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Ann. I. H. Poincaré – AN 28 (2011) 551–564

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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A local symmetry result for linear elliptic problems with solutions changing sign

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Received 24 September 2010; received in revised form 25 March 2011; accepted 25 March 2011

Available online 2 April 2011

Abstract

We prove that the only domain Ω such that there exists a solution to the following problem $\Delta u + \omega^2 u = -1$ in Ω, $u = 0$ on $\partial\Omega$, and $\frac{1}{|\partial\Omega|}\int_{\partial\Omega} \partial_{\mathbf{n}} u = c$, for a given constant c, is the unit ball B_1, if we assume that Ω lies in an appropriate class of Lipschitz domains.

1. Introduction

Let us consider the following problem: for $\omega \in \mathbb{R}$, is it true that the only domain Ω such that there exists a solution u to the problem

$$\begin{cases} \Delta u + \omega^2 u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with

$$\partial_{\mathbf{n}} u = c \quad \text{on } \partial\Omega, \tag{1.2}$$

is a ball? Here Ω is a sufficiently smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, $\partial_{\mathbf{n}} u$ is the external normal derivative to the boundary $\partial\Omega$, and c is a given constant. By using the Alexandrov method of moving planes J. Serrin [20] has proved that if there exists a solution u to (1.1), (1.2), and if u has a *sign* in Ω, then $\Omega = B_1$ (for example for $\omega = 0$, by the maximum principle it follows that u is positive in Ω). For the particular case $\omega = 0$ see also the proofs of H. Weinberger [23], based on a Rellich-type identity and on the maximum principle, and M. Choulli, A. Henrot [7], which use the technique of domain derivative. We point out that Serrin in [20] has studied the same type of problem for more general nonlinear elliptic equations. For further references concerning symmetry (and non-symmetry) results for overdetermined elliptic problems, see also [1–4,8–19,21,22]. All these results need hypothesis on the sign of u. In [5] the authors have given a positive answer to the above question by supposing that

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doi:10.1016/j.anihpc.2011.03.005

(i) $\omega^2 \notin \{\lambda_n\}_{n\geqslant 1}$ ($\{\lambda_n\}_{n\geqslant 1}$ being the sequence, in increasing order, of eigenvalues of $-\Delta$ in B_1 with Dirichlet boundary conditions),
(ii) $\omega \notin \Lambda$, where Λ is an enumerable set of $\mathbb{R}^+$, whose limit points are the values λ_{1m}, for some integer $m \geqslant 1$, λ_{1m} being the mth-zero of the first-order Bessel function I_1,
(iii) Ω is such that the $\ker(\Delta + \omega^2) = \{0\}$ in Ω,
(iv) the boundary $\partial\Omega$ is a Lipschitz perturbation of the unit sphere ∂B_1 of $\mathbb{R}^N$.

We point out that in [5] no hypothesis are required on the sign of the solution u. We can say that paper [6] can be considered as preparatory of [5] (in the sense that some ideas developed in [6] are used in [5]). In the present paper we give a new proof of the result proved in [5], which let us permit to avoid hypothesis (i)–(iii) above.

We recall that if let us denote by $(\lambda_n)_{n\geqslant 1}$ the sequence, in increasing order, of eigenvalues of $-\Delta$ in B_1 with Dirichlet boundary conditions, we have that the eigenvalue λ_n, for some $n \in \mathbb{N}$, coincides, for some integers $\ell \geqslant 0$ and $m \geqslant 1$, with $\lambda_{\ell m}^2$. Here and in what follows $\lambda_{\ell m}$ will denote the mth-zero of the so-called N-dimensional ℓ-order Bessel function of the first kind I_ℓ, i.e. $I_\ell(\lambda_{\ell m}) = 0$ (see Section 2). We recall in particular that (see [5, Lemma 3.5])

$$I_0' = -I_1 \quad \text{in } \mathbb{R}.$$

From these remarks it follows that the function $u^{(0)}$ given by

$$u^{(0)}(x) = \frac{1}{\omega^2}\left(\frac{I_0(\omega r)}{I_0(\omega)} - 1\right) \quad \text{in } B_1, \tag{1.3}$$

solves (1.1), (1.2) when $\Omega = B_1$. Here $r = |x|$, $|\cdot|$ denoting the Euclidean norm in $\mathbb{R}^N$. We observe that if the constant ω is smaller or equal than λ_{11}, the solution $u^{(0)}$ is positive in B_1, while if ω is bigger than λ_{11}, then $u^{(0)}$ changes sign. In the rest of the paper we will assume $\omega \geqslant 0$. The same conclusions hold true for $\omega < 0$, since the coefficient ω^2 is even in (1.1). We stress out that in order that (1.3) makes sense, in the rest of the paper we will suppose that

$$\omega \notin \{\lambda_{0m}\}_{m\geqslant 1}.$$

Here and in what follows $c = \partial_{\mathbf{n}} u^{(0)}$ on ∂B_1. By (1.3), we obtain that

$$c = \frac{I_0'(\omega)}{\omega I_0(\omega)}. \tag{1.4}$$

In the present paper we prove the following

Theorem 1.1. *For $\omega \notin \{\lambda_{0m}\}_{m\geqslant 1}$, there exists a class $\mathcal{D}$ of $C^{2,\alpha}$-domains such that if u is a solution to* (1.1) *verifying*

$$\frac{1}{|\partial\Omega|}\int_{\partial\Omega} \partial_{\mathbf{n}} u = c,$$

with $\Omega \in \mathcal{D}$, and c given by (1.4)*, then $\Omega = B_1$, and $u = u^{(0)}$.*

The idea underlying the proof of Theorem 1.1 is the following. Let E be the vector space of $C^{2,\alpha}$ functions defined on the unit sphere ∂B_1, i.e.

$$E = \left\{k \in C^{2,\alpha}(\partial B_1)\right\},$$

$0 < \alpha < 1$. For $k \in E$, let Ω_k be the domain whose boundary $\partial\Omega_k$ can be written as perturbation of ∂B_1, i.e.

$$\partial\Omega_k = \left\{x = (1+k)y,\ y \in \partial B_1\right\}$$

(in particular for $k \equiv 0$ on ∂B_1, $\Omega_0 = B_1$). We denote by Φ the following operator

$$\Phi : E \mapsto \mathbb{R},$$

defined by

$$\Phi(k) = \int_{\partial\Omega_k} \partial_{\mathbf{n}} u_p - c \int_{\partial\Omega_k},$$

where u_p is a particular solution to (1.1), when $\Omega = \Omega_k$ (u_p will be defined in Section 3 below). We observe that Φ has not a sign in a neighborhood of 0 in E (i.e. Φ is neither positive nor negative). In fact $\Phi(0) = 0$ (since $u_p = u^{(0)}$ when $\Omega = B_1$). Moreover since the unit sphere centered at the point $x_0 \in \mathbb{R}^N$ is parametrized by

$$\partial B_1(x_0) = \{x = (1 + k')y,\ y \in \partial B_1\},$$

where k' is given by

$$k'(y) = x_0 \cdot y - 1 + \sqrt{1 + |x_0 \cdot y|^2 - |x_0|^2}, \tag{1.5}$$

we have that $\Phi(k') = 0$, with

$$k' \to 0 \quad \text{in } E, \quad \text{as } x_0 \to 0.$$

So the best one can expect is that Φ is different to 0 in $\mathcal{O} \setminus \{k \in E;\ k = k'\}$, for some neighborhood $\mathcal{O}$ of 0 in E. By studying the behavior of the operator Φ at 0, we prove that if $\omega \notin \{\lambda_{\ell m}\}_{\ell \geqslant 2, m \geqslant 1}$, with $\lambda_{\ell m} \neq \lambda_{1m'}$, for all $m' \geqslant 1$, then Φ is differentiable at zero in E. On the other hand if $\omega = \lambda_{\ell m}$, for some $\ell \geqslant 2$, and $m \geqslant 1$ (with $\lambda_{\ell m} \neq \lambda_{1m'}$, for all $m' \geqslant 1$), then Φ is differentiable at zero in the vector space

$$E_\ell = \{k \in E;\ k_{\ell q} = 0,\ k_{pq'} = 0,\ p \in I\} \tag{1.6}$$

of functions $k \in E$ which don't have either the frequency ℓ or the frequency p, I being a (eventually empty) finite set of positive integer such that $I_p(\lambda_{\ell m}) = 0$ (the cardinality of I depending on the multiplicity of the eigenvalue $\lambda_{\ell m}^2$, see Section 2 for more details). Here and in what follows $k_{st} = \frac{1}{|\partial B_1|} \int_{\partial B_1} k Y_{st}$ is the s-order (Fourier) coefficient of k, and Y_{st} is the spherical harmonic of degree s, with $t = 1, \ldots, d_s$. More precisely we have that the differential at zero in the direction k has a sign if $k_0 \neq 0$ (see Lemma 3.3), k_0 being the zeroth-order coefficient of k (i.e. $k_0 = \frac{1}{|\partial B_1|} \int_{\partial B_1} k$). We can show then that there exists a neighborhood $\mathcal{O}$ of 0 in E such that Φ is positive in $\mathcal{O} \cap E^+$, and Φ is negative in $\mathcal{O} \cap E^-$, where E^+ and E^- are two circular sectors respectively in the subset $\{k \in E;\ k_0 < 0\}$, and $\{k \in E;\ k_0 > 0\}$. Now, since if there exists a solution u to (1.1), when $\Omega = \Omega_k$, verifying $\frac{1}{|\partial \Omega_k|} \int_{\partial \Omega_k} \partial_{\mathbf{n}} u = c$, one can prove that $\Phi(k) = 0$, we obtain that $k = 0$, if we assume that $k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{0\})$. Finally, since the operator Φ is invariant up to isometries, we obtain that the class $\mathcal{D}$ in Theorem 1.1 is defined as

$$\mathcal{D} = \{\Omega;\ \Omega = \sigma(\Omega_k)\},$$

for some $\sigma \in \Sigma$, and some $\Omega_k \in \mathcal{G}$, where Σ is the set of isometries of $\mathbb{R}^N$, and

$$\mathcal{G} = \{\Omega_k;\ k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{0\})\}.$$

We stress out that E through the paper is the space of functions of class $C^{2,\alpha}$ on ∂B_1 (this means that we consider only regular perturbations of the unit sphere), but, up to obvious changes, the same conclusions hold true in the case where E is the space of functions of class $C^{0,1}$ on ∂B_1, i.e. the boundary $\partial \Omega_k$ is of Lipschitz class. The paper is organized as follows: in the next section we give some notations used through the paper, in Section 3 we give the first-order approximation of the operator Φ in a neighborhood of 0, and in Section 4 we prove Theorem 1.1, and we consider the Lipschitz case. Finally in Section 5 counter-examples to Theorem 1.1 are given.

2. Preliminaries and notations

Let us denote by B_1 the ball of radius 1 in $\mathbb{R}^N$ centered at zero. By $\overline{B}_1$ we define the Euclidean closure of B_1. Let us denote by I_ℓ the so-called N-dimensional ℓ-order Bessel function of the first kind, i.e.

$$I_\ell(r) = r^{-\nu} J_{\nu+\ell}(r),$$

where $\nu = \frac{N}{2} - 1$, and $J_{\nu+\ell}$ is the well-known $(\nu + \ell)$-order Bessel function of the first kind (we observe that for $N = 2$, I_ℓ coincides with the ℓ-order Bessel function of the first kind J_ℓ). I_ℓ solves the following Bessel equation

$$I_\ell'' + \frac{N-1}{r} I_\ell' + \left(1 - \frac{\ell(\ell + N - 2)}{r^2}\right) I_\ell = 0 \quad \text{in } \mathbb{R}.$$

Let $\lambda_{\ell m}$ be the mth-zero of the ℓ-order Bessel function I_ℓ. Let $(\lambda_n)_{n\geqslant 1}$ be the sequence, in increasing order, of eigenvalues of $-\Delta$ in B_1 with Dirichlet boundary conditions. An eigenvalue λ_n, for some $n \in \mathbb{N}$, coincides, for some integer $\ell \geqslant 0$, and $m \geqslant 1$, with $\lambda_{\ell m}^2$. The corresponding eigenfunctions can be written as (in polar coordinates)

$$\begin{aligned}
\varphi_1 &= I_\ell(\lambda_{\ell m} r) Y_{\ell 1}(\theta), \\
\vdots\ &\ \vdots \qquad \vdots \\
\varphi_{d_\ell} &= I_\ell(\lambda_{\ell m} r) Y_{\ell d_\ell}(\theta), \\
\varphi_{p_q} &= I_p(\lambda_{\ell m} r) Y_{pq}(\theta),
\end{aligned}$$

where $p \in I$, and I is a (eventually empty) finite set (by Fredholm theorem) of integer such that $I_p(\lambda_{\ell m}) = 0$, i.e.

$$I = \{p \in \mathbb{N},\ p \neq \ell;\ I_p(\lambda_{\ell m}) = 0\}. \tag{2.1}$$

Here Y_{st} is the spherical harmonic of degree s, with $t = 1, \dots, d_s$, and

$$d_s = \begin{cases} 1 & \text{if } s = 0, \\ \frac{(2s+N-2)(s+N-3)!}{s!(N-2)!} & \text{if } s \geqslant 1. \end{cases}$$

We will use the following convention: we say that a function f has the frequency s, if the s-order coefficient of f, i.e. $f_{st} = \frac{1}{|\partial B_1|} \int_{\partial B_1} f Y_{st}$, is different to zero. And similarly we say that a function f doesn't have the frequency s, if the s-order coefficient of f vanishes.

Let $\tilde{k}$ be a $C^{2,\alpha}$-extension of k into $\overline{B}_1$. Let us call A the Jacobian matrix of change of variable

$$x = (1 + k(y))y, \quad y \in \overline{B}_1 \tag{2.2}$$

(where we denote $\tilde{k}$ by k). The matrix A is given by

$$A_{ij} = \begin{bmatrix} 1 + k + y_1\partial_1 k & y_1\partial_2 k & \cdots & y_1\partial_N k \\ y_2\partial_1 k & 1 + k + y_2\partial_2 k & \cdots & y_2\partial_N k \\ \vdots & \vdots & \vdots & \vdots \\ y_N\partial_1 k & \cdots & \cdots & 1 + k + y_N\partial_N k \end{bmatrix}.$$

Let $G = A^T A$. The matrix G can be written as

$$G = I_N + G^{(1)} + o(\|k\|),$$

where I_N is the N-order identity matrix, and the matrix $G^{(1)}$ depends linearly on k and ∇k. Following [5], the matrix $G^{(1)}$ is given by

$$G_{ij}^{(1)} = 2kI_N + \begin{bmatrix} 2x_1\partial_1 k & x_1\partial_2 k + x_2\partial_1 k & \cdots & x_1\partial_N k + x_N\partial_1 k \\ x_1\partial_2 k + x_2\partial_1 k & 2x_2\partial_2 k & \cdots & x_2\partial_N k + x_N\partial_2 k \\ \vdots & \vdots & \vdots & \vdots \\ x_1\partial_N k + x_N\partial_1 k & \cdots & \cdots & 2x_N\partial_N k \end{bmatrix}. \tag{2.3}$$

3. The first-order expansion of the operator Φ

A function $k \in E$ can be written, in Fourier series expansion, as

$$k = k_0 + \sum_{p\geqslant 1} \sum_{q=1}^{d_p} k_{pq} Y_{pq} \quad \text{on } \partial B_1.$$

We recall that problem (1.1) cannot have solutions or, if a solution exists, it cannot be unique. This happens all times the kernel $\ker(\Delta + \omega^2) \neq \{0\}$ in Ω. More precisely by Fredholm theorem there exists a solution to (1.1) if and only if

$$-1 \in \ker(\Delta + \omega^2)^\perp \quad \text{in } \Omega.$$

We can write a solution u as

$$u = u_p + u_h,$$

where u_p is a particular solution to (1.1) such that

$$u_p \in \ker\big(\Delta + \omega^2\big)^{\perp} \quad \text{in } \Omega, \tag{3.1}$$

and u_h solves the corresponding homogeneous problem. We observe that u_p is unique and can be written as

$$u_p = \sum_{p \in I^C} \sum_{q=1}^{n_p} \alpha_{pq} \psi_{pq},$$

where $\alpha_{pq} = \frac{\int_\Omega \psi_{pq}}{\mu - \lambda_p}$ is the p-order Fourier coefficient of u. Here λ_p and ψ_{pq} are respectively the pth-eigenvalue and a corresponding eigenfunction of $-\Delta$ in Ω (with Dirichlet boundary conditions), and n_p is the dimension of the corresponding eigenspace. I is a finite set of integer (by Fredholm theorem), and I^C is the complementary of I. On the other hand if the kernel $\ker(\Delta + \omega^2) = \{0\}$, then a solution u exists and is unique. For example for $\omega = \lambda_{\ell m}$, for some $\ell, m \geqslant 1$, then $u_p = \frac{1}{\lambda_{\ell m}^2}(\frac{I_0(\lambda_{\ell m} r)}{I_0(\lambda_{\ell m})} - 1)$ is a particular solution to (1.1) when $\Omega = B_1$ (lying in the $\ker(\Delta + \lambda_{\ell m}^2)^{\perp}$ in B_1), and u_h has the form (in polar coordinates)

$$u_h = \sum_{q=1}^{d_\ell} \alpha_{\ell q} I_\ell(\lambda_{\ell m} r) Y_{\ell q}(\theta) + \sum_{p \in I} \sum_{q=1}^{d_p} \alpha_{pq} I_p(\lambda_{\ell m} r) Y_{pq}(\theta),$$

where I is defined in (2.1), and $\alpha_{\ell 1}, \dots, \alpha_{\ell d_\ell}, \alpha_{pq} \in \mathbb{R}$. We denote by Φ the following operator

$$\Phi : E \mapsto \mathbb{R},$$

defined by

$$\Phi(k) := \int_{\partial \Omega_k} \partial_{\mathbf{n}} u_p - c \int_{\partial \Omega_k},$$

where u_p is a particular solution to (1.1), verifying (3.1), when $\Omega = \Omega_k$. The operator Φ is well-defined, since we suppose that a solution u exists for k lying in some neighborhood of 0 in E. Using (2.2), we have that the function $\tilde{u}$ defined by

$$\tilde{u}(y) = u\big((1+k)y\big) \quad \text{in } \overline{B}_1,$$

solves

$$\begin{cases} \operatorname{div}(\sqrt{g} G^{-1} \nabla \tilde{u}) + \omega^2 \sqrt{g}\, \tilde{u} = -\sqrt{g} & \text{in } B_1, \\ \tilde{u} = 0 & \text{on } \partial B_1, \end{cases} \tag{3.2}$$

where $g = |\det G|$. Following [5], the external normal derivative of u at the point $x = (1+k)y \in \partial \Omega_k$ is given by

$$\partial_{\mathbf{n}} u\big((1+k)y\big) = \big(G^{-1} y \cdot y\big)^{-1/2} G^{-1} \nabla \tilde{u} \cdot y.$$

The operator Φ then becomes

$$\Phi(k) = \int_{\partial B_1} \big(G^{-1} y \cdot y\big)^{-1/2} G^{-1} \nabla \tilde{u}_p \cdot y \sqrt{\tilde{g}} - c \int_{\partial B_1} \sqrt{\tilde{g}},$$

where $\tilde{u}_p(y) = u_p((1+k)y)$, and $\sqrt{\tilde{g}}$ is the surface element of the new variable y. Let us denote $\tilde{u}_p$ by u_p, and y by x. We begin by proving the following

Lemma 3.1. *We have*

$$u_p \to u^{(0)} \quad \text{as } k \to 0.$$

Proof of Lemma 3.1. Let $z = u_p - u^{(0)}$. By writing the matrix $\sqrt{g} G^{-1}$ in (3.2) as

$$\sqrt{g} G^{-1} = I_N + K, \tag{3.3}$$

it follows that z solves

$$\begin{cases} \Delta w + \omega^2 w = (1-\sqrt{g})(\omega^2 u_p + 1) - \operatorname{div}(K\nabla u_p) & \text{in } B_1, \\ w = 0 & \text{on } \partial B_1. \end{cases} \tag{3.4}$$

Let assume that the $\ker(\Delta+\omega^2) = \{0\}$ in B_1. The solution w to (3.4) can be written as

$$w = \sum_{p=1}^{+\infty}\sum_{q=1}^{n_p}\alpha_{pq}\psi_{pq},$$

where the p-order Fourier coefficient

$$\alpha_{pq} = \frac{\int_{B_1}((1-\sqrt{g})(\omega^2 u_p + 1) - \operatorname{div}(K\nabla u_p))\psi_{pq}}{\omega^2 - \lambda_p}.$$

Since

$$\sqrt{g} = 1 + Nk + x\cdot\nabla k + o(\|k\|), \tag{3.5}$$

we obtain

$$w \to 0 \quad \text{as } k \to 0.$$

On the other hand, if the $\ker(\Delta+\omega^2) \neq \{0\}$ in B_1, i.e. $\omega^2 = \lambda_n$, for some $n \geqslant 2$ (we recall that $\lambda_n \notin \{\lambda_{0m}^2\}_{m\geqslant 1}$), then a solution w to (3.4) can be written as

$$w = w_p + w_h,$$

where

$$w_p = \sum_{p\in I^C}\sum_{q=1}^{n_p}\alpha_{pq}\psi_{pq}.$$

We claim that $w_p = z$. We have that the function $w_p - z$ solves

$$\begin{cases} \Delta(w_p - z) + \lambda_n(w_p - z) = 0 & \text{in } B_1, \\ w_p - z = 0 & \text{on } \partial B_1. \end{cases}$$

So we obtain

$$w_p - z = \sum_{p\in I}\sum_{q=1}^{n_p}\beta_{pq}\psi_{pq},$$

i.e.

$$u_p = u^{(0)} + w_p + \sum_{p\in I}\sum_{q=1}^{n_p}\beta_{pq}\psi_{pq},$$

for all $\beta_{pq}\in\mathbb{R}$. Since u_p is a solution to (3.2), it follows that

$$\begin{aligned} -\sqrt{g} &= \operatorname{div}(\sqrt{g}G^{-1}\nabla u_p) + \lambda_n\sqrt{g}u_p \\ &= \operatorname{div}(\sqrt{g}G^{-1}\nabla(u^{(0)} + w_p)) + \lambda_n\sqrt{g}(u^{(0)} + w_p) \\ &\quad + \sum_{p\in I}\sum_{q=1}^{n_p}\beta_{pq}\operatorname{div}(\sqrt{g}G^{-1}\nabla\psi_{pq}) + \lambda_n\sqrt{g}\sum_{p\in I}\sum_{q=1}^{n_p}\beta_{pq}\psi_{pq} \\ &= -\sqrt{g} + \sum_{p\in I}\sum_{q=1}^{n_p}\beta_{pq}(\operatorname{div}(\sqrt{g}G^{-1}\nabla\psi_{pq}) + \lambda_n\sqrt{g}\psi_{pq}). \end{aligned}$$

In particular we obtain

$$\beta_{pq}\big(\operatorname{div}\big(\sqrt{g}G^{-1}\nabla\psi_{pq}\big)+\lambda_n\sqrt{g}\psi_{pq}\big)=0.$$

We claim that

$$\operatorname{div}\big(\sqrt{g}G^{-1}\nabla\psi_{pq}\big)+\lambda_n\sqrt{g}\psi_{pq}\not\equiv 0 \quad \text{in } B_1.$$

By contradiction let assume that there exists a $p\in I$ and a $q\in\{1,\dots,n_p\}$ such that

$$\operatorname{div}\big(\sqrt{g}G^{-1}\nabla\psi_{pq}\big)+\lambda_n\sqrt{g}\psi_{pq}=0 \quad \text{in } B_1.$$

By defining by $y=y(x)$ the inverse of the change of variable (2.2), we obtain that

$$\tilde\psi_{pq}(x)=\psi_{pq}\big(y(x)\big), \quad x\in\Omega_k,$$

solves

$$\Delta\tilde\psi_{pq}+\lambda_n\tilde\psi_{pq}=0 \quad \text{in } \Omega_k, \qquad \tilde\psi_{pq}=0 \quad \text{on } \partial\Omega_k.$$

This implies that λ_n is an eigenvalue of $-\Delta$ in Ω_k. Then u_p doesn't lie in $\ker(\Delta+\lambda_n)^\perp$ in Ω_k, which yields a contradiction. This yields that $\beta_{pq}=0$, for all $p\in I$, and $q=1,\dots,n_p$, and then $u_p=u^{(0)}+w_p$. $\square$

By (3.3) it follows that

$$\sqrt{g}I_N-G=KG=\big(K^{(1)}+o\big(\|k\|\big)\big)\big(I_N+G^{(1)}+o\big(\|k\|\big)\big),$$

where $K^{(1)}$ denotes the one-order term of the matrix K (the matrix $G^{(1)}$ is given by (2.3)). In particular the matrix

$$K^{(1)}=g^{(1)}I_N-G^{(1)}, \tag{3.6}$$

where $g^{(1)}$, the one-order term of $\sqrt{g}$, is given by

$$g^{(1)}=Nk+x\cdot\nabla k. \tag{3.7}$$

By (3.5) we have

$$\frac{1}{\sqrt{g}}=1-Nk-x\cdot\nabla k+o\big(\|k\|\big),$$

and by (3.3), (3.6), and (3.7), we obtain

$$\begin{aligned} G^{-1}&=\frac{I_N}{\sqrt{g}}+\frac{1}{\sqrt{g}}K^{(1)}+\cdots\\ &=I_N-G^{(1)}+o\big(\|k\|\big). \end{aligned} \tag{3.8}$$

Lemma 3.2. *If $\omega\notin\{\lambda_{\ell m}\}_{\ell\geqslant 2,m\geqslant 1}$, with $\lambda_{\ell m}\neq\lambda_{1m'}$, for all $m'\geqslant 1$, then u_p has the form*

$$u_p=u^{(0)}+u^{(1)}+o\big(\|k\|\big) \quad \text{in } E, \tag{3.9}$$

where $u^{(1)}$ solves

$$\begin{cases} \Delta u^{(1)}+\omega^2u^{(1)}=f^{(1)} & \text{in } B_1,\\ u^{(1)}=0 & \text{on } \partial B_1, \end{cases} \tag{3.10}$$

and $f^{(1)}$ is given by

$$f^{(1)}=-(Nk+x\cdot\nabla k)\big(1+\omega^2u^{(0)}\big)-\operatorname{div}\big(K^{(1)}\nabla u^{(0)}\big).$$

If $\omega=\lambda_{\ell m}$, for some $\ell\geqslant 2$, and $m\geqslant 1$ (with $\lambda_{\ell m}\neq\lambda_{1m'}$, for all $m'\geqslant 1$), the same holds true by changing E with E_ℓ, where E_ℓ is defined in (1.6).

To prove Lemma 3.2, we observe that if the $\ker(\Delta+\omega^2)=\{0\}$ in B_1, then u_p admits a one-order expansion in E. The same holds true if the $\ker(\Delta+\omega^2)\neq\{0\}$ in B_1, with $\omega=\lambda_{1m}$, for some $m\geqslant 1$. On the other hand, if the $\ker(\Delta+\omega^2)=\{0\}$ in B_1, i.e. $\omega=\lambda_{\ell m}$, for some $\ell\geqslant 2$, and $m\geqslant 1$, then u_p admits a one-order expansion in the vector space E_ℓ of functions $k\in E$ which don't have either the frequency ℓ or the frequency p, with $p\in I$, the set I being defined in (2.1).

Proof of Lemma 3.2. Let $\omega\notin\{\lambda_{\ell m}\}_{\ell\geqslant 2,m\geqslant 1}$, with $\lambda_{\ell m}\neq\lambda_{1m'}$, for all $m'\geqslant 1$. Let assume that u_p can be written as in (3.9). Then u_p solves

$$\begin{cases}\Delta u_p+\operatorname{div}(K\nabla u_p)+\omega^2\sqrt{g}u_p=-\sqrt{g} & \text{in } B_1,\\ u_p=0 & \text{on } \partial B_1.\end{cases}\tag{3.11}$$

We have

$$\begin{aligned}\operatorname{div}(K\nabla u_p)+\sqrt{g}\big(\omega^2u_p+1\big)&=\operatorname{div}\big(K^{(1)}\big(\nabla u^{(0)}+\nabla u^{(1)}\big)\big)\\&\quad+(1+Nk+x\cdot\nabla k)\big(\omega^2\big(u^{(0)}+u^{(1)}\big)+1\big)+\cdots.\end{aligned}\tag{3.12}$$

The one-order terms in (3.12) are given by

$$(Nk+x\cdot\nabla k)\big(1+\omega^2u^{(0)}\big)+\omega^2u^{(1)}+\operatorname{div}\big(K^{(1)}\nabla u^{(0)}\big).$$

By taking the one-order terms in (3.11), we obtain that $u^{(1)}$ solves (3.10). By a direct calculation $u^{(1)}$ has the form

$$u^{(1)}=\frac{I_0'(\lambda_{1m}r)}{\lambda_{1m}I_0(\lambda_{1m})}rk,$$

if $\omega=\lambda_{1m}$, since $I_0'=-I_1$. Otherwise, for $\omega\neq\lambda_{1m}$, then $u^{(1)}$ has the form

$$u^{(1)}=\frac{I_0'(\omega r)}{\omega I_0(\omega)}rk+\bar u,$$

where $\bar u$ solves

$$\begin{cases}\Delta\bar u+\omega^2\bar u=0 & \text{in } B_1,\\ \bar u=\frac{I_1(\omega)}{\omega I_0(\omega)}k & \text{on } \partial B_1.\end{cases}$$

The solution $\bar u$ (in polar coordinates) can be written as

$$\bar u(r,\theta)=-c\left(k_0I_0(\omega r)/I_0(\omega)+\sum_{p\geqslant 1}\sum_{q=1}^{d_p}k_{pq}I_p(\omega r)/I_p(\omega)Y_{pq}(\theta)\right).\tag{3.13}$$

Now obviously (3.13) is well-defined for all $\omega\notin\{\lambda_{\ell m}\}_{\ell\geqslant 2,m\geqslant 1}$. Let us define by

$$w=u_p-u^{(0)}-u^{(1)}.$$

The function w solves

$$\begin{cases}\Delta w+\omega^2w=(1-\sqrt{g})(\omega^2u_p+1)-\operatorname{div}(K\nabla u_p)-f^{(1)} & \text{in } B_1,\\ w=0 & \text{on } \partial B_1.\end{cases}$$

By writing u_p as

$$u_p=u^{(0)}+f,$$

with $f(k)=o(1)$ as $k\to 0$ in E, we obtain

$$(1-\sqrt{g})\big(\omega^2u_p+1\big)-\operatorname{div}(K\nabla u_p)-f^{(1)}=o\big(\|k\|\big).$$

By standard $C^{2,\alpha}$-estimates we obtain

$$\|w\|_{C^{2,\alpha}(B_1)}=o\big(\|k\|\big).$$

Now if $\omega=\lambda_{\ell m}$, for some $\ell\geqslant 2$, and $m\geqslant 1$, then (3.13) makes sense if and only if $k\in E_\ell$, and the same above conclusions hold true, by substituting E with E_ℓ. $\square$

Lemma 3.3. *If* $\omega \notin \{\lambda_{\ell m}\}_{\ell \geqslant 2, m \geqslant 1}$, *with* $\lambda_{\ell m} \neq \lambda_{1m'}$, *for all* $m' \geqslant 1$, *then the operator* Φ *is differentiable at* 0 *in* E, *and*

$$\langle d\Phi(0) \mid k \rangle = -k_0 \left(\frac{I_1'(\omega)}{I_0(\omega)} + \frac{I_0'(\omega)^2}{I_0(\omega)^2} \right) |\partial B_1|.$$

Otherwise if $\omega = \lambda_{\ell m}$, *for some* $\ell \geqslant 2$, *and* $m \geqslant 1$, *the same holds true by changing* E *with* E_ℓ.

The previous lemma means that if $\omega = \lambda_{\ell m}$, for some $\ell \geqslant 2$, and $m \geqslant 1$, then Φ is not differentiable at 0 in k, with k having the form

$$k = \sum_{m=1}^{d_\ell} k_{\ell m} Y_{\ell m}(\theta) + \sum_{p \in I} \sum_{q=1}^{d_p} k_{pq} Y_{pq}(\theta). \tag{3.14}$$

Proof of Lemma 3.3. By (2.3), (3.8), and (3.9), we obtain

$$\begin{aligned}
\Phi(k) &= \int_{\partial B_1} \left(G^{-1}x \cdot x\right)^{-1/2} G^{-1} \nabla u_p \cdot x \sqrt{\tilde{g}} - c \int_{\partial B_1} \sqrt{\tilde{g}} \\
&= \int_{\partial B_1} \left(G^{-1}x \cdot x\right)^{-1/2} G^{-1} \nabla u^{(0)} \cdot x \sqrt{\tilde{g}} - c \int_{\partial B_1} \sqrt{\tilde{g}} + \int_{\partial B_1} \left(G^{-1}x \cdot x\right)^{-1/2} G^{-1} \nabla u^{(1)} \cdot x \sqrt{\tilde{g}} + \cdots \\
&= c \int_{\partial B_1} (1 - 2k - 2\partial_{\mathbf{n}} k)^{1/2} \sqrt{\tilde{g}} - c \int_{\partial B_1} \sqrt{\tilde{g}} \\
&\quad + \int_{\partial B_1} (1 - 2k - 2\partial_{\mathbf{n}} k)^{-1/2} \left(\partial_{\mathbf{n}} u^{(1)} - G^{(1)} \nabla u^{(1)} \cdot x\right) \sqrt{\tilde{g}} + \cdots .
\end{aligned} \tag{3.15}$$

Since the surface element $\sqrt{\tilde{g}}$ can be written as

$$\sqrt{\tilde{g}} = 1 + o\left(\|k\|\right),$$

by taking the one-order terms in (3.15), we obtain

$$\langle d\Phi(0) \mid k \rangle = -c \int_{\partial B_1} (k + \partial_{\mathbf{n}} k) + \int_{\partial B_1} \partial_{\mathbf{n}} u^{(1)}.$$

Since

$$\partial_{\mathbf{n}} u^{(1)} = \left(\frac{I_0''(\omega)}{I_0(\omega)} + c \right) k + c \partial_{\mathbf{n}} k + \partial_{\mathbf{n}} \bar{u},$$

and

$$\partial_{\mathbf{n}} \bar{u} = -c\omega \left(k_0 I_0'(\omega)/I_0(\omega) + \sum_{p \geqslant 1} \sum_{q=1}^{d_p} k_{pq} I_p'(\omega)/I_p(\omega) Y_{pq}(\theta) \right),$$

we obtain

$$\begin{aligned}
\langle d\Phi(0) \mid k \rangle &= -c \int_{\partial B_1} (k + \partial_{\mathbf{n}} k) + \left(c - \frac{I_1'(\omega)}{I_0(\omega)} \right) \int_{\partial B_1} k + c \int_{\partial B_1} \partial_{\mathbf{n}} k + \int_{\partial B_1} \partial_{\mathbf{n}} \bar{u} \\
&= -\frac{I_1'(\omega)}{I_0(\omega)} \int_{\partial B_1} k - c\omega \frac{I_0'(\omega)}{I_0(\omega)} k_0 |\partial B_1| \\
&= -k_0 \left(\frac{I_1'(\omega)}{I_0(\omega)} + \frac{I_0'(\omega)^2}{I_0(\omega)^2} \right) |\partial B_1|,
\end{aligned}$$

being $c = \frac{I_0'(\omega)}{\omega I_0(\omega)}$. $\square$

Lemma 3.4. *The number*

$$\frac{I_1'(\omega)}{I_0(\omega)} + \frac{I_0'(\omega)^2}{I_0(\omega)^2} > 0. \tag{3.16}$$

Proof of Lemma 3.4. We have

$$\Phi(k_0) = \int\limits_{\partial B_{1+k_0}} \partial_{\mathbf{n}} u_p - c \int\limits_{\partial B_{1+k_0}} = \left(\frac{I_0'((1+k_0)\omega)}{I_0((1+k_0)\omega)} - \frac{I_0'(\omega)}{I_0(\omega)}\right)\frac{|\partial B_{1+k_0}|}{\omega}.$$

Now since the function

$$\frac{I_0'(\omega)}{I_0(\omega)}$$

is decreasing in ω, it follows that for $k_0 > 0$ sufficiently small, the function

$$\frac{I_0'((1+k_0)\omega)}{I_0((1+k_0)\omega)} - \frac{I_0'(\omega)}{I_0(\omega)} < 0.$$

So Φ is decreasing in the direction tk_0, for some $t \in I$, and then

$$\langle \mathrm{d}\Phi(0) \mid k_0 \rangle < 0,$$

which yields (3.16). □

4. Proof of Theorem 1.1

Before proceeding with the proof of Theorem 1.1, we need the following

Lemma 4.1. *There exists a neighborhood $\mathcal{O}$ of the origin in E, such that if $k \in \mathcal{O} \cap E_1^C$, then the mass center $\bar{x}$ of Ω_k is different to zero.*

Here E_1 is the vector space

$$E_1 = \{k \in E;\ k_{1q} = 0\},$$

of functions $k \in E$ which don't have the frequency 1, and

$$E_1^C = \{k \in E;\ k_{1q} \neq 0 \text{ for some } q = 1, \dots, N\},$$

the complementary of E_1, is the set of functions k which have the frequency 1. We recall that the mass center of a domain Ω is the point $\bar{x}$ of coordinates

$$\bar{x}_i = \frac{1}{|\Omega|}\int\limits_{\Omega} x_i, \quad i = 1, \dots, N.$$

Proof of Lemma 4.1. For $i = 1, \dots, N$, let us denote by F_i the following operator

$$F_i : E \to \mathbb{R},$$

defined by

$$F_i(k) = \frac{1}{|\Omega_k|}\int\limits_{\Omega_k} x_i,$$

i.e. the operator F_i associates to k the ith component of the mass center $\bar{x}$ of the domain Ω_k. By the change of variable (2.2), we obtain

$$F_i(k) = \frac{1}{|\Omega_k|}\int_{\Omega_k} x_i = \frac{1}{\int_{B_1}\sqrt{g}}\int_{B_1}(1+k)x_i\sqrt{g}$$
$$= \int_{B_1}(1 - Nk - x\cdot\nabla k + \cdots)\int_{B_1}\big(x_i + (N+1)kx_i + x\cdot\nabla k x_i + \cdots\big)$$
$$= \int_{B_1}(1 - Nk - x\cdot\nabla k + \cdots)\int_{B_1}\big((N+1)kx_i + x\cdot\nabla k x_i + \cdots\big).$$

By taking the one-order terms, we have that the differential of F_i at zero in k is given by

$$\big\langle \mathrm{d}F_i(0) \mid k\big\rangle = (N+1)\sum_{p\geqslant 1}\sum_{q=1}^{d_p} k_{pq}\int_0^1 r^{p+N}\int_{\partial B_1} Y_{pq}Y_{1i} + \sum_{p\geqslant 1}\sum_{q=1}^{d_p} pk_{pq}\int_0^1 r^{p+N-1}\int_{\partial B_1} Y_{pq}Y_{1i}$$
$$= (N+1)k_{1i}\int_0^1 r^{N+1} + k_{1i}\int_0^1 r^N$$
$$= \left(1 + \frac{1}{(N+2)(N+1)}\right)k_{1i}.$$

Let $k \in E_1^C$. Then there exists at least a $q \in \{1,\dots,N\}$ such that $k_{1q} \neq 0$. So there exists a neighborhood $\mathcal{O}$ of the origin in E such that F_q is increasing (or decreasing) in $\mathcal{O}\cap E_1^C$. Now, since $F_i(0) = 0$, we obtain that $\bar{x}_q \neq 0$. □

The previous lemma implies in particular that if the mass center of Ω_k is at the point zero, then k doesn't have the frequency 1, i.e. $k_{1q} = 0$ for all $q = 1,\dots,N$. This means that a domain Ω_k, with $k \in \mathcal{O}\cap E_1$ is either a domain with mass center at 0, or $\Omega_k = \sigma(\Omega_{\tilde{k}})$, for some $\sigma \in \Sigma$, and some domain $\Omega_{\tilde{k}}$, where Σ is the set of isometries of $\mathbb{R}^N$, and $\Omega_{\tilde{k}}$ has mass center at zero. Now since the operator Φ is invariant up to isometries, we obtain that Φ has a sign in a neighborhood $\mathcal{O}$ of 0 in E, if Φ has a sign in $\mathcal{O}\cap E_1$. For this reason in what follows we will concentrate our attention on the space E_1. We observe for example that the function

$$k' = x_0\cdot y - 1 + \sqrt{1 + |x_0\cdot y|^2 - |x_0|^2},$$

which parametrizes the sphere $\partial B_1(x_0)$ centered at x_0, has the frequency 1, which is equal to x_0, i.e. $k' \in E_1^C$. In fact the function

$$h(y) = \sqrt{1 + |x_0\cdot y|^2 - |x_0|^2}$$

is even in the variable y, and then the function hY_{1m} is odd, which implies that $\int_{\partial B_1} hY_{1m} = 0$, for all $m = 1,\dots,N$.

Proof of Theorem 1.1. *Step 1*. Let assume that $\omega \notin \{\lambda_{\ell m}\}_{\ell\geqslant 2, m\geqslant 1}$, with $\lambda_{\ell m} \neq \lambda_{1m'}$, for all $m' \geqslant 1$. Let us define by

$$E_\epsilon^+ = \big\{k \in E_1;\ \|k\| = 1,\ k_0 \leqslant -\epsilon\big\},$$

and by

$$E_\epsilon^- = \big\{k \in E_1;\ \|k\| = 1,\ k_0 \geqslant \epsilon\big\},$$

for some positive constant $\epsilon < 1$. We have

$$\big\langle \mathrm{d}\Phi(0) \mid k\big\rangle \geqslant \epsilon C|\partial B_1| \quad \text{for all } k \in E_\epsilon^+,$$

and

$$\big\langle \mathrm{d}\Phi(0) \mid k\big\rangle \leqslant -\epsilon C|\partial B_1| \quad \text{for all } k \in E_\epsilon^-,$$

where $C = \frac{I_1'(\omega)}{I_0(\omega)} + \frac{I_0'(\omega)^2}{I_0(\omega)^2}$. So there exists a sufficiently small interval I of 0 in $\mathbb{R}^+$ such that Φ is *positive* in

$$E^+ = \{tk;\ t \in I,\ k \in E_\epsilon^+\}, \tag{4.1}$$

and Φ is *negative* in

$$E^- = \{tk;\ t \in I,\ k \in E_\epsilon^-\}. \tag{4.2}$$

Let $\mathcal{O}$ be a neighborhood of 0 in E such that $\mathcal{O} \cap E^+ \cup \{0\}$ is contained in $E^+ \cup \{0\}$, and $\mathcal{O} \cap E^- \cup \{0\}$ is contained in $E^- \cup \{0\}$. Now if $\omega = \lambda_{\ell m}$, for some $\ell \geqslant 2$, and $m \geqslant 1$, the same above conclusions hold true by changing E_1 with the subspace

$$E_\ell = \{k \in E_1;\ k_{\ell q} = 0,\ k_{pq'} = 0,\ p \in I\}$$

of E_1. Now since for example Φ is positive in $E^+ \cap E_\ell$ and is continuous in E^+, and E_ℓ is finite dimensional, it follows that Φ is positive in E^+.

Step 2. Let $\mathcal{D}$ be the class of $C^{2,\alpha}$-domains defined as

$$\mathcal{D} = \{\Omega;\ \Omega = \sigma(\Omega_k)\},$$

for some $\sigma \in \Sigma$, and some $\Omega_k \in \mathcal{G}$, where Σ is the set of isometries of $\mathbb{R}^N$, and

$$\mathcal{G} = \{\Omega_k;\ k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{0\})\}.$$

Let assume that there exists a $\Omega \in \mathcal{D}$ such that $\frac{1}{|\partial\Omega|}\int_{\partial\Omega} \partial_{\mathbf{n}} u = c$. Since the problem is invariant up to isometries we have that $\frac{1}{|\partial\Omega_k|}\int_{\partial\Omega_k} \partial_{\mathbf{n}} u = c$, for some $k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{0\})$.

Step 3. Let assume that the kernel $\ker(\Delta + \omega^2) = \{0\}$ in Ω_k. Then u coincides with u_p, and

$$\Phi(k) = 0.$$

Let assume that $k \in \mathcal{O} \cap E^+ \cup \{0\}$. This yields that $k = 0$, since Φ is positive in $\mathcal{O} \cap E^+$. Now if the kernel $\ker(\Delta + \omega^2) \neq \{0\}$ in Ω_k, then u can be written as

$$u = u_p + u_h \quad \text{in } \Omega_k.$$

Since by Fredholm theorem $-1 \in \ker(\Delta + \omega^2)^\perp$, by divergence theorem we obtain

$$0 = \int_{\Omega_k} u_h = -\frac{1}{\omega^2}\int_{\Omega_k} \Delta u_h = -\frac{1}{\omega^2}\int_{\partial\Omega_k} \partial_{\mathbf{n}} u_h.$$

Then we have

$$\Phi(k) = \int_{\partial\Omega_k} \partial_{\mathbf{n}} u_p - c\int_{\partial\Omega_k} = \int_{\partial\Omega_k} \partial_{\mathbf{n}} u - c\int_{\partial\Omega_k} = 0. \qquad \square$$

We conclude this section by examining briefly the Lipschitz case. Let us define by

$$E = \{k \in C^{0,1}(\partial B_1)\}.$$

Let $u \in H^1(\Omega_k)$ be a weak solution to (1.1), when $\Omega = \Omega_k$, and $k \in E$. Then u solves

$$\int_{\Omega_k} \nabla u \cdot \nabla\phi - \omega^2\int_{\Omega_k} u\phi = \int_{\Omega_k} \phi,$$

for all $\phi \in C_c^\infty(\Omega_k)$. Since, by regularity results, $u \in C^{0,1}(\overline{\Omega}_k)$, the operator Φ is well-defined in E. By repeating the same arguments as in the regular case, one can prove the following

Theorem 4.2. *For $\omega \notin \{\lambda_{0m}\}_{m\geqslant 1}$, there exists a class $\mathcal{D}$ of Lipschitz domains, such that if $u \in H^1(\Omega)$ is a weak solution to* (1.1) *verifying*

$$\frac{1}{|\partial\Omega|}\int\limits_{\partial\Omega} \partial_{\mathbf{n}} u = c,$$

with $\Omega \in \mathcal{D}$, and c given by (1.4)*, then $\Omega = B_1$, and $u = u^{(0)}$.*

5. Concluding remark

We recall that by the proof of Theorem 1.1 it follows that Φ is positive in the circular sector E^+ in $\{k \in E;\ k_0 < 0\}$, and is negative in the circular sector E^- in $\{k \in E;\ k_0 > 0\}$. So the operator Φ must vanish somewhere. In fact let $\epsilon > 0$ be fixed. Let $k \in E^-$. Then $\Phi(k)$ is negative. Now the domain $\tilde{\Omega}_k$, whose boundary is given by

$$\partial\tilde{\Omega}_k = \bigl\{x = \bigl(1 + (a+k)\bigr)y,\ y \in \partial B_1\bigr\},$$

with $-1 < a < 0$, is a contraction of the domain Ω_k. We can find then a value a such that $a + k \in E^+$. But $\Phi(a+k)$ is positive. Then there exists a $\bar{k}$ such that $\Phi(\bar{k}) = 0$. By repeating the same argument for all $\epsilon > 0$, and for all $k \in E^-$, we can find a variety $\mathcal{M}$ in E_1 (whose tangent space at 0 is contained or coincides with $E_0 = \{k;\ k_0 = 0\}$), such that Φ vanishes identically on $\mathcal{M}$. In particular we obtain that all domains Ω lying in the class

$$\mathcal{D} = \bigl\{\Omega;\ \Omega = \sigma(\Omega_k)\bigr\},$$

for some $\sigma \in \Sigma$, and some $k \in \mathcal{M}$, are counter-examples to Theorem 1.1.

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