oplus W_0^{1,q}(D_0)$. Therefore, $W_*^{1,q}(\Omega)$ can be considered as a subspace of $\mathbf{R} \oplus W_0^{1,q}(D_0)$ consisting of functions that are constant on each \bar{D}_i ($i = 1, 2, \dots, k$).

A closer look of the proof of Morrey’s estimate in [14, pp. 266–268] yields that, if $2 < q < \infty$, for all $\varphi \in W^{1,q}(\Omega)$,

$$|\varphi(x) - \varphi(y)| \leq C_q |x - y|^{1 - \frac{2}{q}} \|\nabla \varphi\|_{L^q(\Omega)} \quad \forall x, y \in \Omega, \tag{6.3}$$

where the constant $C_q = \frac{12\pi}{4\pi - 3\sqrt{3}} \left(\frac{q-2}{q-1}\right)^{\frac{1-q}{q}}$. Note that the estimate (6.3) holds also when $q = \infty$ with C_∞ defined by

$$C_\infty = \lim_{q \rightarrow \infty} C_q = \frac{12\pi}{4\pi - 3\sqrt{3}}.$$

Therefore, if $2 < q \leq \infty$, each function $\varphi \in W_*^{1,q}(\Omega)$ can be viewed as a continuous function in Hölder’s space $C^{1-\frac{2}{q}}(\bar{\Omega})$ and so we automatically consider $W_*^{1,q}(\Omega) \subset C(\bar{\Omega})$ if $q > 2$.

Lemma 6.1. *Let $1 \leq q \leq \infty$ and $Z \in L^q(\Omega; \mathbf{R}^2)$. Then $Z \in \mathcal{S}_0^q(\Omega)$ if and only if there exists a $\varphi \in W_*^{1,q}(\Omega)$ (unique up to adding constants) such that $Z = (\varphi_{x_2}, -\varphi_{x_1}) = -(\nabla\varphi)^\perp$. Moreover, if $2 < q \leq \infty$, we have*

$$|\varphi(x) - \varphi(y)| \leq C_q |x - y|^{1-\frac{2}{q}} \|Z\|_{L^q(\Omega)} \quad \forall x, y \in \Omega. \tag{6.4}$$

Proof. First, assume $Z = (\varphi_{x_2}, -\varphi_{x_1})$ for some $\varphi \in W_*^{1,q}(\Omega)$. Let $c_i = \varphi|_{\Gamma_i}$ for $i = 0, 1, \dots, k$ and let $\tilde{\varphi} \in W_0^{1,q}(D_0)$ be the function equal to $\varphi - c_0$ in Ω and $c_i - c_0$ on \bar{D}_i . Define $\tilde{Z} = \nabla\tilde{\varphi} \in L^q(D_0; \mathbf{R}^2)$. Then $\tilde{Z} = Z$ on Ω and $\tilde{Z} = 0$ on $D_0 \setminus \Omega$ and hence we easily have

$$\int_{\Omega} Z \cdot \nabla\zeta \, dx = \int_{D_0} \tilde{Z} \cdot \nabla\zeta \, dx = \int_{D_0} (\tilde{\varphi}_{x_2}\zeta_{x_1} - \tilde{\varphi}_{x_1}\zeta_{x_2}) \, dx = 0$$

for all $\zeta \in C_0^1(\mathbf{R}^2)$. By definition, $Z \in \mathcal{S}_0^q(\Omega)$. We now assume $Z \in \mathcal{S}_0^q(\Omega)$. Let \tilde{Z} be the extension of Z by zero onto $\mathbf{R}^2 \setminus \Omega$. Then $\tilde{Z} \in L^2(\mathbf{R}^2; \mathbf{R}^2)$ satisfies $\text{div } \tilde{Z} = 0$ in the sense of distributions. Hence, there exists $f \in W_{loc}^{1,2}(\mathbf{R}^2)$ such that $\tilde{Z} = (f_{x_2}, -f_{x_1})$ on \mathbf{R}^2 . Note that $\nabla f = 0$ on $\mathbf{R}^2 \setminus \Omega$ and thus f is constant on each component of $\mathbf{R}^2 \setminus \Omega$. Hence $f|_{\Gamma_j} = d_j$ is a constant for each $j = 0, 1, \dots, k$. Since $\tilde{Z} \in L^q(\mathbf{R}^2; \mathbf{R}^2)$, we have $f \in W_{loc}^{1,q}(\mathbf{R}^2)$. Let $\varphi = f|_{\Omega}$. Then $\varphi \in W_*^{1,q}(\Omega)$ satisfies $Z = (\varphi_{x_2}, -\varphi_{x_1})$ in Ω . Since Ω is connected, it is easily seen that φ is unique up to constants. If $2 < q \leq \infty$, the estimate (6.4) follows from (6.3). This completes the proof. \square

We prove the first part of Theorem 1.4 in the following theorem.

Theorem 6.2. *Any minimizer \bar{G} of (1.1) determined in Proposition 4.1 is an absolute minimizer of $\rho(\beta, H)$ as defined in Definition 1.1.*

Proof. Let \bar{G} be the weak limit of G_{q_j} determined in Proposition 4.1 and $E \Subset \Omega$ be an open set with connected $\Omega \setminus E$. Assume $G \in \mathcal{S}_\beta(\Omega)$ satisfies that $\tilde{\delta}_E(G) = \tilde{\delta}_E(\bar{G})$. We want to show

$$\left\| \frac{\bar{G} + H}{\sigma} \right\|_{L^\infty(E)} \leq \left\| \frac{G + H}{\sigma} \right\|_{L^\infty(E)}. \tag{6.5}$$

Since the proof is long, we split it into several steps.

Step 1. Since $G_{q_j} - \bar{G} \in \mathcal{S}_0^{q_j}(\Omega)$, by Lemma 6.1, there exists a function $\varphi^j \in W_*^{1,q_j}(\Omega)$ such that $G_{q_j} = \bar{G} + (\varphi^j_{x_2}, -\varphi^j_{x_1})$. We make φ^j unique by assuming $\varphi^j|_{\Gamma_0} = 0$. Since $G_{q_j} \rightharpoonup \bar{G}$ in $L^r(\Omega; \mathbf{R}^2)$ for each $r > 1$, we have $\nabla\varphi^j \rightharpoonup 0$ in $L^r(\Omega; \mathbf{R}^2)$ for each $r > 1$. By (6.4), $\{\varphi^j\}$ is a uniformly bounded and equi-continuous sequence of continuous functions on $\bar{\Omega}$ and hence there exists a subsequence of $\{\varphi^j\}$ which converges uniformly to a continuous function φ^0 on $\bar{\Omega}$. Obviously, $\varphi^0|_{\Gamma_0} = 0$ and $\nabla\varphi^0 = 0$ in the sense of distributions in Ω ; hence, $\varphi^0 \equiv 0$ on $\bar{\Omega}$. Without loss of generality, we assume that the whole sequence $\{\varphi^j\}$ converges uniformly to zero on $\bar{\Omega}$ as $j \rightarrow \infty$.

Step 2. Consider $Z = (G - \bar{G})\chi_E \in L^\infty(\Omega; \mathbf{R}^2)$. Since $\tilde{\delta}_E(G) = \tilde{\delta}_E(\bar{G})$, it follows that $Z \in \mathcal{S}_0^\infty(\Omega)$. Hence, by Lemma 6.1, there exists a (Lipschitz continuous) function $\varphi \in W_*^{1,\infty}(\Omega)$ such that $Z = (\varphi_{x_2}, -\varphi_{x_1})$ on Ω . Since $\nabla\varphi = 0$ a.e. on $\Omega \setminus E$ and $\Omega \setminus E$ is connected, we have that $\varphi(x)$ is constant on $\Omega \setminus E$. By adding a constant, without loss of generality, we assume $\varphi \equiv 0$ on $\Omega \setminus E$. So $\varphi(x) = 0$ on ∂E . We remark that the connectedness of $\Omega \setminus E$ plays an important role here since otherwise we would only assert that φ is constant but perhaps different on each

component of $\Omega \setminus E$ and so we may not assume $\varphi = 0$ on ∂E . The condition $\varphi = 0$ on ∂E seems critical in Step 5 below.

Step 3. Let $I_j(u; A) = \|\frac{\nabla u + \tilde{H}}{\sigma}\|_{L^{q_j}(A)}$, $I(u; A) = \|\frac{\nabla u + \tilde{H}}{\sigma}\|_{L^\infty(A)}$, where $\tilde{H} = -(\bar{G} + H)^\perp$. Then (6.5) is equivalent to

$$I(0; E) \leq I(\varphi; E). \tag{6.6}$$

The following steps are devoted to the proof of (6.6). Note that $I_j(u; A) = \|\frac{G+H}{\sigma}\|_{L^{q_j}(A)}$ if $G = \bar{G} - (\nabla u)^\perp$. The minimality of $G_{q_j} = \bar{G} - (\nabla \varphi^j)^\perp$ can be written as

$$I_j(\varphi^j; \Omega) = \min_{u \in W_*^{1,q_j}(\Omega)} I_j(u; \Omega) \quad \forall j = 1, 2, \dots \tag{6.7}$$

We decompose the set $E = E_+ \cup E_0 \cup E_-$, where

$$E_\pm = \{x \in E \mid \pm\varphi(x) > 0\}, \quad E_0 = \{x \in E \mid \varphi(x) = 0\}.$$

Note that $I(0; E) = \max\{I(0; E_+), I(0; E_0), I(0; E_-)\}$. We prove (6.6) in different cases. First assume $I(0; E_0) > \max\{I(0; E_+), I(0; E_-)\}$. In this case, $I(0; E) = I(0; E_0)$ and $|E_0| > 0$. On E_0 , $\varphi = 0$ and hence $\nabla \varphi = 0$ for a.e. $x \in E_0$. So $I(0; E_0) = I(\varphi; E_0) \leq I(\varphi; E)$ and (6.6) follows.

Step 4. We now assume $I(0; E) = \max\{I(0; E_+), I(0; E_-)\}$. Without loss of generality, assume $I(0; E) = I(0; E_+)$. Let $\epsilon_k > 0$ be a decreasing sequence converging to zero such that each of the following open sets has finite perimeter in Ω :

$$E_+^k = \{x \in E \mid \varphi(x) > \epsilon_k\}, \tag{6.8}$$

$$E_+^{j,k} = \left\{x \in E \mid \varphi(x) > 0, \varphi^j(x) < \varphi(x) - \frac{\epsilon_k}{2}\right\}. \tag{6.9}$$

These sets will satisfy the requirements similar to (4.13)–(4.15) mentioned above in Proposition 4.4 above. Note that $E_+ = \bigcup_{k=1}^\infty E_+^k$. Given k , since $\varphi^j(x) \rightarrow 0$ uniformly on $x \in \bar{\Omega}$, we have $E_+^k \subset E_+^{j,k}$ for all sufficiently large j . Let

$$\tilde{\varphi}_{j,k}(x) = \left(\varphi(x) - \frac{\epsilon_k}{2}\right)\chi_{E_+^{j,k}}(x) + \varphi^j(x)\chi_{\bar{\Omega} \setminus E_+^{j,k}}(x) \quad (x \in \bar{\Omega}). \tag{6.10}$$

Step 5. We claim that for each given k the function $\tilde{\varphi}_{j,k}$ defined by (6.10) belongs to $W_*^{1,q_j}(\Omega)$ for all sufficiently large j . Since both φ^j and φ are in $W_*^{1,q_j}(\Omega)$, this claim amounts to proving $\tilde{\varphi}_{j,k}$ is continuous on $\bar{\Omega}$. Let $\bar{x} \in \partial(E_+^{j,k})$ and we would like to show $\varphi^j(\bar{x}) = \varphi(\bar{x}) - \frac{\epsilon_k}{2}$ and hence $\tilde{\varphi}_{j,k}$ is continuous on $\bar{\Omega}$. Note that $\bar{x} \in \partial(E_+^{j,k})$ implies $\varphi^j(\bar{x}) \leq \varphi(\bar{x}) - \frac{\epsilon_k}{2}$. Suppose $\varphi^j(\bar{x}) < \varphi(\bar{x}) - \frac{\epsilon_k}{2}$. In this case, since $\bar{x} \in \partial(E_+^{j,k})$, we will have the following possibilities: (a) $\bar{x} \in \partial E$; (b) $\bar{x} \notin \partial E$, $\varphi(\bar{x}) = 0$. In either case, we have $\varphi(\bar{x}) = 0$. This would imply that $\varphi^j(\bar{x}) < -\frac{\epsilon_k}{2}$, which is impossible for sufficiently large j since $\varphi^j \rightarrow 0$ uniformly on $\bar{\Omega}$ as $j \rightarrow \infty$. Hence $\varphi^j(\bar{x}) = \varphi(\bar{x}) - \frac{\epsilon_k}{2}$ at any point $\bar{x} \in \partial(E_+^{j,k})$ for all sufficiently large j . Therefore $\tilde{\varphi}_{j,k} \in W_*^{1,q_j}(\Omega)$ for all sufficiently large j .

Step 6. We use $\tilde{\varphi}_{j,k}$ as a test in (6.7) and thus it follows that $I_j(\varphi^j; \Omega) \leq I_j(\tilde{\varphi}_{j,k}; \Omega)$ for all sufficiently large j . Canceling the common terms on $\Omega \setminus E_+^{j,k}$, we have $I_j(\varphi^j; E_+^{j,k}) \leq I_j(\varphi - \frac{\epsilon_k}{2}; E_+^{j,k}) = I_j(\varphi; E_+^{j,k})$. Hence, for each given k and all sufficiently large j ,

$$I_j(\varphi^j; E_+^k) \leq I_j(\varphi^j; E_+^{j,k}) \leq I_j(\varphi; E_+^{j,k}) \leq I(\varphi; E_+) |E_+|^{\frac{1}{q_j}}.$$

Letting $j \rightarrow \infty$ and using (4.6), it follows that

$$I(0; E_+^k) \leq \liminf_{j \rightarrow \infty} I_j(\varphi^j; E_+^k) \leq I(\varphi; E_+).$$

Since E_+^k increases to E_+ , this proves $I(0; E_+) = \lim_{k \rightarrow \infty} I(0; E_+^k) \leq I(\varphi; E_+)$, from which (6.6) follows. This completes the proof of Theorem 6.2. \square

The rest of this section is devoted to the second part of Theorem 1.4 in the special case where $H = 0$ and $\sigma = 1$; we thus study the problem

$$\min_{G \in \mathcal{S}_\beta(\Omega)} \|G\|_{L^\infty(\Omega)}, \quad (6.11)$$

where $\Omega \subset \mathbf{R}^2$ is described as above and $\beta \in L^\infty(\partial\Omega)$ is a given function to be specified below.

We need to consider Lipschitz functions and their extensions relative to domain Ω . First of all, following [19], we define the distance-function $d_\Omega : \bar{\Omega} \times \bar{\Omega} \rightarrow \mathbf{R}$ relative to $\bar{\Omega}$ by

$$d_\Omega(x, y) = \liminf_{\substack{(a,b) \in \Omega \times \Omega \\ (a,b) \rightarrow (x,y)}} \left(\inf_{\substack{\gamma \in C^1([0,1]; \Omega) \\ \gamma(0)=a, \gamma(1)=b}} \int_0^1 |\dot{\gamma}(t)| dt \right). \quad (6.12)$$

(The dot here and below means differentiation with respect to the given parameter.) Given any nonempty set $S \subset \bar{\Omega}$ and any function $u : S \rightarrow \mathbf{R}$, we say u is *Lipschitz on S* with respect to d_Ω and write $u \in \text{Lip}_\Omega(S)$ provided that

$$\text{Lip}_S(u) := \sup_{x \neq y \in S} \frac{|u(x) - u(y)|}{d_\Omega(x, y)} < \infty. \quad (6.13)$$

It is well known that any Lipschitz function u on S admits a *minimal* Lipschitz extension v on $\bar{\Omega}$; that is, $v = u$ on S and $\text{Lip}_{\bar{\Omega}}(v) = \text{Lip}_S(u)$. A minimal Lipschitz extension v of u is called an *absolutely minimizing Lipschitz extension* (AMLE) if $\text{Lip}_V(v) = \text{Lip}_{\partial V}(v)$ for every open set $V \subset \bar{\Omega} \setminus S$. For existence and uniqueness of AMLE and other related results, we refer to [6,19,23].

We now make further specific assumptions on $\partial\Omega$ and β . Let us parametrize Γ_0 *counter-clockwise* and other Γ_i 's ($i = 1, \dots, k$) *clockwise* using the arc-length parameter s on each curve. We assume that the parametric equation so obtained

$$\mathbf{x}^i(s) = (x_1^i(s), x_2^i(s)), \quad 0 \leq s < L_i,$$

for each $i = 0, 1, \dots, k$, is one-to-one and Lipschitz continuous from $[0, L_i)$ onto Γ_i . Then for almost every $s \in [0, L_i)$ the unit tangent vector τ^i and outward unit normal vector ν^i at $x = \mathbf{x}^i(s) \in \Gamma_i$ are given by

$$\tau^i = \dot{\mathbf{x}}^i(s) = (\dot{x}_1^i(s), \dot{x}_2^i(s)), \quad \nu^i = -(\tau^i)^\perp = (\dot{x}_2^i(s), -\dot{x}_1^i(s)). \quad (6.14)$$

For each $i = 0, 1, \dots, k$, let

$$b_i(s) = \beta_i(\mathbf{x}^i(s)), \quad a_i(s) = \int_0^s b_i(t) dt, \quad 0 \leq s < L_i. \quad (6.15)$$

Then $a_i \in W^{1,\infty}(0, L_i)$ and $\|\dot{a}_i\|_{L^\infty} \leq \|\beta_i\|_{L^\infty(\Gamma_i)}$, $\|a_i\|_{L^\infty} \leq L_i \|\beta_i\|_{L^\infty(\Gamma_i)}$. Define a function α_i on each Γ_i by setting $\alpha_i(x) = a_i(s)$ if $x \in \Gamma_i$ is represented by $x = \mathbf{x}^i(s)$ for some $s \in [0, L_i)$. Let $\alpha = \sum_{i=0}^k \alpha_i \chi_{\Gamma_i}$. We make the following assumption:

$$\alpha_i \in \text{Lip}_\Omega(\Gamma_i) \quad \forall i = 0, 1, \dots, k. \quad (6.16)$$

A necessary condition of (6.16) is that $a_i(L_i^-) = \int_{\Gamma_i} \beta_i d\mathcal{H}^1 = 0$ for all $i = 0, 1, \dots, k$, which is stronger than the usual assumption $\int_{\partial\Omega} \beta d\mathcal{H}^1 = 0$.

Proposition 6.3. *Under the assumption (6.16), we have that $\alpha \in \text{Lip}_\Omega(\partial\Omega)$ and that there exists a function $\psi \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ such that $\psi = \alpha$ on $\partial\Omega$ and $\|\nabla\psi\|_{L^\infty(\Omega)} = \text{Lip}_{\partial\Omega}(\alpha)$. Furthermore, $F = (\psi_{x_2}, -\psi_{x_1}) \in \mathcal{S}_\beta(\Omega)$.*

Proof. Let $x \neq y \in \partial\Omega$. Assume $x \in \Gamma_i$ and $y \in \Gamma_j$. If $i \neq j$, then $|\alpha(x) - \alpha(y)| = |\alpha_i(x) - \alpha_j(y)| \leq M$ and $|x - y| \geq m > 0$, where

$$M = 2 \max_{0 \leq i \leq k} \{L_i \|\beta_i\|_{L^\infty(\Gamma_i)}\}, \quad m = \min_{i \neq j} \{|a - b| : a \in \Gamma_i, b \in \Gamma_j\}.$$

Hence $|\alpha(x) - \alpha(y)| \leq \frac{M}{m}|x - y| \leq \frac{M}{m}d_\Omega(x, y)$ if $i \neq j$. Now assume $i = j$; then $x, y \in \Gamma_i$. By assumption (6.16), it follows also that $|\alpha(x) - \alpha(y)| \leq Kd_\Omega(x, y)$, where

$$K = \max_{0 \leq i \leq k} \text{Lip}_{\Gamma_i}(\alpha_i) < \infty.$$

This proves that $\alpha \in \text{Lip}_\Omega(\partial\Omega)$. The existence of a minimal Lipschitz extension ψ follows from, e.g., Lemma 1.6 and Theorem 1.8 of [19]. In fact, for any $\alpha \in \text{Lip}_\Omega(\partial\Omega)$, such a function ψ can be taken to be

$$\psi(x) = \inf_{y \in \partial\Omega} (\alpha(y) + \text{Lip}_{\partial\Omega}(\alpha) d_\Omega(x, y)) \quad \forall x \in \bar{\Omega}. \tag{6.17}$$

We prove the last statement. Let $F = (\psi_{x_2}, -\psi_{x_1})$. Given any $\zeta \in C^1(\mathbf{R}^2)$, by an easy density argument, we have

$$\int_\Omega F \cdot \nabla \zeta \, dx = \int_\Omega (\psi_{x_2} \zeta_{x_1} - \psi_{x_1} \zeta_{x_2}) \, dx = \int_{\partial\Omega} (\alpha \zeta_{x_1} \nu_2 - \alpha \zeta_{x_2} \nu_1) \, d\mathcal{H}^1,$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit normal on the boundary of Ω . Split $\partial\Omega$ into the union of Γ_i and use the formula for outward unit normal on each Γ_i given in (6.14) and we obtain

$$\begin{aligned} \int_\Omega F \cdot \nabla \zeta \, dx &= \int_{\partial\Omega} (\alpha \zeta_{x_1} \nu_2 - \alpha \zeta_{x_2} \nu_1) \, d\mathcal{H}^1 \\ &= \sum_{i=0}^k \int_{\Gamma_i} (\alpha_i \zeta_{x_1} \nu_2 - \alpha_i \zeta_{x_2} \nu_1) \, d\mathcal{H}^1 = - \sum_{i=0}^k \int_0^{L_i} (a_i \zeta_{x_1} \dot{x}_1^i + a_i \zeta_{x_2} \dot{x}_2^i) \, ds \\ &= - \sum_{i=0}^k \int_0^{L_i} a_i(s) \frac{d}{ds} (\zeta(\mathbf{x}^i(s))) \, ds = \sum_{i=0}^k \int_0^{L_i} \zeta(\mathbf{x}^i(s)) \frac{d}{ds} (a_i(s)) \, ds \\ &= \sum_{i=0}^k \int_0^{L_i} \zeta(\mathbf{x}^i(s)) b_i(s) \, ds = \sum_{i=0}^k \int_{\Gamma_i} \zeta(x) \beta_i(x) \, d\mathcal{H}^1(x). \end{aligned}$$

This proves that $\int_\Omega F \cdot \nabla \zeta \, dx = \int_{\partial\Omega} \beta \zeta \, d\mathcal{H}^1$ for all $\zeta \in C^1(\mathbf{R}^2)$. Hence, by definition, $F \in \mathcal{S}_\beta(\Omega)$. The proof is now complete. \square

We have the following characterization of the set $\mathcal{S}_\beta(\Omega)$.

Proposition 6.4. *$G \in \mathcal{S}_\beta(\Omega)$ if and only if $G = (\varphi_{x_2}, -\varphi_{x_1})$ in Ω for a unique function $\varphi \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ satisfying $\varphi = \alpha_0$ on Γ_0 and $\varphi = c_i + \alpha_i$ on Γ_i for $i = 1, \dots, k$, where $\mathbf{c} = (c_1, \dots, c_k) \in \mathbf{R}^k$ and $\alpha_i \in \text{Lip}_\Omega(\Gamma_i)$ is the function defined above. Furthermore, letting*

$$\alpha_{\mathbf{c}} = \alpha_0 \chi_{\Gamma_0} + \sum_{i=1}^k (c_i + \alpha_i) \chi_{\Gamma_i} = \alpha + \sum_{i=1}^k c_i \chi_{\Gamma_i}, \tag{6.18}$$

it follows that $\|G\|_{L^\infty(\Omega)} \geq \text{Lip}_{\partial\Omega}(\alpha_{\mathbf{c}})$.

Proof. Let $F = (\psi_{x_2}, -\psi_{x_1})$, where $\psi \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ is any function determined in Proposition 6.3. Given any $G \in L^\infty(\Omega; \mathbf{R}^2)$, $G \in \mathcal{S}_\beta(\Omega)$ if and only if $G - F \in \mathcal{S}_0^\infty(\Omega)$. By Lemma 6.1, this condition is equivalent to $G - F = (\phi_{x_2}, -\phi_{x_1})$ for some $\phi \in W_*^{1,\infty}(\Omega)$, which becomes $G = (\varphi_{x_2}, -\varphi_{x_1})$, where $\varphi = \psi + \phi \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ satisfies $\varphi|_{\Gamma_i} = c_i + \alpha_i$ for all $i = 0, 1, \dots, k$. We make φ unique by taking $c_0 = 0$; that is, $\varphi = \alpha_{\mathbf{c}}$ on $\partial\Omega$ for some constant vector $\mathbf{c} \in \mathbf{R}^k$. Finally, the inequality $\|G\|_{L^\infty(\Omega)} \geq \text{Lip}_{\partial\Omega}(\alpha_{\mathbf{c}})$ follows from Lemma 1.6 of [19]. \square

Let $L(\mathbf{c}) = \text{Lip}_{\partial\Omega}(\alpha_{\mathbf{c}})$. Then, for any $\mathbf{c} = (c_1, \dots, c_k) \in \mathbf{R}^k$,

$$L(\mathbf{c}) = \max_{0 \leq i \leq j \leq k} \left\{ \sup_{\substack{x \in \Gamma_i, y \in \Gamma_j \\ x \neq y}} \frac{|c_i - c_j + \alpha_i(x) - \alpha_j(y)|}{d_\Omega(x, y)} \right\}, \tag{6.19}$$

where $c_0 = 0$. It is easily seen that L is convex on \mathbf{R}^k and $L(\mathbf{c}) \geq l_0|\mathbf{c}| - l_1$ for all $\mathbf{c} \in \mathbf{R}^k$, where $l_0 > 0$ and l_1 are certain constants. Therefore the following set is a nonempty compact convex set in \mathbf{R}^k :

$$\Sigma = \operatorname{argmin}(L) = \left\{ \mathbf{c} \in \mathbf{R}^k \mid L(\mathbf{c}) = \min_{\mathbf{c} \in \mathbf{R}^k} L(\mathbf{c}) \right\}. \tag{6.20}$$

Finally, we reformulate and prove the second part of Theorem 1.4 as follows.

Theorem 6.5. *It follows that*

$$\min_{G \in \mathcal{S}_\beta(\Omega)} \|G\|_{L^\infty(\Omega)} = \min_{\mathbf{c} \in \mathbf{R}^k} L(\mathbf{c}) = \min_{\mathbf{c} \in \mathbf{R}^k} \operatorname{Lip}_{\partial\Omega}(\alpha_{\mathbf{c}}). \tag{6.21}$$

Moreover, $\bar{G} \in \mathcal{S}_\beta(\Omega)$ is a minimizer if and only if there exists a unique function $\bar{\varphi} \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ such that $\bar{G} = (\bar{\varphi}_{x_2}, -\bar{\varphi}_{x_1})$ in Ω and $\bar{\varphi} = \alpha_{\bar{\mathbf{c}}}$ on $\partial\Omega$ for some $\bar{\mathbf{c}} \in \Sigma$, where Σ is the set defined above by (6.20). Furthermore, \bar{G} is an absolute minimizer if and only if $\bar{\varphi}$ is the AMLE of $\alpha_{\bar{\mathbf{c}}}$ on $\bar{\Omega}$. Therefore, there exists a unique absolute minimizer \bar{G} if the set Σ is a singleton and there exist infinitely many absolute minimizers \bar{G} if Σ contains more than one points.

Proof. By Proposition 6.4, we easily see that

$$\min_{G \in \mathcal{S}_\beta(\Omega)} \|G\|_{L^\infty(\Omega)} \geq \min_{\mathbf{c} \in \mathbf{R}^k} \operatorname{Lip}_{\partial\Omega}(\alpha_{\mathbf{c}}) = \min_{\mathbf{c} \in \mathbf{R}^k} L(\mathbf{c}).$$

Let $\bar{\mathbf{c}} \in \Sigma$; that is, $L(\bar{\mathbf{c}}) = \min_{\mathbf{c} \in \mathbf{R}^k} L(\mathbf{c})$. Using the formula (6.17), there exists a function $\psi \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ such that $\psi = \alpha_{\bar{\mathbf{c}}}$ on $\partial\Omega$ and $\|\nabla\psi\|_{L^\infty(\Omega)} = \operatorname{Lip}_{\partial\Omega}(\alpha_{\bar{\mathbf{c}}}) = L(\bar{\mathbf{c}})$. Let $G = (\psi_{x_2}, -\psi_{x_1})$. Then $G \in \mathcal{S}_\beta(\Omega)$ and $\|G\|_{L^\infty(\Omega)} = L(\bar{\mathbf{c}})$. This proves (6.21); the proof also shows that $\bar{G} \in \mathcal{S}_\beta(\Omega)$ is a minimizer if and only if there exists a unique function $\bar{\varphi} \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ such that $\bar{G} = (\bar{\varphi}_{x_2}, -\bar{\varphi}_{x_1})$ in Ω and $\bar{\varphi} = \alpha_{\bar{\mathbf{c}}}$ on $\partial\Omega$ for some $\bar{\mathbf{c}} \in \Sigma$. Assume $\bar{G}, \bar{\varphi}$ are given this way. We would like to show that \bar{G} is an absolute minimizer if and only if $\bar{\varphi}$ is the AMLE of $\alpha_{\bar{\mathbf{c}}}$ on $\bar{\Omega}$. We split this proof into two steps.

Step 1. Assume $\bar{G} = (\bar{\varphi}_{x_2}, -\bar{\varphi}_{x_1})$ is an absolute minimizer of (6.11). We show that $\bar{\varphi}$ is the AMLE of $\alpha_{\bar{\mathbf{c}}}$. By the many equivalent descriptions of AMLE in [6, Theorem 4.1 and Proposition 4.5], we only need to show that, for any open disk $V \Subset \Omega$ (hence $\Omega \setminus V$ is connected) and any function $\phi \in W^{1,\infty}(\Omega)$ with $\phi = \bar{\varphi}$ on ∂V , it follows that

$$\|\nabla\bar{\varphi}\|_{L^\infty(V)} \leq \|\nabla\phi\|_{L^\infty(V)}. \tag{6.22}$$

To prove (6.22), let $\tilde{\varphi} = \phi\chi_V + \bar{\varphi}\chi_{\Omega \setminus V}$ and $\tilde{G} = (\tilde{\varphi}_{x_2}, -\tilde{\varphi}_{x_1})$. Then $\tilde{\varphi} \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ and $\tilde{\varphi} = \alpha_{\bar{\mathbf{c}}}$ on $\partial\Omega$; hence, by Proposition 6.4, $\tilde{G} \in \mathcal{S}_\beta(\Omega)$. Also $\tilde{\delta}_V(\tilde{G}) = \tilde{\delta}_V(\bar{G})$ since $\tilde{\varphi} = \bar{\varphi}$ on ∂V . Hence (6.22) follows from the absolute minimality of \bar{G} .

Step 2. Assume $\bar{\varphi}$ is the AMLE of $\alpha_{\bar{\mathbf{c}}}$. We show that $\bar{G} = (\bar{\varphi}_{x_2}, -\bar{\varphi}_{x_1})$ is an absolute minimizer of (6.11). Let $G \in \mathcal{S}_\beta(\Omega)$ and let $E \subset \Omega$ be an open set with $\Omega \setminus E$ connected. Assume $\tilde{\delta}_E(G) = \tilde{\delta}_E(\bar{G})$. By Proposition 6.4, $G = (\phi_{x_2}, -\phi_{x_1})$, where $\eta = \phi - \bar{\varphi} \in W^{1,\infty}_*(\Omega)$ satisfies $\eta = \alpha_{\mathbf{c}}$ for some $\mathbf{c} \in \mathbf{R}^k$. Since $\tilde{\delta}_E(G) = \tilde{\delta}_E(\bar{G})$, it follows that $\nabla\eta = 0$ on $\Omega \setminus E$ and hence, by the connectedness of $\Omega \setminus E$ and continuity of η , it follows that η is constant on $\bar{\Omega} \setminus \bar{E}$. However, since $\Gamma_0 \cup \partial E \subset \bar{\Omega} \setminus \bar{E}$ and $\eta = 0$ on Γ_0 , we have $\eta = 0$ on ∂E and thus $\phi = \bar{\varphi}$ on ∂E . Therefore, from the equivalent descriptions of the AMLE, we have

$$\|\bar{G}\|_{L^\infty(E)} = \|\nabla\bar{\varphi}\|_{L^\infty(E)} \leq \|\nabla\phi\|_{L^\infty(E)} = \|G\|_{L^\infty(E)}.$$

This proves that \bar{G} is an absolute minimizer. The proof is complete. \square

Example 6.1 (Special case of Example 5.1 in two dimensions). Let $\Omega = \{x \in \mathbf{R}^2 \mid r < |x| < R\}$ be the annulus. Let $H = 0$, $h = \operatorname{div} H = 0$ and $\beta = g : \partial\Omega \rightarrow \mathbf{R}$ as before; i.e., $\beta(x) = 0$ on $|x| = r$, $\beta(x) = x_1$ on $|x| = R$. In this case, the function $\alpha : \partial\Omega \rightarrow \mathbf{R}$ defined above is given by $\alpha(x) = 0$ on $|x| = r$ and $R^2 - Rx_2$ on $|x| = R$. The function $L(c) = \operatorname{Lip}_{\partial\Omega}(\alpha_c)$ with $c \in \mathbf{R}$ defined by (6.19) can be computed to be

$$L(c) = \frac{|c - R^2| + R^2}{R - r}.$$

Note that the set $\Sigma = \operatorname{argmin}(L) = \{R^2\}$ is a singleton and hence

$$\rho(\beta, 0) = \min_{G \in \mathcal{S}_\beta(\Omega)} \|G\|_{L^\infty(\Omega)} = \min_{c \in \mathbf{R}} L(c) = \frac{R^2}{R - r},$$

which agrees with the general result obtained in Proposition 5.5. However, by Theorem 6.5, we know that the problem for $\rho(\beta, 0)$ has a unique absolute minimizer \bar{G} , which is given by $\bar{G} = (\bar{\varphi}_{x_2}, -\bar{\varphi}_{x_1})$, where $\bar{\varphi}$ is the absolute minimizing Lipschitz extension onto Ω of the boundary function $\alpha_{R^2}(x) = R^2$ on $|x| = r$ and $R^2 - Rx_2$ on $|x| = R$. Therefore, for this problem, the functions \bar{G} and \bar{F} determined as weak limits in Proposition 4.1 and Theorem 3.4 are unique, which implies the *whole* sequences $\{G_q\}$ and $\{|\nabla u_p|^{p-2} \nabla u_p\}$ defined there converge in the respective cases as $q \rightarrow \infty$ and $p \rightarrow 1$. Recall that, by Example 5.1, the limit \bar{u} of $\{u_p\}$ is also unique.

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