Infinite-dimensional attractors for parabolic equations with p-Laplacian in heterogeneous medium

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Dedicated to Professor Alain Haraux on the occasion of his 60th birthday

Abstract

In this paper we give a detailed study of the global attractors for parabolic equations governed by the p-Laplacian in a heterogeneous medium. Not only the existence but also the infinite dimensionality of the global attractors is presented by showing that their ε-Kolmogorov entropy behaves as a polynomial of the variable $1/\varepsilon$ as ε tends to zero, which is not observed for non-degenerate parabolic equations. The upper and lower bounds for the Kolmogorov ε-entropy of infinite-dimensional attractors are also obtained.
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1. Introduction

In order to describe the long-time behaviour of many dissipative systems generated by the evolution of PDEs in mathematical physics, one often uses the notion of the so-called global attractor, which is a compact invariant set in the phase space X which attracts the images of all bounded subsets under the temporal evolution. If the global attractor exists, its properties guarantee that the dynamical system reduced to the attractor $\mathcal{A}$ contains all the nontrivial dynamics of the initial system. One of the important questions in this theory is in what sense the dynamics reduced to the attractor are finite or infinite dimensional. It is well known that usually for regular (non-degenerate) dissipative autonomous PDEs in a bounded domain Ω, the Kolmogorov ε-entropy $H_\varepsilon(\mathcal{A}, X)$ of their attractors $\mathcal{A}$ has the asymptotic property such as:

$$C_1|\Omega| \log_2(1/\varepsilon) \leqslant H_\varepsilon(\mathcal{A}, X) \leqslant C_2|\Omega| \log_2(1/\varepsilon),$$

where $|\Omega|$ denotes the volume of Ω and C_i $(i = 1, 2)$ are some constants independent of $|\Omega|$.

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In contrast to non-degenerate parabolic PDEs, not so much is known on the long-time behaviour for the degenerate case (see [6,12,8,5]). In particular, the degenerate parabolic equations of the form

$$(E) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta_p u(x,t) + u(x,t), & (x,t) \in \Omega \times [0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

with $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, $p > 2$ are investigated in our previous paper [3], where the following features are revealed:

(a) infinite dimensionality of attractor,
(b) polynomial asymptotics of its ε-Kolmogorov entropy,
(c) difference of the asymptotics of the ε-Kolmogorov entropy depending on the choice of the underlying phase spaces,

which one cannot observe in non-degenerate cases.

Remark 1. It is also known that some non-degenerate parabolic equations in unbounded domains may possess infinite-dimensional global attractors. For these cases, however, the asymptotics of their Kolmogorov entropy are always logarithmic (see [4]).

Remark 2. It should be noted that the usual method for obtaining lower bounds of the Kolmogorov entropy (or dimension) of attractors is based on the instability index of hyperbolic equilibria (see [13]), which in turn requires a differentiability of the associated semigroup with respect to the initial data. However, this method may not be applicable for degenerate parabolic equations, since the associated semigroups are usually not differentiable.

The main purpose of the present paper is to give a detailed study of the global attractors for much more wider class of parabolic equations with p-Laplacian in a heterogeneous medium, that is

$$(E)_p \quad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta_p u(x,t) - g\big(x, u(x,t)\big) + h(x), & (x,t) \in \Omega \times [0,\infty), & \text{(E1)} \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), & \text{(E2)} \\ u(x,0) = u_0(x), & x \in \Omega, & \text{(E3)} \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

The present paper is composed as follows. In Section 1, we first prepare basic results on the existence, uniqueness and regularity of solutions of (E1)–(E3) and under these preparations, the existence of the global attractor for the semigroup generated by (E1)–(E3) will be presented in Section 2. The infinite dimensionality of the global attractors and the asymptotics of their Kolmogorov entropy is given in Section 3. In particular, we show that its Kolmogorov entropy admits polynomial asymptotics which shed light on completely new phenomena (see Remark 1). We here note that one cannot apply the direct approach developed in [3] for $(E)_p$, so to achieve our goal, it requires new ideas. To carry this through, we rely on some comparison results and some special scale transformations which will play a very important role in our arguments. Thus parabolic equations with p-Laplacian of the form (E1) give natural examples of dissipative equations of mathematical physics in heterogeneous media with infinite-dimensional attractors. We especially emphasize that the method developed in this paper has a general nature and can be applied to other classes of degenerate evolution equations.

2. Existence of global solutions and a priori estimates

In this section, we investigate the solvability of the initial–boundary value problem $(E)_p$. To this end, we assume that $g(x,\xi)$ can be decomposed into two parts, the monotone part $g_0(x,\xi)$ and the non-monotone part $g_1(x,\xi)$, i.e., $g(x,\xi) = g_0(x,\xi) + g_1(x,\xi)$ and we further assume that g_0, g_1 satisfy the following conditions:

(a) $g_0(x, 0) = 0$, $g_0(x, \xi) \in C(\overline{\Omega} \times \mathbb{R}^1)$ and $g_0(x, \xi)$ is monotone increasing with respect to ξ for all $x \in \Omega$.
(b) $g_1(x, \xi) \in C(\overline{\Omega} \times \mathbb{R}^1)$ and $g_1(x, \xi)$ is a globally Lipschitz function with respect to ξ, i.e., there exists a constant $L > 0$ such that

$$\sup_{x \in \Omega} |g_1(x, \xi) - g_1(x, \eta)| \leqslant L|\xi - \eta| \quad \forall \xi, \eta \in \mathbb{R}^1. \tag{1}$$

Then $(\text{E})_p$ admits a unique global solution in the following sense.

Theorem 1. *Assume that* (a) *and* (b) *are satisfied. Then, for any* $u_0, h \in L^2(\Omega)$*, there exists a unique solution* u *of* $(\text{E})_p$ *satisfying*

$$u \in C([0, T]; L^2(\Omega)), \qquad \sqrt{t}u_t \in L^2(0, T; L^2(\Omega)), \qquad \Delta_p u \in L^2_{\text{loc}}((0, T]; L^2(\Omega)),$$

$$u \in L^p(0, T; W_0^{1,p}(\Omega)), \qquad \mathcal{G}_0(u) \in L^1(0, T),$$

$$t\|\nabla u\|_{L^p(\Omega)}^p, t\mathcal{G}_0(u) \in L^\infty(0, T), \quad \forall T \in (0, \infty),$$

where $\mathcal{G}_0(u) = \int_\Omega G_0(x, u(x, t)) \, dx$, $G_0(x, \xi) = \int_0^\xi g_0(x, s) \, ds$.
Furthermore $S(t) : u_0(\cdot) \mapsto u(\cdot, t)$ *is continuous in the strong topology of* $L^2(\Omega)$.

Proof. By assumption (a), it is immediate to see that $u \mapsto -\Delta_p u + g_0(\cdot, u)$ is monotone in $L^2(\Omega)$. However, from (a) alone it is impossible to say whether it is maximal monotone in $L^2(\Omega)$. Hence, to show the existence of solutions for $(\text{E})_p$, we cannot rely on the solvability of abstract evolution equations governed by maximal monotone operators with Lipschitz perturbations (see [1]). To cope with this difficulty, we introduce some approximation procedure and make use of L^∞-Energy Method (see [10,11]) and the smoothing effect of the p-Laplacian in L^∞-space.

Step 1. Let $u_0, h \in L^\infty(\Omega)$ and consider

$$(\text{E})_p^M \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u + \partial I_M(u) + g(x, u) \ni h, & (x, t) \in \Omega \times [0, \infty), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times [0, \infty), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{2}$$

where $I_M(\cdot)$ is the indicator function of the closed convex set $\{u \in L^2(\Omega); \ |u| \leqslant M \text{ a.e. } x \in \Omega\}$:

$$I_M(u) = \begin{cases} 0 & \text{if } |u(x)| \leqslant M \text{ a.e. } x \in \Omega, \\ +\infty & \text{otherwise} \end{cases}$$

and $\partial I_M(\cdot)$ denotes its subdifferential operator given by

$$\partial I_M(u) = \begin{cases} 0 & \text{if } |u(x)| < M, \\ [0, +\infty] & \text{if } u(x) = M, \\ [-\infty, 0] & \text{if } u(x) = -M. \end{cases} \tag{3}$$

Here we define $\varphi_M(\cdot)$ by

$$\varphi_M(u) = \begin{cases} \frac{1}{p}\|\nabla u\|_{L^p}^p + I_M(u) & \text{if } |u(x)| \leqslant M \text{ a.e. } x \in \Omega, \ u \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\varphi_M(\cdot)$ is a lower semicontinuous convex functional from $L^2(\Omega)$ into $[0, +\infty]$ and its subdifferential $\partial \varphi_M(\cdot)$ satisfies

$$\partial \varphi_M(u) = -\Delta_p u + \partial I_M(u),$$

$$D(\varphi_M) = \{u \in W_0^{1,p}(\Omega); \ |u(x)| \leqslant M \text{ a.e. } x \in \Omega\},$$

$$D(\partial \varphi_M) = \{u \in D(\varphi_M); \ \Delta_p u \in L^2(\Omega)\}.$$

By putting $B(u) = g(\cdot, u(\cdot))$, we can reduce our approximate equation $(\text{E})_p^M$ to the following abstract equation:

$$(\text{AE})_p^M \quad \frac{du}{dt}(t) + \partial \varphi_M(u(t)) + B(u(t)) \ni h, \qquad u(0) = u_0.$$

In order to assure the existence of global solutions of $(\text{AE})_p^M$, we apply Theorem III and Corollary IV of [9] by taking $H = L^2(\Omega)$, $\varphi^t(\cdot) = \varphi_M(\cdot)$, $B(t, \cdot) = B(\cdot) = g(\cdot, u(\cdot))$. To this end, we need to check compactness condition (A1), demiclosedness condition (A2) and boundedness conditions (A5) and (A6) given in [9]. In fact, for any $L > 0$, the level set $\{u;\ \varphi_M(u) + \|u\|_H^2 \leqslant L\}$ is compact in $L^2(\Omega)$ by virtue of Rellich's compact embedding theorem, which assures (A1). The demiclosedness of the operator $B : u \mapsto B(u) = g(\cdot, u(\cdot))$ in $L^2(\Omega) \times L^2(\Omega)$ (i.e., the graph $G(B)$ of B is closed in $L_s^2(\Omega) \times L_w^2(\Omega)$ endowed with the strong topology L_s^2 and the weak topology L_w^2) is easily derived from (a) and (b), whence (A2) follows. As for the boundedness condition, since $u \in D(\partial \varphi_M)$ implies that $|u(x)| \leqslant M$, we easily get

$$\big\| B(u) \big\|_{L^2(\Omega)} \leqslant C_M \quad \forall u \in D(\partial \varphi_M),$$

whence follow (A5) and (A6). Thus we see that for any $u_0 \in \overline{D(\partial \varphi_M)} = \{u;\ |u(x)| \leqslant M\}$ and $h \in L^2(\Omega)$, $(\text{AE})_p^M$ admits a global solution u satisfying $u \in C([0, T]; L^2(\Omega))$, $\sqrt{t} u_t, \sqrt{t} \Delta_p u \in L^2(0, T; L^2(\Omega))$ $\forall T > 0$. Furthermore, the uniqueness follows easily from the monotonicity of $u \mapsto -\Delta_p u + g_0(\cdot, u(\cdot))$ in $L^2(\Omega)$ and the Lipschitz continuity of $u \mapsto g_1(\cdot, u(\cdot))$ in $L^2(\Omega)$.

A priori estimate 1. We multiply $(\text{E})_p^M$ by $|u|^{r-2} u$ and integrate over Ω to get

$$\|u(t)\|_{L^r}^{r-1} \frac{d}{dt} \|u(t)\|_{L^r} + (r-1) \int_\Omega |u|^{r-2} |\nabla u|^p\, dx + \int_\Omega \partial I_M(u) |u|^{r-2} u\, dx$$

$$+ \int_\Omega g_0(x, u) |u|^{r-2} u\, dx \leqslant \int_\Omega \big\{ \big| g_1(x, u) \big| + \big| h(x) \big| \big\} |u|^{r-2} u\, dx$$

$$\leqslant L \|u\|_{L^r}^r + C \big(1 + \|h\|_{L^\infty} \big) \|u\|_{L^r}^{r-1}.$$

Here we used (1) and the fact that $\partial I_M(u) |u|^{r-2} u \geqslant 0$, $g_0(x, u) |u|^{r-2} u \geqslant 0$, which are assured by (3) and (a). Hence, by Gronwall's inequality, we have

$$\sup_{0 \leqslant t \leqslant T} \big\| u(t) \big\|_{L^r} \leqslant \big(\|u_0\|_{L^r} + C \big(1 + \|h\|_{L^\infty} \big) T \big) e^{LT}.$$

Here letting $r \to \infty$, we obtain (see [10,11])

$$\sup_{0 \leqslant t \leqslant T} \big\| u(t) \big\|_{L^\infty} \leqslant \big(\|u_0\|_{L^\infty} + C \big(1 + \|h\|_{L^\infty} \big) T \big) e^{LT} =: C_T. \tag{4}$$

For each $u_0 \in L^\infty(\Omega)$ and $T > 0$, fix M so that $C_T < M$, then by virtue of (3) and (4), we find that $\partial I_M(u(t)) = 0$ $\forall t \in [0, T]$, which implies that $u(t)$ gives a solution of $(\text{E})_p$ on $[0, T]$.

Step 2. Let $u_0^n, h^n \in L^\infty(\Omega)$, $u_0^n \to u_0$, $h^n \to h$ in $L^2(\Omega)$ as $n \to \infty$, and let u^n be the global solution of $(\text{E})_p$ with $u_0 = u_0^n$, $h = h^n$, i.e.,

$$(\text{E})_{p,n} \quad u_t^n - \Delta_p u^n + g_0\big(x, u^n\big) = -g_1\big(x, u^n\big) + h^n, \qquad u_0^n(0) = u_0^n.$$

A priori estimate 2. We first note that the monotonicity of $g_0(\cdot, \xi)$ and the definition of G_0 imply

$$G_0(x, \xi) = \int_0^\xi g_0(x, s)\, ds \leqslant g_0(x, \xi) \xi.$$

Hence, multiplying $(\text{E})_{p,n}$ by u^n, we get

$$\frac{1}{2} \frac{d}{dt} \big\| u^n(t) \big\|_{L^2}^2 + \big\| \nabla u^n(t) \big\|_{L^p}^p + \mathcal{G}_0\big(u^n(t)\big) \leqslant \big(\big\| g_1\big(\cdot, u^n\big) \big\|_{L^2} + \big\| h^n \big\|_{L^2} \big) \big\| u^n(t) \big\|_{L^2}$$

$$\leqslant L \big\| u^n(t) \big\|_{L^2}^2 + \big(C + \big\| h^n \big\|_{L^2} \big) \big\| u^n(t) \big\|_{L^2}, \tag{5}$$

whence it follows that

$$\sup_{0 \leqslant t \leqslant T} \|u^n(t)\|_{L^2}^2 + \int_0^T \left(\|\nabla u^n(t)\|_{L^p}^p + \mathcal{G}_0(u^n(t)) \right) dt \leqslant C_0 \left(\|u_0\|_{L^2}, \|h\|_{L^2} \right). \tag{6}$$

Next multiply $(E)_{p,n}$ by $t(u^n)_t$ and integrate over $[0, T]$ with respect to t, then we have

$$t\|(u^n)_t\|_{L^2}^2 + t\frac{d}{dt}\left(\frac{1}{p} \|\nabla u^n(t)\|_{L^p}^p + \mathcal{G}_0(u^n(t)) \right)$$

$$\leqslant \left(\|g_1(u^n(t))\|_{L^2} + \|h^n\|_{L^2} \right) t \|(u^n)_t\|_{L^2} \leqslant \frac{1}{2} t \|(u^n)_t\|_{L^2}^2 + C_0,$$

$$\sup_{0 \leqslant t \leqslant T} t\left(\frac{1}{p} \|\nabla u^n(t)\|_{L^p}^p + \mathcal{G}_0(u^n(t)) \right) + \int_0^T t \|(u^n)_t\|_{L^2}^2 \, dt \leqslant C_0. \tag{7}$$

Hence, from Eq. $(E)_{p,n}$, we derive the a priori bound

$$\left\| \sqrt{t} \left(-\Delta_p u^n + g_0(x, u^n) \right) \right\|_{L^2(Q)} \leqslant C_0, \quad Q = \Omega \times (0, T). \tag{8}$$

However, this does not assure the boundedness of $\|\sqrt{t}\Delta_p u^n\|_{L^2(Q)}$. Nevertheless, by applying a comparison theorem, which will be given in Lemma 1, between u^n and $v^\pm$, solutions of (17) satisfying (20), we can derive some a priori estimate for u^n in $L^\infty(\Omega)$ as follows. We first note that $v^\pm$ also satisfy (15), so by the same reasoning as used for (12), we can obtain the a priori bound for $\|v^\pm\|_{L^2(\Omega)}$. Then integrating (15) on $[0, T]$, we derive, by Poincaré's inequality, the a priori bound for $\|v^\pm\|_{L^p(Q)}$ depending only on $\|u_0\|_{L^2(\Omega)}$, $\|h\|_{L^2(\Omega)}$. Hence the right-hand side of (20) is bounded by some constant independent of n on $[\delta, T]$ for each $\delta > 0$, which together with (18) yields the following estimate:

$$\forall \delta > 0, \ \exists C_\delta = C(\delta, C_0) \quad \text{s.t.} \quad \sup_{\delta \leqslant t \leqslant T} \|u^n(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C_\delta,$$

whence, by (8), it follows that

$$\sup_{\delta \leqslant t \leqslant T} \|g_0(\cdot, u^n(t))\|_{L^\infty(\Omega)} \leqslant C_\delta, \qquad \|\Delta_p u^n\|_{L^2(\delta, T; L^2(\Omega))} \leqslant C_\delta. \tag{9}$$

Convergence. $w = u^m - u^n$ satisfies

$$w_t - \left(\Delta_p u^m - \Delta_p u^n \right) + \left(g_0(x, u^m) - g_0(x, u^n) \right) = - \left(g_1(x, u^m) - g_1(x, u^n) \right). \tag{10}$$

Multiplying (10) by w and using the monotonicity of $u \mapsto -\Delta_p u + g_0(\cdot, u)$ in $L^2(\Omega)$ and condition (b), we get

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{L^2}^2 \leqslant L \|w(t)\|_{L^2}^2,$$

whence it follows that

$$\|w(t)\|_{L^2} \leqslant \|u_0^m - u_0^n\|_{L^2} e^{Lt},$$

which implies that $\{u^n\}$ forms a Cauchy sequence in $L^2(\Omega)$. Thus, in view of (6), (7) and (9), we find that there exists a subsequence of $\{u^n\}$ denoted again by $\{u^n\}$ such that

$$u^n \to u \quad \text{strongly in } C\left([0, T]; L^2(\Omega)\right), \text{ and a.e. } (x, t) \in Q,$$

$$g_1(x, u^n) \to g_1(x, u) \quad \text{strongly in } C\left([0, T]; L^2(\Omega)\right),$$

$$\nabla u^n \to \nabla u \quad \text{weakly in } L^p\left([0, T]; L^p(\Omega)\right),$$

$$\sqrt{t} u_t^n \to \sqrt{t} u_t \quad \text{weakly in } L^2(Q),$$

$$g_0(x, u^n) \to g_0(x, u) \quad \text{weakly star in } L^\infty(Q_\delta),$$

$$\Delta_p u^n \to \Delta_p u \quad \text{weakly in } L^2(Q_\delta), \qquad Q_\delta = \Omega \times [\delta, T], \quad \forall \delta > 0.$$

Here we used the demiclosedness of $g_0(\cdot, u)$, $-\Delta_p u$, i.e., their graphs are closed in $\mathcal{H}_s \times \mathcal{H}_w$, where $\mathcal{H}_s$ and $\mathcal{H}_w$ denote $L^2(Q_\delta)$ endowed with the strong topology and the weak topology respectively. Furthermore, by virtue of the lower-semicontinuity of $\|\nabla u\|_{L^p}^p$, $\mathcal{G}_0(u)$, we easily see that $\|t|\nabla u|^p\|_{L^\infty(0,T;L^1(\Omega))}$, $\|\mathcal{G}_0(u)\|_{L^1(0,T)}$, $\|t\mathcal{G}_0(u)\|_{L^\infty(0,T)}$ are all bounded. $\quad\square$

Remark 3. If $g(x, u) \in C(\overline{\Omega} \times \mathbb{R}^1)$ satisfies $g(x, 0) = 0$, $g_\tau'(x, \tau) \geqslant -K$ $\forall (x, \tau) \in \Omega \times \mathbb{R}^1$, then by putting $g_0(x, \tau) = g(x, \tau) + K\tau$ $((g_0)_\tau'(x, \tau) \geqslant 0)$, $g_1(x, \tau) = -K\tau$, we find that $g(\cdot, u)$ falls within our class.

In particular, $g(x, \tau) = C_1(x)|\tau|^{q_1-2}\tau - C_2(x)|\tau|^{q_2-2}\tau$ $(2 \leqslant q_2 < q_1 < +\infty)$ satisfies (a) and (b), provided that $C_1(x), C_2(x) \in L^\infty(\Omega)$ and $0 < c_1 \leqslant C_1(x)$ $\forall x \in \Omega$. Since $g_\tau'(x, \tau) = |\tau|^{q_2-1}(C_1(x)(q_1 - 1)|\tau|^{q_1-q_2} - C_2(x)(q_2 - 1)) \geqslant -K$ $\forall (x, \tau) \in \Omega \times \mathbb{R}^1$.

As for the a priori bounds for solutions of (E1)–(E3), we obtain the following result.

Theorem 2. *Assume that* (a) *and* (b) *are satisfied and let* $\frac{2N}{N+2} < p < \infty$, $u_0 \in L^2(\Omega)$ *and* $h \in L^\infty(\Omega)$. *For the case* $\frac{2N}{N+2} < p \leqslant 2$, *we further assume*

$$|g_0(s)| \geqslant k_0|s|^{1+\theta} - k_1 \quad (\theta, k_0, k_1 > 0). \tag{11}$$

Then, for any $0 < \delta_1 < \delta_2 < \delta_3 \leqslant 1$, *there exist constants* C_1, C_2, C_3 *depending on* δ_1, δ_2, δ_3 *but not on the initial data* $u_0 \in L^2(\Omega)$ *such that every solution of* (E1)–(E3) *satisfies*

$$\|u(t)\|_{L^2(\Omega)} \leqslant C_1 \quad \forall t \in [\delta_1, +\infty), \tag{12}$$

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant C_2 \quad \forall t \in [\delta_2, +\infty), \tag{13}$$

$$\|u(t)\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant C_3 \quad \forall t \in [\delta_3, +\infty). \tag{14}$$

Proof. Multiply (E1) by u, then the same argument as for (5) gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u\|_{L^p}^p + \mathcal{G}_0(u) \leqslant L\|u(t)\|_{L^2}^2 + (C + \|h\|_{L^2})\|u\|_{L^2}. \tag{15}$$

Then by Poincaré's inequality, we get for the case $p > 2$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \gamma_1\|u\|_{L^2}^p \leqslant \gamma_2, \tag{16}$$

for some $\gamma_1, \gamma_2 > 0$. Hence Ghidaglia-type estimate (see [13]) assures (12). For the case $\frac{2N}{N+2} \leqslant p \leqslant 2$, (11) yields the same estimate as (16) with $p = 2 + \theta$, whence follows (12).

In order to derive the L^∞-estimate, we need the following lemma.

Lemma 1. *Let* $v^\pm$ *be the unique solution of*

$$v_t^\pm = \Delta_p v^\pm - \widetilde{k}_0|v^\pm|^\theta v^\pm - g_1(v^\pm) \pm (|h| + \widetilde{k}_1), \qquad v^\pm|_{\partial\Omega} = 0 \tag{17}$$

with the initial condition $v^\pm(x, 0) = \pm|u_0(x)|$ *respectively, where* $\widetilde{k}_i = 0$ *for* $p > 2$ *and* $\widetilde{k}_i = k_i$ *for* $p \leqslant 2$ $(i = 0, 1)$. *Then the solution* u *of* $(E)_p$ *satisfies*

$$v^-(x, t) \leqslant u(x, t) \leqslant v^+(x, t) \quad \text{for a.e. } (x, t) \in \Omega \times [0, +\infty). \tag{18}$$

Proof. Let $u^+(x, t)$ be the unique solution of $(E)_p$ with h and u_0 replaced by $|h|$ and $|u_0|$ respectively.

Then it is easy to see that $u^+ \geqslant 0$ and $(u^+ - v^+)$ satisfies

$$(u^+ - v^+)_t = \Delta_p u^+ - \Delta_p v^+ - g_0(u^+) + \widetilde{k}_0|v^+|^\theta v^+ - \widetilde{k}_1 - g_1(u^+) + g_1(v^+). \tag{19}$$

Multiply (19) by $[u^+ - v^+]^+ = \max(u^+(x, t) - v^+(x, t), 0)$. Then noting that

$$\big(\Delta_p u^+ - \Delta_p v^+, [u^+ - v^+]^+\big) = -\big\|\nabla\big([u^+ - v^+]^+\big)\big\|_{L^p}^p \leqslant 0,$$

$$\big(-g_0(u^+) + \widetilde{k}_0|v^+|^\theta v^+ - \widetilde{k}_1, [u^+ - v^+]^+\big) \leqslant -\widetilde{k}_0\big(|u^+|^\theta u^+ - |v^+|^\theta v^+, [u^+ - v^+]^+\big) \leqslant 0,$$

$$\big(-g_1(u^+) + g_1(v^+), [u^+ - v^+]^+\big) \leqslant L\big\|[u^+ - v^+]^+\big\|_{L^2}^2,$$

we get

$$\big\|[u^+ - v^+]^+(t)\big\|_{L^2}^2 \leqslant \big\|[u^+ - v^+]^+(0)\big\|_{L^2}^2 e^{2Lt},$$

whence follows $0 \leqslant u^+ \leqslant v^+$. By much the same argument as above, we also find $u \leqslant u^+$. Thus repeating this procedure for u^- (the solution of $(E)_p$ with h and u_0 replaced by $-|h|$ and $-|u_0|$) and v^-, we obtain (18). $\quad\square$

Proof of Theorem 2 (continued). Since Eqs. (17) have the simple form, the L^∞-estimate for $v^\pm$ has been fully investigated by many authors. For instance, Theorem 3.2 of Chapter 5 of [2] gives the estimate:

$$\sup_{x\in\Omega}\big|v^\pm(x,t)\big| \leqslant \max\left(1, C\Big(t^{\frac{p}{N}} + \frac{1}{t}\Big)^{\frac{1}{q-\delta}}\bigg(\int\limits_0^t\!\!\int\limits_\Omega |v^\pm|^\delta\, dx\, d\tau\bigg)^{\frac{p}{N(q-\delta)}}\right) \tag{20}$$

where $\delta = 2 + \theta$ for $p \leqslant 2$, $\delta = p$ for $p > 2$ and $q = p(N+2)/N$.

Here [2] requires the condition $p \leqslant \delta < p\frac{N+2}{N}$, which is obviously satisfied for $p > 2$. As for the case where $\frac{2N}{N+2} < p \leqslant 2$, since $\frac{2N}{N+2} < p$ implies $2 < p\frac{N+2}{N}$, we can choose a sufficiently small θ_0 such that $p \leqslant 2 + \theta_0 < p\frac{N+2}{N}$. If $\theta_0 < \theta$, it is clear that (11) is satisfied with θ and k_1 replaced by θ_0 and $k_0 + k_1$ respectively. As is seen in the proof of Theorem 1, a priori estimate 2, $v^\pm$ satisfy (16) with $p = p$ or $p = 2 + \theta$. Hence, integrating this on $[t, t+1]$, we get, by (12)

$$\sup_{t\geqslant 1}\big\|v^\pm\big\|_{L^\delta(t,t+1;L^\delta(\Omega))} \leqslant C_1. \tag{21}$$

Thus the estimate (13) is derived from (18), (20) and (21).

Now we can rewrite (E1) as $u_t(x,t) = \Delta_p u(x,t) + \widetilde{h}(x,t)$, where $\widetilde{h}(x,t) = -g_0(x, u(x,t)) - g_1(x, u(x,t)) + h(x)$. Note that (13) assures $\widetilde{h} \in L^\infty(\Omega \times [\delta_2, +\infty))$. Consequently, by virtue of Theorem 1.2 in Chapter 10 of [2], we can derive the $C^{1,\alpha}(\overline{\Omega})$-bound for u on $[\delta_3, +\infty)$. $\quad\square$

3. Global attractor and Kolmogorov's ε-entropy

Let Φ be a Banach space. The set $\mathcal{A} \subset \Phi$ is called a global attractor of the semigroup $S(t)$ in Φ if the following conditions are satisfied:

(1) The set $\mathcal{A}$ is a compact subset of the phase space Φ.
(2) It is strictly invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$.
(3) For every bounded subset $B \subset \Phi$, $\lim_{t\to\infty} \mathrm{dist}(S(t)B, \mathcal{A}) = 0$, where $\mathrm{dist}(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_\Phi$.

Our existence result for global attractors of (E1)–(E3) in $\Phi = L^2(\Omega)$ can be stated as follows.

Theorem 3. *Let all the assumptions in Theorem 2 be satisfied. Then the semigroup $S(t)$ associated with Eqs.* (E1)–(E3) *possesses a global attractor $\mathcal{A}$ in the phase space $L^2(\Omega)$ which is globally bounded in $C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0, 1]$ and has the following structure*: $\mathcal{A} := \mathcal{K}_0 := \{u(0); \{u(t)\}_{t\in\mathbb{R}^1} \in \mathcal{K}\}$, *where $\mathcal{K}$ is the set of all bounded solutions of* (E1)–(E2) *defined on $\mathbb{R}^1$, i.e.,*

$$\mathcal{K} = \left\{ \{u(t)\}_{t\in\mathbb{R}^1};\ u(t) \text{ is a solution of } (E1)\text{–}(E2) \text{ on } \mathbb{R}^1,\ \sup_{t\in\mathbb{R}^1}\big\|u(t)\big\|_{L^2} < +\infty \right\}.$$

Proof. In order to prove the existence of the global attractor $\mathcal{A}$ for (E1)–(E3), it suffices to show that the semigroup $S(t)$ associated with (E1)–(E3) is continuous in the topology of $L^2(\Omega)$ for each $t > 0$ and that there exists a pre-compact absorbing set $\mathcal{B}$ in $L^2(\Omega)$ such that for every $x \in L^2(\Omega)$, there exists $T = T(x) > 0$ such that $S(t)x \in \mathcal{B}$

$\forall t \in [T, +\infty)$. (See [13, Theorem 1.1].) For our case, the first property is assured by Theorem 1 and the second by Theorem 2. The characterization of $\mathcal{A}$ in terms of $\mathcal{K}$ is derived by standard arguments. $\quad \square$

Next we present lower bounds for Kolmogorov's ε-entropy of the attractor $\mathcal{A}$ in $\Phi = L^p(\Omega)$ $(1 \leqslant p \leqslant \infty)$, denoted by $\mathcal{H}_\varepsilon(\mathcal{A}, \Phi)$, which is defined by the base 2 logarithm of $\mathcal{N}_\varepsilon(\mathcal{A}, \Phi)$, that is, $\mathcal{H}_\varepsilon(\mathcal{A}, \Phi) := \log_2 \mathcal{N}_\varepsilon(\mathcal{A}, \Phi)$. Here we denote by $\mathcal{N}_\varepsilon(\mathcal{A}, \Phi)$ the minimal number of ε-balls in Φ that covers $\mathcal{A}$ (recall that $\mathcal{A}$ is a compact set in Φ). From now on we assume that $p > 2$, $h \equiv 0$ and $g(\cdot, u)$ satisfies the following assumption.

$(I)_g$ There exist an open bounded subset ω of Ω and $\alpha > 0$ such that $g^\alpha(x, u) = g(x, u) + \alpha u$ satisfies:

$\quad$ $(I)_1$ There exist $a(s)$, $\rho \geqslant 0$ and $h(x, s, v)$ satisfying

$$s^{\frac{-p+1}{p-2}} g^\alpha\left(x, s^{\frac{1}{p-2}} v\right) = a(s)|v|^\rho h(x, s, v), \quad (x, s, v) \in \omega \times [0, 1] \times \mathbb{R}^1,$$

$$h(x, s, v), h'_v(x, s, v) \in C\left(\overline{\omega} \times [0, 1] \times \mathbb{R}^1\right), \qquad h(x, s, 0) = 0,$$

$$a \geqslant 0, \qquad a \in L^1(0, 1), \qquad sa^2 \in L^1(0, 1).$$

$\quad$ $(I)_2$ There exist $C > 0$ and $\delta > 0$ such that

$$|v|^\rho \big| h(x, s, v) \big| \leqslant C|v|^{1+\delta}, \quad (x, s, v) \in \omega \times [0, 1] \times [0, 1].$$

Then our result on the infinite dimensionality of global attractors for (E1)–(E3) is as follows.

Theorem 4. *Let* $(I)_g$ *be satisfied and assume that* (E1)–(E3) *possess a global attractor* $\mathcal{A}$ *in the topology of* $L^2(\Omega)$. *Then the fractal dimension of* $\mathcal{A}$ *is infinite.*

The presentation of condition $(I)_g$ might be somewhat obscure. In order to clarify the meaning of this condition, we show below that it covers a very large class of nonlinearity.

(Ex.1). Let $g(x, u) = -\alpha u + b_1(x)|u|^{q_1-2}u - b_2(x)|u|^{q_2-2}u$, $\alpha > 0$, $2 < q_2 < q_1$, $b_1, b_2 \in C(\overline{\omega})$, then $g(\cdot, u)$ satisfies $(I)_g$.

In fact, since $g^\alpha(x, u) = b_1(x)|u|^{q_1-2}u - b_2(x)|u|^{q_2-2}u$, we get

$$s^{\frac{-p+1}{p-2}} g^\alpha\left(x, s^{\frac{1}{p-2}} v\right) = s^{\frac{-p+1}{p-2}} \left\{ b_1(x) s^{\frac{q_1-1}{p-2}} |v|^{q_1-2}v - b_2(x) s^{\frac{q_2-1}{p-2}} |v|^{q_2-2}v \right\}$$

$$= s^{\frac{q_2-p}{p-2}} |v|^{q_2-2} \left\{ b_1(x) s^{\frac{q_1-q_2}{p-2}} |v|^{q_1-q_2}v - b_2(x)v \right\}.$$

Hence we can put

$$a(s) = s^{\frac{q_2-p}{p-2}}, \qquad \rho = q_2 - 2 > 0, \qquad h(x, s, v) = b_1(x) s^{\frac{q_1-q_2}{p-2}} |v|^{q_1-q_2}v - b_2(x)v.$$

Then it is easy to see that

$$h(x, s, v), h'_v(x, s, v) \in C\left(\overline{\Omega} \times [0, 1] \times \mathbb{R}^1\right), \qquad h(x, s, 0) = 0,$$

$$|a|_{L^1} = \int_0^1 s^{\frac{q_2-p}{p-2}} \, ds = \frac{p-2}{q_2-2}, \qquad |sa^2|_{L^1} = \int_0^1 ss^{\frac{2q_2-2p}{p-2}} \, ds = \frac{p-2}{2q_2-4},$$

$$|v|^\rho \big| h(x, s, v) \big| \leqslant C|v|^{q_2-1} = C|v|^{1+\delta}, \quad (x, s, v) \in \omega \times [0, 1] \times [0, 1],$$

$$\delta = q_2 - 2 > 0.$$

(Ex.2). Let $g(x, u), g'_u(x, u), g''_u(x, u) \in C(\overline{\omega} \times \mathbb{R}^1)$, $g'_u(x, 0) = -\alpha < 0$, $g(x, 0) = 0$, then $g(\cdot, u)$ satisfies $(I)_g$.

In fact, we first note that

$$g(x, u) = g(x, 0) + g'_u(x, 0)u + \int_0^u (u - t)g''_u(x, t) \, dt, \qquad g^\alpha(x, u) = \int_0^u (u - t)g''_u(x, t) \, dt.$$

Then we get

$$
\begin{aligned}
s^{\frac{-p+1}{p-2}} g^{\alpha}\left(x, s^{\frac{1}{p-2}} v\right) &= s^{\frac{-p+1}{p-2}} \int_0^{s^{\frac{1}{p-2}} v} \left(s^{\frac{1}{p-2}} v - t\right) g_v''(x, t)\, dt \\
&= s^{\frac{-p+1}{p-2}} \int_0^v \left(s^{\frac{1}{p-2}} v - s^{\frac{1}{p-2}} t\right) g_v''\left(x, s^{\frac{1}{p-2}} t\right) s^{\frac{1}{p-2}}\, dt \\
&= s^{\frac{3-p}{p-2}} \int_0^v (v - t) g_v''\left(x, s^{\frac{1}{p-2}} t\right) dt,
\end{aligned}
$$

and we put

$$
\rho = 0, \qquad h(x, s, v) = \int_0^v (v - t) g_v''\left(x, s^{\frac{1}{p-2}} t\right) dt, \qquad a(s) = s^{\frac{3-p}{p-2}}.
$$

Hence we obtain

$$
h(x, s, v), h_v'(x, s, v) = \int_0^v g''\left(x, s^{\frac{1}{p-2}} t\right) dt \in C\left(\overline{\omega} \times [0, 1] \times \mathbb{R}^1\right), \qquad h(x, s, 0) = 0,
$$

$$
|a|_{L^1} = \int_0^1 s^{\frac{3-p}{p-2}}\, ds = (p - 2), \qquad |sa^2|_{L^1} = \int_0^1 ss^{\frac{6-2p}{p-2}}\, ds = \frac{p-2}{2},
$$

$$
|v|^\rho |h(x, s, v)| = \left| \int_0^v (v-t) g_v''\left(x, s^{\frac{1}{p-2}} t\right) dt \right| \leqslant \int_0^{|v|} \left| 1 - \frac{t}{v} \right| \left| g_v''\left(x, s^{\frac{1}{p-2}} t\right) \right| dt\, |v|
$$

$$
\leqslant \max_{(x,s) \in \overline{\omega} \times [0,1]} |g''(x, s)| |v|^2, \quad (x, s, v) \in \omega \times [0, 1] \times [0, 1],\ \delta = 1.
$$

In order to establish the estimate from below for ε-Kolmogorov entropy of our global attractor $\mathcal{A}$, we rely on the following fact.

Lemma 2. *Let* $\mathcal{K}^-$ *be the set of all bounded solutions of* (E1)–(E2) *on* $\mathbb{R}^1_-$, *i.e.,* $\mathcal{K}^- = \{\{u(t)\}_{t \in \mathbb{R}^1_-};\ u(t)$ *satisfies* (E1)– (E2) *on* $\mathbb{R}^1_-$, $\sup_{t \in \mathbb{R}^1_-} \|u(t)\|_{L^2} < +\infty\}$, *where* $\mathbb{R}^1_- := (-\infty, 0]$ *and let* $\mathcal{K}^-(t)$ *be the section of* $\mathcal{K}^-$ *at* $t = t \in \mathbb{R}^1_-$, *i.e.,* $\mathcal{K}^-(t) = \{u(t);\ \{u(t)\}_{t \in \mathbb{R}^1_-} \in \mathcal{K}^-\}$. *Then* $\mathcal{K}^-(0) \subset \mathcal{A}$ *holds true.*

Proof. Put $\mathcal{B} = \bigcup_{t \in \mathbb{R}^1_-} \mathcal{K}^-(t)$. Then, since $\mathcal{B}$ is bounded in $L^2(\Omega)$, for arbitrary $\eta > 0$, there exists $T > 0$ such that $\text{dist}(S(T)\mathcal{B}, \mathcal{A}) < \eta$. For any $a_0 \in \mathcal{K}^-(0)$, there exists $a_T \in \mathcal{K}^-(-T) \subset \mathcal{B}$ such that $S(T)a_T = a_0$. Hence we get

$$
\text{dist}(a_0, \mathcal{A}) = \text{dist}\left(S(T)a_T, \mathcal{A}\right) \leqslant \text{dist}\left(S(T)\mathcal{B}, \mathcal{A}\right) < \eta, \quad \forall \eta > 0,
$$

which implies $\text{dist}(a_0, \mathcal{A}) = 0$, i.e., $a_0 \in \bar{\mathcal{A}} = \mathcal{A}\ \forall a_0 \in \mathcal{K}^-(0)$. Thus $\mathcal{K}^-(0) \subset \mathcal{A}$ is derived. $\quad \square$

Before we proceed to the proof of Theorem 4, we prepare a couple of results on the following auxiliary equation:

$$
(\text{E})_p^t \quad \begin{cases} w_t = p_\alpha\left(\Delta_p w - a(t)|w|^\rho h(x, t, w)\right), & (x, t) \in \omega \times (0, 1), \\ w|_{\partial \omega} = 0, \quad t \in [0, 1], \qquad w(x, 0) = w_0(x), \quad x \in \omega, \end{cases} \tag{22}
$$

where $p_\alpha = \frac{1}{\alpha(p-2)}$. As for the solvability of this equation, the following result holds.

Lemma 3. *Let* $(I)_g$ *be satisfied. Then for every* $w_0 \in L^\infty(\omega)$, *there exists* $T_0 = T_0(\|w_0\|_{L^\infty}) > 0$ *such that* $(E)_p^t$ *admits a unique solution* w *on* $[0, T_0]$ *satisfying*

$$w \in C([0, T_0]; L^2(\omega)) \cap C((0, T_0]; W_0^{1,p}(\omega)) \cap L^\infty(\omega \times (0, T_0)),$$
$$\sqrt{t}w_t, \sqrt{t}\Delta_p w \in L^2(\omega \times (0, T_0)). \tag{23}$$

Furthermore there exists a (sufficiently small) $\varepsilon_0 > 0$ *such that if* $\|w_0\|_{L^\infty} \leqslant \varepsilon_0$, *then the solution* w *of* $(E)_p^t$ *given above can be continued up to* $[0, 1]$ *and satisfies* $\sup_{t \in [0,1]} \|w(t)\|_{L^\infty} \leqslant 1$.

Proof. We again apply the L^∞-Energy Method as in the proof of Theorem 1 (Step 1). Take $0 \leqslant a_n(t) \in C([0, 1])$ so that $a_n(t) \to a(t)$ in $L^1(0, T)$ and $\sqrt{t}a_n(t) \to \sqrt{t}a(t)$ in $L^2(0, T)$ as $n \to \infty$. Put $M = \|w_0\|_{L^\infty} + 1$ and consider the following equations:

$$(E)_{p,n}^{t,M} \quad \begin{cases} w_t^n \in p_\alpha(\Delta_p w^n - \partial I_M(w^n) - a_n(t)|w^n|^\rho h(x, t, w^n)) & \text{in } \omega \times (0, 1), \\ w^n|_{\partial\omega} = 0, \quad t \in [0, 1], \qquad w^n(x, 0) = w_0(x), \quad x \in \omega, \end{cases} \tag{24}$$

where $\partial I_M(\cdot)$ is the subdifferential of $I_M(\cdot)$, given by (3). As in the proof of Theorem 1, we can reduce $(E)_{p,n}^{t,M}$ to some abstract Cauchy problem such as $(AE)_p^M$, and apply Theorem III and Corollary IV of [9] by taking $H = L^2(\Omega)$, $\varphi^t(\cdot) = p_\alpha \varphi_M(\cdot)$, $B(t, w) = p_\alpha a_n(t)|w|^\rho h(x, t, w)$. Here we note that $w_0 \in \overline{D(\partial\varphi_M)} = \{u; |u(x)| \leqslant M = \|w_0\|_{L^\infty} + 1\}$. Thus the existence of solutions of (24) is verified.

A priori estimates. We first note that

$$|h(x, s, v)| = \left| \int_0^v h_\tau'(x, s, \tau) \, d\tau \right| \leqslant \ell_0(|v|)|v|,$$
$$\ell_0(r) = \max\{|h_v'(x, s, v)|; \ x \in \overline{\omega}, \ s \in [0, 1], \ |v| \leqslant r\}.$$

Here $\ell_0(r)$ is monotone increasing in $r \in [0, +\infty)$. Then, as in the proof of Theorem 1 (a priori estimate 1), by multiplying $(E)_{p,n}^{t,M}$ by $|u|^{r-2}u$, we get

$$\frac{1}{r}\frac{d}{dt}\|w^n(t)\|_{L^r}^r \leqslant p_\alpha a_n(t)\|w^n(t)\|_{L^\infty}^\rho \ell_0(\|w^n(t)\|_{L^\infty})\|w^n(t)\|_{L^r}^r,$$
$$\|w^n(t)\|_{L^r} \leqslant \|w_0\|_{L^r} + \int_0^t a_n(s)\ell_1(\|w^n(s)\|_{L^\infty})\|w^n(s)\|_{L^r} \, ds, \quad \text{with } \ell_1(r) = p_\alpha r^\rho \ell_0(r).$$

Then letting $r \to \infty$, we obtain (see [10,11])

$$\|w^n(t)\|_{L^\infty} \leqslant \|w_0\|_{L^\infty} + \int_0^t a_n(s)\ell(\|w^n(s)\|_{L^\infty}) \, ds, \quad \text{with } \ell(r) = \ell_1(r)r. \tag{25}$$

Here define a positive number δ and choose T_0 such that

$$\delta = \frac{1}{\ell(M) + 1}, \qquad \int_0^{T_0} a(s) \, ds < \frac{\delta}{2}. \tag{26}$$

Then, since $a_n \to a$ in $L^1(0, 1)$, there exists N such that

$$\int_0^{T_0} a_n(s) \, ds < \delta \quad \forall n \geqslant N. \tag{27}$$

We claim that

$$\left\| w^n(t) \right\|_{L^\infty} < \|w_0\|_{L^\infty} + 1 = M \quad \forall t \in [0, T_0], \ \forall n \geqslant N. \tag{28}$$

To see this, we put

$$z^n(t) = \|w_0\|_{L^\infty} + \int_0^t a_n(s)\ell\big(\|w^n(s)\|_{L^\infty}\big) \, ds.$$

Then it is clear that $z^n(t) \in C([0, 1])$ and $\|w^n(t)\|_{L^\infty} \leqslant z^n(t) \ \forall t \in [0, 1]$. Hence, since $\ell(\cdot)$ is monotone increasing, $z^n(\cdot)$ satisfies

$$z^n(t) \leqslant \|w_0\|_{L^\infty} + \int_0^t a_n(s)\ell\big(z^n(s)\big) \, ds \quad \forall t \in [0, 1], \tag{29}$$

and hence

$$z^n(t) < \|w_0\|_{L^\infty} + 1 = M \quad \forall t \in [0, T_0], \ \forall n \geqslant N. \tag{30}$$

In fact, if (30) does not hold, there exists $t_0 \in (0, T_0)$ such that $z^n(t_0) \geqslant M$, then since $z^n(t)$ is continuous on $[0, 1]$ and $z^n(0) = \|w_0\|_{L^\infty} < M$, there exists $t_1 \in (0, t_0)$ such that

$$z^n(t_1) = M, \qquad z^n(t) < M \quad \forall t \in [0, t_1).$$

Hence, by (26), (27) and (29), we get

$$M = z^n(t_1) = \|w_0\|_{L^\infty} + \int_0^{t_1} a_n(s)\ell\big(z^n(s)\big) \, ds \leqslant \|w_0\|_{L^\infty} + \ell(M) \int_0^{t_1} a_n(s) \, ds$$

$$\leqslant \|w_0\|_{L^\infty} + \ell(M)/\big\{\ell(M) + 1\big\} < \|w_0\|_{L^\infty} + 1 = M,$$

which leads to a contradiction. This yields (30), whence follows (28). Hence $\partial I_M(w^n(t)) = 0 \ \forall t \in [0, T_0], \ \forall n \geqslant N$, so $w^n(t)$ satisfies

$$w_t^n \in p_\alpha\big(\Delta_p w^n - a_n(t)|w^n|^\rho h(x, t, w^n)\big), \quad (x, t) \in \omega \times (0, T_0), \ n \geqslant N. \tag{31}$$

By virtue of (28), multiplication of (31) by w^n gives

$$\frac{1}{2}\frac{d}{dt}\left\| w^n(t) \right\|_{L^2}^2 + p_\alpha \left\| \nabla w^n(t) \right\|_{L^p}^p \leqslant C_0 a_n(t), \quad t \in (0, T_0), \ n \geqslant N,$$

where C_0 denotes a general constant depending only on $\|w_0\|_{L^\infty}$. Hence we get

$$\sup_{0 \leqslant t \leqslant T_0} \left\| w^n(t) \right\|_{L^2}^2 + \int_0^{T_0} \left\| \nabla w^n(t) \right\|_{L^p}^p \, dt \leqslant C_0 \quad \forall n \geqslant N. \tag{32}$$

Furthermore, the multiplication of (31) by $t w_t^n$ gives

$$t \left\| w_t^n(t) \right\|_{L^2}^2 + \frac{p_\alpha t}{p}\frac{d}{dt} \left\| \nabla w^n(t) \right\|_{L^p}^p \leqslant C_0 a_n(t) t \left\| w_t^n(t) \right\|_{L^2} \leqslant \frac{1}{2} t \left\| w_t^n(t) \right\|_{L^2}^2 + \frac{1}{2} C_0^2 t |a_n(t)|^2.$$

Then integrating this on $[0, T_0]$, by (32) we get

$$\left\| \sqrt{t} w_t^n \right\|_{L^2(\omega \times (0, T_0))} + \left\| \sqrt{t} \Delta_p w^n \right\|_{L^2(\omega \times (0, T_0))} + \sup_{0 \leqslant t \leqslant T_0} t \left\| \nabla w^n(t) \right\|_{L^p}^p \leqslant C_0. \tag{33}$$

Convergence. Since $-\Delta_p$ is monotone in L^2, we easily see that $U(t) = w^n(t) - w^m(t)$ satisfies

$$\frac{1}{2}\frac{d}{dt}\left\| U(t) \right\|_{L^2}^2 \leqslant p_\alpha\big(a_n(t)\widetilde{h}\big(x, t, w^n(t)\big) - a_m(t)\widetilde{h}\big(x, t, w^m(t)\big), U(t)\big)_{L^2},$$

where $\widetilde{h}(x, s, v) = |v|^\rho h(x, s, v)$ satisfies

$$\widetilde{h}'_v(x, s, v) = \rho |v|^{\rho-2} v h(x, s, v) + |v|^\rho h'_v(x, s, v)$$

$$= |v|^\rho \left(\rho \int_0^1 h'_v(x, s, \tau v) \, d\tau + h'_v(x, s, v) \right) \in C(\overline{\omega} \times [0, 1] \times \mathbb{R}^1).$$

Then, in view of (28), we have, for all $n, m \geqslant N$

$$\left| \widetilde{h}(x, t, w^n(x, t)) - \widetilde{h}(x, t, w^m(x, t)) \right| \leqslant C_0 \left| w^n(x, t) - w^m(x, t) \right|, \quad (x, t) \in \omega \times [0, T_0].$$

Hence, again by (28), we get

$$\frac{1}{2} \frac{d}{dt} \| U(t) \|_{L^2}^2 \leqslant \left| a_n(t) - a_m(t) \right| \left\| \widetilde{h}(\cdot, t, w^n(\cdot, t)) \right\|_{L^2} \| U(t) \|_{L^2} + C_0 a_m(t) \| U(t) \|_{L^2}^2,$$

$$\frac{d}{dt} \| U(t) \|_{L^2} \leqslant C_0 \left| a_n(t) - a_m(t) \right| + C_0 a_m(t) \| U(t) \|_{L^2}, \quad n, m \geqslant N.$$

Then we obtain

$$\sup_{0 \leqslant t \leqslant T_0} \| U(t) \|_{L^2} \leqslant C_0 \left(\int_0^{T_0} \left| a_n(t) - a_m(t) \right| \, dt \right) e^{\int_0^{T_0} a_m(t) \, dt}, \quad n, m \geqslant N, \tag{34}$$

which implies that $\{w^n(\cdot, t)\}_{n \geqslant N}$ forms a Cauchy sequence in $C([0, T_0]; L^2(\omega))$. Therefore, by virtue of (33), we can extract a subsequence of $\{w^n(t)\}$ denoted again by $\{w^n(t)\}$ such that

$$w^n \to w \quad \text{strongly in } C([0, T_0]; L^2(\omega)), \text{ and a.e. } (x, t) \in \omega \times [0, T_0],$$

$$\nabla w^n \to \nabla w \quad \text{weakly in } L^p([0, T_0]; L^p(\omega)),$$

$$\sqrt{t} w_t^n \to \sqrt{t} w_t \quad \text{weakly in } L^2(\omega \times [0, T_0]),$$

$$\sqrt{t} \Delta_p w^n \to \sqrt{t} \Delta_p w \quad \text{weakly in } L^2(\omega \times [0, T_0]),$$

$$\sqrt{t} a_n(t) |w^n|^\rho h(x, t, w^n(x, t)) \to \sqrt{t} a(t) |w|^\rho h(x, t, w(x, t)) \quad \text{weakly in } L^2(\omega \times [0, T_0]).$$

Furthermore, the fact that $t |\nabla w(t)|^p \in L^\infty(0, T_0; L^1(\omega))$ assures the continuity of $w(t)$ on $(0, T_0]$ in the weak topology of $W_0^{1,p}(\omega)$ and the fact that $\sqrt{t} w_t, \sqrt{t} \Delta_p w \in L^2(\omega \times [0, T_0])$ assures the absolute continuity of $\| \nabla w(t) \|_{L^p(\omega)}^p$ on $(0, T_0]$. Hence, by virtue of the uniform convexity of $W_0^{1,p}$, we find that $w(t) \in C((0, T_0]; W_0^{1,p}(\omega))$.

The uniqueness of the solution is derived by exactly the same arguments used for (34) with $a_n(\cdot) = a_m(\cdot) = a(\cdot)$.

Global existence. We first note that by assumption (I)$_2$ and the same verification as for (25) with $a_n(t)$, $w_n(t)$ replaced by $a(t)$, $w(t)$, we can obtain

$$\| w(t) \|_{L^\infty} \leqslant \| w_0 \|_{L^\infty} + \int_0^t a(s) p_\alpha C \| w(s) \|_{L^\infty}^{1+\delta} \, ds, \tag{35}$$

as long as $\sup_{0 \leqslant s \leqslant t} \| w(s) \|_{L^\infty} \leqslant 1$ holds true. Here we define $\varepsilon_0 \in (0, 1/2)$ by

$$\varepsilon_0 = \frac{1}{2} \left(\frac{1}{2^{1+\delta} C p_\alpha |a|_{L^1(0,1)} + 1} \right)^{1/\delta} \tag{36}$$

and claim that if $\| w_0 \|_{L^\infty} \leqslant \varepsilon_0$, then

$$\sup_{0 \leqslant t \leqslant 1} \| w(t) \|_{L^\infty} \leqslant 2\varepsilon_0 < 1, \tag{37}$$

which assures the existence of global solutions. To see this, we put

$$z(t) = \|w_0\|_{L^\infty} + \int_0^t a(s) p_\alpha C \|w(s)\|_{L^\infty}^{1+\delta} ds,$$

then $z(t)$ is continuous and satisfies $\|w(t)\|_{L^\infty} \leqslant z(t)$ and

$$z(t) \leqslant \|w_0\|_{L^\infty} + \int_0^t a(s) p_\alpha C z(s)^{1+\delta} ds. \tag{38}$$

In order to prove (37), it suffices to show that $\sup_{0 \leqslant t \leqslant 1} z(t) \leqslant 2\varepsilon_0$, provided that $\|w_0\|_{L^\infty} \leqslant \varepsilon_0$. Suppose that this does not hold, then there exists $t_1 \in (0, 1)$ such that

$$z(t_1) = 2\varepsilon_0, \qquad z(t) < 2\varepsilon_0 \quad \forall t \in [0, t_1).$$

Hence, by (36) and (38), we get

$$2\varepsilon_0 = z(t_1) \leqslant \|w_0\|_{L^\infty} + \int_0^{t_1} a(s) p_\alpha C z(s)^{1+\delta} ds \leqslant \varepsilon_0 + |a|_{L^1(0,t_1)} p_\alpha C 2^{1+\delta} \varepsilon_0^\delta \varepsilon_0 < 2\varepsilon_0,$$

which leads to a contradiction. $\quad\square$

Here we prepare a comparison result which enables us to compare solutions of $(\mathrm{E})_p^t$ with solutions of simplified equations.

Lemma 4. *Let w be a positive solution of $(\mathrm{E})_p^t$ on $[0, 1]$ satisfying $\|w(x, t)\|_{L^\infty(\omega \times [0,1])} \leqslant 1$ and (23) with $T_0 = 1$, and let $w^\pm$ satisfy (23) with $T_0 = 1$ and*

$$w_t^- \leqslant p_\alpha \big(\Delta_p w^- - Ca(t)w^-\big), \qquad p_\alpha \big(\Delta_p w^+ + Ca(t)w^+\big) \leqslant w_t^+, \quad t \in (0, 1),$$
$$w^-(x, 0) \leqslant w(x, 0) \leqslant w^+(x, 0).$$

Then it holds that $w^-(x, t) \leqslant w(x, t) \leqslant w^+(x, t)$ for a.e. $x \in \omega$, $\forall t \in [0, 1]$.

Proof. Since $\|w(x, t)\|_{L^\infty(\omega \times [0,1])} \leqslant 1$, by $(\mathrm{I})_2$, it is easy to see that w satisfies

$$p_\alpha \big(\Delta_p w - Ca(t)w\big) \leqslant w_t \leqslant p_\alpha \big(\Delta_p w + Ca(t)w\big) \quad \forall t \in (0, 1).$$

Hence we get

$$\frac{1}{p_\alpha} \big(w(t) - w^+(t)\big)_t \leqslant \Delta_p w(t) - \Delta_p w^+(t) + Ca(t) \big(w(t) - w^+(t)\big).$$

Multiplying this by $[w - w^+]^+(t) = \max(w(t) - w^+(t), 0)$, we have

$$\frac{1}{2p_\alpha} \frac{d}{dt} \big\|[w - w^+]^+(t)\big\|_{L^2}^2 \leqslant Ca(t) \big\|[w - w^+]^+(t)\big\|_{L^2}^2 \quad \text{a.e. } t \in (0, 1).$$

Then, integrating this on $[\delta, t]$ with $\delta > 0$ and applying Gronwall's inequality, we obtain

$$\big\|[w - w^+]^+(t)\big\|_{L^2}^2 \leqslant \big\|[w - w^+]^+(\delta)\big\|_{L^2}^2 e^{2p_\alpha C \int_0^t a(\xi) d\xi}. \tag{39}$$

Since $\|[w - w^+]^+(+0)\|_{L^2} = 0$, letting $\delta \to 0$ in (39), we conclude that $w(x, t) \leqslant w^+(x, t)$ for a.e. $x \in \omega$, $\forall t \in [0, 1]$. The assertion $w^-(x, t) \leqslant w(x, t)$ can be verified by much the same arguments as above. $\quad\square$

Now we are ready to prove Theorem 4.

Proof of Theorem 4. We introduce a new time scale $s^\pm = s^\pm(t)$ by

$$(\text{S})^\pm \quad \begin{cases} \dfrac{d}{dt} s^\pm(t) = e_p^\pm(t) := p_\alpha \big(e^{\mp \int_0^t p_\alpha Ca(\xi)\, d\xi} \big)^{2-p}, & t \in (0, 1), \\ s^\pm(0) = 0. \end{cases} \tag{40}$$

Since $e_p^\pm(t)$ is strictly positive and bounded on $[0, 1]$, there exist unique solutions $s^\pm(t)$ of $(\text{S})^\pm$, which are strictly increasing on $[0, 1]$. Define

$$T_1^\pm := s^\pm(1) = \int_0^1 e_p^\pm(\xi)\, d\xi, \tag{41}$$

then $0 < T_1^- < p_\alpha < T_1^+$ and $s^\pm(t) \in [0, T_1^\pm]\ \forall t \in [0, 1]$.

Consider

$$(\text{P}) \quad \frac{\partial}{\partial s} w(x, s) = \Delta_p w(x, s), \quad (x, s) \in \mathbb{R}^N \times (0, +\infty).$$

Then the following facts are well known:

(P1) L^r-norms are Lyapunov functions for (P), i.e., every solution $w(s)$ of (P) satisfies $\|w(s)\|_{L^r} \leqslant \|w(0)\|_{L^r}\ \forall s \in [0, +\infty),\ \forall r \in [1, \infty]$.

(P2) (P) admits the following Barenblatt-type solutions

$$w_{\delta, \gamma}(x, s) = (s + \delta)^{-k} \left[\gamma - q \left(\frac{|x|}{(s + \delta)^{k/N}} \right)^{\frac{p}{p-1}} \right]_+^{\frac{p-1}{p-2}}, \quad [r]_+ = \max(r, 0),$$

$$k = \left(p - 2 + \frac{p}{N} \right)^{-1}, \qquad q = \frac{p-2}{p} \left(\frac{k}{N} \right)^{\frac{1}{p-1}}, \quad \delta, \gamma > 0.$$

(P3) $\operatorname{supp} w_{\delta, \gamma}(x, s)$ is monotone increasing in s.

(P4) $(w_{\delta, \gamma}(x, s))_s,\ \Delta_p w_{\delta, \gamma}(x, s)$ belong to $C(\mathbb{R}^n \times [0, +\infty))$.

Let K be the unit ball in $\mathbb{R}^N$ centered at the origin, then by (P2), fixing the parameters δ, γ suitably, we can choose a solution $w_1(x, s)$ of (P) so that

$$\begin{aligned} &0 \leqslant w_1(x, s) \leqslant 1 \quad \forall(x, s) \in \mathbb{R}^n \times [0, T_1^+], \\ &\operatorname{supp} w_1(\cdot, s) \subset K \quad \forall t \in [0, T_1^+], \\ &\big\| w_1(\cdot, T_1^+) \big\|_{L^r} \geqslant \delta_0 \quad \forall r \in [1, \infty], \end{aligned} \tag{42}$$

where δ_0 is a positive constant independent of r. (Since we can assume that $\operatorname{supp} w_1(\cdot, T_1^+) = K_1$ is small enough to satisfy $|K_1| \leqslant 1$ without loss of generality, if we take $\|w_1(\cdot, T_1^+)\|_{L^1} \geqslant \delta_0$, then (42) is satisfied for all $r \in [1, \infty]$.)

Furthermore it is easy to see that $w_\varepsilon(x, s) := \varepsilon w(\varepsilon^{\frac{2-p}{p}} x, s)$ is a solution of (P) and satisfies

$$\begin{aligned} &0 \leqslant w_\varepsilon(x, s) \leqslant \varepsilon \quad \forall(x, s) \in \mathbb{R}^n \times [0, T_1^+], \\ &\operatorname{supp} w_\varepsilon(\cdot, s) \subset K_\varepsilon := \varepsilon^{\frac{p-2}{p}} K \quad \forall t \in [0, T_1^+]. \end{aligned} \tag{43}$$

Moreover $w_\varepsilon(x - x_i, s)$ also gives a solution of (P) and for sufficiently small ε, there exists a finite set $R_\varepsilon := \{x_i\} \subset \omega$ such that

(R1) $\qquad (x_i + K_\varepsilon) \cap (x_j + K_\varepsilon) = \emptyset, \quad \forall x_i, x_j \in R_\varepsilon,\ i \neq j,$

(R2) $\qquad \#R_\varepsilon \geqslant C_\omega \left(\dfrac{1}{\varepsilon} \right)^{\frac{N(p-2)}{p}},$

(R3) $\qquad \displaystyle\bigcup_{i=1}^{\#R_\varepsilon} (x_i + K_\varepsilon) \Subset \omega.$

Consequently, for every $\vec{m} \in \{0, 1\}^{\#R_\varepsilon} := \{\vec{m} = (m_1, m_2, \ldots, m_{\#R_\varepsilon}); \ m_j = 0 \text{ or } 1, \ j = 1, 2, \ldots, m_{\#R_\varepsilon}\}$, the function

$$w_{\vec{m},\varepsilon}(x, s) = \sum_{i=1}^{\#R_\varepsilon} m_i w_\varepsilon(x - x_i, s) \tag{44}$$

solves (P) and is supported in ω. On the other hand, for $\vec{m}^1 \neq \vec{m}^2$, we obviously have

$$\left\| w_{\vec{m}^1,\varepsilon}(x, s) - w_{\vec{m}^2,\varepsilon}(x, s) \right\|_{L^\infty} = \varepsilon \left\| w_1(x, s) \right\|_{L^\infty} \quad \forall s \in \left[0, T_1^+\right]. \tag{45}$$

Thus we find $2^{\#R_\varepsilon}$ different solutions of (P) supported in ω having the form (44). Furthermore, as for the measurement in the topology of L^r $(1 \leqslant r < \infty)$, instead of (45), we get

$$\left\| w_{\vec{m}^1,\varepsilon}(x, s) - w_{\vec{m}^2,\varepsilon}(x, s) \right\|_{L^r} \geqslant \varepsilon^{\frac{pr+N(p-2)}{pr}} \left\| w_1(x, s) \right\|_{L^r} \quad \forall s \in \left[0, T_1^+\right]. \tag{46}$$

Here we define new functions $V_{\vec{m},\varepsilon}^\pm(x, t)$ via new time scales $s^\pm(t)$ defined by (40) as follows:

$$V_{\vec{m},\varepsilon}^\pm(x, t) := w_{\vec{m},\varepsilon}\left(x, s^\pm(t)\right), \quad t \in [0, 1]. \tag{47}$$

Then, by (40), we easily find

$$\begin{cases} \dfrac{\partial}{\partial t} V_{\vec{m},\varepsilon}^\pm(x, t) = \dfrac{d}{dt} s^\pm(t) \dfrac{\partial}{\partial s^\pm} w_{\vec{m},\varepsilon}\left(x, s^\pm\right) = e_p^\pm(t) \Delta_p V_{\vec{m},\varepsilon}^\pm(x, t), \\[2mm] V_{\vec{m},\varepsilon}^\pm(x, 0) = w_{\vec{m},\varepsilon}(x, 0), \\[2mm] \text{supp } w_{\vec{m},\varepsilon}(\cdot, 0) \subset \text{supp } V_{\vec{m},\varepsilon}^\pm(\cdot, t) \subset \text{supp } w_{\vec{m},\varepsilon}\left(\cdot, T_1^\pm\right) \quad \forall t \in [0, 1]. \end{cases} \tag{48}$$

We further introduce new functions $\widetilde{w}_{\vec{m},\varepsilon}^\pm$ by

$$\widetilde{w}_{\vec{m},\varepsilon}^\pm(x, t) := e^\pm(t) V_{\vec{m},\varepsilon}^\pm(x, t), \qquad e^\pm(t) := e^{\pm \int_0^t p_\alpha C a(\xi)\, d\xi}, \quad t \in [0, 1]. \tag{49}$$

Then, by (48), it is easy to see

$$\frac{\partial}{\partial t} \widetilde{w}_{\vec{m},\varepsilon}^\pm(x, t) = e^\pm(t) \frac{\partial}{\partial t} V_{\vec{m},\varepsilon}^\pm(x, t) \pm p_\alpha C a(t) e^\pm(t) V_{\vec{m},\varepsilon}^\pm(x, t)$$

$$= p_\alpha \left(e^\pm(t)\right)^{p-1} \Delta_p V_{\vec{m},\varepsilon}^\pm(x, t) \pm p_\alpha C a(t) \widetilde{w}_{\vec{m},\varepsilon}^\pm(x, t).$$

Thus we find

$$\begin{cases} \dfrac{\partial}{\partial t} \widetilde{w}_{\vec{m},\varepsilon}^\pm(x, t) = p_\alpha \left(\Delta_p \widetilde{w}_{\vec{m},\varepsilon}^\pm(x, t) \pm C a(t) \widetilde{w}_{\vec{m},\varepsilon}^\pm(x, t)\right), \quad t \in [0, 1], \\[2mm] \widetilde{w}_{\vec{m},\varepsilon}^\pm(x, 0) = w_{\vec{m},\varepsilon}(x, 0), \\[2mm] \text{supp } w_{\vec{m},\varepsilon}(\cdot, 0) \subset \text{supp } \widetilde{w}_{\vec{m},\varepsilon}^\pm(\cdot, t) \subset \text{supp } w_{\vec{m},\varepsilon}\left(\cdot, T_1^\pm\right) \quad \forall t \in [0, 1]. \end{cases} \tag{50}$$

Let $\widetilde{w}_{\vec{m},\varepsilon}(x, t)$ be the unique solution of $(E)_p^t$ on $[0, 1]$ with $\widetilde{w}_{\vec{m},\varepsilon}(x, 0) = w_{\vec{m},\varepsilon}(x, 0)$, whose existence is assured by Lemma 3 for all $\varepsilon \in (0, \varepsilon_0]$. Then, by the comparison theorem given in Lemma 4 (note that $\widetilde{w}_{\vec{m},\varepsilon}^\pm(x, t)$ satisfy the regularity required there), we observe

$$\begin{cases} \dfrac{\partial}{\partial t} \widetilde{w}_{\vec{m},\varepsilon}(x, t) = p_\alpha \left(\Delta_p \widetilde{w}_{\vec{m},\varepsilon}(x, t) - a(t) |\widetilde{w}_{\vec{m},\varepsilon}|^\rho h(x, t, \widetilde{w}_{\vec{m},\varepsilon})\right), \\[2mm] \widetilde{w}_{\vec{m},\varepsilon}(x, 0) = w_{\vec{m},\varepsilon}(x, 0), \\[2mm] \widetilde{w}_{\vec{m},\varepsilon}^-(x, t) \leqslant \widetilde{w}_{\vec{m},\varepsilon}(x, t) \leqslant \widetilde{w}_{\vec{m},\varepsilon}^+(x, t) \quad \text{a.e. } \omega, \ \forall t \in [0, 1], \\[2mm] \text{supp } w_{\vec{m},\varepsilon}(\cdot, 0) \subset \text{supp } \widetilde{w}_{\vec{m},\varepsilon}(\cdot, t) \subset \text{supp } w_{\vec{m},\varepsilon}\left(\cdot, T_1^\pm\right) \quad \forall t \in [0, 1]. \end{cases} \tag{51}$$

Hence, by virtue of the choice of R_ε, supp $\widetilde{w}_{\vec{m},\varepsilon}(\cdot, t)$ does not touch the boundary of ω for all $t \in [0, 1]$. Therefore, the zero extension of $\widetilde{w}_{\vec{m},\varepsilon}(\cdot, t)$ to Ω, denoted again by $\widetilde{w}_{\vec{m},\varepsilon}(\cdot, t)$, becomes a solution of $(E)_p^t$ with ω replaced by Ω.

We introduce another time scale $\tau = \tau(t) \in (-\infty, 0]$ by

$$\tau(t) = p_\alpha \log t, \quad t \in (0, 1] \quad \Longleftrightarrow \quad t(\tau) = e^{\alpha(p-2)\tau}, \quad \tau \in (-\infty, 0], \qquad \frac{dt}{d\tau} = \alpha(p-2)t$$

and define a new function $\widetilde{W}_{\vec{m},\varepsilon}(x,\tau)$ by

$$\widetilde{W}_{\vec{m},\varepsilon}(x,\tau) = \widetilde{w}_{\vec{m},\varepsilon}\big(x, t(\tau)\big), \quad (x,\tau) \in \Omega \times (-\infty, 0].$$

Then we have

$$\begin{aligned}
\frac{\partial}{\partial \tau} \widetilde{W}_{\vec{m},\varepsilon}(x,\tau) &= \frac{dt}{d\tau} \frac{\partial}{\partial t} \widetilde{w}_{\vec{m},\varepsilon}(x,t) \\
&= \alpha(p-2)t p_\alpha \big(\Delta_p \widetilde{w}_{\vec{m},\varepsilon} - a(t)|\widetilde{w}_{\vec{m},\varepsilon}|^\rho h\big(x,t,\widetilde{w}_{\vec{m},\varepsilon}(t)\big)\big) \\
&= t\big(\Delta_p \widetilde{w}_{\vec{m},\varepsilon} - t^{\frac{-p+1}{p-2}} g^\alpha\big(x, t^{\frac{1}{p-2}} \widetilde{w}_{\vec{m},\varepsilon}(t)\big)\big).
\end{aligned}$$

Therefore we get

$$\begin{cases}
\dfrac{\partial}{\partial \tau} \widetilde{W}_{\vec{m},\varepsilon}(x,\tau) = e^{\alpha(p-2)\tau} \Delta_p \widetilde{W}_{\vec{m},\varepsilon}(x,\tau) - e^{-\alpha\tau} g^\alpha\big(x, e^{\alpha\tau} \widetilde{W}_{\vec{m},\varepsilon}(x,\tau)\big), \\
\widetilde{W}_{\vec{m},\varepsilon}(x,0) = \widetilde{w}_{\vec{m},\varepsilon}(x,1), \\
\operatorname{supp} w_{\vec{m},\varepsilon}(\cdot,0) \subset \operatorname{supp} \widetilde{W}_{\vec{m},\varepsilon}(\cdot,\tau) \subset \operatorname{supp} w_{\vec{m},\varepsilon}\big(\cdot, T_1^\pm\big) \quad \forall \tau \in (-\infty, 0].
\end{cases}$$

Now define

$$W_{\vec{m},\varepsilon}(x,\tau) = e^{\alpha\tau} \widetilde{W}_{\vec{m},\varepsilon}(x,\tau), \quad (x,\tau) \in \Omega \times (-\infty, 0],$$

then

$$\begin{aligned}
\frac{\partial}{\partial \tau} W_{\vec{m},\varepsilon}(x,\tau) &= \alpha W_{\vec{m},\varepsilon}(x,\tau) + e^{\alpha\tau} \frac{\partial}{\partial \tau} \widetilde{W}_{\vec{m},\varepsilon}(x,\tau) \\
&= \alpha W_{\vec{m},\varepsilon}(x,\tau) + e^{\alpha(p-1)\tau} \Delta_p \widetilde{W}_{\vec{m},\varepsilon}(x,\tau) - g^\alpha\big(x, W_{\vec{m},\varepsilon}(x,\tau)\big),
\end{aligned}$$

whence it follows that

$$\begin{cases}
\dfrac{\partial}{\partial \tau} W_{\vec{m},\varepsilon}(x,\tau) = \Delta_p W_{\vec{m},\varepsilon}(x,\tau) - g(x, W_{\vec{m},\varepsilon}), \quad (x,\tau) \in \Omega \times (-\infty, 0], \\
W_{\vec{m},\varepsilon}(x,0) = \widetilde{w}_{\vec{m},\varepsilon}(x,1), \\
\operatorname{supp} w_{\vec{m},\varepsilon}(\cdot,0) \subset \operatorname{supp} W_{\vec{m},\varepsilon}(\cdot,\tau) \subset \operatorname{supp} w_{\vec{m},\varepsilon}\big(\cdot, T_1^\pm\big) \quad \forall \tau \in (-\infty, 0].
\end{cases} \tag{52}$$

Thus we observe that

$$\big\{ W_{\vec{m},\varepsilon}(x,\tau) \big\}_{\tau \in \mathbb{R}_-^1} \in \mathcal{K}^- \quad \text{and} \quad W_{\vec{m},\varepsilon}(x,0) = \widetilde{w}_{\vec{m},\varepsilon}(x,1) \in \mathcal{K}^-(0). \tag{53}$$

Next we put $\vec{e}_i = (0,\ldots,1,\ldots,0)$, i.e., the j-th component of $\vec{e}_i = \delta_{ij}$ $(j = 1,2,\ldots,\#R_\varepsilon)$ and define

$$\widetilde{w}_\varepsilon(x-x_i,t) = \widetilde{w}_{\vec{e}_i,\varepsilon}(x,t), \quad t \in [0,1].$$

Then, by virtue of (43), (R1) and (51), we see that

$$\operatorname{supp} \widetilde{w}_\varepsilon(x-x_i,t) \cap \operatorname{supp} \widetilde{w}_\varepsilon(x-x_j,t) = \emptyset, \quad i \neq j, \ \forall t \in [0,1].$$

Hence we can write

$$\widetilde{w}_{\vec{m},\varepsilon}(x,t) = \sum_{i=1}^{\#R_\varepsilon} m_i \widetilde{w}_\varepsilon(x-x_i,t), \quad t \in [0,1].$$

Therefore, if $\vec{m}_1$ and $\vec{m}_2$ differ at the i-th component, we have, by (53)

$$\begin{aligned}
\big\| W_{\vec{m}_1,\varepsilon}(x,0) - W_{\vec{m}_2,\varepsilon}(x,0) \big\|_{L^r} &= \big\| \widetilde{w}_{\vec{m}_1,\varepsilon}(x,1) - \widetilde{w}_{\vec{m}_2,\varepsilon}(x,1) \big\|_{L^r} \\
&\geqslant \big\| \widetilde{w}_\varepsilon(x-x_i,1) \big\|_{L^r}.
\end{aligned} \tag{54}$$

Recalling (51), (47) and (49), we get

$$\widetilde{w}_\varepsilon(x-x_i,1) \geqslant \widetilde{w}_\varepsilon^-(x-x_i,1) = e^-(1) w_\varepsilon\big(x-x_i, T_1^-\big).$$

Hence, since $0 < T_1^- < T_1^+$, from (P1) and (42), we deduce that

$$
\begin{aligned}
\big\|\widetilde{w}_\varepsilon(x-x_i,1)\big\|_{L^r} &\geqslant e^-(1)\big\|w_\varepsilon\big(x-x_i,T_1^-\big)\big\|_{L^r} \\
&\geqslant e^-(1)\big\|w_\varepsilon\big(x-x_i,T_1^+\big)\big\|_{L^r} \\
&= e^-(1)\varepsilon^{\frac{pr+N(p-2)}{pr}}\big\|w_1\big(x,T_1^+\big)\big\|_{L^r} \\
&\geqslant e^-(1)\delta_0\varepsilon^{\frac{pr+N(p-2)}{pr}}.
\end{aligned}
\tag{55}
$$

Then, combining (54) with (55), we obtain

$$
\big\|W_{\vec{m}_1,\varepsilon}(x,0)-W_{\vec{m}_2,\varepsilon}(x,0)\big\|_{L^r}\geqslant e^-(1)\delta_0\varepsilon^{\frac{pr+N(p-2)}{pr}},
$$

which can be rewritten as

$$
\big\|W_{\vec{m}_1,(k\varepsilon)^\beta}(x,0)-W_{\vec{m}_2,(k\varepsilon)^\beta}(x,0)\big\|_{L^r}\geqslant 2\varepsilon,\quad \forall \varepsilon\in\big(0,(\varepsilon_0/k)^{1/\beta}\big),
$$

$$
\beta=\frac{pr}{pr+N(p-2)},\qquad k=\frac{2}{e^-(1)\delta_0}.
$$

This estimate implies that balls in L^r with radius $\varepsilon>0$ can contain at most one element belonging to $\{W_{\vec{m},(k\varepsilon)^\beta}(x,0);$ $\vec{m}\in\{0,1\}^{\#R_{(k\varepsilon)^\beta}}\}\subset\mathcal{K}^-(0)$ and its cardinality can be estimated, by (R2) and (53), as follows

$$
\#\big\{W_{\vec{m},(k\varepsilon)^\beta}(x,0);\ \vec{m}\in\{0,1\}^{\#R_{(k\varepsilon)^\beta}}\big\}\geqslant 2^{\#R_{(k\varepsilon)^\beta}},
$$

$$
\#R_{(k\varepsilon)^\beta}\geqslant C_\omega\bigg(\frac{1}{(k\varepsilon)^\beta}\bigg)^{\frac{N(p-2)}{p}}=C_\omega\bigg(\frac{1}{k\varepsilon}\bigg)^{\frac{Nr(p-2)}{pr+N(p-2)}}.
$$

Hence, by the definition of $\mathcal{N}_\varepsilon$, $\mathcal{H}_\varepsilon$ and Lemma 2, we get

$$
\mathcal{N}_\varepsilon\big(\mathcal{K}^-(0),L^r\big)\geqslant 2^{\#R_{(k\varepsilon)^\beta}},
$$

$$
\mathcal{H}_\varepsilon\big(\mathcal{A},L^r\big)\geqslant C_\omega\bigg(\frac{1}{k\varepsilon}\bigg)^{\frac{Nr(p-2)}{pr+N(p-2)}},\quad r\in[1,\infty].
$$

Moreover, since Kolmogorov's ε-entropy of a bounded set of $C^1(\overline{\Omega})$ in the topology of L^∞ is estimated from above by $C(\frac{1}{\varepsilon})^N$ (see [7]), we obtain

$$
C_\omega\bigg(\frac{1}{k\varepsilon}\bigg)^{\frac{N(p-2)}{p+N(p-2)}}\leqslant C_\omega\bigg(\frac{1}{k\varepsilon}\bigg)^{\frac{Nr(p-2)}{pr+N(p-2)}}\leqslant\mathcal{H}_\varepsilon\big(\mathcal{A},L^r\big)\leqslant C\bigg(\frac{1}{\varepsilon}\bigg)^N\quad\forall r\in[1,\infty].
\tag{56}
$$

To complete the proof of Theorem 4, it remains to recall that $\dim_F(\mathcal{A},L^r)$, the fractal dimension of $\mathcal{A}$, can be expressed in terms of Kolmogorov's ε-entropy via the definition: $\dim_F(\mathcal{A},L^r):=\limsup_{\varepsilon\to 0}\frac{\mathcal{H}_\varepsilon(\mathcal{A},L^r)}{\log_2\frac{1}{\varepsilon}}$. Letting $\varepsilon\to 0$ in (56), we conclude

$$
\dim_F\big(\mathcal{A},L^r\big)=\infty\quad\forall r\in[1,\infty].\quad\square
$$

Concluding remarks. (1) For the sake of the simplicity of the presentation, the monotone part $g_0(x,\xi)$ of $g(x,\xi)$ is here assumed to be single-valued and continuous with respect to ξ. However, Theorems 1, 2 and 3 hold true with obvious modifications for any $g_0(x,\xi)$, which is a (possibly multivalued) maximal monotone graph in $\mathbb{R}^2$ for a.e. $x\in\Omega$ such that

$$
0\in g_0(x,0),\qquad \sup\big\{|z|;\ z\in g_0(x,\xi),\ x\in\Omega,\ |\xi|\leqslant M\big\}\leqslant C_M\quad\forall M>0.
$$

(2) It is clear that Theorem 4 holds true for unbounded domains Ω, since the arguments in the proof are always localized in a bounded domain ω.

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