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Ann. I. H. Poincaré – AN 28 (2011) 499–527

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Nash–Moser iteration and singular perturbations

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Received 27 July 2009; received in revised form 2 November 2010; accepted 3 March 2011

Available online 19 May 2011

Abstract

We present a simple and easy-to-use Nash–Moser iteration theorem tailored for singular perturbation problems admitting a formal asymptotic expansion or other family of approximate solutions depending on a parameter $\varepsilon \to 0$. The novel feature is to allow loss of powers of ε as well as the usual loss of derivatives in the solution operator for the associated linearized problem. We indicate the utility of this theorem by describing sample applications to (i) systems of quasilinear Schrödinger equations, and (ii) existence of small-amplitude profiles of quasilinear relaxation systems in the degenerate case that the velocity of the profile is a characteristic mode of the hyperbolic operator.

1. Introduction

Because the expansions themselves furnish arbitrarily accurate approximate solutions, and because the associated linearized estimates are often stiff in terms of amplitude and or smoothness, Nash–Moser iteration appears particularly well-adapted to the verification of asymptotic expansions such as arise in various singular perturbation problems depending on a small parameter $\varepsilon \to 0$. However, standard Nash–Moser theorems allow only for loss of derivatives and not loss of powers of ε in the estimates on the linearized solution operator, so that to apply Nash–Moser iteration to problems that do lose powers of ε would appear to require a careful accounting of constants throughout the entire Nash–Moser iteration to check that the argument closes.

The purpose of this article therefore is to present a simple and general-purpose theorem carrying out this accounting, which can be applied as an easy-to-use black box to this type of problem. We conclude by presenting two sample applications for which both loss of derivatives and of powers of ε naturally occur for the linearized problem, one for systems of quasilinear Schrödinger equations and one in existence of small-amplitude profiles of quasilinear relaxation systems. The latter, due to Métivier, Texier and Zumbrun, was treated in [17] by the approach presented here; though special cases may be treated by other methods [23,9,10,3], we do not know of any other solution in the generality considered there.

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1 Research of B.T. was partially supported under NSF grant number DMS-0505780. B.T. acknowledges support from the Agence Nationale de la Recheche through grant ANR-08-BLAN-0301-01.
2 Research of K.Z. was partially supported under NSF grants No. DMS-0300487 and DMS-0801745.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2011.05.001

Our approach follows a very simple proof given by Xavier Saint-Raymond [22] of a (parameter-independent) Nash–Moser implicit function theorem [8,18] in a Sobolev space setting. A novel aspect is our treatment of uniqueness, which we have not seen elsewhere – in particular the incorporation of a phase condition in the case that the linearized operator has a kernel. (See Theorem 2.20.)

We note that a parameter-dependent Nash–Moser scheme was recently used by Alvarez-Samaniego and Lannes [2] to prove local-in-time well-posedness of model equations in oceanography. Alvarez-Samaniego and Lannes do not allow losses in ε in the linearized solution operator, which is the main point here.

In [5], Iooss, Plotnikov and Toland use a parameter-dependent Nash–Moser to prove the existence of periodic standing water-waves in the case of an infinite depth. They deal with a situation in which the linearized operator might not be invertible on small sets on ε, and proceed by taking out corresponding bad regions in ε. Here we consider everywhere invertible linearized operators, with losses (materialized by inverse powers of ε in the estimates) for all values of the parameter ε, and proceed by keeping track of these losses.

An important reference on Nash–Moser-type theorems is Hamilton [4]. Another good reference is Alinhac and Gérard's book [1].

Plan of the paper and scheme of the main proof. In Section 2, we state the main theorem on ε-dependent Nash–Moser iteration, giving the proof afterward in Section 3.

The proof classically combines an iteration step (Newton's method) with a regularization procedure (in Sobolev spaces, high-frequency truncation). It is largely based on Xavier Saint-Raymond's elegant and short proof [22]. The key in our context is to show that the bounds can be made uniform in the small parameter. Our main observation is that this can be achieved under the condition that the regularizing sequence is diverging to $+\infty$ fast enough, depending on the small parameter (see (3.19)).

In Section 4, we describe applications to systems of quasilinear Schrödinger equations, and in Section 5 to existence of small-amplitude profiles of quasilinear relaxation systems in the degenerate case that the velocity of the relaxation profile is a characteristic mode of the hyperbolic operator.

2. A simple Nash–Moser theorem

2.1. Setting

Consider two families of Banach spaces $\{E_s\}_{s\in\mathbb{R}}$, $\{F_s\}_{s\in\mathbb{R}}$, and a family of equations

$$\Phi^\varepsilon\big(u^\varepsilon\big) = 0, \quad u^\varepsilon \in E_s, \tag{2.1}$$

indexed by $\varepsilon \in (0,1)$, where for some $m \geqslant 0$, $s_0, s_1 \in \mathbb{R}$, with $s_0 + 2m \leqslant s_1$, for all ε,

$$\Phi^\varepsilon \in C^2(E_s, F_{s-m}), \quad \text{for all } s_0 + m \leqslant s \leqslant s_1. \tag{2.2}$$

Let $|\cdot|_s$ denote the norm in E_s and $\|\cdot\|_s$ denote the norm in F_s. The norms $|\cdot|_s$ and $\|\cdot\|_s$ and spaces E_s and F_s are possibly ε-dependent, as in our applications in Sections 4 and 5. We assume that the embeddings

$$E_{s'} \hookrightarrow E_s, \qquad F_{s'} \hookrightarrow F_s, \quad s \leqslant s', \tag{2.3}$$

hold, and have norms less than one:

$$|\cdot|_s \leqslant |\cdot|_{s'}, \qquad \|\cdot\|_s \leqslant \|\cdot\|_{s'}, \quad s \leqslant s'. \tag{2.4}$$

We assume the interpolation property:

$$|\cdot|_{s+\sigma} \lesssim |\cdot|_s^{\frac{\sigma'-\sigma}{\sigma'}} |\cdot|_{s+\sigma'}^{\frac{\sigma}{\sigma'}}, \quad 0 < \sigma < \sigma', \tag{2.5}$$

where $|u|_s \lesssim |v|_{s'}$ stands for $|u|_s \leqslant C|v|_{s'}$, for some $C > 0$ possibly depending on s and s' but not on ε, nor on u and v. We assume in addition the existence of a family of regularizing operators

$$S_\theta : E_s \to E_s, \quad \theta > 0,$$

such that for all $s \leqslant s'$,

$$|S_\theta u - u|_s \lesssim \theta^{s-s'} |u|_{s'}, \tag{2.6}$$
$$|S_\theta u|_{s'} \lesssim \left(1 + \theta^{s'-s}\right)|u|_s. \tag{2.7}$$

Example 2.1. Let

$$E_s = H^s\left(\mathbb{R}^d\right), \qquad F_s = H^s\left(\mathbb{R}^d\right) \times H^{s+1}\left(\mathbb{R}^d\right),$$

with ε-dependent norms defined by

$$|v|_s := \|v\|_{H^s_\varepsilon} := \left\| \left(1 + |\varepsilon\xi|^2\right)^{s/2} (\mathcal{F}v)(\xi) \right\|_{L^2(\mathbb{R}^d_\xi)}, \qquad \left\|(f, g)\right\|_s = |f|_s + |g|_{s+1}, \tag{2.8}$$

where $\mathcal{F}v$ denotes the Fourier transform of v. Then (2.3), (2.4) and (2.5) hold. A family of regularizing operators $E_s \to E_s$ is given by

$$S_\theta : u \to S_\theta(u) := \mathcal{F}^{-1}\left(\chi\left(\theta^{-1}\xi\right)\hat{u}\right),$$

where $\chi : \mathbb{R}^d \to [0, 1]$ is a smooth truncation function, identically equal to 1 for $|\xi| \leqslant 1$, and identically equal to 0 for $|\xi| \geqslant 2$. The family $\{S_\theta\}_{\theta>0}$ satisfies (2.6) and (2.7).

Remark 2.2. The family of norms $\|\cdot\|_{H^s_\varepsilon}$ satisfies Moser's inequality

$$\|uv\|_{H^s_\varepsilon} \lesssim |u|_{L^\infty} \|v\|_{H^s_\varepsilon} + \|u\|_{H^s_\varepsilon} |v|_{L^\infty}, \quad s \geqslant 0,\ u, v \in H^s \cap L^\infty,$$

and, if F is smooth and satisfies $F(0) = 0$, for some non-decreasing $C : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\left\|F(u)\right\|_{H^s_\varepsilon} \leqslant C\left(|u|_{L^\infty}\right)\|u\|_{H^s_\varepsilon}, \quad s > d/2,\ u \in H^s.$$

Example 2.3. Consider the family of maps

$$\Phi^\varepsilon(u) = \left(\sum_{1 \leqslant j \leqslant d} A_j(u)\varepsilon\partial_{x_j} u, u \right),$$

where $A_j : u \in \mathbb{R}^n \to A_j(u) \in \mathbb{R}^{n\times n}$ is smooth. By Moser's inequality (Remark 2.2), for $s > 1 + d/2$, the application Φ^ε maps smoothly H^s to $H^{s-1} \times H^s$, that is, with the notation of Example 2.1, E_s to F_{s-1}.

2.2. First assumption: tame direct bounds

We assume bounds for Φ^ε and its first two derivatives.

Assumption 2.4. For some $\gamma_0 \geqslant 0$, $\gamma_1 \geqslant 0$, for all s such that $s_0 + m \leqslant s \leqslant s_1 - m$, for all $u, v, w \in E_{s+m}$, there holds

$$\left\|\Phi^\varepsilon(u)\right\|_s \leqslant \mathbf{C}_0\left(1 + |u|_{s+m}\right), \tag{2.9}$$
$$\left\|\left(\Phi^\varepsilon\right)'(u) \cdot v\right\|_s \leqslant \mathbf{C}_1\left(\varepsilon^{-\gamma_1}|v|_{s_0+m}|u|_{s+m} + |v|_{s+m}\right), \tag{2.10}$$
$$\left\|\left(\Phi^\varepsilon\right)''(u) \cdot (v, w)\right\|_s \leqslant \mathbf{C}_2\left(\varepsilon^{-2\gamma_1}|v|_{s_0+m}|w|_{s_0+m}|u|_{s+m} + \varepsilon^{-\gamma_1}|w|_{s_0+m}|v|_{s+m} + \varepsilon^{-\gamma_1}|v|_{s_0+m}|w|_{s+m}\right), \tag{2.11}$$

where the functions $\mathbf{C}_j = \mathbf{C}_j(\varepsilon, (|u|_{s_0+\tilde{m}})_{0\leqslant \tilde{m} \leqslant m})$ satisfy

$$\sup\left\{\mathbf{C}_j,\ j = 0, 1, 2,\ \varepsilon \in (0, 1),\ |u|_{s_0+m} \lesssim \varepsilon^{\gamma_0}\right\} < +\infty.$$

The simplest example is given by a product map:

Example 2.5. Consider the map $\Phi_0 : u \in H^s(\mathbb{R}^d; \mathbb{R}) \to (u^2, u) \in H^s \times H^s(\mathbb{R}^d; \mathbb{R})$, for $d/2 < s$. There holds, by Remark 2.2,

$$\left\|\Phi_0(u)\right\|_{H^s_\varepsilon \times H^s_\varepsilon} \lesssim |u|_{L^\infty}\|u\|_{H^s_\varepsilon} + \|u\|_{H^s_\varepsilon},$$

where the weighted norm $\|\cdot\|_{H^s_\varepsilon}$ is defined in (2.8). For small ε, the classical Sobolev embedding $H^s \hookrightarrow L^\infty$, for $d/2 < s_0 \leqslant s$, has a large norm if H^s is equipped with the weighted norm $\|\cdot\|_{H^s_\varepsilon}$:

$$|u|_{L^\infty} \lesssim \varepsilon^{-d/2}\|u\|_{H^{s_0}_\varepsilon},$$

so that

$$\big\|\Phi_0(u)\big\|_{H^s_\varepsilon \times H^s_\varepsilon} \lesssim \big(1 + \varepsilon^{-d/2}\|u\|_{H^{s_0}_\varepsilon}\big)\|u\|_{H^s_\varepsilon},$$

and

$$\big\|\Phi_0'(u)\cdot v\big\|_{H^s_\varepsilon \times H^s_\varepsilon} \lesssim \big(1 + \varepsilon^{-d/2}\|u\|_{H^{s_0}_\varepsilon}\big)\|v\|_{H^s_\varepsilon} + \varepsilon^{-d/2}\|v\|_{H^{s_0}_\varepsilon}\|u\|_{H^s_\varepsilon}.$$

In particular, Assumption 2.4 holds with $m = 0$, $\gamma_0 = \gamma_1 = d/2$.

Another simple example is given by the map defined in Example 2.3:

Example 2.6. Consider the map $\Phi^\varepsilon : E_s \to F_{s-1}$, introduced in Example 2.3, where the families of functional spaces E_s and F_s and their ε-dependent norms were introduced in Example 2.1. Let $d/2 < s_0 < 1 + d/2$ be given. Just like in the above remark, for small ε, the classical Sobolev embedding

$$H^{N+s_0}\big(\mathbb{R}^d\big) \hookrightarrow W^{N,\infty}\big(\mathbb{R}^d\big), \quad N \in \mathbb{N},$$

has a large norm

$$|v|_{W^{N,\infty}} \lesssim \varepsilon^{-N-d/2}|v|_{N+s_0}, \tag{2.12}$$

where $|\cdot|_s = \|\cdot\|_{H^s_\varepsilon}$ is defined in (2.8). By Remark 2.2, for $s \geqslant s_0$,

$$\big|A_j(u)\varepsilon\partial_{x_j}u\big|_s \leqslant C_{A_j}\big(|u|_{L^\infty}\big)\big(|u|_{s+1} + |\varepsilon\partial_{x_j}u|_{L^\infty}|u|_s\big),$$

where $C_{A_j} : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing and depends only on A_j, s and d. By (2.12), for $s \geqslant s_0 + 1$,

$$\big|A_j(u)\varepsilon\partial_{x_j}u\big|_s \leqslant C_{A_j}\big(\varepsilon^{-d/2}|u|_{s_0}\big)\big(1 + \varepsilon^{-d/2}|u|_{s_0+1}\big)|u|_{s+1},$$

and (2.9) holds with

$$\mathbf{C}_0 = 1 + \sum_{1\leqslant j\leqslant d} C_{A_j}\big(\varepsilon^{-d/2}|u|_{s_0}\big)\big(1 + \varepsilon^{-d/2}|u|_{s_0+1}\big). \tag{2.13}$$

Besides,

$$\begin{aligned}\big\|\big(\Phi^\varepsilon\big)'(u)v\big\|_s &\leqslant \sum_{1\leqslant j\leqslant d}\big|\big(A_j'(u)\cdot v\big)\varepsilon\partial_{x_j}u\big|_s + \big|A_j(u)\varepsilon\partial_{x_j}v\big|_s + |v|_{s+1}\\ &\leqslant \sum_{1\leqslant j\leqslant d} C_{A_j,A_j'}\big(|u|_{L^\infty}\big)\big(\varepsilon^{-d/2}|v|_{s_0+1}|u|_{s+1} + \big(1 + \varepsilon^{-d/2}|u|_{s_0+1}\big)|v|_{s+1}\big)\end{aligned}$$

and (2.10) holds with $\gamma_0 = \gamma_1 = d/2$ and

$$\mathbf{C}_1 = 1 + \sum_{1\leqslant j\leqslant d} C_{A_j,A_j'}\big(\varepsilon^{-d/2}|u|_{s_0}\big)\big(1 + \varepsilon^{-d/2}|u|_{s_0+1}\big).$$

The bound for $(\Phi^\varepsilon)''$ is similar. We conclude that Assumption 2.4 holds with $\gamma_0 = \gamma_1 = d/2$.

Remark 2.7. In connection with (2.12), where the embedding constant blows up to $+\infty$ as ε decreases to 0, the "constants" $\mathbf{C}_j$ in Assumption 2.4 are *not* assumed to be non-decreasing in their arguments; in Example 2.6 this is reflected in (2.13).

Example 2.8. Given $T > 0$, consider the functional spaces

$$E_s = C^1\big([0,T], H^{s-1}\big(\mathbb{R}^d\big)\big) \cap C^0\big([0,T], H^s\big(\mathbb{R}^d\big)\big), \qquad F_{s-1} = C^0\big([0,T], H^{s-1}\big(\mathbb{R}^d\big)\big) \times H^s\big(\mathbb{R}^d\big),$$

with norms

$$|u|_s := \sup_{0\leqslant t\leqslant T}\left(\left\|\varepsilon\partial_t u(t)\right\|_{H^{s-1}_\varepsilon} + \left\|u(t)\right\|_{H^s_\varepsilon}\right), \qquad \left\|(f_1, f_2)\right\|_{s-1} := \sup_{0\leqslant t\leqslant T}\left\|f_1(t)\right\|_{H^{s-1}_\varepsilon} + \|f_2\|_{H^s_\varepsilon},$$

where the weighted Sobolev $\|\cdot\|_{H^s_\varepsilon}$ norms are defined in (2.8).

Let s_0 and s_1 be given such that $d/2 < s_0 < 1 + d/2 < s_1$, with $s_0 + 2 \leqslant s_1$. Let a bounded family $(f^\varepsilon)_{\varepsilon\in(0,1)} \subset F_{s_1-1}$; meaning $f^\varepsilon \in F_{s_1-1}$ for all ε, with $\sup_{\varepsilon\in(0,1)} \|f^\varepsilon\|_{s_1-1} < \infty$.

Consider the family of maps defined by

$$\Phi^\varepsilon(u) := \left(\varepsilon\partial_t u + \sum_{1\leqslant j\leqslant d} A_j(u)\varepsilon\partial_{x_j} u - f_1^\varepsilon,\ u_{|t=0} - f_2^\varepsilon\right), \tag{2.14}$$

where the maps $A_j : u \in \mathbb{R}^n \to A_j(u) \in \mathbb{R}^{n\times n}$ are smooth. Then, Φ^ε maps E_s to F_{s-1} for $s_0 + 1 \leqslant s \leqslant s_1$, and we can check exactly as in Example 2.6 that Assumption 2.4 holds with $\gamma_0 = \gamma_1 = d/2$.

Remark 2.9. Assumption 2.4 is stable by shifts that are both smooth and small, in the following sense: if Φ^ε satisfy Assumption 2.4, with associated parameters m, s_0, s_1, and γ_0, and if $(a^\varepsilon)_{\varepsilon\in(0,1)} \subset E_{s_1}$, with $\sup_{\varepsilon\in(0,1)} \varepsilon^{-\gamma_0}|a^\varepsilon|_{s_0+m} < \infty$, then $\tilde{\Phi}^\varepsilon := \Phi^\varepsilon(a^\varepsilon + \cdot)$ satisfies Assumption 2.4 with the same parameters.

2.3. Second assumption: tame inverse bounds

Our second and key assumption states that if u is small enough in some "low" norm, then $(\Phi^\varepsilon)'(u)$ has a right inverse, with a right-inverse bound (2.16) that is possibly stiff with respect to ε and may show a loss of derivatives:

Assumption 2.10. For some $r \geqslant 0$, $r' \geqslant 0$, there holds $s_0 + m \leqslant s_1 - \max(m, r)$, and for some $\gamma \geqslant 0$, $\kappa \geqslant 0$, for all s such that $s_0 + m \leqslant s \leqslant s_1 - \max(m, r)$, for all $u \in E_{s+r}$ such that

$$|u|_{s_0+m+r} \lesssim \varepsilon^\gamma, \tag{2.15}$$

the map $(\Phi^\varepsilon)'(u) : E_{s+m} \to F_s$ has a right inverse $\Psi^\varepsilon(u)$:

$$\left(\Phi^\varepsilon\right)'(u)\Psi^\varepsilon(u) = \mathrm{Id} : F_s \to F_s,$$

satisfying, for all $g \in F_{s+r'}$,

$$\left|\Psi^\varepsilon(u)g\right|_s \leqslant \mathbf{C}\varepsilon^{-\kappa}\left(\|g\|_{s_0+m+r'}|u|_{s+r} + \|g\|_{s+r'}\right), \tag{2.16}$$

where $\mathbf{C} = \mathbf{C}(\varepsilon, |u|_{s_0+m+r})$ satisfies

$$\sup\left\{\mathbf{C},\ \varepsilon \in (0, 1),\ |u|_{s_0+m+r} \lesssim \varepsilon^\gamma\right\} < \infty.$$

Remark 2.11 *(On stiffness of the right-inverse bound (2.16)).* The right-inverse bound (2.16) is stiff with respect to ε if $\kappa > 0$. The case $\kappa = 0$ corresponds to no loss in ε and is covered for instance by the result of Alvarez-Samaniego and Lannes [2].

Remark 2.12 *(On losses of derivatives in the right-inverse bound (2.16)).* The loss of derivatives is parameterized by r and r'. Estimate (2.16) states indeed that we can solve the linearized equation $(\Phi^\varepsilon)'(u)v = g$, with a bound for the solution $v = \Psi^\varepsilon(u)g$ that gives a control of the low norm $|v|_s$ in terms of high norms, $|u|_{s+r}$ and $\|g\|_{s+r'}$, of the background and source.

The case $r = r' = 0$ corresponds to no loss and can typically be handled by Picard iteration, as in the classical existence proof for quasilinear symmetric systems.

Estimate (2.16) in Assumption 2.10 is *tame* with respect to r and r'. This is essential: one may check that the proof of Theorem 2.19 (given in Section 3) collapses if it is not.

The distinction between r and r' is somewhat illusory, since we can always redefine F_s and m, so that $r' = 0$. We also note that the proof with $r' \neq 0$ is the same as with $r' = 0$.

A simple example of a family of maps satisfying Assumption 2.10 is given by systems of symmetric quasilinear equations in a semi-classical setting, as detailed in the following example. (This example is meant to test our theory in the favorable context of symmetric quasilinear systems, where a Lax iteration scheme is certainly preferable to a Nash–Moser scheme.)

Example 2.13. Consider the functional spaces and the family of maps introduced in Example 2.8. Again, $T > 0$ and $d/2 < s_0 < 1 + d/2 < s_1$, are given, with s_1 measuring the regularity of $(f_1^\varepsilon, f_2^\varepsilon)$, and $s_0 + 2 \leqslant s_1$. Then, as discussed in Example 2.8, Assumption 2.4 holds with $\gamma_0 = \gamma_1 = d/2$ and $m = 1$.

Given s such that $s_0 + 1 \leqslant s \leqslant s_1 - 1$, and $u \in E_{s+1}$, $g = (g_1, g_2) \in F_s$, consider the equation $(\Phi^\varepsilon)'(u)v = g$, corresponding to the linearized initial-value problem

$$\begin{cases} \varepsilon\partial_t v + \sum_{1\leqslant j\leqslant d} A_j(u)\varepsilon\partial_{x_j} v = g_1 - \sum_{1\leqslant j\leqslant d} A_j'(u)v\varepsilon\partial_{x_j} u, \\ v_{|t=0} = g_2. \end{cases} \tag{2.17}$$

Assume that the maps A_j take values in the symmetric matrices. Then, by the classical linear hyperbolic theory, there exists a unique $v \in E_s$ solution of (2.17). We now show that v satisfies an estimate of the form (2.16).

The classical commutator estimate gives, for $0 \leqslant |\alpha| \leqslant s$ and $w \in H^s$,

$$\Re e\big((\varepsilon\partial_x)^\alpha A_j(u)\varepsilon\partial_{x_j} w, (\varepsilon\partial_x)^\alpha w\big)_{L^2} \leqslant C\big(|u|_{L^\infty}\big)\big(|\varepsilon\partial_x u|_{L^\infty}\|w\|_{H_\varepsilon^s}^2 + \|u\|_{H_\varepsilon^s}|w|_{L^\infty}\|w\|_{H_\varepsilon^s}\big), \tag{2.18}$$

where the weighted Sobolev norm $\|\cdot\|_{H_\varepsilon^s}$ is defined in (2.8). Besides, by Remark 2.2,

$$\big\|A_j'(u)w\varepsilon\partial_{x_j} u\big\|_{H_\varepsilon^s} \leqslant C\big(|u|_{L^\infty}\big)\big(|\varepsilon\partial_{x_j} u|_{L^\infty}\big(\|w\|_{H_\varepsilon^s} + \|u\|_{H_\varepsilon^s}|w|_{L^\infty}\big) + \|u\|_{H_\varepsilon^{s+1}}|w|_{L^\infty}\big). \tag{2.19}$$

If we now use (2.12) with $N = 0, 1$, together with estimates (2.18) and (2.19), we find that the solution v to (2.17) satisfies

$$\begin{aligned} \big\|v(t)\big\|_{H_\varepsilon^s}^2 \leqslant \|g_2\|_{H_\varepsilon^s}^2 &+ \int_0^t \varepsilon^{-1}\|g_1\|_{H_\varepsilon^s}\|v\|_{H_\varepsilon^s}\,dt' + \int_0^t C\big(\varepsilon^{-1-d/2}\|u\|_{H_\varepsilon^{s_0+1}}\big)\|v\|_{H_\varepsilon^s}^2\,dt' \\ &+ \int_0^t C\big(\varepsilon^{-d/2}\|u\|_{H_\varepsilon^{s_0+1}}\big)\varepsilon^{-1-d/2}\|u\|_{H_\varepsilon^{s+1}}\|v\|_{H_\varepsilon^{s_0}}\|v\|_{H_\varepsilon^s}\,dt'. \end{aligned} \tag{2.20}$$

We now restrict to a background $u \in H^{s+1}$ satisfying

$$|u|_{s_0+2} \lesssim \varepsilon^{1+d/2}. \tag{2.21}$$

Then, by Gronwall's Lemma, estimate (2.20) used with $s = s_0$, implies the bound

$$\max_{[0,T]}\|v\|_{H_\varepsilon^{s_0}} \lesssim \|g_2\|_{H_\varepsilon^{s_0}} + \varepsilon^{-1}\max_{[0,T]}\|g_1\|_{H_\varepsilon^{s_0}}, \tag{2.22}$$

which we use back in (2.20) to obtain

$$\begin{aligned} \max_{[0,T]}\|v\|_{H_\varepsilon^s} \lesssim{}& \|g_2\|_{H_\varepsilon^s} + \varepsilon^{-1}\max_{[0,T]}\|g_1\|_{H_\varepsilon^s} \\ &+ \varepsilon^{-1-d/2}\Big(\|g_2\|_{H_\varepsilon^{s_0}} + \varepsilon^{-1}\max_{[0,T]}\|g_1\|_{H_\varepsilon^{s_0}}\Big)\max_{[0,T]}\|u\|_{H_\varepsilon^{s+1}}. \end{aligned} \tag{2.23}$$

By using Eq. (2.17)(i) directly, we find the bound

$$\|\varepsilon\partial_t v\|_{H_\varepsilon^{s-1}} \lesssim C\big(|u|_{L^\infty}, |\varepsilon\partial_x u|_{L^\infty}\big)\big(\|v\|_{H_\varepsilon^s} + \|u\|_{H_\varepsilon^s}\big(|v|_{L^\infty} + |\varepsilon\partial_x v|_{L^\infty}\big)\big). \tag{2.24}$$

Note in (2.24) the occurrence of $|\varepsilon\partial_x v|_{L^\infty}$, which cannot be controlled by (2.22). This forces us to go back to (2.20) with $s = s_0 + 1$. At this point we make full use of (2.21), while a bound for $|u|_{s_0+1}$ sufficed in order to estimate $\|v\|_{H_\varepsilon^s}$, and obtain

$$\max_{[0,T]}\|v\|_{H_\varepsilon^{s_0+1}} \lesssim \|g_2\|_{H_\varepsilon^{s_0+1}} + \varepsilon^{-1}\max_{[0,T]}\|g_1\|_{H_\varepsilon^{s_0+1}}. \tag{2.25}$$

Combining (2.23), (2.24) and (2.25), we obtain that, if u satisfies (2.21), then there holds

$$|v|_s \lesssim \varepsilon^{-2-d/2}\|g\|_{s_0+1}|u|_{s+1} + \varepsilon^{-1}\|g\|_s.$$

We can conclude that if for all j, A_j is symmetric, then the family of maps and functional spaces defined in Example 2.8 satisfy Assumption 2.10 with $\gamma = 1 + d/2$, $\kappa = 2 + d/2$, $r = 1$ and $r' = 0$.

Remark 2.14. If Φ^ε satisfies Assumption 2.10 with parameters γ, κ, r, r', given a family $(a^\varepsilon)_{\varepsilon\in(0,1)} \subset E_{s_1}$, if $\sup_{\varepsilon\in(0,1)} \varepsilon^{-\gamma}|a^\varepsilon|_{s_0+m+r} < \infty$, then $\tilde{\Phi}^\varepsilon := \Phi^\varepsilon(a^\varepsilon + \cdot)$ satisfies Assumption 2.10, with the same parameters as Φ^ε.

2.4. Third assumption: existence of an approximate solution

We consider a family of maps $\Phi^\varepsilon : E_s \to F_{s-m}$ satisfying Assumptions 2.4 and 2.10, with associated parameters m, s_0, s_1, and γ_0, γ_1, γ, r, r', and κ.

Assumption 2.15. Let k be such that

$$\max(\kappa + \gamma_0, 2\kappa + \gamma_1, \kappa + \gamma) < k. \tag{2.26}$$

For some positive function $\bar{p} = \bar{p}(m, r, r', \gamma_0, \gamma_1, \gamma, \kappa, k) \geqslant 2m + \max(r, r')$ specified in Remark 2.23, there holds

$$s_0 + m + \max(r, r') + \bar{p} < s_1, \tag{2.27}$$

and, for some s satisfying

$$s_0 + m + \max(r, r') \leqslant s < s_1 - \bar{p}, \tag{2.28}$$

there holds

$$\|\Phi^\varepsilon(0)\|_s \lesssim \varepsilon^k. \tag{2.29}$$

We first comment on (2.27):

Remark 2.16. In Example 2.13, the index s_1 measures the regularity of the source f_1^ε and the initial datum f_2^ε; in this view inequality (2.27) should be understood as a regularity requirement on the data. In particular, as discussed in Remark 2.23, as k approaches from above the limiting value $\max(\kappa + \gamma_0, 2\kappa + \gamma_1, \kappa + \gamma)$, the parameter $\bar{p}$ blows up to $+\infty$, meaning that we require the data to be infinitely regular in this limit.

In our examples, inequality (2.29) reflects the existence of a family of approximate solutions to $\Phi^\varepsilon = 0$:

Remark 2.17. Consider a family of maps $\Phi^\varepsilon \in C^2(E_s, F_{s-m})$ satisfying Assumptions 2.4 and 2.10, and an associated family of approximate solutions $u_a^\varepsilon \in E_{s_1}$ to the equations $\Phi^\varepsilon = 0$, in the sense that $\|\Phi^\varepsilon(u_a^\varepsilon)\|_s \lesssim \varepsilon^k$, with k and s satisfying (2.26)–(2.28). Then, the maps $\tilde{\Phi}^\varepsilon := \Phi^\varepsilon(u_a^\varepsilon + \cdot)$ satisfy Assumption 2.15.

For quasilinear systems, small initial data give crude examples of approximate solutions:

Example 2.18. Consider the initial-value problem

$$\partial_t u + \sum_{1\leqslant j\leqslant d} A_j(u)\partial_{x_j} u = 0, \quad u_{|t=0} = \varepsilon^\sigma a(x), \tag{2.30}$$

and the associated family of maps Φ^ε defined in (2.14), with $(f_1^\varepsilon, f_2^\varepsilon) \equiv (0, \varepsilon^\sigma a)$. Assume that $a \in H^{s_1}$, with s_1 satisfying (2.27). As described in Examples 2.8 and 2.13, if A_j is symmetric for all j, then Assumptions 2.4 and 2.10 hold for Φ^ε in the functional spaces E_s, F_s introduced in Example 2.8, for any $T > 0$, for s satisfying (2.28).

If the maps $u \to A_j(u)$ satisfy the bounds $|A_j(u)| \lesssim |u|^\ell$, for $\ell \geqslant 0$, in particular if they are ℓ-homogeneous, then there holds $\|\Phi^\varepsilon(\varepsilon^\sigma a)\|_{s_1-1} \lesssim \varepsilon^{\sigma(1+\ell)+1}$. Using Remark 2.17 above, and the specific values of γ_0, γ_1, γ and κ given in Examples 2.8 and 2.13, this implies that if $\sigma(1+\ell) > 3(1+d/2)$, then Assumption 2.15 is satisfied by $\tilde{\Phi}^\varepsilon := \Phi^\varepsilon(\varepsilon^\sigma a + \cdot)$.

2.5. Results

Our main result gives existence in E_{s+m}, where s satisfies (2.28), of a solution to Eq. (2.1).

Theorem 2.19 *(Existence). Under Assumptions* 2.4, 2.10 *and* 2.15*, for ε small enough, there exists a real sequence θ_j^ε, satisfying $\theta_j^\varepsilon \to +\infty$ as $j \to +\infty$ and ε is held fixed, such that the sequence*

$$u_0^\varepsilon := 0, \qquad u_{j+1}^\varepsilon := u_j^\varepsilon + S_{\theta_j^\varepsilon} v_j^\varepsilon, \qquad v_j^\varepsilon := -\Psi^\varepsilon\big(u_j^\varepsilon\big)\Phi^\varepsilon\big(u_j^\varepsilon\big), \tag{2.31}$$

is well defined and converges, as $j \to \infty$ and ε is held fixed, to a solution u^ε of (2.1) *in $s + m$ norm, with s as in* (2.28)*, which satisfies the bound*

$$\big|u^\varepsilon\big|_s \lesssim \varepsilon^{k-\kappa}. \tag{2.32}$$

In applications (Sections 4 and 5), we apply Theorem 2.19 to a map $\Phi^\varepsilon(u_a^\varepsilon + \cdot)$, with the notation of Remark 2.17, so that in practice Theorem 2.19 is not a result about small solutions: the smallness condition (2.32) bears on the perturbation variable u, the full solution to $\Phi^\varepsilon = 0$ being $u_a^\varepsilon + u$. The smallness condition (2.29) is an accuracy condition bearing on the approximate solution u_a^ε; we show in Remark 2.22 that this condition is sharp in the usual implicit function theorem setting without losses in derivatives.

We supplement the above existence result by the following local uniqueness theorem. Contrary to Theorem 2.19, Theorem 2.20 does not rely on a Nash–Moser iterative scheme.

Theorem 2.20 *(Local uniqueness). Under Assumptions* 2.4, 2.10 *and* 2.15*, for ε small enough, if $(\Phi^\varepsilon)'$ is invertible, i.e., Ψ^ε is also a left inverse, then the solution described in Theorem* 2.19 *is unique in a ball of radius $o(\varepsilon^{\max(\kappa+\gamma_1,\gamma_0,\gamma)})$ in $s_0 + 2m + r'$ norm. More generally, if $\hat u^\varepsilon$ is a second solution within this ball, then $(\hat u^\varepsilon - u^\varepsilon)$ is approximately tangent to* $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$*, in the sense that its distance in s_0 norm from* $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$ *is $o(|\hat u^\varepsilon - u^\varepsilon|_{s_0})$. In particular, if* $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$ *is finite-dimensional, then u is the unique solution in the ball satisfying the additional "phase condition"*

$$\Pi_{u^\varepsilon}\big(\hat u^\varepsilon - u^\varepsilon\big) = 0, \tag{2.33}$$

where Π_{u^ε} is any uniformly bounded projection onto $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$. (*In a Hilbert space, any orthogonal projection onto* $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$.)

In a non-Hilbertian context, the existence of such a projection Π_{u^ε} is discussed in Remark 2.25 below.

2.6. Remarks

We first remark that the proofs use only part of the information contained in (2.16) and (2.11):

Remark 2.21. An examination of the proofs of Theorems 2.19 and 2.20 reveals that for existence we require estimate (2.16) only for f in the image of Φ^ε or $(\Phi^\varepsilon)''$, since Ψ^ε is estimated only in composition with one or the other of these operators, while for uniqueness we need only the estimate for $\Psi^\varepsilon(u)(\Phi^\varepsilon)''(u)$ that would result by composing estimates (2.16) and (2.11).

Next we remark that the approximation rate is sharp by comparison with the standard Newton scheme:

Remark 2.22. For $\gamma_0, \gamma \leqslant \kappa + \gamma_1$, corresponding to a critical value $k_c = 2\kappa + \gamma_1$ in (2.26), Theorem 2.19 states that a loss of $\varepsilon^{-\kappa}$ in the linear estimates means that, with the notation of Remark 2.17, $\|\Phi^\varepsilon(u_a^\varepsilon)\|_s \lesssim \varepsilon^{2\kappa+\gamma_1+\eta}$, any $\eta > 0$, is the accuracy needed on the approximate solution.

This condition is sharp even for convergence of a standard Newton iteration scheme

$$u_{n+1}^\varepsilon := u_n^\varepsilon - \Psi^\varepsilon\big(u_n^\varepsilon\big)\Phi^\varepsilon\big(u_n^\varepsilon\big), \qquad u_0^\varepsilon := u_a^\varepsilon, \quad \big|u_a^\varepsilon\big|_{s+m} \lesssim \varepsilon^{\gamma_1},$$

for problems with no loss of derivatives ($r = r' = 0$), corresponding by the computation

$$\begin{aligned}\|\Phi^\varepsilon(u_1^\varepsilon)\|_s &= \left\| \int_0^1 (1-t)(\Phi^\varepsilon)''\big(u_0^\varepsilon + t(u_1^\varepsilon - u_0^\varepsilon)\big)\cdot\big(u_1^\varepsilon - u_0^\varepsilon, u_1^\varepsilon - u_0^\varepsilon\big)\right\|_s \\ &\lesssim \big(\varepsilon^{-2\gamma_1}\big(\|u_0^\varepsilon\|_s + \|u_1^\varepsilon - u_0^\varepsilon\|_s\big) + \varepsilon^{-\gamma_1}\big)\|u_1^\varepsilon - u_0^\varepsilon\|_s^2 \\ &\lesssim \varepsilon^{-\gamma_1}\|u_1^\varepsilon - u_0^\varepsilon\|_s^2 \\ &\lesssim \varepsilon^{-\gamma_1}\big|\Psi^\varepsilon(u_0^\varepsilon)\Phi^\varepsilon(u_0^\varepsilon)\big|_s^2 \\ &\lesssim \varepsilon^{-(2\kappa+\gamma_1)}\|\Phi^\varepsilon(u_0^\varepsilon)\|_s^2 \\ &\lesssim \varepsilon^{\eta}\|\Phi^\varepsilon(u_0^\varepsilon)\|_s\end{aligned}$$

in the case $m = 0$ to the condition that error $(\|\Phi^\varepsilon(u_n^\varepsilon)\|_s)_{n\in\mathbb{N}}$ decreases at the first step.

We make precise the parameter $\bar{p}$ that appears in Assumption 2.15:

Remark 2.23. From (3.23), we find that $\bar{p}$ is

$$\bar{p} = m + \inf_{N>N_0} \frac{(N+1)(m + \max(r, r') + M)}{1 - \frac{\kappa}{k} - M}, \tag{2.34}$$

with

$$N_0 := \frac{\kappa + km'}{k - (2\kappa + \gamma_1)}, \qquad M := \max\left(\frac{\gamma_0}{k}, \frac{\gamma}{k}, \frac{1}{2}\left(1 + \frac{\gamma_1}{k} + \frac{m'}{N} + \frac{\kappa}{kN}\right)\right), \qquad m' := \max\big(m + r', r\big).$$

We observe the following asymptotic behavior as k approaches from above the critical value $k_c := \max(\kappa + \gamma_0, 2\kappa + \gamma_1, \kappa + \gamma)$ given in (2.26):

- If $k_c = 2\kappa + \gamma_1$, then $\bar{p}$ blows up like $(k - k_c)^{-2}$ as $k \downarrow k_c$.
- If $k_c = \kappa + \gamma_0$, or $k_c = \kappa + \gamma$, then $\bar{p}$ blows up like $(k - k_c)^{-1}$ as $k \downarrow k_c$.

The phase condition (2.33) can in some situations be made explicit:

Remark 2.24. Let Π_u be a bounded projection onto $\ker(\Phi^\varepsilon)'(u)$, as in (2.33). If the map $(u, v) \to \Pi_u v$ is continuous in $E_{s_0} \times E_{s_0}$, uniformly in ε, then the implicit phase condition (2.33) can be replaced by the explicit

$$\Pi_0\big(\hat{u}^\varepsilon - u^\varepsilon\big) = 0.$$

See [21, Section 2], for related discussions of uniqueness up to phase conditions.

We discuss the existence of the projection mentioned in Theorem 2.20:

Remark 2.25. We first remark that if $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$ is finite-dimensional, then a bounded projection exists by the Hahn–Banach Theorem; see, e.g., [19].

In the infinite-dimensional case, we note that a projection Π onto a subspace S of a Banach space is bounded if and only if the distance from $s \in S$ to $\operatorname{Ker}\Pi$ is greater than or equal to $|s|/C$ for some uniform $C > 0$. (Indeed, $|s| = |\Pi(s - t)| \leqslant C|s - t|$ for all $s \in S$, $t \in \operatorname{Ker}\Pi$ is equivalent to the statement that Π is bounded, since $s - t$ runs over the entire Banach space as s and t are varied.)

This implies that if there is an isometry between spaces F_s^ε and a common set of spaces F_s^0, and if that $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$, considered (under mapping by this isometry) as a subset of F_s^0 is finite-dimensional, with a limit as $\varepsilon \to 0$, then, there exist a family of projections Π^ε onto $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$ that are uniformly bounded with respect to ε in each F_s^ε, for ε sufficiently small.

Indeed, by the above note (second paragraph of the present remark), there exists a bounded projection Π^0 onto the limit as $\varepsilon \to 0$ of $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$. Denote by $\tilde{F} := \operatorname{Ker}\Pi^0$ the associated complementary subspace. Defining Π^ε to be the projection along $\tilde{F}$ onto $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$, we find by the Hahn–Banach Theorem, compactness of the intersection of the unit ball with $\operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$, and continuity, that Π^ε is bounded for ε sufficiently small.

We finally describe a somewhat artificial application for orientation:

Example 2.26. Consider the quasilinear initial-value problem (2.30), with associated maps Φ^ε. The datum $\varepsilon^\sigma a$ is assumed to belong to H^{s_1}, with s_1 as in (2.27).

By Examples 2.8 and 2.13, if the matrices A_j are symmetric, then Assumptions 2.4 and 2.10 hold in E_s, F_s, for any $T > 0$, for s satisfying (2.28).

By Remarks 2.9 and 2.14, if the initial datum is small enough: $\sigma > d/2$, then Assumptions 2.4 and 2.10 still hold, with the same parameters, for the translated maps $\tilde{\Phi}^\varepsilon := \Phi^\varepsilon(\varepsilon^\sigma a + \cdot)$.

By Example 2.18, if $\sigma > \sigma_c := \max(d/2, 3(1+\ell)^{-1}(1+d/2))$, then the translated maps $\tilde{\Phi}^\varepsilon$ satisfy in addition Assumption 2.15.

We can conclude that Theorem 2.19 yields existence in $C^0([0,T], H^s(\mathbb{R}^d))$ of a solution to (2.30). If $\ell = 1$, the smallness condition for the initial datum if $\sigma > 3/2 + 3d/4$.

We thus partly recovered a classical small-amplitude, existence result of the quasilinear hyperbolic theory. Note that, as mentioned in Remark 2.16, if $\sigma - \sigma_c$ is small, then $\bar{p}$ is large, hence s, satisfying (2.28), is much smaller than s_1, meaning that the solution is much less regular than the datum.

For quasilinear symmetric systems, the Lax iteration scheme gives an existence result with no smallness assumption on the datum, and with the sharp regularity criterion $s > 1 + d/2$. In this view, it is much better suited for the resolution of quasilinear symmetric systems than the Nash–Moser scheme described above.

3. Proofs of Theorems 2.19 and 2.20

We write Φ for Φ^ε, θ_j for θ_j^ε, etc. Let θ_0 be such that

$$\theta_0^{-\alpha} \leqslant \varepsilon^{\max(\gamma_0,\gamma_1,\gamma)}, \tag{3.1}$$

for some $\alpha > 0$ to be chosen later. Introduce the family of inequalities $\mathcal{C}_1(j)$, for $j \in \mathbb{N}$,

$$\mathcal{C}_1(j; q, \alpha): \quad |v_j|_{s+q} \lesssim \theta_j^{-\alpha}$$

depending on α and some $q \geqslant m$ and s to be chosen later. We assume that Assumptions 2.4, 2.10, and 2.15 hold, and start by proving three lemmas.

Lemma 3.1. *Assume that $s_0 < s \leqslant s_1 - q$, and*

- *the sequence u_j is well defined,*
- $\lim_{j\to+\infty} \|\Phi(u_j)\|_s = 0$,
- *condition $\mathcal{C}_1(j; q, \alpha)$ holds for all j,*
- *the series $\theta_j^{-\alpha}$ is convergent, with*

$$\sum_{j=0}^{+\infty} \theta_j^{-\alpha} \lesssim \theta_0^{-\alpha}. \tag{3.2}$$

Then u_j converges, in $s + q$ norm, to a solution of (2.1) *which satisfies*

$$|u|_{s+q} \lesssim \theta_0^{-\alpha}. \tag{3.3}$$

Proof. If $\mathcal{C}_1(j)$ holds for all j, then the sequence u_j converges, in $s + q$ norm, to $u \in E_{s+q}$, and we have the estimate

$$|u_j|_{s+q} \lesssim \sum_{j=0}^{j-1} \theta_j^{-\alpha}, \tag{3.4}$$

which implies (3.3). There holds

$$\big\|\Phi(u)\big\|_s \leqslant \big\|\Phi(u_j)\big\|_s + \left\| \int_0^1 \Phi'\big(u_j + t(u-u_j)\big)\cdot(u-u_j)\,dt \right\|_s, \tag{3.5}$$

and the first term in the upper bound tends to 0 as $j \to \infty$. We note that, by (3.2), (3.3), (3.4), and (3.1), there holds $|u_j|_{s+m} + |u|_{s+m} \lesssim \varepsilon^{\gamma_0}$. Hence, by the tame direct bound (2.10) in Assumption 2.4,

$$\left\| \int_0^1 \Phi'\big(u_j + t(u-u_j)\big)\cdot(u-u_j)\,dt \right\|_s \lesssim |u-u_j|_{s+m}.$$

The upper bound tends to 0 as $j \to +\infty$. With (3.5), this implies that u solves (2.1). □

Let p be such that

$$q + \max\big(r, r'+m\big) \leqslant p, \qquad s_0 + m + \max\big(r, r'\big) + p \leqslant s_1. \tag{3.6}$$

Introduce the family of inequalities $\mathcal{C}_2(j)$, for $j \in \mathbb{N}$ and $N \geqslant 0$:

$$\mathcal{C}_2(j; q, \alpha, p, N): \quad \begin{cases} |u_j|_{s+q} \lesssim \theta_0^{-\alpha}, \\ \big\|\Phi(u_j)\big\|_s \lesssim \theta_j^{-1}, \\ |u_j|_{s+p} \lesssim \theta_j^N. \end{cases}$$

Lemma 3.2. *Assume that $s_0 + m + \max(r, r') \leqslant s \leqslant s_1 - p$, and*

- *for all $j' \leqslant j$, $u_{j'}$ is well defined and condition $\mathcal{C}_1(j')$ holds,*
- *there holds*

$$\sum_{j'=0}^{j} \theta_j^{-\alpha} \lesssim \theta_0^{-\alpha}, \tag{3.7}$$

- *condition $\mathcal{C}_2(j)$ holds, with parameters satisfying*

$$\theta_j^{m-q-\alpha} + \varepsilon^{-\gamma_0}\theta_j^{-2\alpha} \leqslant \theta_{j+1}^{-1}, \tag{3.8}$$

$$\varepsilon^{-\kappa}\theta_j^{\max(m+r',r)}\theta_j^N \leqslant \theta_{j+1}^N. \tag{3.9}$$

Then v_{j+1} is well defined in E_{s+q} and $\mathcal{C}_2(j+1)$ holds.

Proof. If conditions $\mathcal{C}_1(j')$ hold for all $j' \leqslant j$ and if (3.7) holds, then

$$|u_{j+1}|_{s+q} \lesssim \theta_0^{-\alpha}. \tag{3.10}$$

Bound (3.10) is $\mathcal{C}_2(j+1)$(i). Besides, (3.10) and (3.1) imply that u_{j+1} also satisfies (2.15), so that, by $\mathcal{C}_2(j+1)$(iii), the first bound in (3.6) and the tame inverse bound (2.16) in Assumption 2.10, v_{j+1} is defined in E_{s+q}.

To prove $\mathcal{C}_2(j+1)$(ii), we use the fact that (2.31) is almost a Newton's scheme:

$$\big\|\Phi(u_{j+1})\big\|_s \leqslant E_1 + E_2,$$

where E_1 is the error due to the regularization:

$$E_1 = \big\|\Phi'(u_j)\cdot(S_{\theta_j}v_j - v_j)\big\|_s,$$

and E_2 is the error due to the scheme:

$$E_2 = \left\| \int_0^1 (1-t)\Phi''(u_j + tS_{\theta_j}v_j)\cdot(S_{\theta_j}v_j, S_{\theta_j}v_j)\,dt \right\|_s.$$

Conditions $(\mathcal{C}_1(j'))_{1\leqslant j'\leqslant j-1}$, together with (3.1) and (3.7), imply $|u_j|_{s+m} \lesssim \varepsilon^{\gamma_0}$. Together with the tame direct bound (2.10) in Assumption 2.4, this gives

$$E_1 \lesssim |S_{\theta_j} v_j - v_j|_{s+m},$$

and with (2.6) and $\mathcal{C}_1(j)$,

$$E_1 \lesssim \theta_j^{m-q-\alpha}. \tag{3.11}$$

By $\mathcal{C}_1(j)$, (2.7), and (3.1), there holds $|S_{\theta_j} v_j|_{s+m} \lesssim \varepsilon^{\gamma_0}$. With the tame direct bound (2.11) in Assumption 2.4, this gives

$$E_2 \lesssim \varepsilon^{-2\gamma_1} |S_{\theta_j} v_j|^2_{s_0+m}\big(|u_j|_{s+m} + |S_{\theta_j} v_j|_{s+m}\big) + \varepsilon^{-\gamma_1} |S_{\theta_j} v_j|_{s+m} |S_{\theta_j} v_j|_{s_0+m},$$

and, bounding s_0+m norms by $s+m$ norms, and using $|u_j|_{s+m} + |v_j|_{s+m} \lesssim \theta_j^{-\alpha}$, a consequence of $(\mathcal{C}_1(j'))_{j'\leqslant j}$, and (3.7), we obtain

$$E_2 \lesssim \varepsilon^{-\gamma_1} \theta_j^{-2\alpha}. \tag{3.12}$$

Bounds (3.11), (3.12) and (3.8) imply $\mathcal{C}_2(j+1)$(ii). Finally, to prove $\mathcal{C}_2(j+1)$(iii), we remark that, by (2.7),

$$\begin{aligned} |u_{j+1}|_{s+p} &\leqslant |u_j|_{s+p} + |S_{\theta_j} v_j|_{s+p} \\ &\lesssim |u_j|_{s+p} + \theta_j^{\max(m+r',r)} |v_j|_{s+p-\max(m+r',r)}. \end{aligned} \tag{3.13}$$

Under (3.6), the tame direct bound (2.9) in Assumption 2.4 and the tame inverse bound (2.16) in Assumption 2.10 imply

$$\begin{aligned} |v_j|_{s+p-\max(m+r',r)} &\lesssim \varepsilon^{-\kappa}\big(|u_j|_{s+p} \|\Phi(u_j)\|_{s_0+m+r'} + \|\Phi(u_j)\|_{s+p-\max(m+r',r)+r'}\big) \\ &\lesssim \varepsilon^{-\kappa}\big(1 + |u_j|_{s+p}\big)\big(1 + \|\Phi(u_j)\|_{s_0+m+r'}\big) \\ &\lesssim \varepsilon^{-\kappa} \theta_j^N. \end{aligned} \tag{3.14}$$

Bounds (3.13), (3.14) and (3.9) imply $\mathcal{C}_2(j+1)$(iii). □

Lemma 3.3. *Let j be such that*

$$\varepsilon^{-\kappa} \theta_j^{-\beta} \leqslant \theta_j^{-\alpha} \tag{3.15}$$

where

$$\beta := \big(p' + \max(r, r')\big)^{-1}\big((p'-q) - N\big(q + \max(r, r')\big)\big), \qquad p' := p - \max\big(m + r', r\big).$$

Then condition $\mathcal{C}_2(j)$ implies $\mathcal{C}_1(j)$.

Proof. Bound $\mathcal{C}_2(j)$(i), together with (3.1), implies that u_j satisfies (2.15). Then, bound $\mathcal{C}_2(j)$(iii) implies that v_j is well defined in $E_{s+p-\max(m+r',r)}$, and we can check, exactly as in the proof of (3.14) in Lemma 3.2, that the bound

$$|v_j|_{s+p-\max(m+r',r)} \lesssim \varepsilon^{-\kappa} \theta_j^N \tag{3.16}$$

holds. Besides, by the tame inverse bound (2.16) in Assumption 2.10,

$$\begin{aligned} |v_j|_{s-\max(r,r')} &\lesssim \varepsilon^{-\kappa}\big(|u_j|_s \|\Phi(u_j)\|_{s_0+m} + \|\Phi(u_j)\|_s\big) \\ &\lesssim \varepsilon^{-\kappa}\big(1 + \theta_0^{-\alpha}\big)\|\Phi(u_j)\|_s \\ &\lesssim \varepsilon^{-\kappa} \theta_j^{-1}. \end{aligned} \tag{3.17}$$

Finally, bounds (3.16), (3.17) and the interpolation property (2.5) imply

$$\begin{aligned} |v_j|_{s+q} &\lesssim |v_j|_{s-r''}^{\frac{p'-q}{p'+r''}} |v_j|_{s+p'}^{\frac{q+r''}{p'+r''}} \\ &\lesssim \varepsilon^{-1} \theta_j^{-\beta}, \end{aligned} \tag{3.18}$$

where $r'' = \max(r, r')$, and the lemma follows, with (3.15). □

End of proof of Theorem 2.19, existence. Let $q = m + \alpha$. Define

$$\theta_0 := \varepsilon^{-k}, \qquad \theta_{j+1} := \theta_j^{\zeta}, \quad j \geqslant 0, \tag{3.19}$$

for some $\zeta > 1$ to be chosen below. Then (3.1) is satisfied if

$$\max\left(\frac{\gamma_0}{\alpha}, \frac{\gamma}{\alpha}\right) \leqslant k, \tag{3.20}$$

and (3.7) is satisfied.

By (3.19) and Assumption 2.15, condition $\mathcal{C}_2(0)$ is satisfied. Condition (3.15) is satisfied as soon as

$$\frac{\kappa}{\beta - \alpha} \leqslant k. \tag{3.21}$$

By Lemma 3.3, (3.21) also implies that condition $\mathcal{C}_1(0)$ is satisfied. With definition (3.19), conditions (3.8) and (3.9) translate respectively into

$$\frac{\gamma_1}{k} < 2\alpha - \zeta, \quad \text{and} \quad \frac{\kappa}{N\zeta - N - \max(m + r', r)} \leqslant k. \tag{3.22}$$

Suppose now that for all $0 \leqslant j' \leqslant j$, $u_{j'}$ is well defined and $\mathcal{C}_1(j')$ and $\mathcal{C}_2(j')$ hold. Then by Lemma 3.2, condition $\mathcal{C}_1(j+1)$ is satisfied if (3.22) holds, and by Lemma 3.3, condition $\mathcal{C}_2(j+1)$ is satisfied if (3.21) holds.

We just proved that, under (3.20), (3.21) and (3.22), conditions $\mathcal{C}_1(j)$ and $\mathcal{C}_2(j)$ hold for all j.

Conditions (3.20), (3.21) and (3.22) are equivalent to

$$M \leqslant \frac{1}{2}\left(\zeta + \frac{\gamma_1}{k}\right) < \alpha \leqslant \left(1 + \frac{1}{p_0}\right)^{-1}\left(1 - \frac{\kappa}{k} - \frac{1}{p_0}\big(m + r''\big)\right) \tag{3.23}$$

with

$$M := \max\left(\frac{\gamma_0}{k}, \frac{\gamma}{k}, \frac{1}{2}\left(1 + \frac{\gamma_1}{k} + \frac{m'}{N} + \frac{\kappa}{kN}\right)\right), \qquad p_0 := \frac{p - m' + \max(r, r')}{N + 1},$$

and $m' := \max(m + r', r)$. Under (2.26), if N and p are large enough, namely

$$\frac{\kappa + km'}{k - (2\kappa + \gamma_1)} =: N_0 < N, \qquad \bar{p} < p, \tag{3.24}$$

where $\bar{p}$ is specified in Remark 2.23, then we can find α and ζ satisfying (3.23).

Let now α, ζ, N, and p be such that (3.23) holds. By (3.19) and $\zeta > 1$, the series $\theta_j^{-\alpha}$ is convergent and satisfies (3.2). Besides, conditions $\mathcal{C}_2(j)$ imply $\|\Phi(u_j)\|_s \to 0$. We can thus apply Lemma 3.1: the sequence u_j converges to a solution u of (2.1) in $s + q$ norm, satisfying (3.3). Besides, as (3.17) holds for all j,

$$|u_j|_s \lesssim \varepsilon^{-\kappa} \sum_{j'=0}^{j} \theta_{j'}^{-\beta} \lesssim \varepsilon^{-\kappa}\theta_0^{-1},$$

hence (2.32). □

Proof of Theorem 2.20, local uniqueness. Suppressing ε, let $\hat{u}$ be a second solution in $E_{s_0+2m+r'}$ of $\Phi(u) = 0$, lying within $o(\varepsilon^{\max(\kappa+\gamma_1,\gamma_0,\gamma)})$ of u (and thus of 0). Then, Taylor expanding, and using Assumption 2.4, we have

$$0 = \Phi(\hat{u}) - \Phi(u) = \Phi'(u)(\hat{u} - u) + B(u, \hat{u}),$$

where

$$B(u, \hat{u}) := \int_0^1 (1 - t)\Phi''\big(tu + (1 - t)\hat{u}\big) \cdot (\hat{u} - u, \hat{u} - u)\, dt.$$

Applying $\Psi(u)$ and using Assumption 2.10, we thus have

$$(\hat{u} - u) + \Psi(u)B(u, \hat{u}) \in \operatorname{Ker}\Phi'(u),$$

where

$$\begin{aligned}\big|\Psi(u)B(u,\hat u)\big|_{s_0} &\lesssim \varepsilon^{-\kappa-\gamma_1}\big(1+\varepsilon^{-\gamma_1}\big(|\hat u|_{s_0+m}+|u|_{s_0+m}\big)\big)|\hat u-u|^2_{s_0+2m+r'}\\ &= o\big(|\hat u-u|_{s_0+2m+r'}\big).\end{aligned}$$

This verifies tangency. Finally, from $\hat u - u + o(|\hat u - u|) \in \operatorname{Ker}(\Phi^\varepsilon)'(u^\varepsilon)$, we have

$$\hat u - u + o\big(|\hat u - u|\big) = \Pi_{u^\varepsilon}\big(\hat u^\varepsilon - u^\varepsilon\big) + o\big(|\Pi_{u^\varepsilon}||\hat u - u|\big),$$

which, with (2.33) and the assumed uniform boundedness of $|\Pi_{u^\varepsilon}|$, gives

$$\hat u - u = o\big(|\hat u - u|\big) + o\big(|\Pi_{u^\varepsilon}||\hat u - u|\big) = o\big(|\hat u - u|\big),$$

and thus $\hat u - u = 0$. □

4. Application: systems of quasilinear Schrödinger equations

Consider systems of quasilinear Schrödinger equations in $v = (v_1, \dots, v_n) \in \mathbb{C}^n$,

$$\partial_t v_j + i\lambda_j \Delta_x v_j = \sum_{1\leqslant j'\leqslant n} b_{jj'}(v,\partial_x)v_{j'} + c_{jj'}(v,\partial_x)\bar v_{j'}, \quad 1\leqslant j\leqslant n,\ t\geqslant 0, x\in\mathbb{R}^d, \tag{4.1}$$

with $d \geqslant 2$. The λ_j are assumed to be real and pairwise distinct, and the coefficients $b_{jj'}$ and $c_{jj'}$ are first-order differential operators: $(b_{jj'}, c_{jj'})(v, \partial_x) = \sum_{1\leqslant k\leqslant d}(b_{kjj'}(v), c_{kjj'}(v))\partial_{x_k}$, where the maps $v \in \mathbb{C} \to (b_{kjj'}, c_{kjj'})(v) \in \mathbb{C}^2$ are smooth and satisfy, for some $\ell \in \mathbb{N}$ with $\ell \geqslant 2$, and some $C > 0$, for all $0 \leqslant |\alpha| \leqslant 2$, for all v,

$$\big|\partial_v^\alpha b_{kjj'}(v)\big| + \big|\partial_v^\alpha c_{kjj'}(v)\big| \leqslant C|v|^{\ell-|\alpha|}. \tag{4.2}$$

We make the following assumption:

Assumption 4.1. For all j, j' such that $\lambda_j + \lambda_{j'} = 0$, there holds $c_{jj'} = c_{j'j}$. For all j, there holds $\Im m\, b_{jj} \equiv 0$.

Assumption 4.1 is a "transparency" condition, similar to Assumptions 2.1, 2.5 and 2.10 in [6] and Assumption 2.15 in [20]. It means that the singular source terms in (4.6) possess some favorable structure (cancellation or symmetry) at the resonances.

Consider a family of initial data

$$v^\varepsilon(0,x) = \varepsilon^\sigma a_\varepsilon(x), \quad \text{with } \sup_{\varepsilon\in(0,1)} \|a_\varepsilon\|_{H_\varepsilon^{s_1}} < \infty, \tag{4.3}$$

where $\sigma > 0$ and a_ε is for instance concentrating: $a_\varepsilon(x) = a^0(\frac{x}{\varepsilon})$, or oscillating: $a_\varepsilon(x) = a^0(x)e^{ix\cdot\xi_0/\varepsilon}$, for some $\xi_0 \in \mathbb{R}^d$; in both cases $a^0 \in H^{s_1}$, for some large s_1.

Our goal is to show that, under Assumption 4.1, for s_1 and σ large enough, any $T > 0$ and ε small enough, we can apply Theorem 2.19 to prove existence over $[0, T]$, in weighted Sobolev spaces, for the initial-value problem (4.1)–(4.3).

Example 4.2. Our assumptions are satisfied in particular by systems

$$\begin{cases}\partial_t v_1 + i\Delta v_1 = b_{12}(v,\partial_x)v_2 + c_{11}(v,\partial_x)\bar v_1 + c(v)\partial_x \bar v_2,\\ \partial_t v_2 - i\Delta v_2 = b_{22}(v,\partial_x)v_2 + c(v)\partial_x\bar v_1 + c_{22}(v,\partial_x)\bar v_2,\end{cases}$$

if b_{22} is real, b_{12}, b_{22}, c_{11}, and c_{22} are first-order differential operators, and all coefficients are ℓ-homogeneous in v, for some integer $\ell \geqslant 2$.

Remark 4.3. Rauch and Métivier give in [13, Theorem 1.5]; see also [12, Theorem 8.1.2] a local existence and uniqueness result for the Cauchy problem for (4.1), under Assumption 4.1, for data in H^s, with $s > 1 + d/2$. There is no small parameter in their setting. We compare Rauch and Métivier's result with ours in Remark 4.7.

4.1. Semi-classical setting

Introducing $u = (v, \bar{v})^T \in \mathbb{C}^{2n}$, we obtain a system

$$\partial_t u + iA(\partial_x)u = B(u, \partial_x)u, \tag{4.4}$$

where A is the diagonal, second-order, constant-coefficient operator

$$A(\partial_x) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n)\Delta_x, \tag{4.5}$$

and B is the first-order operator

$$B = \begin{pmatrix} \mathcal{B} & \mathcal{C} \\ \bar{\mathcal{C}} & \bar{\mathcal{B}} \end{pmatrix}, \qquad \mathcal{B} := (b_{jj'})_{1 \leqslant j,j' \leqslant n}, \qquad \mathcal{C} := (c_{jj'})_{1 \leqslant j,j' \leqslant n}.$$

Let

$$J := \{(j, j'),\ \lambda_j + \lambda_{j'} = 0\},$$

and $\chi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ be a frequency truncation, such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ for $|\xi| \leqslant 1/2$ and $\chi \equiv 0$ for $|\xi| \geqslant 1$. The source B in (4.4) decomposes into the sum of a resonant, a non-resonant term, and a low-frequency term: $B = B_r + B_{nr} + B_{lf}$, where

- the resonant term is
$$B_r := \operatorname{diag}(b_{11}, \ldots, b_{nn}, \bar{b}_{11}, \ldots \bar{b}_{nn}) + \begin{pmatrix} 0 & \mathcal{C}_J \\ \bar{\mathcal{C}}_J & 0 \end{pmatrix},$$
with the notation $(\mathcal{C}_J)_{jj'} := c_{jj'}$ if $(j, j') \in J$, and $(\mathcal{C}_J)_{jj'} := 0$ otherwise. The key is that, under Assumption 4.1, for all v, ξ, the matrix $B_r(v, \xi)$ is hermitian;
- the non-resonant term is
$$B_{nr} := \begin{pmatrix} \mathcal{B}^1 & \mathcal{C}^1 \\ \bar{\mathcal{C}}^1 & \bar{\mathcal{B}}^1 \end{pmatrix},$$
where $(\mathcal{B}^1)_{jj'} := (1 - \chi)b_{jj'}$ if $j \neq j'$, $(\mathcal{B}^1)_{jj'} := 0$ otherwise; $(\mathcal{C}^1)_{jj'} := (1 - \chi)c_{jj'}$ if $(j, j') \notin J$, $(\mathcal{C}^1)_{jj'} := 0$ otherwise;
- the low-frequency term is
$$B_{lf} := \begin{pmatrix} \mathcal{B}^0 & \mathcal{C}^0 \\ \bar{\mathcal{C}}^0 & \bar{\mathcal{B}}^0 \end{pmatrix},$$
where $(\mathcal{B}^0)_{jj'} := \chi b_{jj'}$ if $j \neq j'$, $(\mathcal{B}^0)_{jj'} := 0$ otherwise; $(\mathcal{C}^0)_{jj'} := \chi c_{jj'}$ if $(j, j') \notin J$, $(\mathcal{C}^0)_{jj'} := 0$ otherwise.

By assumption, B is homogeneous degree one in ξ. Taking into account the dependence of the datum in x/ε, and using the homogeneity of A and B, we work with weighted derivatives, and rewrite (4.4) as

$$\partial_t u + \frac{i}{\varepsilon^2} A(\varepsilon\partial_x)u = \frac{1}{\varepsilon} B(u, \varepsilon\partial_x)u. \tag{4.6}$$

The family of initial-value problems (4.6)–(4.3) corresponds to the equation $\Phi^\varepsilon(u) = 0$ for the family of maps

$$\Phi^\varepsilon(u) := \begin{pmatrix} \varepsilon^2\partial_t u + iA(\varepsilon\partial_x)u - \varepsilon B(u, \varepsilon\partial_x)u \\ u_{|t=0} - \varepsilon^\sigma a_\varepsilon \end{pmatrix}. \tag{4.7}$$

Given $T > 0$, consider the functional spaces

$$E_s = H^s(\mathbb{R}^d), \qquad F_{s-2} = C^0([0, T], H^{s-2}(\mathbb{R}^d)) \times H^s(\mathbb{R}^d), \tag{4.8}$$

with norms

$$|u|_s := \sup_{0 \leqslant t \leqslant T} \left(\left\| \varepsilon^2 \partial_t u(t) \right\|_{H_\varepsilon^{s-2}} + \left\| u(t) \right\|_{H_\varepsilon^s} \right), \qquad \left\| (f_1, f_2) \right\|_{s-2} := \sup_{0 \leqslant t \leqslant T} \left\| f_1(t) \right\|_{H_\varepsilon^{s-2}} + \| f_2 \|_{H_\varepsilon^s},$$

where the weighted Sobolev norms $\| \cdot \|_{H_\varepsilon^s}$ are defined in (2.8). By definition, Φ^ε belongs to $C^2(E_s, F_{s-2})$, for all s such that $s_0 + 2 \leqslant s \leqslant s_1$, for any $d/2 < s_0 < 1 + d/2$.

4.2. Tame direct bounds

Given $a^0 \in H^{s_1}$, there holds $\sup_{\varepsilon\in(0,1)} \|a^0(x/\varepsilon)\|_{H^{s_1}_\varepsilon} < \infty$, $\sup_{\varepsilon\in(0,1)} \|a^0(x)e^{ix\cdot\xi_0/\varepsilon}\|_{H^{s_1}_\varepsilon} < \infty$. We assume that $s_0+4 \leqslant s_1$. Let $s_0+2 \leqslant s \leqslant s_1-2$, and $u \in H^{s+2}$. There holds

$$\big\|\Phi^\varepsilon(u)\big\|_s \leqslant \varepsilon^\sigma C\big(a^0,s\big) + C(\lambda_j)|u|_{s+2} + \varepsilon\big\|B(u,\varepsilon\partial_x)u\big\|_{H^s_\varepsilon}, \tag{4.9}$$

for some $C(a^0,s) > 0$ and $C(\lambda_j) > 0$.

Lemma 4.4. *The family Φ^ε defined in* (4.7) *satisfies Assumption* 2.4 *with $(\gamma_0,\gamma_1) \in \mathbb{R}_+\times\mathbb{R}_+$ such that*

$$\left.\begin{aligned} 1-\frac{d\ell}{2}+\gamma_0\ell+\min(0,\gamma_1-\gamma_0) \geqslant 0 \\ 1-d+\min(\gamma_0+\gamma_1,2\gamma_1)\geqslant 0\end{aligned}\right\} \quad \text{if } \ell = 2,$$

$$1-\frac{d\ell}{2}+\gamma_0\ell+\min\big(0,\gamma_1-\gamma_0,2(\gamma_1-\gamma_0)\big)\geqslant 0, \quad \text{if } \ell\geqslant 3. \tag{4.10}$$

Proof. We start from (4.9) and bound the differential operator $B(u,\varepsilon\partial_x)u$ as in Example 2.6. By (4.2), for $s > d/2$ and $|u|_{L^\infty} \leqslant M_0$, there holds $\|d(u)\|_{H^s_\varepsilon} \lesssim C(M_0)|u|^{\ell-1}_{L^\infty}\|u\|_{H^s_\varepsilon}$, with $d = b_{jj'}, c_{jj'}$. Thus we obtain

$$\varepsilon\big\|B(u,\varepsilon\partial_x)u\big\|_{H^s_\varepsilon} \lesssim \varepsilon^{1-d\ell/2}\|u\|^\ell_{H^{s_0+1}_\varepsilon}\|u\|_{H^{s+1}_\varepsilon}. \tag{4.11}$$

It follows that (2.9) holds for any γ_0 such that $1-d\ell/2+\gamma_0\ell \geqslant 0$. The first derivative $(\Phi^\varepsilon)'(u)$ involves $(\partial_u B(u,\varepsilon\partial_x)\cdot v)u + B(u,\varepsilon\partial_x)v$, where $(\partial_u B(u,\varepsilon\partial_x)\cdot v)u$ is a differential operator acting on u, with coefficients depending on v, and satisfies

$$\varepsilon\big\|\big(\partial_u B(u,\varepsilon\partial_x)\cdot v\big)u\big\|_{H^s_\varepsilon} \lesssim \varepsilon^{1-d\ell/2}\|u\|^{\ell-1}_{H^{s_0+1}_\varepsilon}\big(\|v\|_{H^{s_0}_\varepsilon}\|u\|_{H^{s+1}_\varepsilon} + \|u\|_{H^{s_0+1}_\varepsilon}\|v\|_{H^s_\varepsilon}\big).$$

The other term in the first derivative, $B(u,\varepsilon\partial_x)v$, is bounded as in (4.11), and we obtain

$$\big\|\big(\Phi^\varepsilon\big)'(u)v\big\|_s \lesssim |v|_{s+2} + \varepsilon^{1-d\ell/2}\|u\|^{\ell-1}_{H^{s_0+1}_\varepsilon}\big(\|u\|_{H^{s_0+1}_\varepsilon}\|v\|_{H^{s+1}_\varepsilon} + \|v\|_{H^{s_0}_\varepsilon}\|u\|_{H^{s+1}_\varepsilon}\big).$$

The bound for the second derivative is similar, and we obtain the bounds of Assumption 2.4 under condition (4.10). □

4.3. Tame inverse bounds

For the linearized system of quasilinear Schrödinger equations $(\Phi^\varepsilon)'(\underline{u})u = (f_1,f_2)$, explicitly

$$\begin{cases} \partial_t u + \dfrac{i}{\varepsilon^2}A(\varepsilon\partial_x)u = \dfrac{1}{\varepsilon}B(\underline{u},\varepsilon\partial_x)u + \big(\partial_u B(\underline{u},\partial_x)\cdot u\big)\underline{u} + \dfrac{1}{\varepsilon^2}f_1, \\ u_{|t=0} = f_2, \end{cases} \tag{4.12}$$

we give a tame bound for u, of the form (2.16), by using the "transparency" condition expressed in Assumption 4.1. The key is that, by Assumption 4.1, the matrix $B_r(\underline{u},\xi)$ is hermitian for all $(\underline{u},\xi)$, while $B_{nr}(\underline{u},\xi)$ corresponds to non-resonant interactions and can be eliminated by a normal form reduction. The other singular term, $(1/\varepsilon)B_{lf}$, is a low-frequency term, hence its singular prefactor does not harm the estimate. The non-singular term $\partial_u B(\underline{u},\partial_x)\cdot u)\underline{u}$ is a differential operator acting on $\underline{u}$; we denote it $D := D(u,\underline{u},\partial_x\underline{u})$.

In the proof of Lemma 4.5, we use the notation and results of Section 4.6 on pseudo-differential symbols and operators.

Lemma 4.5. *Given $T > 0$, $s_0+2\leqslant s\leqslant s_1-2$, $f \in F_{s+2}$ and $\underline{u} \in E_{s+2}$, if $\underline{u}$ satisfies*

$$|\underline{u}|_{s_0+4} \lesssim \varepsilon^\gamma, \qquad \gamma = \frac{d}{2}+\frac{d}{2(\ell-1)}, \tag{4.13}$$

then there exists a unique $u \in E_s$ satisfying (4.12)*, and there holds*

$$|u|_{H^s_\varepsilon} \lesssim \varepsilon^{-2}\|f\|_s + \varepsilon^{-3}\|f\|_{s_0+2}|\underline{u}|_{s+2}. \tag{4.14}$$

Proof. Our goal is to prove estimates over $[0, T]$ for (4.12); existence and uniqueness then follow by classical arguments. We do not expect the estimates to be uniform in ε, and aim for polynomials prefactors in $\varepsilon^{-\kappa} e^{Ct}$, for some $C > 0$; in this view, the only obstacle is the singular term $(1/\varepsilon)B(\underline{u}, \varepsilon\partial_x)$ in the right-hand side, which, by direct bounds and Gronwall's Lemma, a priori contributes $e^{Ct/\varepsilon}$.

We look for a pseudo-differential matrix symbol $M = M(\underline{u}, \xi) = (M_{jj'}(\underline{u}, \xi))_{1\leqslant j,j'\leqslant 2n}$ that belongs to the class Γ_s^{-1} defined below in Section 4.6, such that, using the notation (4.20) for pseudo-differential operators in semi-classical quantization, the map

$$\check{u} := \big(\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon(M)\big)^{-1} u \tag{4.15}$$

satisfies an equation that will allow an estimate of the form (4.14). If u solves (4.12), then $\check{u}$ solves $\partial_t \check{u} = \mathcal{A}\check{u} + g$, where

$$\mathcal{A} := \big(\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon(M)\big)^{-1}\bigg(-\frac{i}{\varepsilon^2}A(\varepsilon\partial_x) + \frac{1}{\varepsilon}B(\underline{u}, \varepsilon\partial_x) + D\bigg)\big(\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon(M)\big),$$

and $g := (\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon(M))^{-1}(\varepsilon^{-2} f_1 - \varepsilon \mathrm{op}_\varepsilon(\partial_t M)\check{u})$. By Lemma 4.11,

$$\mathcal{A} = -\frac{i}{\varepsilon^2}A(\varepsilon\partial_x) + \frac{1}{\varepsilon}(B_r + B_{lf})(\underline{u}, \varepsilon\partial_x)u + \frac{1}{\varepsilon}\mathrm{op}_\varepsilon(H) + D + \mathrm{op}_\varepsilon(E),$$

where

$$H(t,x,\xi) := B_{nr}\big(\underline{u}(t,x), i\xi\big) - i\big[A(i\xi), M\big(\underline{u}(t,x), \xi\big)\big],$$

and the remainder E is

$$\begin{aligned}\mathrm{op}_\varepsilon(E) := {} & \mathrm{op}_\varepsilon(\tilde{M})\big(-iA(\varepsilon\partial_x) + \varepsilon(B_r + B_{lf})(\underline{u}, \varepsilon\partial_x) + \varepsilon^2 D\big)\big(\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon(M)\big) - R(A, M)\\ & + \mathrm{op}_\varepsilon(M)\big(-iA(\varepsilon\partial_x) + \varepsilon(B_r + B_{lf})(\underline{u}, \varepsilon\partial_x) + \varepsilon^2 D\big)\mathrm{op}_\varepsilon(M),\end{aligned}$$

with $\varepsilon^2 \mathrm{op}_\varepsilon(\tilde{M}) := (\mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon(M))^{-1} - \mathrm{Id} + \varepsilon \mathrm{op}_\varepsilon(M)$. We used above the notation R for remainders introduced in Lemma 4.11.

By the diagonal structure of A, the matrix commutator $[A(i\xi), M]$ is

$$\big[A(i\xi), M\big] = \big((\Lambda_j - \Lambda_{j'})M_{jj'}\big)_{1\leqslant j,j'\leqslant 2n},$$

where

$$\Lambda_j = -\lambda_j|\xi|^2, \quad \text{if } 1 \leqslant j \leqslant n, \qquad \Lambda_j = \lambda_{j-n}|\xi|^2, \quad \text{if } n+1 \leqslant j \leqslant 2n,$$

in accordance with (4.5). We note that, since the λ_j are pairwise distinct, $\Lambda_j - \Lambda_{j'} = 0$ if and only if $1 \leqslant j \leqslant n$ and $n+1 \leqslant j' \leqslant 2n$ with $(j, j'-n) \in J$, or $n+1 \leqslant j \leqslant 2n$ and $1 \leqslant j' \leqslant n$, with $(j-n, j') \in J$. By definition of B_{nr} in Section 4.1, for such couples $(j, j'-n)$ and $(j-n, j')$, there holds $(B_{nr})_{jj'} \equiv 0$. Besides, by definition of B_{nr}, for small ξ there holds $B_{nr} \equiv 0$. This implies that

$$M_{jj'}\big(\underline{u}(t,x), \xi\big) := \begin{cases} -i(\Lambda_j(\xi) - \Lambda_{j'}(\xi))^{-1}(B_{nr})_{jj'}(\underline{u}(t,x), i\xi), & \text{if } \Lambda_j - \Lambda_{j'} \neq 0,\\ 0, & \text{if } \Lambda_j - \Lambda_{j'} = 0,\end{cases}$$

defines a symbol $M \in \Gamma_s^{-1}$. With this choice of M, there holds $H \equiv 0$, and the equation in $\check{u}$ simplifies into

$$\partial_t \check{u} + \frac{i}{\varepsilon^2}A(\varepsilon\partial_x)\check{u} = \frac{1}{\varepsilon}(B_r + B_{lf})(\underline{u}, \varepsilon\partial_x)\check{u} + D + \mathrm{op}_\varepsilon(E)\check{u} + g. \tag{4.16}$$

We now perform direct estimates on the reduced equation (4.16). By reality of the λ_j,

$$\Re e\, \frac{i}{\varepsilon^2}\big(A(\varepsilon\partial_x)\check{u}, \check{u}\big)_{H^s_\varepsilon} = 0.$$

By the hermitian structure of $B_r(\underline{u}, \xi)$ and Lemma 4.12,

$$\Re e\, \frac{1}{\varepsilon}\big(B_r(\underline{u}, \varepsilon\partial_x)\check{u}, \check{u}\big)_{H^s_\varepsilon} \lesssim |\underline{u}|_{L^\infty}^{\ell-1}|\underline{u}|_{W^{1,\infty}}\|\check{u}\|^2_{H^s_\varepsilon} + \varepsilon^{-1-d/2}|\underline{u}|_{L^\infty}^{\ell-1}\|\check{u}\|_{H^{s_0+1}_\varepsilon}\|\underline{u}\|_{H^s_\varepsilon}\|\check{u}\|_{H^s_\varepsilon}.$$

By Lemma 4.10 with $m = s_0 - s$, under (4.13),

$$\frac{1}{\varepsilon}\big\|B_{lf}(\underline{u}, \varepsilon\partial_x)\check{u}\big\|_{H^s_\varepsilon} \lesssim \varepsilon^{-1}\|\check{u}\|_{H^{s_0}_\varepsilon}\big(|\underline{u}|^{\ell}_{L^\infty} + \varepsilon^{-d/2}|\underline{u}|^{\ell-1}_{L^\infty}\|\underline{u}\|_{H^s_\varepsilon}\big).$$

The zeroth-order term D satisfies

$$\|D\|_{H^s_\varepsilon} \lesssim |\underline{u}|^{\ell-1}_{L^\infty}|\partial_x\underline{u}|_{L^\infty}\|\check{u}\|_{H^s_\varepsilon} + \varepsilon^{-d/2}|\underline{u}|^{\ell-2}_{L^\infty}|\underline{u}|_{W^{1,\infty}}\|\check{u}\|_{H^{s_0}_\varepsilon}\|\underline{u}\|_{H^{s+1}_\varepsilon}.$$

The change of variable M satisfies, for all $w \in H^{s-1}$,

$$\big\|\mathrm{op}_\varepsilon(M)w\big\|_{H^s_\varepsilon} \lesssim C\big(|\underline{u}|_{L^\infty}\big)\big(|\underline{u}|^{\ell}_{L^\infty}\|w\|_{H^{s-1}_\varepsilon} + \varepsilon^{-d/2}|\underline{u}|^{\ell-1}_{L^\infty}\|w\|_{H^{s_0}_\varepsilon}\|\underline{u}\|_{H^s_\varepsilon}\big).$$

Let us now restrict to a background $\underline{u}$ satisfying (4.13). Then, the above bounds become

$$\begin{aligned}
&\Re e\,\frac{1}{\varepsilon}\big(B_r(\underline{u}, \varepsilon\partial_x)\check{u}, \check{u}\big)_{H^s_\varepsilon} \lesssim \big(\varepsilon^{-1+\gamma}\|\check{u}\|_{H^s_\varepsilon} + \varepsilon^{-1}\|\check{u}\|_{H^{s_0+1}_\varepsilon}\|\underline{u}\|_{H^s_\varepsilon}\big)\|\check{u}\|_{H^s_\varepsilon},\\
&\frac{1}{\varepsilon}\big\|B_{lf}(\underline{u}, \varepsilon\partial_x)\check{u}\big\|_{H^s_\varepsilon} \lesssim \big(\varepsilon^{-1+\gamma}\|\check{u}\|_{H^{s_0}_\varepsilon} + \varepsilon^{-1}\|\check{u}\|_{H^{s_0}_\varepsilon}\|\underline{u}\|_{H^s_\varepsilon}\big)\|\check{u}\|_{H^s_\varepsilon},\\
&\|D\|_{H^s_\varepsilon} \lesssim \varepsilon^{-1+\gamma}\|\check{u}\|_{H^s_\varepsilon} + \varepsilon^{-1}\|\check{u}\|_{H^{s_0}_\varepsilon}\|\underline{u}\|_{H^{s+1}_\varepsilon},\\
&\big\|\mathrm{op}_\varepsilon(M)w\big\|_{H^s_\varepsilon} \lesssim \varepsilon^{\gamma}\|w\|_{H^{s-1}_\varepsilon} + \|w\|_{H^{s_0}_\varepsilon}\|\underline{u}\|_{H^s_\varepsilon},
\end{aligned}$$

for all $w \in H^{s-1}$. In particular, for all $w \in H^{s_0}$,

$$\big\|\mathrm{op}_\varepsilon(M)w\big\|_{H^{s_0}_\varepsilon} \lesssim \varepsilon^{\gamma}\|w\|_{H^{s_0}_\varepsilon}. \tag{4.17}$$

A consequence of (4.17) is that, given $k \geqslant 2$, the operator $\mathrm{op}_\varepsilon(M)^k$ maps H^s to $H^{\max(s-k,s_0)}$, and for all $w \in H^{\max(s-k,s_0)}$,

$$\begin{aligned}
\big\|\mathrm{op}_\varepsilon(M)^k w\big\|_{H^s} &\lesssim \mathbf{M}_0^{-1}(M)^k\|w\|_{H^{s-k}_\varepsilon} + \varepsilon^{-d/2}\sum_{0\leqslant k'\leqslant k-1}\mathbf{M}_0^{-1}(M)^{k'}N^{-1}_{s-k'}(M)\|w\|_{H^{s_0}_\varepsilon}\\
&\lesssim |\underline{u}|^{\ell k}_{L^\infty}\|w\|_{H^{s-k}_\varepsilon} + \varepsilon^{-d/2}\sum_{0\leqslant k'\leqslant k-1}|\underline{u}|^{\ell(k'+1)-1}_{L^\infty}\|\underline{u}\|_{H^{s-k'}_\varepsilon}\|w\|_{H^{s_0}_\varepsilon}\\
&\lesssim \varepsilon^{\gamma k}\|w\|_{H^{s-k}_\varepsilon} + k\|\underline{u}\|_{H^s_\varepsilon}\|w\|_{H^{s_0}_\varepsilon}.
\end{aligned}$$

It follows that $\mathrm{op}_\varepsilon(\tilde{M}) = \sum_{k\geqslant 2}(-\varepsilon)^{k-2}\mathrm{op}_\varepsilon(M)^k$ maps H^s to H^{s-2}, and for all $w \in H^{s-2}$,

$$\big\|\mathrm{op}_\varepsilon(\tilde{M})w\big\|_{H^s_\varepsilon} \lesssim \varepsilon^{2(\gamma-1)}\|w\|_{H^{s-2}_\varepsilon} + \|\underline{u}\|_{H^s_\varepsilon}\|w\|_{H^{s_0}}.$$

The above bounds and Lemma 4.11 imply

$$\big\|\mathrm{op}_\varepsilon(E)\check{u}\big\|_{H^s_\varepsilon} \lesssim \|\check{u}\|_{H^s_\varepsilon} + \varepsilon^{-1}\|\check{u}\|_{H^{s_0+1}_\varepsilon}\|\underline{u}\|_{H^{s+2}_\varepsilon}.$$

The remainder g satisfies

$$\begin{aligned}
\|g\|_{H^s_\varepsilon} &\lesssim \varepsilon^{-2}\|f_1\|_{H^s_\varepsilon} + \varepsilon^{-2}\|f_1\|_{H^{s_0}_\varepsilon}\|\underline{u}\|_{H^s_\varepsilon} + \varepsilon|\underline{u}|^{\ell-1}_{L^\infty}|\partial_t\underline{u}|_{L^\infty}\|\check{u}\|_{H^{s-1}_\varepsilon}\\
&\quad + \varepsilon^{1-d/2}|\underline{u}|^{\ell-2}_{L^\infty}\|\check{u}\|_{H^{s_0+1}_\varepsilon}\big(|\underline{u}|_{L^\infty}\|\partial_t\underline{u}\|_{H^s_\varepsilon} + |\partial_t\underline{u}|_{L^\infty}\|\underline{u}\|_{H^s_\varepsilon}\big)\\
&\lesssim \varepsilon^{-2}\|f_1\|_{H^s_\varepsilon} + \varepsilon^{-2}\|f_1\|_{H^{s_0}_\varepsilon}\|\underline{u}\|_{H^s_\varepsilon} + \varepsilon^{-1+\gamma}\|\check{u}\|_{H^{s-1}_\varepsilon} + \varepsilon^{-1+\gamma}\|\check{u}\|_{H^{s_0+1}_\varepsilon}|\underline{u}|_{s+2}.
\end{aligned}$$

Collecting the above bounds, we obtain the estimate

$$\begin{aligned}
\partial_t\|\check{u}\|^2_{H^s_\varepsilon} &\lesssim \|\check{u}\|^2_{H^s_\varepsilon} + \big(\varepsilon^{-1+\gamma}\|\check{u}\|_{H^{s_0+1}_\varepsilon} + \varepsilon^{-1}\|\check{u}\|_{H^{s_0+1}_\varepsilon}\|\underline{u}\|_{H^{s+2}_\varepsilon}\\
&\quad + \varepsilon^{-2}\|f_1\|_{H^s_\varepsilon} + \varepsilon^{-2}\|f_1\|_{H^{s_0}_\varepsilon}\|\underline{u}\|_{H^s_\varepsilon}\big)\|\check{u}\|_{H^s_\varepsilon},
\end{aligned} \tag{4.18}$$

valid for any $s_0 + 2 \leqslant s \leqslant s_1 - 2$. We now let $s = s_0 + 2$ in (4.18), and obtain

$$\|\check{u}\|_{H^{s_0+2}_\varepsilon} \lesssim \|f_2\|_{H^{s_0+2}_\varepsilon} + \varepsilon^{-2}\|f_1\|_{H^{s_0+2}_\varepsilon},$$

which we plug back in (4.18) to get

$$\|\check{u}\|_{H^s_\varepsilon} \lesssim \|f_2\|_{H^s_\varepsilon} + \varepsilon^{-2}\|f_1\|_{H^s_\varepsilon} + \varepsilon^{-1}\big(\|f_2\|_{H^{s_0+2}_\varepsilon} + \varepsilon^{-2}\|f_1\|_{H^{s_0+2}_\varepsilon}\big)\|\underline{u}\|_{H^{s+2}_\varepsilon}.$$

In order to estimate $\varepsilon^2\partial_t\check{u}$, we use (4.12) directly, and, via (4.15) and the estimate for the operator norm of $\mathrm{op}_\varepsilon(M)$, we finally obtain (4.14). □

4.4. Result

Introduce $\sigma_a \geqslant 0$, such that $\|a_\varepsilon\|_{H^{s_1}_\varepsilon} = O(\varepsilon^{\sigma_a})$. For instance, in the concentrating case: $a_\varepsilon(x) = a^0(x/\varepsilon)$, we have $\sigma_a = d/2$, and in the oscillating case: $a_\varepsilon(x) = a^0(x)e^{ix\cdot\xi_0/\varepsilon}$, we have $\sigma_a = 0$. Introduce also the critical index k_c defined by

$$k_c = \max(\kappa + \gamma_0, 2\kappa + \gamma_1, \kappa + \gamma),$$

where γ_0, γ_1 are given by Lemma 4.4, and γ, κ by Lemma 4.5, so that k_c depends only on d and ℓ.

Theorem 4.6. *Under Assumption* 4.1*, if the initial datum* (4.3) *is small enough and smooth enough, meaning that* s_1 *satisfies* (2.27) *and*

$$1 + \sigma(\ell + 1) + \sigma_a > k_c, \tag{4.19}$$

then, for any $T > 0$*, if* ε *is small enough, the initial-value problem* (4.1)–(4.3) *has a solution* $v \in C^1([0, T], H^{s-2}(\mathbb{R}^d)) \cap C^0([0, T], H^s(\mathbb{R}^d))$*, for* s *satisfying* (2.28).

The regularity condition on the datum, (2.27), is meant here with $m = r = 2$, $r' = 0$, γ_0, γ_1 given by Lemma 4.4, and γ, κ given by Lemma 4.5.

Proof. Let $d/2 < s_0 < 1 + d/2$. The map Φ^ε defined in (4.7) belongs to $C^2(E_s, F_{s-2})$, where the functional spaces are defined in (4.8), for s such that $s_0 + 2 \leqslant s \leqslant s_1$, where s_1 is the assumed regularity of the datum a^0.

For the values of the parameters given just above, we saw in Section 4.2 that Φ^ε satisfies Assumption 2.4; besides, Lemma 4.5 states that Assumption 2.10 holds.

Let a_f be the solution to the free Schrödinger system, and $\tilde{\Phi}^\varepsilon$ the family of shifted maps:

$$a_f(t, x) := \varepsilon^\sigma \exp\left(-i\frac{t}{\varepsilon^2}A(\varepsilon\partial_x)\right)a_\varepsilon, \qquad \tilde{\Phi}^\varepsilon := \Phi^\varepsilon(a_f + \cdot).$$

The family $\tilde{\Phi}^\varepsilon$ satisfies Assumptions 2.4 and 2.10, with the same parameters as Φ^ε, by Remarks 2.9 and 2.14. There holds $\tilde{\Phi}^\varepsilon(0) := (-\varepsilon B(a_f, \varepsilon\partial_x)a_f, 0)$, so that

$$\begin{aligned}\big\|\tilde{\Phi}^\varepsilon(0)\big\|_s &\lesssim \varepsilon|a_f|^{\ell-1}_{L^\infty}\big(|a_f|_{L^\infty}\|a_f\|_{H^{s+1}_\varepsilon} + |\varepsilon\partial_x a_f|_{L^\infty}\|a_f\|_{H^s_\varepsilon}\big)\\ &\lesssim \varepsilon^{1+\sigma(\ell+1)}\|a_\varepsilon\|_{H^{s+1}_\varepsilon}.\end{aligned}$$

Condition (2.26) here takes the form (4.19). Under this condition, $\tilde{\Phi}^\varepsilon$ also satisfies Assumption 2.15, and we conclude by application of Theorem 2.19. □

4.5. Discussion and examples

Condition (4.19) relates the size of the datum in L^∞ and $H^{s_1}_\varepsilon$ to the space dimension and the homogeneity of the differential operators in the system of quasilinear Schrödinger equations (4.1).

The following remark explains how Theorem 4.6 extends Rauch and Métivier's result mentioned in Remark 4.3:

Remark 4.7. In Rauch and Métivier's result, Theorem 1.5 of [13], or Theorem 8.1.2 of [12], the existence time T^*_s is a decreasing function of the initial Sobolev norm $\|\varepsilon^\sigma a_\varepsilon\|_{H^s}$; there holds $T^*_s \to 0$ as $\varepsilon \to 0$ if the datum tends to $+\infty$ in H^s norm as $\varepsilon \to 0$, and $T^*_s \to +\infty$ as $\varepsilon \to 0$ if the datum tends to 0 in H^s norm; besides, $T^*_{s'} \geqslant T^*_s$ if $s' \geqslant s$. This features are shared with first-order quasilinear symmetric systems.

The datum in (4.3) satisfies in the concentrating case $\|a^0(x/\varepsilon)\|_{H^{1+d/2}} = O(\varepsilon^{-1})$, and in the oscillating case $\|a^0(x)e^{ix\cdot\xi_0/\varepsilon}\|_{H^{1+d/2}} = O(\varepsilon^{-1-d/2})$. Let $\sigma_1 = 1$ in the concentrating case and $\sigma_1 = 1 + d/2$ in the oscillating case, so that $\|\varepsilon^\sigma a_\varepsilon\|_{H^{1+d/2}} = O(\varepsilon^{\sigma-\sigma_1})$ in both cases. Let s and s_1 be as in Theorem 4.6, and assume that $a^0 \in H^{s_1}$.

Given any $s_0 > d/2$, if $\sigma < \sigma_1$, the datum $\varepsilon^\sigma a_\varepsilon$ is large in H^{1+s_0}, hence large in H^{s_1}; in particular, $T^*_{s_1} \to 0$ as $\varepsilon \to 0$. Assume now that in addition to $\sigma < \sigma_1$, condition (4.19) holds. Then the datum is small in $H^{1+s_0}_\varepsilon$, but the equation written in $\varepsilon\partial_x$ derivatives, (4.6), has large source terms. Under Assumption 4.1, these terms are not present in the normal form of the equation, and Theorem 4.6 grants an arbitrarily long existence time in H^s_ε. Thus, Theorem 4.6 extends Theorem 1.5 of [13] (Theorem 8.1.2 of [12]) in the case that both $\sigma < \sigma_1$ and condition (4.19) hold, for very regular data (indeed, $s_1 \gg 1 + d/2$ in practice, see Remark 2.23).

For concentrating or oscillating data, the values of ℓ and σ that allow both conditions $\sigma < \sigma_1$ and (4.19) to hold are described in the following:

Example 4.8. In two space dimensions, $d = 2$:

- in the concentrating case, conditions $\sigma < \sigma_1$ and (4.19) are incompatible if $\ell = 2$ and $\ell = 3$, and they hold for $\frac{9}{2(\ell+1)} < \sigma < 1$ if $\ell \geqslant 4$;
- in the oscillating case, conditions $\sigma < \sigma_1$ and (4.19) are incompatible if $\ell = 2$; they hold for $\frac{5}{\ell+1} < \sigma < 2$ if $\ell \geqslant 3$.

Example 4.9. In three space dimensions, $d = 3$:

- in the concentrating case, conditions $\sigma < \sigma_1$ and (4.19) are incompatible if $\ell = 2$ and $\ell = 3$, and they hold for $\frac{4}{\ell+1} < \sigma < 1$ if $\ell \geqslant 4$;
- in the oscillating case, conditions $\sigma < \sigma_1$ and (4.19) hold for $2 < \sigma < 5/2$ if $\ell = 2$; they hold for $\frac{11}{2(\ell+1)} < \sigma < 5/2$ if $\ell \geqslant 3$.

4.6. Pseudo-differential symbols and operators

Given $m, s \in \mathbb{R}$, we define the class Γ^m_s as the space of symbols σ defined on $\mathbb{R}^d_x \times \mathbb{R}^d_\xi$, such that, for all $k \in \mathbb{N}$, $\sigma \in C^k(\mathbb{R}^d_\xi; H^s(\mathbb{R}^d_x))$, and

$$\mathbf{N}^m_{k,s}(\sigma) := \sup_{|\beta| \leqslant k} \sup_\xi \left(1 + |\xi|^2\right)^{(|\beta|-m)/2} \left\|\partial_\xi^\beta \sigma(\cdot,\xi)\right\|_{H^s_\varepsilon} < \infty,$$

where $\|\cdot\|_{H^s_\varepsilon}$ is defined in (2.8). Symbols in $S^m_{1,0}$ that do not depend on x are called Fourier multipliers of order m. To a symbol $\sigma \in \Gamma^m_s$, we associate the pseudo-differential operator $\mathrm{op}_\varepsilon(\sigma)$ defined by its action on $\mathcal{S}(\mathbb{R}^d)$ as

$$\mathrm{op}_\varepsilon(\sigma)u := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(x, \varepsilon\xi)\hat{u}(\xi)\, d\xi. \tag{4.20}$$

Let

$$\mathbf{M}^m_{k,k'}(\sigma) := \sup_{|\beta| \leqslant k} \sup_{|\beta'| = k'} \sup_\zeta \left(1 + |\xi|^2\right)^{(|\beta|-m)/2} \left|\partial_\xi^\beta \partial_x^{\beta'} \sigma(\cdot,\xi)\right|_{L^\infty}.$$

Given a symbol $\sigma \in \Gamma^m_s$, if $s > k' + d/2$, there holds $\mathbf{M}^m_{k,k'}(\sigma) < \infty$.

The following three lemmas describe the action, composition, and adjoints of operators with symbols in Γ^m_s, based on the results of [14,7], and the identity

$$\mathrm{op}_\varepsilon(\sigma)u = (h_\varepsilon)^{-1}\mathrm{op}_1(\tilde{\sigma})h_\varepsilon, \qquad \tilde{\sigma}(x,\xi) := \sigma(\varepsilon x, \xi), \tag{4.21}$$

relating classical and semi-classical quantizations, where $(h_\varepsilon f)(x) := \varepsilon^{d/2} f(\varepsilon x)$, so that $\|h_\varepsilon f\|_{1,s} = \|f\|_{\varepsilon,s}$. In the statements of these results, we shorten $\mathbf{N}^m_{k,s}$ and $\mathbf{M}^m_{k,k'}$ into $\mathbf{N}^m_s$ and $\mathbf{M}^m_{k'}$, where it is understood that a certain number of derivatives in ξ, depending only on d, are involved in the semi-norms.

Lemma 4.10. *Given $m \in \mathbb{R}$, $s \geqslant s_0 > d/2$, and $\sigma \in \Gamma_s^m$, for all $u \in H^{s+m}$, there holds*

$$\left\| \mathrm{op}_\varepsilon(\sigma) u \right\|_{H^s_\varepsilon} \leqslant \mathbf{M}_0^m(\sigma) \|u\|_{H_\varepsilon^{s+m}} + \varepsilon^{-d/2} \mathbf{N}_s^m(\sigma) \|u\|_{H_\varepsilon^{s_0+m}}.$$

Proof. Use Theorem 1 in [7], and (4.21). □

Lemma 4.11. *Let $m_1, m_2, s_2 \in \mathbb{R}$, and $s_0 > d/2$. Let σ_1 be a Fourier multiplier of order m_1, and $\sigma_2 \in \Gamma_{s_2}^{m_2}$. If $s_2 \geqslant s_0 + \max(m_1, 0) + 1$, there holds*

$$\mathrm{op}_\varepsilon(\sigma_1)\mathrm{op}_\varepsilon(\sigma_2) - \mathrm{op}_\varepsilon(\sigma_1\sigma_2) = \varepsilon R(\sigma_1, \sigma_2),$$

where for all $s_0 \leqslant s \leqslant s_2 - \max(m_1, 0)$, for all $u \in H^{s+m_1+m_2-1}$,

$$\begin{aligned} \left\| R(\sigma_1, \sigma_2) u \right\|_{H^s_\varepsilon} \lesssim{}& \mathbf{M}_0^{m_1}(\sigma_1)\mathbf{M}_1^{m_2}(\sigma_2) \|u\|_{H_\varepsilon^{s+m_1+m_2-1}} \\ &+ \varepsilon^{-1-d/2} \mathbf{M}_0^{m_1}(\sigma_1) \mathbf{N}^{m_2}_{s+\max(m_1,0)}(\sigma_2) \|u\|_{H_\varepsilon^{s_0+m_1+m_2-\max(m_1,0)}}. \end{aligned} \tag{4.22}$$

Proof. Use Theorem 3(ii) in [7], and (4.21). □

Lemma 4.12. *Given $m \in \mathbb{R}$, $s \geqslant 1 + s_0 > 1 + d/2$, and $\sigma \in \Gamma_s^m$, there holds for all $u \in H^{s+m-1}$,*

$$\left\| \left(\mathrm{op}_\varepsilon(\sigma)^* - \mathrm{op}_\varepsilon\left(\sigma^*\right)\right) u \right\|_{H^s_\varepsilon} \lesssim \varepsilon \mathbf{M}_1(\sigma) \|u\|_{H_\varepsilon^{s+m-1}} + \varepsilon^{-d/2} \mathbf{N}_s(\sigma) \|u\|_{H_\varepsilon^{s_0+m}}.$$

Proof. A direct consequence of Lemma 4.10 and Proposition B.22 in [14]. □

5. Application: small-amplitude shock profiles for quasilinear relaxation equations with characteristic velocities

We consider finally the problem of existence of relaxation profiles

$$U(x,t) = \bar{U}(x - st), \qquad \lim_{z \to \pm\infty} \bar{U}(z) = U_\pm \tag{5.1}$$

of a relaxation system

$$\partial_t U + A(U)\partial_x U = Q(U),$$

with

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 \\ q \end{pmatrix}, \tag{5.2}$$

in one spatial dimension, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^r$, where, for some smooth v_* and f, some $\theta > 0$,

$$q\big(u, v_*(u)\big) \equiv 0, \qquad \Re e\, \sigma\big(\partial_v q\big(u, v_*(u)\big)\big) \leqslant -\theta < 0, \tag{5.3}$$

$\sigma(\cdot)$ denoting spectrum, and

$$\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} = \begin{pmatrix} \partial_u f & \partial_v f \end{pmatrix}. \tag{5.4}$$

Here, we are thinking particularly of the case n bounded and $r \gg 1$ arising through discretization or moment closure approximation of the Boltzmann equation or other kinetic models; that is, we seek estimates and proof independent of the dimension of v. Recall, for Boltzmann's equation and its finite approximants, that $n = 5$ is the dimension of the equilibrium (u) system corresponding to standard gas-dynamical flow, whereas the total dimension $n + r$ may be arbitrarily large: for example, it is infinite for the continuous Boltzmann equations and 13 for the Grad 13-moment approximation, with an increasing number of moments as the desired level of accuracy is increased.

For fixed n, r, the existence problem was treated in [23,9] under the additional assumption $\det(A - sI) \neq 0$ corresponding to non-degeneracy of the traveling-wave ODE, using standard center-manifold techniques for amplitudes $U_+ - U_-$ sufficiently small. However, as pointed out in [9,10], this assumption is satisfied in general only (by considerations coming from the subcharacteristic condition) for 2×2 models $r = n = 1$, and is unrealistic for larger models ($n > 1$ or $r > 1$). Moreover, it is not satisfied for the (infinite-dimensional) Boltzmann equations, for which the eigenvalues of A are constant particle speeds of all values, hence cannot be *uniformly* satisfied for discrete velocity or moment closure approximations as the number of modes goes to infinity, at least if they are faithful (consistent) models of Boltzmann. For, the set of characteristic speeds, given by the eigenvalues of A, in that case must approach a dense set in the limit as the number of modes goes to infinity, and so A cannot be uniformly invertible. Thus, the region of validity for such center manifold arguments as in [23,9] in general shrinks to zero as the number of modes goes to infinity.

A different argument for small-amplitude stability based on Chapman–Enskog expansion and Picard iteration was presented in [15] for the semilinear case $A \equiv$ constant. This yields results independent of dimension; indeed, with slight modifications, it has been applied to the infinite-dimensional Boltzmann equation itself [16]. However, in the quasilinear case, there seems to be an unavoidable loss of derivatives in the iteration process, and so the argument of [15] does not close. This has been remedied in [17] using the Nash–Moser iteration of the present paper. We describe this application here in a simplified case that illustrates the main issues while avoiding some technical details; for the general case, see [17].

5.1. Assumptions

Let $f, A, Q \in C^\infty$. We assume the following:

(i) f *scalar*, corresponding to $n = 1$, $u \in \mathbb{R}$.
(ii) A symmetric.
(iii) $Q = \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} \end{pmatrix}$ block diagonal, with $\Re e\, Q_{22} := \frac{1}{2}(Q_{22} + Q_{22}^T)$ negative definite and $v_*(u) \equiv 0$.
(iv) A_{12} non-vanishing.
(v) $f_*(u) := f(u, 0)$ genuinely nonlinear in the sense of Lax, that is $d^2 f_*(u) \neq 0$.

In the general case, (ii) and (iii) can be achieved by coordinate transformations [17]. Under (ii) and (iii), condition (iv) is the Kawashima genuine coupling condition, a consequence of which is that the skew matrix

$$K := \begin{pmatrix} 0 & A_{12} \\ -A_{21} & 0 \end{pmatrix}$$

satisfies

$$\Re e\,(KA - Q) \geqslant c\mathrm{Id}, \tag{5.5}$$

for some $c > 0$, uniformly in $x \in \mathbb{R}$. Associated with (5.2) is a scalar viscous conservation law

$$\partial_t u + \partial_x f_*(u) = \partial_x\big(b_*(u)\partial_x u\big), \tag{5.6}$$

obtained by Chapman–Enskog expansion (described partly below), with f_* defined in (v) above, and

$$b_*(u) := -A_{12} Q_{22}^{-1} A_{21}(u, 0). \tag{5.7}$$

By our structural assumptions,

$$\Re e\, b_* \geqslant \theta > 0. \tag{5.8}$$

Taking without loss of generality $s = 0$, we study the traveling-wave ODE

$$A(U)U' = Q(U). \tag{5.9}$$

5.2. Chapman–Enskog approximation

Integrating the first equation of (5.9), we obtain

$$\begin{aligned} &f(u, v) = f_*(u_\pm), \\ &A_{21}(u, v)u' + A_{22}(u, v)v' = q(u, v), \end{aligned} \tag{5.10}$$

where f_* is defined in (v), Section 5.1. Taylor expanding the first equation, we obtain

$$f_*(u) + f_v(u,0)v + O\big(v^2\big) = f_*(u_\pm). \tag{5.11}$$

Taylor expanding the second equation and inverting $\partial_v q$, we obtain

$$v = \partial_v q(u,0)^{-1} A_{21}(u,0)u' + O\big(|v|^2\big) + O\big(|v|\big|u'\big|\big) + O\big(\big|v'\big|\big). \tag{5.12}$$

Substituting (5.12) into (5.11) and rearranging, we obtain the approximate viscous profile ODE

$$b_*(u)u' = f_*(u) - f_*(u_\pm) + O\big(v^2\big) + O\big(|v|\big|u'\big|\big) + O\big(\big|v'\big|\big). \tag{5.13}$$

Motivated by (5.12)–(5.13), we define an approximate solution (u_{CE}, v_{CE}) of (5.10) by choosing u_{CE} as a solution of

$$b_*(u_{CE})u'_{CE} = f_*(u_{CE}) - f_*(u_\pm), \tag{5.14}$$

and v_{CE} as the first approximation given by (5.12)

$$v_{CE} = c_*(u_{CE})u'_{CE}. \tag{5.15}$$

Here, (5.14) can be recognized as the traveling-wave ODE associated with approximating scalar viscous conservation law (5.6), with $s = 0$. From standard scalar ODE considerations (normal forms), we obtain the following description of solutions.

Proposition 5.1. *Under the assumptions of Section* 5.1*, for u_0 such that $df_*(u_0) = 0$, in a neighborhood of (u_0, u_0) in $\mathbb{R}^1 \times \mathbb{R}^1$, there is a smooth curve $\mathcal{S}$ passing through (u_0, u_0), such that for $(u_-, u_+) \in \mathcal{S}$ with amplitude $\varepsilon := |u_+ - u_-| > 0$ sufficiently small, the zero speed shock profile equation* (5.14) *has a unique* (*up to translation*) *solution u_{CE} local to u_0. The shock profile is necessarily of* Lax type: *i.e., with $df_*(u_-) > 0 > df_*(u_+)$.*

Moreover, there is $\theta > 0$ and for all k there is C_k independent of (u_-, u_+) and ε, such that

$$\big|\partial_x^k(u_{CE} - u_\pm)\big| \leqslant C_k \varepsilon^{k+1} e^{-\theta\varepsilon|x|}, \quad x \gtrless 0. \tag{5.16}$$

We denote by $\mathcal{S}_+$ the set of $(u_-, u_+) \in \mathcal{S}$ with amplitude $\varepsilon := |u_+ - u_-| > 0$ sufficiently small that the profile u_{CE} exists. Given $(u_-, u_+) \in \mathcal{S}_+$ with associated profile u_{CE}, we define v_{CE} by (5.15) and

$$U_{CE} := (u_{CE}, v_{CE}). \tag{5.17}$$

It is an approximate solution of (5.10) in the following sense:

Corollary 5.2. *For fixed u_- and amplitude $\varepsilon := |u_+ - u_-|$ sufficiently small,*

$$\begin{aligned} &\mathcal{R}_u := f(u_{CE}, v_{CE}) - f_*(u_\pm) = O\big(\big|u'_{CE}\big|^2\big) = O\big(\varepsilon^4 e^{-\theta\varepsilon|x|}\big), \\ &\mathcal{R}_v := g(u_{CE}, v_{CE})' - q(u_{CE}, v_{CE}) = O\big(\big|u''_{CE}\big|\big) = O\big(\varepsilon^3 e^{-\theta\varepsilon|x|}\big) \end{aligned} \tag{5.18}$$

satisfy

$$\begin{aligned} &\big|\partial_x^k \mathcal{R}_u(x)\big| \leqslant C_k \varepsilon^{k+4} e^{-\theta\varepsilon|x|}, \\ &\big|\partial_x^k \mathcal{R}_v(x)\big| \leqslant C_k \varepsilon^{k+3} e^{-\theta\varepsilon|x|}, \quad x \gtrless 0, \end{aligned} \tag{5.19}$$

where C_k is independent of (u_-, u_+) and $\varepsilon = |u_+ - u_-|$.

Proof. For $k = 0$, bounds (5.19) follow by expansions (5.11) and (5.12), definitions (5.14) and (5.15), and bounds (5.16). Bounds for $k > 0$ follow similarly. □

Remark 5.3. One may continue this process to obtain Chapman–Enskog approximations (u^N_{CE}, v^N_{CE}) to all orders, with truncation errors $(\partial_x^k \mathcal{R}_u^N, \partial_x^k \mathcal{R}_v^N) \sim (\varepsilon^{N+k+4}, \varepsilon^{N+k+3})$ [17].

5.3. Statement of the main theorem

We are now ready to state the main result. Define a base state $U_0 = (u_0, 0)$ with $df_*(u_0) = 0$, and a neighborhood $\mathcal{U}$ of U_0.

Theorem 5.4. *Under the assumptions of Section* 5.1*, there are $\varepsilon_0 > 0$ and $\delta > 0$ such that for $(u_-, u_+) \in \mathcal{S}_+$ with amplitude $\varepsilon := |u_+ - u_-| \leqslant \varepsilon_0$, the standing-wave equation* (5.9) *has a solution $\bar{U}$ in $\mathcal{U}$, with associated Lax-type equilibrium shock (u_-, u_+), satisfying for all k:*

$$\begin{aligned}&\left|\partial_x^k(\bar{U} - U_{CE})\right| \leqslant C_k\varepsilon^{k+2}e^{-\delta\varepsilon|x|},\\ &\left|\partial_x^k(\bar{u} - u_\pm)\right| \leqslant C_k\varepsilon^{k+1}e^{-\delta\varepsilon|x|},\\ &\left|\partial_x^k(\bar{v} - v_*(\bar{u})\right| \leqslant C_k\varepsilon^{k+2}e^{-\delta\varepsilon|x|}, \quad x \gtrless 0,\end{aligned} \tag{5.20}$$

where $U_{CE} = (u_{CE}, v_{CE})$ is the approximating Chapman–Enskog profile defined in (5.14)*, and C_k is independent of ε. Moreover, up to translation, this solution is unique within a ball of radius $c\varepsilon$ about U_{CE} in norm*

$$\varepsilon^{-1/2}\|\cdot\|_{L^2} + \varepsilon^{-3/2}\|\partial_x\cdot\|_{L^2} + \cdots + \varepsilon^{-11/2}\left\|\partial_x^5\cdot\right\|_{L^2}, \tag{5.21}$$

for $c > 0$ sufficiently small and K sufficiently large.

That is, behavior of profiles is well-described by Chapman–Enskog approximation. By (iii), the equilibrium v_* in (5.20) is $v_* \equiv 0$. Note that $U_{CE} - U_\pm$ is order $O(\varepsilon)$ in the norm (5.21), by (5.20)(ii)–(iii). A consequence of the bounds (5.20), via [11], is that the Chapman–Enskog profiles are spectrally stable; see [17].

5.4. Functional equation and spaces

Defining the perturbation variable $U := \bar{U} - U_{CE}$, where U_{CE} is defined in (5.17), we obtain from (5.10) the nonlinear perturbation equations $\Phi^\varepsilon(U) = 0$, where

$$\Phi^\varepsilon(U) := \begin{pmatrix} f(U_{CE} + U) - f_*(u_-) \\ A_{21}(U_{CE} + U)(u_{CE} + u)' + A_{22}(U_{CE} + U)(v_{CE} + v)' - q(U_{CE} + U)\end{pmatrix}. \tag{5.22}$$

Formally linearizing Φ^ε about a background profile $\underline{U}$, we obtain

$$(\Phi^\varepsilon)'(\underline{U})U = \begin{pmatrix} A_{11}u + A_{12}v \\ A_{21}u' + A_{22}v' + b_2U - \partial_v q v\end{pmatrix}, \tag{5.23}$$

where

$$A = A(U_{CE} + \underline{U}), \qquad \partial_v q = \partial_v q(U_{CE} + \underline{U}),$$

and

$$b_2U = \big(\partial_u(A_{21} + A_{22})(U_{CE} + \underline{U}) \cdot u + \partial_v(A_{21} + A_{22})(U_{CE} + \underline{U}) \cdot v\big)(U_{CE} + \underline{U})'.$$

The associated linearized equation for a given forcing term $h = (h_1, h_2)$ is

$$\big(\Phi^\varepsilon\big)'(\underline{U})U = h. \tag{5.24}$$

The coefficients and the error term $\mathcal{R}$ from Corollary 5.2 are smooth functions of U_{CE} and its derivatives, so behave like smooth functions of εx. Thus, it is natural to solve the equations in spaces which reflect this scaling. We observe that

$$\big\|f(\varepsilon\cdot)\big\|_{L^2} = \varepsilon^{-1/2}\|f\|_{L^2}, \qquad \big\|f(\varepsilon\cdot)\big\|_{H^s} = \varepsilon^{-1/2}\sum_{k=0}^{s}\varepsilon^k\big\|\partial_x^k f\big\|_{L^2}, \tag{5.25}$$

in one space dimension, for $s \in \mathbb{N}$. We do not introduce explicitly the change of variables $\tilde{x} = \varepsilon x$, but introduce exponentially weighted norms which correspond to usual weighted H^s norms in the $\tilde{x}$ variable: for $s \in \mathbb{N}$ and $\delta \geqslant 0$, we let, in accordance with (5.25),

$$\|f\|_{\varepsilon,\delta,s} := \varepsilon^{1/2}\sum_{0\leqslant k\leqslant s}\varepsilon^{-k}\big\|e^{\delta\varepsilon(1+|\cdot|^2)^{1/2}}\partial_x^k f\big\|_{L^2}, \tag{5.26}$$

the exponential weight accounting for the exponential decay of the source and the solution. For fixed δ, we introduce the spaces $E_s := H^s(\mathbb{R})$, and $F_s := H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$, with norms

$$|h|_{E_s} := \|h\|_{\varepsilon,\delta,s}, \qquad \big|(h_1, h_2)\big|_{F_s} := \|h_1\|_{\varepsilon,\delta,s+1} + \|h_2\|_{\varepsilon,\delta,s}.$$

In particular, the Chapman–Enskog approximate solution of Section 5.2 satisfies, by (5.16),

$$\big|\partial_x^j U_{CE}\big|_{L^\infty} \leqslant \varepsilon^{j+1} C_j, \qquad \big|\partial_x^{j+1} U_{CE}\big|_{E_s} \leqslant \varepsilon^{j+2} C_{j,s}, \quad \text{for } j \geqslant 0, \tag{5.27}$$

where the constants $C_j > 0$, $C_{j,s} > 0$ do not depend on ε, for all $s \in \mathbb{N}$.

Remark 5.5. Moser's inequality in the weighted norms (5.26) is

$$\|fg\|_{\varepsilon,\delta,s} \lesssim |f|_{L^\infty} \|g\|_{\varepsilon,\delta,s} + \|f\|_{\varepsilon,\delta,s} |g|_{L^\infty}, \quad s \geqslant 0,\ f, g \in L^\infty \cap H^s,$$

and the Sobolev embedding has norm

$$\big|\partial_x^k f\big|_{L^\infty} \lesssim \varepsilon^{-1/2} \|f\|_{\varepsilon,\delta,k+1+[d/2]}, \quad k \geqslant 0,\ f \in H^{k+1+[d/2]}.$$

5.5. Nash–Moser iteration scheme

Lemma 5.6. *The application Φ^ε, defined in* (5.22)*, maps smoothly E_s to F_{s-1}, for any s. It satisfies Assumption* 2.4 *with $s_0 = 1$, $\gamma_0 = \gamma_1 = 1/2$, $s_1 = +\infty$, and Assumption* 2.15*, with $k = N$.*

Proof. The bounds of Assumption 2.4, describing the action of Φ^ε and its first two derivatives, follow directly from Moser's inequality and the definition of the weighted Sobolev norms. The bound on $\Phi^\varepsilon(0)$ is immediate from (5.19) and (5.26). □

Proposition 5.7. *Under the assumptions of Theorem* 5.4*, for ε and δ small enough, the map Φ^ε satisfies Assumption* 2.10 *with $r = 1$, $r' = 0$, $\gamma = 1$, and $\kappa = 1$.*

The proof of this proposition is carried out in Section 5.6. Once it is established, existence and uniqueness follow by Theorems 2.19 and 2.20:

Proof of Theorem 5.4 (Existence). The profile U_{CE} exists if ε is small enough. Comparing, we find that Lemma 5.6, Proposition 5.7, and Corollary 5.2 verify, respectively, Assumptions 2.4, 2.10, and 2.15 of our Nash–Moser iteration scheme, with $s_0 = 3$, $\gamma_0 = \gamma_1 = 1/2$, $\gamma = 1$, $m = r = 1$, $r' = 0$, arbitrary s_1, and $k = N$ large enough. Taking s_1 sufficiently large, and applying Theorem 2.19, we thus obtain existence of a solution U^ε of (5.22) with $|U^\varepsilon|_{H^{s+1}_{\varepsilon,\delta}} \leqslant C\varepsilon^2$. Defining $\bar U^\varepsilon := U^\varepsilon_{CE} + U^\varepsilon$, and noting by Sobolev embedding that $|h|_{H^{s+1}_{\varepsilon,\delta}}$ controls $|e^{\delta\varepsilon|x|} h|_{L^\infty}$, we obtain the result. □

Proof of Theorem 5.4 (Uniqueness). Applying Theorem 2.20 for $s_0 = 3$, $\gamma_0 = \gamma_1 = 1/2$, $\gamma = 1$, $k = 3$, $m = r = 1$, $r' = 0$, we obtain uniqueness in a ball of radius $c\varepsilon$ in $H^4_{\varepsilon,0}$, $c > 0$ sufficiently small, under the additional phase condition (2.33). We obtain unconditional uniqueness from this weaker version by the observation that phase condition (2.33) may be achieved for any solution $\bar U = U_{CE} + U$ with

$$\big\|U'\big\|_{L^\infty} \leqslant c\varepsilon^2 \ll U'_{CE}(0) \sim \varepsilon^2$$

by translation in x, yielding $\bar U_a(x) := \bar U(x + a) = U_{CE}(x) + U_a(x)$ with

$$U_a(x) := U_{CE}(x + a) - U_{CE}(x) + U(x + a)$$

so that, defining $\phi := \bar U'/|\bar U'|$, we have $\partial_a \langle \phi, U_a \rangle \sim \langle \phi, U'_{CE} + U' \rangle = \langle \phi, (1 + o(1))\bar U' + U' \rangle = (1 + o(1))|\bar U'| \sim \varepsilon^2$ and so (by the implicit function theorem applied to $h(a) := \varepsilon^{-2}\langle \phi, U_a \rangle$, together with the fact that $\langle \phi, U_0 \rangle = o(\varepsilon)$ and that $\langle \phi, \bar U'_{NS} \rangle \sim |\bar U'_{NS}| \sim \varepsilon^2$) the inner product $\langle \phi, U_a \rangle$, hence also ΠU_a may be set to zero by appropriate choice of $a = o(\varepsilon^{-1})$ leaving U_a in the same $o(\varepsilon)$ neighborhood, by the computation $U_a - U_0 \sim \partial_a U \cdot a \sim o(\varepsilon^{-1})\varepsilon^2$. □

5.6. Linearized estimates

We here carry out the main step in the proof of obtaining corresponding a priori estimates; see Proposition 5.12 below. The remaining step of demonstrating existence for the linearized problem can be carried out by the vanishing viscosity method as in [16], with viscosity coefficient $\eta > 0$, obtaining existence for each positive η by standard boundary-value theory, and noting that the a priori bounds (5.40) of Proposition 5.12 persist under regularization for sufficiently small viscosity $\eta > 0$, so that we can obtain a weak solution in the limit by extracting a weakly convergent subsequence. We omit this step, referring the reader to [15, Section 8], for details. The asserted estimates then follow in the limit by continuity.

The rest of this subsection is devoted to establishing the asserted a priori estimates.

5.6.1. Internal and high frequency estimates

Let $s \in \mathbb{N}$, and some background profile $\underline{U} \in H^s$. We consider Eq. (5.24), and its differentiated form:

$$\big(AU' - dQ + b\big)U = \big(h_1', h_2\big), \tag{5.28}$$

in which $bU := (b_1 U, b_2 U)$, where b_2 is defined in Section 5.4, and b_1 is defined similarly, by differentiating the coefficients A_{11}, A_{12} in the first line of (5.24). The coefficients A, b, and dQ are smooth functions of $U_{CE} + \underline{U}$. The bound for U_{CE}, (5.27), and the assumed bound for $\underline{U}$ imply the coefficient bounds

$$\begin{cases} \big|\partial_x^{j+1} C\big|_{L^\infty} + \big|\partial_x^j b\big|_{L^\infty} \leqslant c_j \varepsilon^{2+j}, & 0 \leqslant j \leqslant s-1, \\ \big\|\partial_x^{k+1} C\big\|_{L^2} + \big\|\partial_x^k b\big\|_{L^2} \leqslant C_k \varepsilon^{1/2+k}\big(\varepsilon + |\underline{U}|_{\varepsilon,0,s+1}\big), & 0 \leqslant k \leqslant s, \end{cases} \tag{5.29}$$

where $C = A, Q, K$, the matrix K being the Kawashima multiplier introduced in Section 5.1. In (5.29), the constants c_j depend on $|\partial_x^{j'}(U_{CE} + \underline{U})|_{L^\infty}$, for $0 \leqslant j' \leqslant j$, while, by the classical Moser's inequality, the constants C_k depend on $|U_{CE} + \underline{U}|_{L^\infty}$.

We give in the following proposition an estimate for the internal variables $U' = (u', v')$ and v.

Proposition 5.8. *For $k \geqslant 1$, for come $C > 0$, for ε and δ small enough, given $h \in F_{k+1}$, if $U \in H^k$ satisfies* (5.28) *with $|\underline{U}|_{E_2} \leqslant \varepsilon$, there holds*

$$\begin{aligned} \big|\partial_x^k U'\big|_{E_0} + \big|\partial_x^k v\big|_{E_0} &\leqslant C\big(\big|\partial_x^k H\big|_{E_0} + \varepsilon^k\big(\big|U'\big|_{E_{k-1}} + \varepsilon|v|_{E_{k-1}} + \varepsilon|u|_{E_0}\big)\big) \\ &\quad + C\varepsilon^{k+1}|\underline{U}|_{E_{k+2}}\big(|v|_{E_1} + \varepsilon|U|_{E_2}\big), \end{aligned} \tag{5.30}$$

where $H = (h_1, h_1', h_1'', h_2, h_2')$.

In order to prove Proposition 5.8, we start with an L^2 estimate for the internal variables:

Lemma 5.9. *For some $C > 0$, for ε sufficiently small, given $(h_1, h_2) \in H^2 \times H^1$, if $U \in H^1$ satisfies* (5.28) *with $\|\underline{U}\|_{\varepsilon,0,2} \leqslant \varepsilon$, there holds*

$$\big\|U'\big\|_{L^2} + \|v\|_{L^2} \leqslant C\big(\|h_1\|_{H^2} + \|h_2\|_{H^1} + \varepsilon\|u\|_{L^2}\big). \tag{5.31}$$

Sketch of proof. The key is to bound the L^2 scalar product $(\mathfrak{S}h, U)_{L^2}$ from above and from below, where $\mathfrak{S}$ is the symmetrizer $\mathfrak{S} = \partial_x^2 + \partial_x \circ K - \lambda$, for an appropriate choice of $\lambda \in \mathbb{R}$, using symmetry of A, and positivity of $KA - Q$ (5.5). A complete proof in given in [17, Section 5.4.1]. □

Proof of Proposition 5.8. We use Lemma 5.9 for $\varepsilon^{1/2} e^{\delta\varepsilon\langle x\rangle} U$, which solves (5.28) with the source term

$$\varepsilon^{1/2} e^{\delta\varepsilon\langle x\rangle}\big(\big(h_1', h_2\big) + \delta\varepsilon\langle x\rangle' \tilde{A} U\big).$$

This gives

$$\big|U'\big|_{E_0} + |v|_{E_0} \leqslant C\big(|H|_{E_0} + \varepsilon|u|_{E_0}\big), \tag{5.32}$$

i.e., estimate (5.30) with $k = 0$. Estimate (5.30) with $k > 0$ is obtained in a similar way, differentiating (5.28) k times. For more details, see [17, Proposition 5.5]. □

5.6.2. Linearized Chapman–Enskog estimate

It remains only to estimate the weighted L^2 norm $|u|_{E_0}$ in order to close the estimates and establish the bound claimed in Proposition (5.7). To this end, we work with the first equation in (5.24) and estimate it by comparison with the Chapman–Enskog approximation of Section 5.2. From the second equation in (5.24), in which, by (5.29), $b = O(\varepsilon^2)$, we find, for small ε,

$$v = (\partial_v q - b_{22})^{-1}\big(A_{21}u' + A_{22}v' + b_{21}u - h_2\big), \tag{5.33}$$

where $b_2 U =: b_{21}u + b_{22}v$. Introducing now (5.33) in the first equation of (5.24), we obtain the linearized profile equation

$$A_{12}(\partial_v q - b_{22})^{-1}A_{21}u' + \big(A_{11} + A_{12}(\partial_v q - b_{22})^{-1}b_{21}\big)u = h^\sharp, \tag{5.34}$$

where $h^\sharp$ depends on the source h and on v', but not on v nor on u:

$$h^\sharp := -A_{12}(\partial_v q - b_{22})^{-1}A_{22}v' + h_1 + A_{12}(\partial_v q - b_{22})^{-1}h_2.$$

Introduce the notation

$$b^\sharp := \big(A_{12}(\partial_v q - b_{22})^{-1}A_{21}\big)(U_{CE} + \cdot),$$
$$f^\sharp := \big(A_{11} + A_{12}(\partial_v q - b_{22})^{-1}b_{21}\big)(U_{CE} + \cdot).$$

Then (5.34) takes the form

$$\big(b^\sharp\partial_x - f^\sharp\big)(\underline{U})u = -h^\sharp. \tag{5.35}$$

We estimate the solution of (5.35) by the following:

Proposition 5.10. *Given $\underline{U} \in H^4$, with $|\underline{U}|_{E_4} \leqslant \varepsilon$, if ε is sufficiently small, then the operator $(b^\sharp\partial_x - f^\sharp)(\underline{U})$ has a right inverse $(b^\sharp\partial_x - f^\sharp)(\underline{U})^\dagger$, satisfying the bound*

$$\big\|\big(b^\sharp\partial_x - f^\sharp\big)(\underline{U})^\dagger h\big\|_{E_0} \leqslant C\varepsilon^{-1}\|h\|_{E_0}, \tag{5.36}$$

and uniquely specified by the property that the solution u to (5.35) *satisfies*

$$\ell_\varepsilon \cdot u(0) = 0 \tag{5.37}$$

for certain unit vector ℓ_ε.

Proof. Working in $\tilde{x} = \varepsilon x$ coordinates, and noting that $\varepsilon^{-1}|f^\sharp(0) - f^\sharp(U_\pm)| \sim e^{-\theta|\tilde{x}|}$, by (5.16), we obtain using $\partial_x = \varepsilon\partial_{\tilde{x}}$ the equation

$$\big(b^\sharp\partial_{\tilde{x}} - \varepsilon^{-1}f^\sharp\big)u = \varepsilon^{-1}h, \qquad u(0) = 0. \tag{5.38}$$

This is a rather standard boundary-value ODE problem with exponentially convergent coefficients at spatial infinity. Using the extra condition $u(0) = 0$, we may break it into a pair of boundary values problems on $(-\infty, 0]$ and $[0, +\infty)$, each of which, by the Lax condition $df_*(u_-) > 0 > df_*(u_+)$, implying that there is a one-dimensional manifold of decaying solutions as $\tilde{x} \to -\infty$ or as $\tilde{x} \to +\infty$, is well-posed, from $H^s_{\varepsilon,\delta}$ to itself, so long as δ is strictly smaller that $\varepsilon^{-1}\min|df_*(u_\pm)|$. Taking account of the ε^{-1} factor in the right-hand side of (5.38), we obtain the result. □

Combining Proposition 5.8 with $k = 1$ and Proposition 5.10, we obtain:

Proposition 5.11. *For some $C > 0$, for ε and δ small enough, given $h \in F_2$, and $\underline{U} \in H^4$ satisfying $|\underline{U}|_{E_4} \leqslant \varepsilon$, if $U = (u, v) \in H^2$ satisfies* (5.24)*, with u satisfying* (5.37)*, there holds*

$$|U|_{E_2} \leqslant C\varepsilon^{-1}|h|_{F_2}. \tag{5.39}$$

Knowing a bound for $\|u\|_{L^2_{\varepsilon,\delta}}$, Proposition 5.8 implies by induction the following final result.

Proposition 5.12. *For $s \geqslant 3$, for some $C > 0$, for ε and δ small enough, given $h \in F_s$ and $\underline{U} \in H^{s+1}$ with $|\underline{U}|_{E_4} \leqslant \varepsilon$, if $U \in H^s$ satisfies* (5.24) *and* (5.37)*, then*

$$|U|_{E_s} \leqslant \varepsilon^{-1} C\big(|\underline{U}|_{E_{s+1}} |h|_{F_2} + |h|_{F_s}\big). \tag{5.40}$$

Proposition 5.12 can be used to establish Proposition 5.7 by a vanishing viscosity argument; see [15].

5.7. Why Nash–Moser?

We conclude by discussing why we seem to need Nash–Moser to close the argument. Recall the standard proof of existence for quasilinear symmetric hyperbolic systems $u_t + A(u)u_x = S$ using energy estimates. One writes an iteration scheme

$$u_t^{n+1} + A\big(u^n\big)u_x^{n+1} = S,$$

which gives H^s bounds $|u^{n+1}|_{H^s} \leqslant C|g|_{H^s}$ so long as $|u^n|_{H^s}$ is small, and contraction in lower norms on small time intervals, giving the result.

But, it is easily checked that this does not work for equations in conservative form $u_t + (A(u)u)_x = S$, for which

$$u_t^{n+1} + \big(A\big(u^n\big)u^{n+1}\big)_x = S,$$

gives H^s bounds $|u^{n+1}|_{H^s} \leqslant C|S|_{H^s}$ rather for $|u^n|_{H^{s+1}}$ small, hence involves loss of derivatives.

Usually, for a conservative equation $u_t + f(u)_x = S$, this is no problem, since we are free to write it in non-conservative form $u_t + df(u)u_x = S$. In the present case, however, it is essential for the key Chapman–Enskog estimation of the macroscopic variable u that we write the first row of our equation in integrated form $f(u, v) = s$, enforcing a linearization $A_{11}u + A_{12}v = \tilde{s}$. But, in the part of our argument in which we control microscopic variables by energy estimates, we differentiate this equation and group it with the second row, thus leading to a partially *conservative* form in which the energy estimates lose a derivative.

That is, the Chapman–Enskog part of our argument does not seem to be compatible with the non-conservative form needed to close energy estimates without losing a derivative. We have not been able to find a direct way around this (using some alternative scheme), and so for the moment Nash–Moser iteration appears essential for the argument.

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