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Spikes in two coupled nonlinear Schrödinger equations

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Abstract

Here we study the interaction and the configuration of spikes in a double condensate by analyzing least energy solutions of two coupled nonlinear Schrödinger equations which model Bose–Einstein condensates of two different hyperfine spin states. When the interspecies scattering length is positive and large enough, spikes of a double condensate repel each other and behave like two separate spikes. In contrast, spikes of a double condensate attract each other and behave like a single spike if the interspecies scattering length is negative and large enough. Our mathematical arguments can prove such physical phenomena. We first use Nehari's manifold to construct least energy solutions, and then use some techniques of singular perturbation problems to derive the asymptotic behavior of least energy solutions. It is shown that the interaction term determines the locations of the two spikes and the asymptotic shape of least energy solutions.

Résumé

Nous étudions l'interaction et la disposition des pics pour un condensat double. Notre méthode utilise l'analyse des solutions avec énergie minimale des deux équations de Schrödinger couplées non linéaires qui modèlent le condensat de Bose et Einstein pour deux états des spins différents hyperfinis. Si la longueur de dispersion entre espèces est positive et suffisamment gande, les pics pour un condensat double se repoussent, et se comportent comme deux pics séparés. En revanche pour un condensat double s'attirent et se comportent comme un seul pic ; la longueur de dispersion entre espèces est négative et suffisamment petite. Nos arguments mathématiques établissent rigoureusement ses phénomènes physiques. Nous utilisons d'une part la variété de Nehari pour la construction de solutions avec énergie minimale et d'autre part les techniques de problèmes de perturbation singulière pour la dérivation du comportement asymptotique de la solutions avec énergie minimale. Nous démontrons que le terme d'interaction détermine la position des pics et la forme asymptotique des solutions avec énergie minimale.

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Keywords: Coupled nonlinear Schrödinger equations; Least energy solutions; Spikes; Sphere-packing

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1. Introduction

We consider the following system of singularly perturbed nonlinear Schrödinger equations

$$\begin{cases} \varepsilon^2\Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \quad \text{in } \Omega, \\ \varepsilon^2\Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in } \Omega, \\ u, v > 0 \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \leqslant 3)$ is a smooth and bounded domain, $\varepsilon > 0$ is a small parameter, $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ are positive constants; $\beta \neq 0$ is a coupling constant. Here $\beta > 0$ means attractive interaction of solutions u and v, on the other hand, $\beta < 0$ means repulsive interaction of solutions u and v. As we will see later, the sign of β has a vital role on the asymptotic behavior of least energy solutions of (1.1).

Problem (1.1) arises in the Hartree–Fock theory for a double condensate i.e. a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (cf. [12]). Physically, u and v are the corresponding condensate amplitudes, $\epsilon^2 = \frac{\hbar^2}{2m}$ and $\mu_j = -(N_j - 1)U_{jj}$, where $\hbar$ is Planck constant, m is the atom mass, N_j is a fixed number of atoms in the hyperfine state $|j\rangle$. Besides, $N_1, N_2 \geqslant 1$, $\beta = -N_2 U_{12}$ and $U_{ij} = 4\pi\frac{\hbar^2}{m}a_{ij}$, where a_{jj}'s and a_{12} are the intraspecies and interspecies scattering lengths. The sign of the scattering length a_{12} determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive or attractive. When $a_{12} > 0$ i.e. $\beta < 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive (cf. [29]). In contrast, when $a_{12} < 0$ i.e. $\beta > 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are attractive. For atoms of the single state $|j\rangle$, when $a_{jj} < 0$ i.e. $\mu_j > 0$, the interactions of the single state $|j\rangle$ are attractive.

Spikes may appear in a single condensate when the s-wave scattering length is negative and large. Due to Feshbach resonance, the s-wave scattering length of a single condensate can be tuned over a very large range by adjusting the externally applied magnetic field. As the s-wave scattering length of a single condensate is negative and large enough, the interactions of atoms are strongly attractive and the associated condensate tends to contract. Donley et al. (cf. [8]) observed anisotropic atom bursts that explode from a single condensate, atoms leaving the condensate in undetected forms, and *spikes* appearing in the condensate wavefunction. It seems spikes may occur in a double condensate when the intraspecies scattering lengths a_{jj}'s are negative and large enough i.e. μ_j's are positive and large enough. However, the interactions of spikes have not yet been observed in a double condensate even though Gupta et al. (cf. [18]) have successfully obtained double condensates of Fermi gas with positive and negative interspecies scattering lengths.

The interspecies scattering length a_{12} plays an important role in a double condensate. Without the effect of trap potentials, when the interspecies scattering length a_{12} is positive and large enough, the states $|1\rangle$ and $|2\rangle$ may repel each other and form segregated domains called phase separation (cf. [19,24,29]). It is natural to believe that spikes of states $|1\rangle$ and $|2\rangle$ may repel each other and behave like two separate spikes as the interspecies scattering length a_{12} is positive and large enough. In contrast, if the interspecies scattering length a_{12} is negative and large enough, spikes of states $|1\rangle$ and $|2\rangle$ may attract each other and behave like one single spike. However, until now, there is not any theoretical and experimental result to support such physical phenomena. The main goal of this paper is to prove such physical phenomena by our mathematical arguments. We study the interaction and the configuration of spikes in a double condensate, especially for positive and large μ_j's, λ_j's and $|\beta|$. The positive constant λ_j is from the chemical potential of the state $|j\rangle$. Without the effect of trap potentials, we may use large chemical potentials λ_j's to persist spikes of a double condensate. By rescaling and some simple assumptions, the problem (1.1) with very large μ_j's, λ_j's and $|\beta|$ is equivalent to the problem (1.1) with μ_j's, λ_j's and $|\beta|$ as positive constants and $\epsilon > 0$ as a small parameter.

In this paper, we study the asymptotic behavior of so-called least-energy solutions which are physically relevant. By this, we mean

(1) $(u_\varepsilon, v_\varepsilon)$ is a solution of (1.1),
(2) $E_\Omega[u_\varepsilon, v_\varepsilon] \leqslant E_\Omega[u, v]$ for any nontrivial solution (u, v) of (1.1),

where $E_\Omega[u, v]$ is the energy functional defined as follows:

$$E_\Omega[u,v] := \frac{\varepsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda_1}{2}\int_\Omega u^2 - \frac{\mu_1}{4}\int_\Omega u^4 + \frac{\varepsilon^2}{2}\int_\Omega |\nabla v|^2 + \frac{\lambda_2}{2}\int_\Omega v^2 - \frac{\mu_2}{4}\int_\Omega v^4 - \frac{\beta}{2}\int_\Omega u^2 v^2, \quad \text{for } u, v \in H_0^1(\Omega). \tag{1.2}$$

As $\varepsilon \downarrow 0$, there are two spikes for both u_ε and v_ε, respectively. If $\beta < 0$, these two spikes repel each other and the boundary of the domain repels these spikes. As a result, the locations of two spikes reach a sphere-packing position in the domain Ω. Physically, in each single state $|j\rangle$, there is a corresponding spike which repels the other spike in the other state if $\beta < 0$. On the other hand, if $\beta > 0$, these two spikes bound each other, and the locations of two spikes reach the innermost part of the domain, due to the repelling effect of the domain.

Now we state our main result of this paper as follows:

Theorem 1.1. *There exists a constant $\beta_0 = \beta_0(N, \lambda_1, \lambda_2, \mu_1, \mu_2) \in (0, \sqrt{\mu_1\mu_2})$ such that the following holds*:

(1) *For any $\beta \in (-\infty, \beta_0)$ and ε sufficiently small,* (1.1) *has a least energy solution $(u_\varepsilon, v_\varepsilon)$. Let P_ε be a local maximum point of u_ε and Q_ε be a local maximum point of v_ε.*
(2) *If $0 < \beta < \beta_0$, then $|P_\varepsilon - Q_\varepsilon|/\varepsilon \to 0$ and*

$$\begin{aligned} &d(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega), \\ &d(Q_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega). \end{aligned} \tag{1.3}$$

Furthermore, $u_\varepsilon(x), v_\varepsilon(x) \to 0$ in $C^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{P_\varepsilon, Q_\varepsilon\})$ and let

$$U_\varepsilon(y) := u_\varepsilon(P_\varepsilon + \varepsilon y), \qquad V_\varepsilon(y) := v_\varepsilon(P_\varepsilon + \varepsilon y)$$

then as $\varepsilon \to 0$, $(U_\varepsilon, V_\varepsilon) \to (U_0, V_0)$ which is a least-energy solution of the following problem in $\mathbb{R}^N$

$$\begin{cases} \Delta U_0 - \lambda_1 U_0 + \mu_1 U_0^3 + \beta U_0 V_0^2 = 0 & \text{in } \mathbb{R}^N, \\ \Delta V_0 - \lambda_2 V_0 + \mu_2 V_0^3 + \beta U_0^2 V_0 = 0 & \text{in } \mathbb{R}^N, \\ U_0(0) = \max_{y\in\mathbb{R}^N} U_0(y), \quad V_0(0) = \max_{y\in\mathbb{R}^N} V_0(y), \\ U_0, V_0 > 0 \quad \text{in } \mathbb{R}^N, \quad U_0, V_0 \to 0 \quad \text{as } |y| \to +\infty. \end{cases} \tag{1.4}$$

(3) *If $\beta < 0$, then we have*

$$\varphi(P_\varepsilon, Q_\varepsilon) \to \max_{(P,Q)\in\Omega^2} \varphi(P, Q) \tag{1.5}$$

where

$$\varphi(P, Q) = \min\bigl\{\sqrt{\lambda_1}\,|P - Q|, \sqrt{\lambda_2}\,|P - Q|, \sqrt{\lambda_1}\,d(P, \partial\Omega), \sqrt{\lambda_2}\,d(Q, \partial\Omega)\bigr\}. \tag{1.6}$$

Furthermore, $u_\varepsilon(x), v_\varepsilon(x) \to 0$ in $C^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{P_\varepsilon, Q_\varepsilon\})$, and if we let

$$U_\varepsilon(y) := u_\varepsilon(P_\varepsilon + \varepsilon y), \qquad V_\varepsilon(y) := v_\varepsilon(Q_\varepsilon + \varepsilon y),$$

then

$$U_\varepsilon(y) \to w_1(y), \qquad V_\varepsilon(y) \to w_2(y)$$

where $w_i(y)$, $i=1,2$, is the unique solution of

$$\begin{cases} \Delta w_i - \lambda_i w_i + \mu_i w_i^3 = 0 \quad \text{in } \mathbb{R}^N, \\ w_i(0) = \max_{y\in\mathbb{R}^N} w_i(y), \quad i=1,2, \\ w_i > 0 \quad \text{in } \mathbb{R}^N, \quad w_i(y) \to 0 \quad \text{as } |y| \to +\infty. \end{cases} \tag{1.7}$$

Some remarks are in order.

Remarks.

1.1. It will be proved in Section 3 that problem (1.4) admits a least-energy solution. By the method of moving plane (using $\beta > 0$), all solutions of (1.4) must be radially symmetric and strictly decreasing. It is an interesting question to study the uniqueness of (U_0, V_0).

1.2. Function $\varphi(P, Q)$ in (1.6) has been introduced in [14], when $\lambda_1 = \lambda_2 = 1$. Condition (1.5) shows that for $\beta < 0$ (repulsive), the two spikes of the least energy solution will reach a sphere-packing position in Ω.

1.3. It is well-known that (1.7) admits a unique radially symmetric solution (see [20]). Moreover, $w_i(y) = \sqrt{\lambda_i/\mu_i}\, w(\sqrt{\lambda_i}\, y)$, where $w(y)$ satisfies

$$\begin{cases} \Delta w - w + w^3 = 0 \quad \text{in } \mathbb{R}^N, \\ w > 0 \quad \text{in } \mathbb{R}^N, \quad w(0) = \max_{y\in\mathbb{R}^N} w(y), \\ w(y) \to 0 \quad \text{as } |y| \to +\infty. \end{cases}$$

1.4. The condition that $\beta < \beta_0 < \sqrt{\mu_1\mu_2}$ reflects the restriction of the techniques we used. We remark that similar condition has also been imposed in the study of systems of competing species [3]. For $\beta \geqslant \sqrt{\mu_1\mu_2}$, we do not know if any least-energy solution exists. For example, if $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$, then problem (1.4) admits a solution

$$\left(\sqrt{\frac{1}{\beta+1}}\, w(y), \sqrt{\frac{1}{\beta+1}}\, w(y)\right).$$

However, we even do not know if such a solution is a least-energy solution.

1.5. Theorem 1.1 can be generalized to multiple-state case:

$$\begin{cases} \varepsilon^2 \Delta u_j - u_j + \mu_j u_j^3 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j = 0 \quad \text{in } \Omega, \\ u_j > 0, \quad j = 1, \ldots, n, \text{ in } \Omega, \\ u_j = 0 \quad \text{on } \partial\Omega, \end{cases}$$

if either $\beta_{ij} < 0$, $\forall i \neq j$, or $\beta_{ij} > 0$, $\forall i \neq j$ but the matrix

$$M = (\beta_{ij}) \quad \text{is positively definite, where } \beta_{ii} = \mu_i.$$

The proof is similar. The other cases are more involved.

1.6. We have restricted ourselves to the spatial dimension $N \leqslant 3$. If $N \geqslant 4$, the nonlinearity u^3, v^3 becomes critical or super-critical. It is unknown if similar results can be obtained for $N \geqslant 4$.

The main idea in proving Theorem 1.1 is by energy expansion, which has been introduced by Ni and Takagi [26,27] in dealing with singularly perturbed Neumann problems and later extended by Ni and Wei [28] in singularly perturbed Dirichlet problems. In the case of $\beta > 0$, since we do not know the uniqueness of solutions to (1.4), we use an idea of [4], where a simplified proof of the results of [27] and [28] is obtained. In the case of $\beta < 0$, the main difficulty is to compute the interactions between the two spikes. Note that both the boundary effect and the interaction between spikes are **exponentially small**. Thus we have to make careful estimates.

We remark that singularly perturbed Neumann or Dirichlet problems have been studied in many papers. A general principle is that the interior spike layer solutions are generated by distance functions. We refer the reader to the articles [1,6,7,9,10,14,15,17,22,31–33] and the references therein. On the other hand, the boundary peaked solutions are related to the boundary mean curvature function. This aspect is discussed in the papers [2,5,16,21,34–36], and the references therein. A good review of the subject is to be found in [25]. Our result here seems to be the first attempt in studying systems of singularly perturbed problems.

The organization of the paper is as follows:

In Section 2, we extend the classical Nehari's manifold approach to a system of semilinear elliptic equations in order to find a least energy solution to (1.1). Hereafter, we need the condition that $\beta < \beta_0$. In Section 3, we use Nehari's manifold approach and symmetrization technique to study limiting ground state solutions to (1.1). It is then shown that for $\beta < 0$, ground state solutions do not exist, while for $0 < \beta < \beta_0$ ground state solution exists and must be radially symmetric and strictly decreasing. Section 4 to Section 8 contains several parts of the proof of Theorem 1.1. The main idea, as we mentioned earlier, is to expand the least energy $c_\epsilon := E_\Omega[u_\epsilon, v_\epsilon]$ where (u_ϵ, v_ϵ) is a least energy solution constructed in Section 2. Section 4 proves the upper bound for c_ϵ as $\beta > 0$, and Section 5 gives the upper bound for c_ϵ as $\beta < 0$. Section 6 is an intermediate step where the first approximation of the least energy solution is given. Section 7 gives the lower bound for c_ϵ as $\beta < 0$, and Section 8 shows the lower bound for c_ϵ as $\beta > 0$.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ϵ, for ϵ sufficiently small. The constant $\sigma \in (0, \frac{1}{100})$ is a fixed small constant.

2. Nehari's manifold approach: existence of a least-energy solution to (1.1)

In this section, we use Nehari's manifold approach to obtain a least energy solution to (1.1). Nehari's manifold approach has been used successfully in the study of single equations. Recently, Conti et al. [3] have used Nehari's manifold to study solutions of competing species systems which are related to an optimal partition problem in N-dimensional domains. Here we use Nehari's manifold to find least energy solution with spikes.

We define the so-called Nehari's manifold as follows:

$$M(\varepsilon, \Omega) = \left\{ (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{l} \varepsilon^2 \int_\Omega |\nabla u|^2 + \lambda_1 \int_\Omega u^2 - \beta \int_\Omega u^2 v^2 = \mu_1 \int_\Omega u^4 \\ \varepsilon^2 \int_\Omega |\nabla v|^2 + \lambda_2 \int_\Omega v^2 - \beta \int_\Omega u^2 v^2 = \mu_2 \int_\Omega v^4 \end{array} \right\}. \tag{2.1}$$

Now we consider the following minimization problem

$$c_\varepsilon := \inf_{\substack{(u,v) \in M(\varepsilon,\Omega),\\ u,v \geqslant 0,\ u,v \not\equiv 0}} E_\Omega[u, v] \tag{2.2}$$

where $E_\Omega[u, v]$ is defined at (1.2).

Note that, for $N \leqslant 3$, by the compactness of Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, $M(\varepsilon, \Omega)$ and c_ε are well-defined. Now we show that

Lemma 2.1. *Suppose that $\beta < \beta_0 < \sqrt{\mu_1 \mu_2}$. Then for ϵ sufficiently small, c_ε can be attained by some $(u_\varepsilon, v_\varepsilon) \in M(\varepsilon, \Omega)$ and it holds that*

$$C_1 \epsilon^N \leqslant \int_\Omega u_\varepsilon^4 \leqslant C_2 \epsilon^N, \qquad C_1 \epsilon^N \leqslant \int_\Omega v_\varepsilon^4 \leqslant C_2 \epsilon^N, \tag{2.3}$$

where C_1, C_2 are two positive constants independent of ϵ and β.

Proof. We first note that if $(u,v)\in M(\varepsilon,\Omega)$, then

$$E_\Omega[u,v]=\frac{1}{4}\left(\varepsilon^2\int_\Omega|\nabla u|^2+\lambda_1\int_\Omega u^2+\varepsilon^2\int_\Omega|\nabla v|^2+\lambda_2\int_\Omega v^2\right)$$
$$=\frac{1}{4}\left[\mu_1\int_\Omega u^4+2\beta\int_\Omega u^2v^2+\mu_2\int_\Omega v^4\right]. \tag{2.4}$$

Let (u_n,v_n) be a minimizing sequence. Then by Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^q(\Omega)$ for $1<q<\frac{2N}{N-2}$, we see that $u_n\to u_\varepsilon$, $v_n\to v_\epsilon$(up to a subsequence) for some functions $u_\varepsilon\geqslant 0$, $v_\epsilon\geqslant 0$ in $L^4(\Omega)$ and hence

$$E_\Omega[u_n,v_n]\to\frac{1}{4}\left[\mu_1\int_\Omega u_\varepsilon^4+2\beta\int_\Omega u_\varepsilon^2v_\epsilon^2+\mu_2\int_\Omega v_\epsilon^4\right]=c_\varepsilon. \tag{2.5}$$

By Fatou's Lemma, we have

$$c_\varepsilon\geqslant\frac{1}{4}\left(\varepsilon^2\int_\Omega|\nabla u_\varepsilon|^2+\lambda_1\int_\Omega u_\varepsilon^2+\varepsilon^2\int_\Omega|\nabla v_\epsilon|^2+\lambda_2\int_\Omega v_\epsilon^2\right), \tag{2.6}$$

$$\varepsilon^2\int_\Omega|\nabla u_\varepsilon|^2+\lambda_1\int_\Omega u_\varepsilon^2\leqslant\mu_1\int_\Omega u_\varepsilon^4+\beta\int_\Omega u_\varepsilon^2v_\epsilon^2, \tag{2.7}$$

$$\varepsilon^2\int_\Omega|\nabla v_\epsilon|^2+\lambda_2\int_\Omega v_\epsilon^2\leqslant\mu_2\int_\Omega v_\epsilon^4+\beta\int_\Omega u_\varepsilon^2v_\epsilon^2. \tag{2.8}$$

We claim that

Claim 1. *The inequalities* (2.3) *holds for* $\beta<\beta_0$ *where* $\beta_0<\sqrt{\mu_1\mu_2}$ *is a generic constant.*

Proof. We first prove the upper bound. For ϵ sufficiently small, we choose test functions u and v such that support$(u)\subset B_\delta(P)$ and support$(v)\subset B_\delta(Q)$, where P and Q are distinct points in Ω and $0<\delta<\frac{1}{4}\min\{d(P,\partial\Omega),d(Q,\partial\Omega),\frac{|P-Q|}{2}\}$. Then it can be shown easily that

$$c_\epsilon\leqslant C\epsilon^N. \tag{2.9}$$

Combining (2.9) with (2.5), we obtain that $\int_\Omega u_\epsilon^4\leqslant C_2\epsilon^N$, $\int_\Omega v_\epsilon^4\leqslant C_2\epsilon^N$.

To prove the lower bound, we divide into two cases.

For $\beta<0$, we have

$$\varepsilon^2\int_\Omega|\nabla u_n|^2+\lambda_1\int_\Omega u_n^2\leqslant\ \mu_1\int_\Omega u_n^4$$

and hence

$$\min(\lambda_1,1)C_\epsilon\sqrt{\left(\int_\Omega u_n^4\right)}\leqslant\mu_1\int_\Omega u_n^4 \tag{2.10}$$

where

$$C_\epsilon:=\inf_{u\in H_0^1(\Omega),\ u\geqslant 0,\ u\not\equiv 0}\frac{\epsilon^2\int_\Omega|\nabla u|^2+\int_\Omega u^2}{(\int_\Omega u^4)^{1/2}}. \tag{2.11}$$

By the results of [28], for ϵ sufficiently small, we have $C_\epsilon \geqslant C_0\epsilon^{N/2}$ for some fixed $C_0 > 0$. This yields that $\int_\Omega u_n^4 \geqslant C_1\epsilon^N$ since $u_n \not\equiv 0$. So we have $\int_\Omega u_\varepsilon^4 \geqslant C_1\epsilon^N$ by Fatou's Lemma. Similarly, for $\beta < 0$, we obtain $\int_\Omega v_\varepsilon^4 \geqslant C_1\epsilon^N$.

For $\beta > 0$, similar arguments leading to (2.10) give

$$\mu_1\|u_n\|_{L^4}^2 + \beta\|v_n\|_{L^4}^2 \geqslant \min(\lambda_1, 1)C_0\epsilon^{N/2}, \tag{2.12}$$

$$\mu_2\|v_n\|_{L^4}^2 + \beta\|u_n\|_{L^4}^2 \geqslant \min(\lambda_2, 1)C_0\epsilon^{N/2}. \tag{2.13}$$

Now we set $\beta_0 \in (0, \sqrt{\mu_1\mu_2})$ as a constant such that $\beta_0 < (\min(\lambda_1, \lambda_2, 1)C_0)/(2C_2^{1/2})$. Then by (2.12) and (2.13), we may obtain that $\int_\Omega u_n^4 \geqslant C_1\epsilon^N$, $\int_\Omega v_n^4 \geqslant C_1\epsilon^N$. By Fatou's Lemma, we have proved (2.3) in the case of $0 < \beta < \beta_0$. □

Next we consider for $s, t > 0$

$$\beta(s,t) = E_\Omega[\sqrt{s}\,u_\varepsilon, \sqrt{t}\,v_\epsilon]. \qquad \square \tag{2.14}$$

We claim that

Claim 2. *$\beta(s,t)$ attains a unique maximum point (s_0, t_0) with $s_0 > 0, t_0 > 0$.*

Proof. It is easy to calculate that

$$\beta(s,t) = s\left[\frac{\varepsilon^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{\lambda_1}{2}\int_\Omega u_\varepsilon^2\right] - \frac{\mu_1}{4}s^2\int_\Omega u_\varepsilon^4 + t\left[\frac{\varepsilon^2}{2}\int_\Omega |\nabla v_\epsilon|^2 + \frac{\lambda_2}{2}\int_\Omega v_\epsilon^2\right] - \frac{\mu_2}{4}t^2\int_\Omega v_\epsilon^4$$
$$- \frac{1}{2}\beta st\int_\Omega u_\varepsilon^2 v_\epsilon^2.$$

We consider the following quadratic form

$$Q(s,t) := \frac{\mu_1}{4}s^2\int_\Omega u_\varepsilon^4 + \frac{\mu_2}{4}t^2\int_\Omega v_\epsilon^4 + \frac{\beta}{2}st\int_\Omega u_\varepsilon^2 v_\epsilon^2. \tag{2.15}$$

If $0 < \beta < \sqrt{\mu_1\mu_2}$, then $Q(s,t)$ is positively definite since the matrix

$$\Sigma := \begin{pmatrix} \frac{\mu_1}{4}\int_\Omega u_\varepsilon^4 & \frac{\beta}{4}\int_\Omega u_\varepsilon^2 v_\epsilon^2 \\ \frac{\beta}{4}\int_\Omega u_\varepsilon^2 v_\epsilon^2 & \frac{\mu_2}{4}\int_\Omega v_\epsilon^4 \end{pmatrix} \tag{2.16}$$

is positively definite. Here we have used Hölder inequality and the fact that both u_ε and v_ϵ are nonzero functions.

If $\beta \leqslant 0$, Σ is still positively definite. Hence by (2.7), (2.8), and $\|u_\varepsilon\|_{L^4}, \|v_\epsilon\|_{L^4} > 0$, we have

$$\mu_1\int_\Omega u_\varepsilon^4 > (-\beta)\int_\Omega u_\varepsilon^2 v_\epsilon^2, \qquad \mu_2\int_\Omega v_\epsilon^4 > (-\beta)\int_\Omega u_\varepsilon^2 v_\epsilon^2,$$

and

$$\det(\Sigma) = \frac{1}{16}\left[\mu_1\mu_2\int_\Omega u_\varepsilon^4\int_\Omega v_\epsilon^4 - \beta^2\left(\int_\Omega u_\varepsilon^2 v_\epsilon^2\right)^2\right] > 0. \tag{2.17}$$

So $Q(s,t)$ is a positively definite form, which implies that $\beta(s,t)$ is concave and hence it has a unique (local) maximum point.

Let (s_0, t_0) be the unique maximum point of $\beta(s,t)$. We need to show that $s_0 > 0$, $t_0 > 0$. Suppose not, without loss of generality, we may assume that $s_0 > 0$, $t_0 = 0$. Then we have $\partial\beta(s_0, 0)/\partial s = 0$, $\partial\beta(s_0, 0)/\partial t \leqslant 0$. That is

$$\varepsilon^2 \int_\Omega |\nabla u_\varepsilon|^2 + \lambda_1 \int_\Omega u_\varepsilon^2 = \mu_1 s_0 \int_\Omega u_\varepsilon^4, \tag{2.18}$$

$$\varepsilon^2 \int_\Omega |\nabla v_\varepsilon|^2 + \lambda_2 \int_\Omega v_\epsilon^2 \leqslant \beta s_0 \int_\Omega u_\varepsilon^2 v_\epsilon^2. \tag{2.19}$$

Hence the constant β must be positive. Using (2.7) and (2.18), we have

$$\mu_1 s_0 \int_\Omega u_\varepsilon^4 \leqslant \mu_1 \int_\Omega u_\varepsilon^4 + \beta \int_\Omega u_\varepsilon^2 v_\varepsilon^2. \tag{2.20}$$

Then by Holder inequality, (2.20) gives

$$s_0 \|u_\epsilon\|_{L^4}^2 \leqslant \|u_\epsilon\|_{L^4}^2 + \frac{\beta}{\mu_1} \|v_\epsilon\|_{L^4}^2. \tag{2.21}$$

Moreover, by (2.19) and (2.21), we obtain

$$\min(\lambda_2, 1) C_\epsilon \leqslant \beta s_0 \|u_\epsilon\|_{L^4}^2 \leqslant \beta\left(1 + \frac{\beta}{\mu_1}\right) C_2^{1/2} \epsilon^{N/2} \tag{2.22}$$

which may get contradiction if $0 < \beta < \beta_0$ and β_0 is sufficiently small such that

$$\beta_0\left(1 + \frac{\beta_0}{\mu_1}\right) C_2^{1/2} \leqslant \frac{1}{2} \min(\lambda_2, 1) C_0,$$

where C_0 is a positive constant such that $C_\epsilon \geqslant C_0 \epsilon^{N/2}$ (cf. [28]). Therefore we have proved that $s_0 > 0$, $t_0 > 0$.

By definition, $(\sqrt{s_0}\, u_\varepsilon, \sqrt{t_0}\, v_\epsilon) \in M(\varepsilon, \Omega)$. That is

$$\varepsilon^2 \int_\Omega |\nabla u_\varepsilon|^2 + \lambda_1 \int_\Omega u_\varepsilon^2 = \mu_1 s_0 \int_\Omega u_\varepsilon^4 + \beta t_0 \int_\Omega u_\varepsilon^2 v_\epsilon^2, \tag{2.23}$$

$$\varepsilon^2 \int_\Omega |\nabla v_\varepsilon|^2 + \lambda_2 \int_\Omega v_\epsilon^2 = \mu_2 t_0 \int_\Omega v_\epsilon^4 + \beta s_0 \int_\Omega u_\varepsilon^2 v_\epsilon^2. \tag{2.24}$$

Consequently, (2.23), (2.24), (2.7) and (2.8) give

$$\mu_1 s_0 \int_\Omega u_\varepsilon^4 + \beta t_0 \int_\Omega u_\varepsilon^2 v_\epsilon^2 \leqslant \mu_1 \int_\Omega u_\varepsilon^4 + \beta \int_\Omega u_\varepsilon^2 v_\epsilon^2, \tag{2.25}$$

$$\mu_2 t_0 \int_\Omega v_\epsilon^4 + \beta s_0 \int_\Omega u_\varepsilon^2 v_\epsilon^2 \leqslant \mu_2 \int_\Omega v_\epsilon^4 + \beta \int_\Omega u_\varepsilon^2 v_\epsilon^2. \tag{2.26}$$

On the other hand

$$E_\Omega[\sqrt{s_0}\, u_\varepsilon, \sqrt{t_0}\, v_\epsilon] \geqslant c_\varepsilon = \frac{1}{4}\left[\mu_1 \int_\Omega u_\varepsilon^4 + 2\beta \int_\Omega u_\varepsilon^2 v_\epsilon^2 + \mu_2 \int_\Omega v_\epsilon^4\right], \tag{2.27}$$

$$E_\Omega[\sqrt{s_0}\, u_\varepsilon, \sqrt{t_0}\, v_\epsilon] = \frac{1}{4}\left[\mu_1 s_0^2 \int_\Omega u_\varepsilon^4 + 2\beta s_0 t_0 \int_\Omega u_\varepsilon^2 v_\epsilon^2 + \mu_2 t_0^2 \int_\Omega v_\epsilon^4\right]. \tag{2.28}$$

Since s_0, $t_0 > 0$, (2.27) and (2.28) imply that (2.25) and (2.26) are equalities, i.e.

$$\mu_1(s_0-1)\int_\Omega u_\varepsilon^4+\beta(t_0-1)\int_\Omega u_\varepsilon^2 v_\epsilon^2=0, \tag{2.29}$$

$$\mu_2(t_0-1)\int_\Omega v_\epsilon^4+\beta(s_0-1)\int_\Omega u_\varepsilon^2 v_\epsilon^2=0. \tag{2.30}$$

Since $\det(\Sigma)>0$, we see immediately that $s_0=1$, $t_0=1$. This implies that $(u_\varepsilon, v_\epsilon)\in M(\varepsilon,\Omega)$, and hence $(u_\varepsilon, v_\epsilon)$ attains the minimum c_ε. □

Next we claim that

Lemma 2.2. $(u_\varepsilon, v_\varepsilon)$ *is a least-energy solution of* (1.1).

Proof. Certainly, $(u_\varepsilon, v_\varepsilon)$ has least-energy, if it is a solution of (1.1). By usual elliptic regularity theory (using the fact that $H_0^1(\Omega)\hookrightarrow L^4(\Omega)$ is compact), we see that there exists two Lagrange multipliers α_1,α_2 such that

$$\nabla E_\Omega[u,v]+\alpha_1\nabla G_1[u,v]+\alpha_2\nabla G_2[u,v]=0 \tag{2.31}$$

where $u\equiv u_\varepsilon$, $v\equiv v_\varepsilon$,

$$G_1[u,v]=\varepsilon^2\int_\Omega|\nabla u|^2+\lambda_1\int_\Omega u^2-\mu_1\int_\Omega u^4-\beta\int_\Omega u^2v^2, \tag{2.32}$$

$$G_2[u,v]=\varepsilon^2\int_\Omega|\nabla v|^2+\lambda_2\int_\Omega v^2-\mu_2\int_\Omega v^4-\beta\int_\Omega u^2v^2. \tag{2.33}$$

Let

$$F(s,t)=E_\Omega[\sqrt{s}\,u,\sqrt{t}\,v]+\alpha_1G_1[\sqrt{s}\,u,\sqrt{t}\,v]+\alpha_2G_2[\sqrt{s}\,u,\sqrt{t}\,v].$$

Then by (2.31), we have

$$\frac{\partial F}{\partial s}(1,1)=\frac{\partial F}{\partial t}(1,1)=0.$$

From the proof of Lemma 2.1, we obtain

$$\frac{\partial}{\partial s}E_\Omega[\sqrt{s}\,u,\sqrt{t}\,v]|_{\{s=t=1\}}=\frac{\partial}{\partial t}E_\Omega[\sqrt{s}\,u,\sqrt{t}\,v]|_{\{s=t=1\}}=0.$$

Hence

$$\alpha_1\left[\varepsilon^2\int_\Omega|\nabla u|^2+\lambda_1\int_\Omega u^2-2\mu_1\int_\Omega u^4-\beta\int_\Omega u^2v^2\right]+\alpha_2\left[-\beta\int_\Omega u^2v^2\right]=0,$$

$$\alpha_1\left[-\beta\int_\Omega u^2v^2\right]+\alpha_2\left[\varepsilon^2\int_\Omega|\nabla v|^2+\lambda_2\int_\Omega v^2-2\mu_2\int_\Omega v^4-\beta\int_\Omega u^2v^2\right]=0,$$

which are equivalent to

$$\Sigma'\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=0 \tag{2.34}$$

where

$$\Sigma' = \begin{pmatrix} \mu_1 \int_\Omega u^4 & \beta \int_\Omega u^2 v^2 \\ \beta \int_\Omega u^2 v^2 & \mu_2 \int_\Omega v^4 \end{pmatrix}. \tag{2.35}$$

Here we have used the fact that $(u, v) = (u_\varepsilon, v_\varepsilon) \in M(\varepsilon, \Omega)$. By the same argument in Lemma 2.1, we have that for $-\infty < \beta < \sqrt{\mu_1 \mu_2}$, $\det(\Sigma') > 0$ and hence $\alpha_1 = \alpha_2 = 0$. This proves that

$$\nabla E_\Omega[u, v] = 0$$

and hence (u, v) is a critical point of $E_\Omega[u, v]$ and satisfies (1.1). By Hopf boundary lemma, it is easy to show that $u > 0$ and $v > 0$, which completes the proof of Lemma 2.2. □

We need another characterization of c_ε, which will be useful later.

Lemma 2.3. *If $-\infty < \beta < 0$, then we have*

$$c_\varepsilon = \inf_{\substack{u,v \in H_0^1(\Omega),\\ u \not\equiv 0,\ v \not\equiv 0}} \sup_{s,t>0} E_\Omega[\sqrt{s}\, u, \sqrt{t}\, v]. \tag{2.36}$$

If $0 < \beta < \beta_0$, then we have

$$c_\varepsilon \geqslant \inf_{\substack{u,v \in H_0^1(\Omega),\\ u \not\equiv 0,\ v \not\equiv 0}} \sup_{s,t>0} E_\Omega[\sqrt{s}\, u, \sqrt{t}\, v]. \tag{2.37}$$

If $0 < \beta < \beta_0$ and the function $E_\Omega[\sqrt{s}\, u, \sqrt{t}\, v]$ has a critical point $s_0 > 0$, $t_0 > 0$, then we have

$$c_\varepsilon \leqslant E_\Omega[\sqrt{s_0}\, u, \sqrt{t_0}\, v]. \tag{2.38}$$

Proof. Let us denote the right-hand side of (2.36) or (2.37) by m_ε. Firstly, by Lemma 2.1, c_ε is attained at $(u_\varepsilon, v_\varepsilon) \in M(\varepsilon, \Omega)$. By Claim 1 in Lemma 2.1, $E_\Omega[\sqrt{s} u_\varepsilon, \sqrt{t}\, v_\varepsilon]$ attains its maximum at $(1, 1)$. Hence

$$m_\varepsilon \leqslant c_\varepsilon = E_\Omega[u_\varepsilon, v_\varepsilon] = \sup_{s,t>0} E_\Omega[\sqrt{s} u_\varepsilon, \sqrt{t} v_\varepsilon]. \tag{2.39}$$

On the other hand, fix $u, v \in H_0^1(\Omega), u \geqslant 0, v \geqslant 0$, and let (s_0, t_0) be a critical point of $\beta(s, t)$. Then $(\sqrt{s_0}\, u, \sqrt{t_0}\, v) \in M(\varepsilon, \Omega)$,

$$c_\varepsilon \leqslant E_\Omega(\sqrt{s_0}\, u, \sqrt{t_0}\, v) \leqslant \sup_{s,t>0} E_\Omega[\sqrt{s}\, u, \sqrt{t}\, v]$$

and hence $c_\varepsilon \leqslant m_\varepsilon$.

Now we need to show that $\beta(s, t)$ has a critical point, which is equivalent to show that the following system has a solution (s, t)

$$\mu_1 s \int_\Omega u^4 + \beta t \int_\Omega u^2 v^2 = \varepsilon^2 \int_\Omega |\nabla u|^2 + \lambda_1 \int_\Omega u^2, \tag{2.40}$$

$$\beta s \int_\Omega u^2 v^2 + \mu_2 t \int_\Omega v^4 = \varepsilon^2 \int_\Omega |\nabla v|^2 + \lambda_2 \int_\Omega v^2. \tag{2.41}$$

If $0 < \beta < \beta_0$, this is our assumption. If $\beta < 0$, then by (2.39), we may assume that

$$0 < \sup_{s,t>0} E_\Omega[\sqrt{s}\, u, \sqrt{t}\, v] < C. \tag{2.42}$$

In this case, the matrix

$$\Sigma' = \begin{pmatrix} \mu_1 \int_\Omega u^4 & \beta \int_\Omega u^2 v^2 \\ \beta \int_\Omega u^2 v^2 & \mu_2 \int_\Omega v^4 \end{pmatrix}$$

must be nondegenerate. In fact, if $\det(\Sigma') = 0$, then

$$E_\Omega[\sqrt{s}\,u, \sqrt{t}\,v] = s\left(\frac{\varepsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda_1}{2}\int_\Omega u^2\right) + t\left(\frac{\varepsilon^2}{2}\int_\Omega |\nabla v|^2 + \frac{\lambda_2}{2}\int_\Omega v^2\right) - \frac{1}{4}\left(s\sqrt{\mu_1\int_\Omega u^4} - t\sqrt{\mu_2\int_\Omega v^4}\right)^2$$

which implies that $\sup_{s,t>0} E_\Omega[\sqrt{s}\,u, \sqrt{t}\,v] = +\infty$, contradicting to (2.42). Therefore (2.40) and (2.41) always have a solution, and we may complete the proof of Lemma 2.3. □

3. Some preliminaries: problems in $\mathbb{R}^N$

In this section, we analyze some problems in $\mathbb{R}^N$. Firstly, we consider w_i, $i = 1, 2$, which are solutions of (1.7). By scaling,

$$w_i(y) = \sqrt{\frac{\lambda_i}{\mu_i}}\, w(\sqrt{\lambda_i}\, y) \tag{3.1}$$

where w is the unique ground state solution of

$$\begin{cases} \Delta w - w + w^3 = 0 \quad \text{in } \mathbb{R}^N, \\ w > 0 \quad \text{in } \mathbb{R}^N, \quad w(0) = \max_{y\in\mathbb{R}^N} w(y), \\ w(y) \to 0 \quad \text{as } |y| \to \infty. \end{cases} \tag{3.2}$$

By Gidas–Ni–Nirenberg's Theorem [13], w is radially symmetric and strictly decreasing. By a theorem of Kwong [20], w is unique. This then implies that w is nondegenerate (see [27]), i.e.

$$\mathrm{Kernel}(\Delta - 1 + 3w^2) = \mathrm{span}\left\{\frac{\partial w}{\partial y_1}, \ldots, \frac{\partial w}{\partial y_N}\right\}. \tag{3.3}$$

Moreover, we have the following asymptotic behavior

$$\left.\begin{aligned} w(r) &= A_N r^{-(N-1)/2} \mathrm{e}^{-r}\left(1 + O\left(\frac{1}{r}\right)\right), && \text{as } r \to +\infty, \\ w'(r) &= -A_N r^{-(N-1)/2} \mathrm{e}^{-r}\left(1 + O\left(\frac{1}{r}\right)\right) && \text{as } r \to +\infty \end{aligned}\right\} \tag{3.4}$$

for some generic constant $A_N > 0$.

We denote the ground state energy of w as

$$I[w] = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 + \frac{1}{2}\int_{\mathbb{R}^N} w^2 - \frac{1}{4}\int_{\mathbb{R}^N} w^4.$$

Then it is easy to see that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w_i|^2 + \frac{\lambda_i}{2}\int_{\mathbb{R}^N} w_i^2 - \frac{\mu_i}{4}\int_{\mathbb{R}^N} w_i^4 = \lambda_i^{(4-N)/2}\mu_i^{-1} I[w]. \tag{3.5}$$

For $-\infty < \beta < \sqrt{\mu_1\mu_2}$, we may define an energy functional given by

$$I[u,v] = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla u|^2 + \frac{\lambda_1}{2}u^2 - \frac{\mu_1}{4}u^4\right) + \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla v|^2 + \frac{\lambda_2}{2}v^2 - \frac{\mu_2}{4}v^4\right) - \frac{\beta}{2}\int_{\mathbb{R}^N} u^2v^2 \tag{3.6}$$

for $u, v \geqslant 0$ and $u, v \in H^1(\mathbb{R}^N)$.

To study the first and the second integrals of (3.6), we let

$$I_{\lambda,\mu}[u] = \int_{\mathbb{R}_N} \left(\frac{1}{2}|\nabla u|^2 + \frac{\lambda}{2}u^2 - \frac{\mu}{4}u^4\right)$$

for $u \geqslant 0$ and $u \in H^1(\mathbb{R}^N)$, where λ and μ are any positive constants. Then we have

Lemma 3.1. $\inf_{u\in\mathcal{N}} I_{\lambda,\mu}[u]$ *is attained only by* $\sqrt{\lambda/\mu}\, w(\sqrt{\lambda}\,|y|)$,
where

$$\mathcal{N} = \left\{u \in H^1(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda \int_{\mathbb{R}^N} u^2 = \mu \int_{\mathbb{R}^N} u^4\right\}.$$

Proof. It is easy to see that $\inf_{u\in\mathcal{N}} I_{\lambda,\mu}[u]$ is equivalent to

$$\inf_{\substack{u\geqslant 0,\ u\not\equiv 0,\\ u\in H^1(\mathbb{R}^N)}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda \int_{\mathbb{R}^N} u^2}{(\int_{\mathbb{R}^N} u^4)^{1/2}}.$$

The rest follows from the standard argument. □

The next lemma is not so trivial.

Lemma 3.2. $\inf_{u\in N'} I_{\lambda,\mu}[u]$ *is also attained only by* $\sqrt{\lambda/\mu}\, w(\sqrt{\lambda} y)$,
where

$$N' = \left\{u \in H^1(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda \int_{\mathbb{R}^N} u^2 \leqslant \mu \int_{\mathbb{R}^N} u^4\right\}.$$

Proof. Let u_n be a minimizing sequence and u_n^* be its Schwartz symmetrization. Then by the property of symmetrization, we have

$$\int_{\mathbb{R}^N} |\nabla u_n^*|^2 + \lambda \int_{\mathbb{R}^N} (u_n^*)^2 \leqslant \int_{\mathbb{R}^N} |\nabla u_n|^2 + \lambda \int_{\mathbb{R}^N} u_n^2,$$
$$\mu \int_{\mathbb{R}^N} u_n^4 = \mu \int_{\mathbb{R}^N} (u_n^*)^4,$$

which gives rise to

$$I_{\lambda,\mu}[u_n^*] \leqslant I_{\lambda,\mu}[u_n]. \tag{3.7}$$

Hence, we may assume that each u_n is radially symmetric and strictly decreasing. Since $u_n \in H^1(\mathbb{R}^N)$, and u_n is strictly decreasing, it is well-known that $u_n(r) \leqslant Cr^{-(N-1)/2}$. So $u_n \to u_0$(up to a subsequence) in $L^4(\mathbb{R}^N)$, where u_0 also radially symmetric and decreasing. By Fatou's Lemma, $u_0 \in N'$. Hence $\inf_{u\in N'} I_{\lambda,\mu}[u]$ can be attained by u_0.

Suppose now that

$$\int_{\mathbb{R}^N} |\nabla u_0|^2 + \lambda \int_{\mathbb{R}^N} u_0^2 < \mu \int_{\mathbb{R}^N} u_0^4. \tag{3.8}$$

Then $u_0 \in (N')^0$ – the interior of N'. By standard theory, u_0 is a critical point of $I_{\lambda,\mu}[u]$, i.e.

$$\nabla I_{\lambda,\mu}[u_0] = 0. \tag{3.9}$$

However,

$$\langle \nabla I_{\lambda,\mu}[u_0], u_0 \rangle = \int_{\mathbb{R}^N} |\nabla u_0|^2 + \lambda \int_{\mathbb{R}^N} u_0^2 - \mu \int_{\mathbb{R}^N} u_0^4 = 0$$

which is a contradiction with (3.8). Hence $u_0 \in \partial N'$, i.e.

$$\int_{\mathbb{R}^N} |\nabla u_0|^2 + \lambda \int_{\mathbb{R}^N} u_0^2 = \mu \int_{\mathbb{R}^N} u_0^4.$$

By Lemma 3.1, $u_0 = \sqrt{\lambda/\mu}\, w(\sqrt{\lambda}|y|)$. $\square$

For $u, v \in H^1(\mathbb{R}^N)$, we define

$$M_1 = \left\{ (u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \;\middle|\; \begin{array}{l} \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda_1 \int_{\mathbb{R}^N} u^2 - \beta \int_{\mathbb{R}^N} u^2 v^2 = \mu_1 \int_{\mathbb{R}^N} u^4 \\ \int_{\mathbb{R}^N} |\nabla v|^2 + \lambda_2 \int_{\mathbb{R}^N} v^2 - \beta \int_{\mathbb{R}^N} u^2 v^2 = \mu_2 \int_{\mathbb{R}^N} v^4 \end{array} \right\}.$$

The following is the main theorem in this section.

Theorem 3.3. *Consider the following problem*

$$I_0 := \inf_{\substack{u,v \geqslant 0,\ (u,v)\in M_1, \\ u \not\equiv 0,\ v \not\equiv 0}} I[u,v]. \tag{3.10}$$

(1) *If $0 < \beta < \beta_0 < \sqrt{\mu_1\mu_2}$, then I_0 is attained. Let (U_0, V_0) be a minimizer of I_0, where (U_0, V_0) satisfies* (1.4), *then U_0, V_0 are radially symmetric and decreasing.*
(2) *If $\beta < 0$, then I_0 is never attained and*

$$I_0 = \left(\lambda_1^{(4-N)/2}\mu_1^{-1} + \lambda_2^{(4-N)/2}\mu_2^{-1}\right) I[w]. \tag{3.11}$$

Proof. (1) Suppose $0 < \beta < \beta_0 < \sqrt{\mu_1\mu_2}$. Let (u_n, v_n) be a minimizing sequence of I_0 and (u_n^*, v_n^*) be the corresponding Schwartz symmetrization. By Theorem 3.4 of Lieb [23],

$$\int_{\mathbb{R}^N} u_n^2(x) v_n^2(x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} (u_n^*)^2 (v_n^*)^2. \tag{3.12}$$

Since $\beta > 0$,

$$I[u_n^*, v_n^*] \leqslant I[u_n, v_n] \tag{3.13}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_n^*|^2 + \lambda_1 \int_{\mathbb{R}^N} (u_n^*)^2 - \beta \int_{\mathbb{R}^N} (u_n^*)(v_n^*)^2 \leqslant \mu_1 \int_{\mathbb{R}^N} (u_n^*)^4, \tag{3.14}$$

$$\int_{\mathbb{R}^N} |\nabla v_n^*|^2 + \lambda_2 \int_{\mathbb{R}^N} (v_n^*)^2 - \beta \int_{\mathbb{R}^N} (u_n^*)^2(v_n^*)^2 \leqslant \mu_2 \int_{\mathbb{R}^N} (v_n^*)^4. \tag{3.15}$$

Therefore

$$I_0 = \inf_{(u,v)\in M_1} I[u,v] \geqslant \inf_{(u,v)\in M_1^*} I[u,v] \equiv I_1 \tag{3.16}$$

where

$$M_1^* = \left\{ (u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \;\middle|\; \begin{array}{l} u, v \text{ radially symmetric and decreasing} \\ \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda_1 \int_{\mathbb{R}^N} u^2 - \beta \int_{\mathbb{R}^N} u^2v^2 \leqslant \mu_1 \int_{\mathbb{R}^N} u^4 \\ \int_{\mathbb{R}^N} |\nabla v|^2 + \lambda_2 \int_{\mathbb{R}^N} v^2 - \beta \int_{\mathbb{R}^N} u^2v^2 \leqslant \mu_2 \int_{\mathbb{R}^N} v^4 \end{array} \right\}. \tag{3.17}$$

Now we want to show that $I_0 = I_1$, and we may study I_1 instead. Since (u, v) radially symmetric and decreasing, we have compactness in $L^4(\mathbb{R}^N)$. As for the proof of Lemma 2.1, we may prove that there exists a minimizer for I_1 called (u_0, v_0) which is radially symmetric and decreasing.

By similar arguments as in the Claim 1 of Lemma 2.1, we deduce that $\int_{\mathbb{R}^N} u_0^4 \geqslant C_1$, $\int_{\mathbb{R}^N} v_0^4 \geqslant C_1$, provided that $\beta < \beta_0$. This implies that $u_0 \not\equiv 0$, $v_0 \not\equiv 0$.

As for the proof of Lemma 3.2, we must have either

$$\int_{\mathbb{R}^N} |\nabla u_0|^2 + \lambda_1 \int_{\mathbb{R}^N} u_0^2 - \beta \int_{\mathbb{R}^N} u_0^2v_0^2 = \mu_1 \int_{\mathbb{R}^N} u_0^4 \tag{3.18}$$

or

$$\int_{\mathbb{R}^N} |\nabla v_0|^2 + \lambda_2 \int_{\mathbb{R}^N} v_0^2 - \beta \int_{\mathbb{R}^N} u_0^2v_0^2 = \mu_2 \int_{\mathbb{R}^N} v_0^4. \tag{3.19}$$

We claim that both (3.18) and (3.19) should hold. Suppose not. We may assume that (3.18) holds but (3.19) does not hold. So we have

$$\nabla I[u_0, v_0] + \lambda \nabla G_1[u_0, v_0] = 0$$

for some Lagrange multiplier λ. We claim that $\lambda = 0$. In fact,

$$\lambda\langle \nabla G_1[u_0, v_0], u_0\rangle = 0, \quad \text{i.e.}$$
$$\lambda\left(2\int_{\mathbb{R}^N} |\nabla u_0|^2 + 2\lambda_1 \int_{\mathbb{R}^N} u_0^2 - 2\beta \int_{\mathbb{R}^N} u_0^2v_0^2 - 4\mu_1 \int_{\mathbb{R}^N} u_0^4\right) = \lambda\left(-2\mu_1 \int_{\mathbb{R}^N} u_0^4\right) = 0$$

and hence $\lambda = 0$. Here we have used the fact that $u_0 \not\equiv 0$. This shows that $(u_0, v_0) \in M_1$. Hence $I_0 = I_1$ can be achieved by a radially symmetric pair (u_0, v_0). As for Lemma 2.1 and 2.2, using the fact that $0 < \beta < \sqrt{\mu_1\mu_2}$, we may prove that (u_0, v_0) must satisfy (1.4).

On the other hand, any minimizer of I_0, called (U_0, V_0), must satisfy

$$\begin{cases} \Delta U_0 - \lambda_1 U_0 + \mu_1 U_0^3 + \beta U_0 V_0^2 = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta V_0 - \lambda_2 V_0 + \mu_2 V_0^3 + \beta U_0^2 V_0 = 0 \quad \text{in } \mathbb{R}^N, \\ U_0, V_0 > 0, \quad U_0, V_0 \in H^1(\mathbb{R}^N). \end{cases} \tag{3.20}$$

As $\beta > 0$, by the moving plane method (see [30]), (U_0, V_0) must be radially symmetric and strictly decreasing. This finishes (1) of Theorem 3.3.

(2) Let $\beta<0$. By choosing $u=\sqrt{s}\,\sqrt{\lambda_1/\mu_1}\,w(\sqrt{\lambda_1}(y-Re_1))$, $v=\sqrt{t}\,\sqrt{\lambda_2/\mu_2}\,w(\sqrt{\lambda_2}\,(y+Re_1))$, where $R\gg 1$,

$$e_1=\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix},$$

we compute the maximum of

$$\beta(s,t)=I[u,v].$$

By choosing R large enough, we see that

$$\int_{\mathbb{R}^N} w^2\big(\sqrt{\lambda_1}\,(y-Re_1)\big)w^2\big(\sqrt{\lambda_2}\,(y+Re_1)\big)\to 0\quad\text{as } R\to+\infty$$

and hence

$$\lim_{R\to+\infty}\max\beta(s,t)\leqslant\big(\lambda_1^{(4-N)/2}\mu_1^{-1}I[w]+\lambda_2^{(4-N)/2}\mu_2^{-1}I[w]\big).$$

So

$$I_0\leqslant\big(\lambda_1^{(4-N)/2}\mu_1^{-1}+\lambda_2^{(4-N)/2}\mu_2^{-1}\big)I[w].$$

Next we claim that

$$I_0\geqslant\big(\lambda_1^{(4-N)/2}\mu_1^{-1}+\lambda_2^{(4-N)/2}\mu_2^{-1}\big)I[w].$$

In fact, let $(u,v)\in M_1$ and $\beta<0$, then

$$I[u,v]\geqslant\frac12\int_{\mathbb{R}^N}|\nabla u|^2+\frac{\lambda_1}{2}\int_{\mathbb{R}^N}u^2-\frac{\mu_1}{4}\int_{\mathbb{R}^N}u^4+\frac12\int_{\mathbb{R}^N}|\nabla v|^2+\frac{\lambda_2}{2}\int_{\mathbb{R}^N}v^2-\frac{\mu_2}{4}\int_{\mathbb{R}^N}v^4\tag{3.21}$$

and

$$\int_{\mathbb{R}^N}|\nabla u|^2+\lambda_1\int_{\mathbb{R}^N}u^2\leqslant\mu_1\int_{\mathbb{R}^N}u^4,$$

$$\int_{\mathbb{R}^N}|\nabla v|^2+\lambda_2\int_{\mathbb{R}^N}v^2\leqslant\mu_2\int_{\mathbb{R}^N}v^4.$$

Hence

$$\begin{aligned}\inf_{(u,v)\in M_1}I[u,v]\geqslant\inf_{(u,v)\in M_2}\bigg[&\frac12\int_{\mathbb{R}^N}|\nabla u|^2+\frac{\lambda_1}{2}\int_{\mathbb{R}^N}u^2-\frac{\mu_1}{4}\int_{\mathbb{R}^N}u^4\\&+\frac12\int_{\mathbb{R}^N}|\nabla v|^2+\frac{\lambda_2}{2}\int_{\mathbb{R}^N}v^2-\frac{\mu_2}{4}\int_{\mathbb{R}^N}v^4\bigg]\end{aligned}\tag{3.22}$$

where

$$M_2=\left\{(u,v)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)\;\middle|\;\begin{matrix}\int_{\mathbb{R}^N}|\nabla u|^2+\lambda_1\int_{\mathbb{R}^N}u^2\leqslant\mu_1\int_{\mathbb{R}^N}u^4\\ \int_{\mathbb{R}^N}|\nabla v|^2+\lambda_2\int_{\mathbb{R}^N}v^2\leqslant\mu_2\int_{\mathbb{R}^N}v^4\end{matrix}\right\}.$$

By Lemma 3.2, the right-hand side of (3.22)

$$\geqslant \left(\lambda_1^{(4-N)/2}\mu_1^{-1} + \lambda_2^{(4-N)/2}\mu_2^{-1}\right)I[w]. \tag{3.23}$$

This proves that $I_0 = (\lambda_1^{(4-N)/2}\mu_1^{-1} + \lambda_2^{(4-N)/2}\mu_2^{-1})I[w]$.

Finally we prove that I_0 is never attained. In fact, suppose I_0 is attained by some (u_0, v_0). Then $(u_0, v_0) \in M_1$. By Maximum principle, $u_0, v_0 > 0$. Then by Lemma 3.2 again

$$\begin{aligned} I_0 = I[u_0, v_0] &> \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_0|^2 + \frac{\lambda_1}{2}\int_{\mathbb{R}^N} u_0^2 - \frac{\mu_1}{4}\int_{\mathbb{R}^N} u_0^4 + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_0|^2 + \frac{\lambda_2}{2}\int_{\mathbb{R}^N} v_0^2 - \frac{\mu_2}{4}\int_{\mathbb{R}^N} v_0^4 \\ &\geqslant \left(\lambda_1^{(4-N)/2}\mu_1^{-1} + \lambda_2^{(4-N)/2}\mu_2^{-1}\right)I[w] \end{aligned}$$

which is a contradiction! This completes the proof of (2) of Theorem 3.1. □

Next theorem concerns a problem on the half-space

$$\mathbb{R}_+^N = \left\{(y_1, \ldots, y_N) \in \mathbb{R}^N \mid y_N > 0\right\}. \tag{3.24}$$

Consider

$$\begin{cases} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta\, u v^2 = 0 \quad \text{in } \mathbb{R}_+^N, \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in } \mathbb{R}_+^N, \\ u, v \geqslant 0 \quad \text{in } \mathbb{R}_+^N,\ u, v \to 0 \quad \text{at } +\infty, \\ u = v = 0 \quad \text{on } \partial\mathbb{R}_+^N. \end{cases} \tag{3.25}$$

Theorem 3.4. *Problem* (3.25) *has no solution unless* $u \equiv 0$, $v \equiv 0$.

Proof. For single equation, similar theorem has been proved in [11]. Now we follow the same idea for the system (3.25). First, by Hopf Boundary Lemma

$$\frac{\partial u}{\partial y_N} < 0, \qquad \frac{\partial v}{\partial y_N} < 0 \quad \text{on } \partial\mathbb{R}_+^N. \tag{3.26}$$

Multiplying the u-equation by $\partial u/\partial y_N$, the v-equation by $\partial v/\partial y_N$, adding them together, and integrating them, we obtain

$$0 = \int_{\mathbb{R}^N} (\Delta u)\frac{\partial u}{\partial y_N} + \int_{\mathbb{R}^N} (\Delta v)\frac{\partial v}{\partial y_N} = -\frac{1}{2}\int_{\mathbb{R}^N} \left[\left(\frac{\partial u}{\partial y_N}\right)^2 + \left(\frac{\partial v}{\partial y_N}\right)^2\right]$$

which contradicts to (3.26). □

Finally, we compute some interaction integrals which will be useful later.

Lemma 3.5. *Assume that* $\lambda_1 \leqslant \lambda_2$ *and* $|P - Q|/\varepsilon \gg 1$. *Then we have*

(1)
$$\begin{aligned} &w\left(\sqrt{\lambda_1}\,\frac{|x-P|}{\varepsilon}\right)w^2\left(\sqrt{\lambda_2}\,\frac{|x-Q|}{\varepsilon}\right) \\ &\quad = w\left(\sqrt{\lambda_1}\,\frac{|P-Q|}{\varepsilon}\right)\left(1+o(1)\right)w^2\left(\sqrt{\lambda_2}\,\frac{|x-Q|}{\varepsilon}\right)\cdot \mathrm{e}^{-\sqrt{\lambda_1}\langle \frac{Q-P}{|Q-P|}, \frac{x-Q}{\varepsilon}\rangle}. \end{aligned}$$

(2) *If* $\lambda_1 < \lambda_2$, *then*

$$\int_{\mathbb{R}^N} w^2\left(\sqrt{\lambda_1}\frac{|x-P|}{\varepsilon}\right)w^2\left(\sqrt{\lambda_2}\,\frac{|x-Q|}{\varepsilon}\right)\mathrm{d}x$$
$$= w^2\left(\sqrt{\lambda_1}\,\frac{|P-Q|}{\varepsilon}\right)\varepsilon^N\big(1+o(1)\big)\int_{\mathbb{R}^N} w^2(\sqrt{\lambda_2}\,y)\,\mathrm{e}^{-2\sqrt{\lambda_1}\langle \frac{Q-P}{|Q-P|},y\rangle}\,\mathrm{d}y. \tag{3.27}$$

(3) *If* $\lambda_1 = \lambda_2$, *then we have*

$$w^{2(1+\sigma/4)}\left(\sqrt{\lambda_1}\frac{|P-Q|}{\varepsilon}\right) \leqslant \varepsilon^{-N}\int_{\mathbb{R}^N} w^2\left(\sqrt{\lambda_1}\,\frac{|x-P|}{\varepsilon}\right)w^2\left(\sqrt{\lambda_2}\,\frac{|x-Q|}{\varepsilon}\right)\mathrm{d}x$$
$$\leqslant w^{2(1-\sigma/4)}\left(\sqrt{\lambda_1}\,\frac{|P-Q|}{\varepsilon}\right). \tag{3.28}$$

Proof. (1) Let $x = Q + \epsilon y$. Since $\lambda_1 \leqslant \lambda_2$, using asymptotic expansion (3.4) of w, we then have

$$w\left(\sqrt{\lambda_1}\,\frac{|x-P|}{\epsilon}\right) = w\left(\sqrt{\lambda_1}\,\frac{|Q-P+\epsilon y|}{\epsilon}\right)$$
$$= \big(1+o(1)\big)w\left(\sqrt{\lambda_1}\,\frac{|Q-P|}{\epsilon}\right)\mathrm{e}^{-\sqrt{\lambda_1}\langle \frac{Q-P}{|Q-P|},y\rangle}. \tag{3.29}$$

(2) If $\lambda_1 < \lambda_2$, then similar to (3.29), we have

$$w^2\left(\sqrt{\lambda_1}\,\frac{|x-P|}{\epsilon}\right) = \big(1+o(1)\big)w^2\left(\sqrt{\lambda_1}\,\frac{|Q-P|}{\epsilon}\right)\mathrm{e}^{-2\sqrt{\lambda_1}\,\langle \frac{Q-P}{|Q-P|},\frac{x-Q}{\varepsilon}\rangle}. \tag{3.30}$$

By Lebesgue's Dominated Convergence Theorem (using the fact that $\lambda_1 < \lambda_2$), we have

$$\int_{\mathbb{R}^N} w^2\left(\sqrt{\lambda_1}\,\frac{|x-P|}{\varepsilon}\right)w^2\left(\sqrt{\lambda_2}\,\frac{|x-Q|}{\varepsilon}\right)\mathrm{d}x$$
$$= \big(1+o(1)\big)w^2\left(\sqrt{\lambda_1}\,\frac{|P-Q|}{\varepsilon}\right)\varepsilon^N\int_{\mathbb{R}^N} w^2(\sqrt{\lambda_2}\,y)\,\mathrm{e}^{-2\sqrt{\lambda_1}\,\langle \frac{Q-P}{|Q-P|},y\rangle}\,\mathrm{d}y.$$

(3) For $\lambda_1 = \lambda_2$, the proof is similar. □

Remark 3.1. If $\lambda_1 = \lambda_2$, $N = 1$, then

$$\int_{R^1} w^2\left(\sqrt{\lambda_1}\left(y-\frac{P}{\varepsilon}\right)\right)w^2\left(\sqrt{\lambda_2}\left(y-\frac{Q}{\varepsilon}\right)\right)\mathrm{d}y = 2\int_0^{+\infty}\left(\frac{1}{\cosh(2\sqrt{\lambda_1}\,y)+\cosh(2\sqrt{\lambda_1}\,|P-Q|/\varepsilon)}\right)\mathrm{d}y.$$

4. Upper bound for c_ϵ when $\beta > 0$

In this section, we study the asymptotic behavior of the least energy solution for the following problem in a ball

$$\begin{cases} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 & \text{in } B_R,\\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } B_R,\\ u, v > 0 \quad \text{in } B_R,\\ u = v = 0 \quad \text{on } \partial B_R, \end{cases} \tag{4.1}$$

where $\beta > 0$, and give an upper bound for c_ε when $\beta > 0$. Hereafter, B_R is the ball of radius R with the center at origin.

The following is the main theorem in this section.

Theorem 4.1. *Let $0 < \beta < \beta_0 < \sqrt{\mu_1\mu_2}$ and consider the following problem*

$$I_R := \inf_{(u,v)\in M_R} I_R[u,v] \tag{4.2}$$

where

$$I_R[u,v] = \frac{1}{2}\int_{B_R} |\nabla u|^2 + \frac{\lambda_1}{2}\int_{B_R} u^2 - \frac{\mu_1}{4}\int_{B_R} u^4 + \frac{1}{2}\int_{B_R} |\nabla v|^2 + \frac{\lambda_2}{2}\int_{B_R} v^2 - \frac{\mu_2}{4}\int_{B_R} v^4 - \frac{\beta}{2}\int_{B_R} u^2v^2,$$

for $u, v \in H_0^1(B_R)$,

and

$$M_R = \left\{(u,v) \in \left(H_0^1(B_R)\right)^2 \;\middle|\; \begin{array}{l} \int_{B_R} |\nabla u|^2 + \lambda_1 \int_{B_R} u^2 - \beta\int_{B_R} u^2v^2 = \mu_1\int_{B_R} u^4 \\ \int_{B_R} |\nabla v|^2 + \lambda_2 \int_{B_R} v^2 - \beta\int_{B_R} u^2v^2 = \mu_2\int_{B_R} v^4 \end{array}\right\}.$$

Then for R sufficiently large, I_R is attained and the minimizer, called (U_R, V_R), is radially symmetric and strictly decreasing and is the least energy solution to (4.1)*. Moreover as $R \to +\infty$, we have*

$$I_R \leqslant I_0 + c_1 \exp\left[-2(1-\sigma)\sqrt{\lambda_1}\, R\right] + c_2 \exp\left[-2(1-\sigma)\sqrt{\lambda_2}\, R\right], \tag{4.3}$$

$$I_R \geqslant I_0 + c_3 \exp\left[-2(1+\sigma)\sqrt{\lambda_1}\, R\right] + c_4 \exp\left[-2(1+\sigma)\sqrt{\lambda_2}\, R\right], \tag{4.4}$$

where $I_0 = I[U_0, V_0]$ which is given in Theorem 3.3*, and $c_1, c_2, c_3, c_4 > 0$ are constants.*

Proof. The first part of the statement follows from the same proof in Theorem 3.3 (1). Let (U_R, V_R) satisfy

$$\begin{cases} \Delta U_R - \lambda_1 U_R + \mu_1 U_R^3 + \beta U_R V_R^2 = 0 & \text{in } B_R, \\ \Delta V_R - \lambda_2 V_R + \mu_2 V_R^3 + \beta U_R^2 V_R = 0 & \text{in } B_R, \\ U_R = U_R(r), \quad V_R = V_R(r), \quad U_R > 0, \quad V_R > 0 & \text{in } B_R, \\ U_R(R) = V_R(R) = 0. \end{cases}$$

Then by standard estimates, we have

$$U_R(r) \leqslant \left(A_0 + o(1)\right) \mathrm{e}^{-(\sqrt{\lambda_1}-\sigma)r}, \qquad V_R(r) \leqslant \left(B_0 + o(1)\right) \mathrm{e}^{-(\sqrt{\lambda_2}-\sigma)r} \tag{4.5}$$

for σ small, where A_0, B_0 are two positive constants.

By comparison principle, one sees that as $R \to +\infty$

$$U_R(R-1), -U_R'(R-1) \sim A_0 R^{-(N-1)/2} \mathrm{e}^{-\sqrt{\lambda_1}(R-1)}, \tag{4.6}$$

$$V_R(R-1), -V_R'(R-1) \sim B_0 R^{-(N-1)/2} \mathrm{e}^{-\sqrt{\lambda_2}(R-1)}. \tag{4.7}$$

To obtain an upper bound for I_R, we use the variational characterization of I_R. To this end, we take (U_0, V_0)-ground-state solution constructed in Theorem 3.3 and a cut-off function

$$\eta(y) = \begin{cases} 1, & |y| < R-1, \\ 0, & |y| > R. \end{cases}$$

Then it is easy to see that for R large the function $I_R[\sqrt{s}\, U_0\eta, \sqrt{t}\, V_0\eta]$ has a local maximum $s_R = 1 + o(1) > 0$, $t_R = 1 + o(1) > 0$. By same argument as in Lemma 2.3, we can compute

$$I_R \leqslant I_R[\sqrt{s_R}\, U_0\eta, \sqrt{t_R}\, V_0\eta] \leqslant I[U_0, V_0] + c_1\, \mathrm{e}^{-2\sqrt{\lambda_1}(1-\sigma)R} + c_2\, \mathrm{e}^{-2\sqrt{\lambda_2}(1-\sigma)R} \tag{4.8}$$

for some positive constants c_1, c_2. Here we have used the fact that

$$\int_{\mathbb{R}^N} \left|\nabla(U_0\eta)\right|^2 + \lambda_1 \int_{\mathbb{R}^N} (U_0\eta)^2 \leqslant \int_{\mathbb{R}^N} |\nabla U_0|^2 + \lambda_1 \int_{\mathbb{R}^N} U_0^2 + c_1\, \mathrm{e}^{-2\sqrt{\lambda_1}(1-\sigma)R},$$

$$\int_{\mathbb{R}^N} \left|\nabla(V_0\eta)\right|^2 + \lambda_2 \int_{\mathbb{R}^N} (V_0\eta)^2 \leqslant \int_{\mathbb{R}^N} |\nabla V_0|^2 + \lambda_2 \int_{\mathbb{R}^N} V_0^2 + c_2\, \mathrm{e}^{-2\sqrt{\lambda_2}(1-\sigma)R}.$$

This proves (4.3).

As for the lower bound (4.4), we extend U_R, V_R to whole $\mathbb{R}^N$ by the following way: let $V_{i,R}, i = 1, 2$ satisfy

$$\begin{cases} \Delta V_{1,R} - \lambda_1 V_{1,R} = 0 & \text{in } \mathbb{R}^N \setminus B_{R-1}, \\ V_{1,R}(R-1) = U_R(R-1), \quad V_{1,R} \to 0 & \text{at } \infty, \end{cases} \tag{4.9}$$

$$\begin{cases} \Delta V_{2,R} - \lambda_2 V_{2,R} = 0 & \text{in } \mathbb{R}^N \setminus B_{R-1}, \\ V_{2,R}(R-1) = V_R(R-1), \quad V_{2,R} \to 0 & \text{at } \infty. \end{cases} \tag{4.10}$$

Let

$$\overline{U}_R = \begin{cases} U_R(r), & r \leqslant R-1, \\ V_{1,R}(r), & r \geqslant R-1, \end{cases} \qquad \overline{V}_R = \begin{cases} V_R(r), & r \leqslant R-1, \\ V_{2,R}(r), & r \geqslant R-1. \end{cases} \tag{4.11}$$

Then we have, for all $s, t > 0$ that

$$\begin{aligned} I[\sqrt{s}\,\overline{U}_R, \sqrt{t}\,\overline{V}_R] &= s\left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla \overline{U}_R|^2 + \frac{\lambda_1}{2}\int_{\mathbb{R}^N} \overline{U}_R^2 - \frac{\mu_1 s}{4}\int_{\mathbb{R}^N} \overline{U}_R^4\right] \\ &\quad + t\left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla \overline{V}_R|^2 + \frac{\lambda_1}{2}\int_{\mathbb{R}^N} \overline{V}_R^2 - \frac{\mu_2 t}{4}\int_{\mathbb{R}^N} \overline{V}_R^4\right] - \frac{\beta}{2} st \int_{\mathbb{R}^N} \overline{U}_R^2 \overline{V}_R^2 \\ &\leqslant I_R[\sqrt{s}\, U_R, \sqrt{t}\, V_R] + \frac{s}{2} \int_{\mathbb{R}^N \setminus B_{R-1}} \left(|\nabla V_{1,R}|^2 + \lambda_1 |V_{1,R}|^2\right) \\ &\quad + \frac{t}{2} \int_{\mathbb{R}^N \setminus B_{R-1}} \left(|\nabla V_{2,R}|^2 + \lambda_2 |V_{2,R}|^2\right) - \frac{s}{2} \int_{B_R \setminus B_{R-1}} \left[|\nabla U_R|^2 + e_1 U_R^2\right] \\ &\quad - \frac{t}{2} \int_{B_R \setminus B_{R-1}} \left[|\nabla V_R|^2 + e_2 V_R^2\right] - \frac{\beta}{2} st \int_{B_R \setminus B_{R-1}} [V_{1,R}^2 V_{2,R}^2] + \frac{\beta}{2} st \int_{B_R \setminus B_{R-1}} [U_R^2 V_R^2] \end{aligned} \tag{4.12}$$

where $e_1 = \lambda_1 - \frac{\mu_1 s}{2} U_R^2$, $e_2 = \lambda_2 - \frac{\mu_2 t}{2} V_R^2$.

Let $Z_{1,R}$ be such that

$$\begin{cases} \Delta Z_{1,R} - \lambda_1 Z_{1,R} = 0, & R-1 < r < R, \\ Z_{1,R}(R-1) = U_R(R-1), & Z_{1,R}(R) = 0. \end{cases}$$

By a standard argument on elliptic partial differential equations, it is easy to check that

$$\int_{B_R \setminus B_{R-1}} |\nabla Z_{1,R}|^2 + \lambda_1 Z_{1,R}^2 \leqslant \int_{B_R \setminus B_{R-1}} \left[|\nabla U_R|^2 + \lambda_1 U_R^2\right].$$

Moreover, by (4.6), we may obtain

$$\int\limits_{B_R\setminus B_{R-1}} |\nabla Z_{1,R}|^2+\lambda_1 Z_{1,R}^2 \leqslant \Bigg[\int\limits_{B_R\setminus B_{R-1}} \big(|\nabla U_R|^2+e_1U_R^2\big)\Bigg]+o(1)\,s\,U_R^2(R-1).$$

In addition,

$$\begin{aligned}
&\frac{s}{2}\Bigg[\int\limits_{\mathbb{R}^N\setminus B_{R-1}} \big(|\nabla V_{1,R}|^2+\lambda_1|V_{1,R}|^2\big)-\int\limits_{B_R\setminus B_{R-1}} \big(|\nabla Z_{1,R}|^2+\lambda_1|Z_{1,R}|^2\big)\Bigg]\\
&\quad=\frac{s}{2}\Bigg[\int\limits_{\partial B_{R-1}} (Z_{1,R}Z_{1,R}'-V_{1,R}V_{1,R}')\Bigg]\\
&\quad=\frac{s}{2}U_R(R-1)\big(Z_{1,R}'(R-1)-V_{1,R}'(R-1)\big)(R-1)^{N-1}|S^{N-1}|\\
&\quad\leqslant-\frac{s}{2}U_R^2(R-1)\alpha_1 \quad \text{for some } \alpha_1>0
\end{aligned}$$

where $|S^{N-1}|$ is the area of the unit-sphere in $\mathbb{R}^N$.

Hence

$$\frac{s}{2}\Bigg[\int\limits_{\mathbb{R}^N\setminus B_{R-1}} \big(|\nabla V_{1,R}|^2+\lambda_1|V_{1,R}|^2\big)-\int\limits_{B_R\setminus B_{R-1}} \big(|\nabla U_R|^2+e_1U_R^2\big)\Bigg]\leqslant-\frac{s}{2}U_R^2(R-1)\alpha_0$$
$$\text{for some } \alpha_0>0. \tag{4.13}$$

Similarly

$$\frac{t}{2}\Bigg[\int\limits_{\mathbb{R}^N\setminus B_{R-1}} \big(|\nabla V_{2,R}|^2+\lambda_2|V_{2,R}|^2\big)-\int\limits_{B_R\setminus B_{R-1}} \big(|\nabla V_R|^2+e_2V_R^2\big)\Bigg]\leqslant-\frac{t}{2}V_R^2(R-1)\beta_0$$
$$\text{for some } \beta_0>0. \tag{4.14}$$

On the other hand, $U_R\leqslant V_{1,R}$, $V_R\leqslant V_{2,R}$, hence

$$\frac{\beta}{2}st\Bigg[\int\limits_{B_R\setminus B_{R-1}} (U_R^2V_R^2-V_{1,R}^2V_{2,R}^2)\Bigg]\leqslant 0.$$

Therefore combing (4.12)–(4.14), we obtain

$$I[\sqrt{s}\,\overline{U}_R;\sqrt{t}\,\overline{V}_R]\leqslant I_R[\sqrt{s}\,U_R,\sqrt{t}\,V_R]-\frac{s}{2}U_R^2(R-1)\alpha_0-\frac{t}{2}V_R^2(R-1)\beta_0.$$

Now we take s_R,t_R such that

$$\begin{aligned}
I_0&\leqslant\sup_{s,t>0} I[\sqrt{s}\,\overline{U}_R,\sqrt{t}\,\overline{V}_R]=I[\sqrt{s_R}\,\overline{U}_R,\sqrt{t_R}\,\overline{V}_R]\\
&\leqslant I_R[\sqrt{s_R}\,U_R,\sqrt{t_R}\,V_R]-c_3\,\mathrm{e}^{-2\sqrt{\lambda_1}(1+\sigma)R}-c_4\,\mathrm{e}^{-2\sqrt{\lambda_2}(1+\sigma)R}\\
&\leqslant\sup_{s,t>0} I_R[\sqrt{s}\,U_R,\sqrt{t}\,V_R]-c_3\,\mathrm{e}^{-2\sqrt{\lambda_1}(1+\sigma)R}-c_4\,\mathrm{e}^{-2\sqrt{\lambda_2}(1+\sigma)R}\\
&\leqslant I_R-c_3\,\mathrm{e}^{-2\sqrt{\lambda_1}(1+\sigma)R}-c_4\,\mathrm{e}^{-2\sqrt{\lambda_2}(1+\sigma)R}
\end{aligned}$$

which gives a lower bound on I_R. Here we have used Eqs. (4.6) and (4.7). □

Finally we consider the attractive case $\beta > 0$ in a general domain.

Lemma 4.2. *For $\beta > 0$ (attractive case), we have*

$$c_\varepsilon \leqslant \varepsilon^N \left\{ I_0 + c_1 \mathrm{e}^{-2\sqrt{\lambda_1}(1-\sigma)R_\epsilon} + c_2 \mathrm{e}^{-2\sqrt{\lambda_2}(1-\sigma)R_\epsilon} \right\} \tag{4.15}$$

where I_0 is given in (3.11) *and $R_\epsilon = \frac{1}{\epsilon} \max_{P \in \Omega} d(P, \partial\Omega)$.*

Proof. Choosing a point $P_0 \in \Omega$ such that $d(P_0, \partial\Omega) = d_0 = \max_{P\in\Omega} d(P, \partial\Omega)$, and letting $R_\epsilon = d_0/\epsilon$, we see that Lemma (4.2) follows directly from Theorem 4.1 since $H_0^1(B_{\epsilon R_\epsilon}(P_0)) \subset H_0^1(\Omega)$. □

5. Upper bounds for c_ε when $\beta < 0$

In this section, we compute some upper bounds for c_ε, using Lemma 2.3, when $\beta < 0$ (repulsive).

Let w_i be the unique solution of (1.7). Fix $P \in \Omega$. We define $w_{i,\varepsilon,P}$ to be the unique solution of the following problem

$$\begin{cases} \varepsilon^2 \Delta w_{i,\varepsilon,P} - \lambda_i w_{i,\varepsilon,P} + \mu_i w_i^3\left(\dfrac{x-P}{\varepsilon}\right) = 0 & \text{in } \Omega, \\ w_{i,\varepsilon,P} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Let

$$w_{i,\varepsilon,P} = w_i\left(\frac{x-P}{\varepsilon}\right) - \varphi_{i,\varepsilon,P}, \qquad \Omega_{\varepsilon,P} = \{y \mid \varepsilon y + P \in \Omega\}. \tag{5.2}$$

The study of $w_{i,\varepsilon,P}$ is contained in Section 4 of [28]. There it is assumed that $\lambda_i = \mu_i = 1$. But an easy scaling argument gives the following lemma.

Lemma 5.1.

(1) $$-\varepsilon \log \varphi_{i,\varepsilon,P}(P) \to 2\sqrt{\lambda_i}\, d(P, \partial\Omega) \quad \text{as } \varepsilon \to 0. \tag{5.3}$$

(2) *Let $V_{\varepsilon,i}(y) = \varphi_{i,\varepsilon,P}(P + \varepsilon y)/\varphi_{i,\varepsilon,P}(P)$. Then as $\varepsilon \to 0$ (up to a subsequence), $V_{\varepsilon,i}(y) \to V_i(y)$, where $V_i(y)$ is a solution of*

$$\begin{cases} \Delta V_i - \lambda_i V_i = 0 & \text{in } \mathbb{R}^N, \\ V_i(0) = 1,\ V_i > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{5.4}$$

(3) $$\sup_{y \in \Omega_{\varepsilon,P}} \left| \mathrm{e}^{-\sqrt{\lambda_i}(1+\sigma)|y|} V_{\varepsilon,i}(y) \right| \leqslant C \quad \text{for any } 0 < \sigma < 1. \tag{5.5}$$

Lemma 5.2.

(1) $$\frac{1}{2}\varepsilon^2 \int_\Omega |\nabla w_{i,\varepsilon,P}|^2 + \frac{\lambda_i}{2} \int_\Omega w_{i,\varepsilon,P}^2 - \frac{\mu_i}{4} \int_\Omega w_{i,\varepsilon,P}^4$$
$$= \varepsilon^N \left[\lambda_i^{(4-N)/2} \mu_i^{-1} I[w] + \big(a_i + o(1)\big) \varphi_{i,\varepsilon,P}(P) \right] \tag{5.6}$$

for some $a_i > 0$, $i = 1, 2$.

(2) $$\int_\Omega w_{1,\varepsilon,P}^2\, w_{2,\varepsilon,Q}^2 = \varepsilon^N \left[\int_{\mathbb{R}^N} w_1^2\left(y - \frac{P}{\varepsilon}\right) w_2^2\left(y - \frac{Q}{\varepsilon}\right) \right] \big(1 + o(1)\big). \tag{5.7}$$

Proof. (1) follows from the computations given in [28]. For (2) we note that

$$\int_\Omega w_{1,\varepsilon,P}^2 w_{2,\varepsilon,Q}^2 \leqslant \int_\Omega w_1^2\left(\frac{x-P}{\varepsilon}\right) w_2^2\left(\frac{x-Q}{\varepsilon}\right) = \varepsilon^N\left[\int_{\mathbb{R}^N} w_1^2\left(y-\frac{P}{\varepsilon}\right) w_2^2\left(y-\frac{Q}{\varepsilon}\right)\right]\big(1+o(1)\big). \tag{5.8}$$

To obtain the other side of the inequality, we note that

$$w_{1,\varepsilon,P} \geqslant \big(1+o(1)\big) w_1\left(\frac{x-P}{\varepsilon}\right), \quad x \in B_{\varepsilon R}(P) \cup B_{\varepsilon R}(Q),$$
$$w_{2,\varepsilon,Q} \geqslant \big(1+o(1)\big) w_2\left(\frac{x-Q}{\varepsilon}\right), \quad x \in B_{\varepsilon R}(P) \cup B_{\varepsilon R}(Q).$$

Hence

$$\int_\Omega w_{1,\varepsilon,P}^2 w_{2,\varepsilon,Q}^2 \geqslant \big(1+o(1)\big) \int_{B_{\varepsilon R}(P)\cup B_{\varepsilon R}(Q)} w_1^2\left(\frac{x-P}{\varepsilon}\right) w_2^2\left(\frac{x-Q}{\varepsilon}\right) \geqslant \big(1+o(1)\big)\varepsilon^N \int_{\mathbb{R}^N} w_1^2\left(y-\frac{P}{\varepsilon}\right) w_2^2\left(y-\frac{Q}{\varepsilon}\right) \mathrm{d}y. \tag{5.9}$$

Combining (5.8) and (5.9), we obtain (5.7). □

Let us denote

$$I_\varepsilon(P,Q) = \int_{\mathbb{R}^N} w_1^2\left(y-\frac{P}{\varepsilon}\right) w_2^2\left(y-\frac{Q}{\varepsilon}\right) \mathrm{d}y, \tag{5.10}$$

$$\delta_\varepsilon(P,Q) = \varphi_{1,\varepsilon,P}(P) + \varphi_{2,\varepsilon,Q}(Q) + I_\varepsilon(P,Q). \tag{5.11}$$

By Lemmas 5.1 and 3.5, we can easily arrive at the following estimate.

Lemma 5.3. *Suppose $d(P,\partial\Omega)/\epsilon \gg 1$, $d(Q,\partial\Omega)/\epsilon \gg 1$, $|P-Q|/\varepsilon \gg 1$. Then we have*

$$\exp\left[-\frac{2^{1+\sigma}\varphi^{1+\sigma}(P,Q)}{\varepsilon}\right] \leqslant \delta_\varepsilon(P,Q) \leqslant \exp\left[-\frac{2^{1-\sigma}\varphi^{1-\sigma}(P,Q)}{\varepsilon}\right]. \tag{5.12}$$

Set

$$\beta(s,t) = E_\Omega[\sqrt{s}\, w_{1,\varepsilon,P}, \sqrt{t}\, w_{2,\varepsilon,Q}]$$

and $(s_\varepsilon, t_\varepsilon)$ be such that $\frac{\partial\beta}{\partial s}(s_\varepsilon,t_\varepsilon) = \frac{\partial\beta}{\partial t}(s_\varepsilon,t_\varepsilon) = 0$. Then we have

$$\varepsilon^2\int_\Omega |\nabla w_{1,\varepsilon,P}|^2 + \lambda_1\int_\Omega w_{1,\varepsilon,P}^2 = \mu_1 s_\varepsilon \int_\Omega w_{1,\varepsilon,P}^4 + \beta t_\varepsilon \int_\Omega w_{1,\varepsilon,P}^2 w_{2,\varepsilon,Q}^2,$$
$$\varepsilon^2\int_\Omega |\nabla w_{2,\varepsilon,Q}|^2 + \lambda_2\int_\Omega w_{2,\varepsilon,Q}^2 = \mu_2 t_\varepsilon \int_\Omega w_{2,\varepsilon,Q}^4 + \beta s_\varepsilon \int_\Omega w_{1,\varepsilon,P}^2 w_{2,\varepsilon,Q}^2.$$

Note that $(s_\varepsilon, t_\varepsilon)$ is unique. Similar to the proof of Lemma 5.3 of [28], we obtain that theorem

$$s_\varepsilon = 1 + O\big(\delta_\varepsilon(P,Q)\big), \tag{5.13}$$
$$t_\varepsilon = 1 + O\big(\delta_\varepsilon(P,Q)\big). \tag{5.14}$$

We are now ready to compute

$$\begin{aligned}
&E_\Omega[\sqrt{s_\varepsilon}\, w_{1,\varepsilon,P}, \sqrt{t_\varepsilon}\, w_{2,\varepsilon,Q}] \\
&= \frac{s_\varepsilon}{2}\left[\varepsilon^2 \int_\Omega |\nabla w_{1,\varepsilon,P}|^2 + \lambda_1 \int_\Omega w_{1,\varepsilon,P}^2\right] - \frac{s_\varepsilon^2}{4}\mu_1 \int_\Omega w_{1,\varepsilon,P}^4 \\
&\quad + \frac{t_\varepsilon}{2}\left[\varepsilon^2 \int_\Omega |\nabla w_{2,\varepsilon,Q}|^2 + \lambda_2 \int_\Omega w_{2,\varepsilon,Q}^2\right] - \frac{t_\varepsilon^2}{4}\mu_2 \int_\Omega w_{2,\varepsilon,Q}^2 - \frac{\beta}{2} s_\varepsilon t_\varepsilon \int_\Omega w_{1,\varepsilon,P}^2 w_{w,\varepsilon,Q}^2 \\
&= E_\Omega[w_{1,\varepsilon,P}, w_{2,\varepsilon,Q}] + \frac{\partial E_\Omega}{\partial s}[\sqrt{s}\, w_{1,\varepsilon,P}, \sqrt{t}\, w_{2,\varepsilon,Q}]\Big|_{\{s=t=1\}} (s_\varepsilon - 1) \\
&\quad + \frac{\partial E_\Omega}{\partial t}[\sqrt{s}\, w_{1,\varepsilon,P}, \sqrt{t}\, w_{2,\varepsilon,Q}]\Big|_{\{s=t=1\}} (t_\varepsilon - 1) + O\big(\varepsilon^N \delta_\varepsilon^2(P,Q)\big) \\
&= \varepsilon^N \Bigg[\sum_{i=1}^{2} \lambda_i^{(4-N)/2} \mu_i^{-1} I[w] + \big(a_1 + o(1)\big)\varphi_{1,\varepsilon,P}(P) + \big(a_2 + o(1)\big)\varphi_{2,\varepsilon,Q}(Q) \\
&\quad - \big(1 + o(1)\big)\frac{\beta}{2} I_\varepsilon(P,Q)\Bigg].
\end{aligned} \tag{5.15}$$

Therefore we have

Lemma 5.4. *For $\beta < 0$ (repulsive case), we have*

$$\begin{aligned}
c_\varepsilon &= \inf_{\substack{u,v \geqslant 0,\ u,v \not\equiv 0,\\ (u,v) \in M(\epsilon,\Omega)}} E_\Omega[u,v] \\
&\leqslant \varepsilon^N \big[\lambda_1^{(4-N)/2} \mu_1^{-1} I[w] + \lambda_2^{(4-N)/2} \mu_2^{-1} I[w] + \big(a_1 + o(1)\big)\varphi_{1,\varepsilon,P}(P) \\
&\quad + \big(a_2 + o(1)\big)\varphi_{2,\varepsilon,Q}(Q) + \big(a_3 + o(1)\big) I_\varepsilon(P,Q)\big]
\end{aligned} \tag{5.16}$$

where a_1, a_2, a_3 are three positive constants. (In fact, $a_3 = -\frac{\beta}{2}$.)

6. Asymptotic behavior of least-energy solution: a first estimate

In this section, we study the asymptotic behavior of $(u_\varepsilon, v_\varepsilon)$ – the least energy solution of E_Ω. Our main result of this section is the following.

Theorem 6.1. *For ε sufficiently small, u_ε has only one local maximum point P_ε and v_ε has only one local maximum point Q_ε. Furthermore*

(1) *if $\beta < 0$ (repulsive case), then we have*

$$\frac{d(P_\varepsilon, \partial\Omega)}{\varepsilon} \to +\infty, \quad \frac{d(Q_\varepsilon, \partial\Omega)}{\varepsilon} \to +\infty, \quad \frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to +\infty, \tag{6.1}$$

and let $U_\varepsilon(y) := u_\varepsilon(P_\varepsilon + \varepsilon y)$, $V_\varepsilon(z) := v_\varepsilon(Q_\varepsilon + \varepsilon z)$, then

$$U_\varepsilon \to w_1(y), \qquad V_\varepsilon(z) \to w_2(z), \tag{6.2}$$
$$\varepsilon|\nabla u_\varepsilon|,\ u_\varepsilon(x) \leqslant C\,\mathrm{e}^{-(1-\sigma)\sqrt{\lambda_1}|x-P_\varepsilon|/\varepsilon}, \tag{6.3}$$
$$\varepsilon|\nabla v_\varepsilon|,\ v_\varepsilon(x) \leqslant C\,\mathrm{e}^{-(1-\sigma)\sqrt{\lambda_2}|x-Q_\varepsilon|/\varepsilon}. \tag{6.4}$$

(2) *if* $0 < \beta < \sqrt{\mu_1\mu_2}$ (*attractive case*), *then we have*

$$\frac{d(P_\varepsilon, \partial\Omega)}{\varepsilon} \to +\infty, \quad \frac{d(Q_\varepsilon, \partial\Omega)}{\varepsilon} \to +\infty, \quad \frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to 0, \tag{6.5}$$

and let $U_\varepsilon(y) := u_\varepsilon(P_\varepsilon + \varepsilon y)$, $V_\varepsilon(y) := (Q_\varepsilon + \varepsilon y)$, *then* $(U_\varepsilon, V_\varepsilon) \to (U_0, V_0)$ *where* (U_0, V_0) *is a least-energy solution of* (1.4), *and* (6.3), (6.4) *hold for* $u_\varepsilon, v_\varepsilon$ *respectively.*

The proof of Theorem 6.1 is divided into two cases: $\beta < 0$ and $\beta > 0$. Let us first consider $\beta < 0$ (repulsive case). We proceed with a few claims

Claim 1 (assuming $\beta < 0$).

$$\frac{d(P_\varepsilon, \partial\Omega)}{\varepsilon} \to +\infty, \qquad \frac{d(Q_\varepsilon, \partial\Omega)}{\varepsilon} \to +\infty.$$

Proof. Suppose not. Without loss of generality, we may assume that $d(P_\varepsilon, \partial\Omega)/\varepsilon \leqslant C$ and that $d(P_\varepsilon, \partial\Omega) = |P_\varepsilon - \overline{P_\varepsilon}|$, $\overline{P_\varepsilon} \in \partial\Omega$. Further, we can assume that $\overline{P_\varepsilon} = 0$ and that the normal derivative at $\overline{P_\varepsilon}$ is pointing at the x_N-direction. Now let $U_\varepsilon(y) := u_\varepsilon(\overline{P_\varepsilon} + \varepsilon y)$, $V_\varepsilon(y) := v_\varepsilon(\overline{P_\varepsilon} + \varepsilon y)$. Then as $\varepsilon \to 0$, $(U_\varepsilon, V_\varepsilon) \to (U, V)$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N_+)$, where (U, V) a solution of

$$\begin{cases} \Delta U - \lambda_1 U + \mu_1 U^3 + \beta U V^2 = 0 & \text{in } \mathbb{R}^N_+, \\ \Delta V - \lambda_2 V + \mu_2 V^3 + \beta U^2 V = 0 & \text{in } \mathbb{R}^N_+, \\ U, V \in H^1(\mathbb{R}^N), \quad U = V = 0 & \text{on } \partial\mathbb{R}^N_+,\ U, V \geqslant 0. \end{cases} \tag{6.6}$$

By Theorem 3.4, $U, V \equiv 0$.

We show that this leads to a contraction. In fact, assuming that $d(P_\varepsilon, \partial\Omega)/\varepsilon \to \eta_0 \geqslant 0$, then since $\mathbf{y}_0 = (0, \ldots, 0, \eta_0)^{\mathrm{T}}$ is a local maximum point for U_ϵ and $U_\epsilon, V_\epsilon \to 0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N_+)$, we have that at $y = \mathbf{y}_0$

$$(-\lambda_1 + \mu_1 U_\epsilon^2 + \beta V_\epsilon^2) U_\epsilon \geqslant 0 \tag{6.7}$$

which implies that either $U_\epsilon(\mathbf{y}_0) = 0$ or $U_\epsilon(\mathbf{y}_0) \geqslant \frac{1}{2}\sqrt{\lambda_1/\mu_1}$. So $U_\epsilon(\mathbf{y}_0) \geqslant \frac{1}{2}\sqrt{\lambda_1/\mu_1}$ and hence $U(\mathbf{y}_0) \geqslant \frac{1}{2}\sqrt{\lambda_1/\mu_1}$. A contraction to $U \equiv 0$. □

Remark 6.1. Claim 1 also works for $\beta > 0$ case.

Next we claim that

Claim 2 (assuming $\beta < 0$).

$$\frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to +\infty.$$

Proof. Suppose not. Let us assume that $|P_\varepsilon - Q_\varepsilon|/\varepsilon \leqslant C$ and let $U_\varepsilon(y) = u_\varepsilon(P_\varepsilon + \varepsilon y)$, $V_\varepsilon = v_\varepsilon(Q_\varepsilon + \varepsilon y)$. Then as $\varepsilon \to 0$, $(U_\varepsilon, V_\varepsilon) \to (U_0, V_0)$ which satisfies

$$\begin{cases} \Delta U_0 - \lambda_1 U_0 + \mu_1 U_0^3 + \beta U_0 V_0^2 = 0 & \text{in } \mathbb{R}^N, \\ \Delta V_0 - \lambda_2 V_0 + \mu_2 V_0^3 + \beta U_0^2 V_0 = 0 & \text{in } \mathbb{R}^N, \\ U_0, V_0 > 0, \quad U_0, V_0 \in H^1(\mathbb{R}^N). \end{cases} \tag{6.8}$$

By Theorem 3.3,

$$I[U_0, V_0] > I_0 = \left[\lambda_1^{(4-N)/2}\mu_1^{-1} + \lambda_2^{(4-N)/2}\mu_2^{-1}\right] I[w].$$

On the other hand

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} E_\varepsilon[u_\varepsilon, v_\varepsilon] \geqslant I[U_0, V_0] > [\lambda_1^{(4-N)/2}\mu_1^{-1} + \lambda_2^{(4-N)/2}\mu_2^{-1}] I[w]$$

which contradicts to the upper bound of $E[u_\varepsilon, v_\varepsilon]$ in (5.16). □

Claim 3 (assuming $\beta < 0$).

$$U_\varepsilon(y) = u_\varepsilon(P_\varepsilon + \varepsilon y) \to w_1(y) \quad \textit{as } \varepsilon \to 0,$$
$$V_\varepsilon(z) = v_\varepsilon(Q_\varepsilon + \varepsilon z) \to w_2(z) \quad \textit{as } \varepsilon \to 0.$$

Proof. By Claim 2, $|P_\varepsilon - Q_\varepsilon|/\varepsilon \to +\infty$. Let $U_\varepsilon(y) = u_\varepsilon(P_\varepsilon + \varepsilon y)$, $\widetilde{V}_\varepsilon(y) = v_\varepsilon(P_\varepsilon + \varepsilon y)$. Then $(U_\varepsilon, \widetilde{V}_\varepsilon) \to (U_0, \widetilde{V}_0)$ and $\widetilde{V}_0 \equiv 0$. Hence U_0 satisfies

$$\begin{cases} \Delta U_0 - \lambda_1 U_0 + \mu_1 U_0^3 = 0 & \text{in } \mathbb{R}^N, \\ U_0 > 0, \quad U_0 \in H^1(\mathbb{R}^N). \end{cases}$$

By the uniqueness of w_1, we see that $U_0 \equiv w_1(y)$. Similarly, $V_\varepsilon(z) = v_\varepsilon(Q_\varepsilon + \varepsilon z) \to w_2(z)$ as $\varepsilon \to 0$. □

Claim 4 (assuming $\beta < 0$). *For ε sufficiently small, P_ε, Q_ε are unique, i.e. u_ε has a unique* (*local*) *maximum point P_ε, and v_ε has a unique* (*local*) *maximum point Q_ε.*

Proof. Suppose not. Suppose u_ε has two local maximum points $P_{\varepsilon,1}$, $P_{\varepsilon,2}$. Then there are two cases to be considered as follows:

Case 4.1: Suppose $|P_{\varepsilon,1} - P_{\varepsilon,2}|/\varepsilon \leqslant C$. This can be excluded by the fact that w_1 has a unique (nondegenerate) maximum point, similar to Section 3 of [28].

Case 4.2: Suppose $|P_{\varepsilon,1} - P_{\varepsilon,2}|/\varepsilon \to +\infty$. In this case, we will have

$$\lim_{\varepsilon\to 0} \varepsilon^{-N}\left[\frac{\varepsilon^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{\lambda_1}{2}\int_\Omega u_\varepsilon^2 - \frac{\mu_1}{4}\int_\Omega u_\varepsilon^4\right] \geqslant 2\lambda_1^{(4-N)/2}\mu_1^{-1} I[w]$$

and

$$\lim_{\varepsilon\to 0} \epsilon^{-N} E[u_\epsilon, v_\epsilon] \geqslant \left(2\lambda_1^{(4-N)/2}\mu_1^{-1} + \lambda_2^{(4-N)/2}\mu_2^{-1}\right) I[w]$$

which contradicts to the upper bound of (5.6).

Similarly, v_ε can only have one local maximum point. □

Claim 5 (assuming $\beta < 0$).

$$u_\varepsilon(x)v_\varepsilon(x) \to 0 \quad \textit{uniformly in } \Omega.$$

Proof. This follows from Claims 3 and 4. In fact, since

$$U_\varepsilon(y) := u_\varepsilon(P_\varepsilon + \varepsilon y) \to w_1(y),$$
$$V_\varepsilon(z) := v_\varepsilon(Q_\varepsilon + \varepsilon z) \to w_2(z),$$
$$\frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to +\infty,$$

we have $u_\varepsilon(x)v_\varepsilon(x) \to 0$ in $B_{\varepsilon R}(P_\varepsilon) \cup B_{\varepsilon R}(Q_\varepsilon)$. Outside $B_{\varepsilon R}(P_\varepsilon) \cup B_{\varepsilon R}(Q_\varepsilon)$, we have $u_\varepsilon(x)v_\varepsilon(x)$ uniformly small. □

Claim 6 (assuming $\beta < 0$). (6.3) *and* (6.4) *hold.*

Proof. In fact, the equation for u_ε becomes

$$\varepsilon^2 \Delta u_\varepsilon - \lambda_1 u_\varepsilon + (\mu_1 u_\varepsilon^2 + \beta v_\varepsilon^2)u_\varepsilon = 0.$$

Outside $B_{\varepsilon R}(P_\varepsilon)$, $u_\varepsilon(x) < \sqrt{\lambda_1 \cdot \sigma/2}/8\sqrt{\mu_1}$. So outside $B_{\varepsilon R}(P_\varepsilon)$, we have

$$\varepsilon^2 \Delta u_\varepsilon - \lambda_1 u_\varepsilon = (-\beta v_\varepsilon^2 - \mu_1 u_\varepsilon^2)u_\varepsilon \geqslant -\lambda_1 \frac{\sigma}{2} u_\varepsilon.$$

Hence

$$u_\varepsilon(x) \leqslant C\,\mathrm{e}^{-\frac{\sqrt{\lambda_1}(1-\sigma/2)}{\varepsilon}|x-P_\varepsilon|}.$$

By Harnack inequality,

$$\varepsilon|\nabla u_\varepsilon| \leqslant C u_\varepsilon(x) \leqslant C\,\mathrm{e}^{(-\sqrt{\lambda_1}(1-\sigma))/\varepsilon|x-P_\varepsilon|}. \qquad \square$$

It remains to consider $\beta > 0$ (attractive) case. Claim 1 still works. So we have $d(P_\varepsilon, \partial\Omega)/\varepsilon$, $d(Q_\varepsilon, \partial\Omega)/\varepsilon \to +\infty$. Next we claim

Claim 7. *Suppose* $\beta > 0$. *Then* $|P_\varepsilon - Q_\varepsilon|/\varepsilon \to 0$.

Proof. We first prove that $|P_\varepsilon - Q_\varepsilon|/\varepsilon \leqslant C$. Suppose not. That is $|P_\varepsilon - Q_\varepsilon|/\varepsilon \to +\infty$. Then as in the proof of Claim 3, we set $U_\varepsilon(y) = u_\varepsilon(P_\varepsilon + \varepsilon y)$, $V_\varepsilon(y) = v_\varepsilon(P_\varepsilon + \varepsilon y)$. Then $(U_\varepsilon, V_\varepsilon) \to (U_0, V_0)$ which solves (6.8) with $U_0 > 0$, $V_0 \geqslant 0$. Similarly, we scale at Q_ε, and set $\overline{U}_\varepsilon(z) = u_\varepsilon(Q_\varepsilon + \varepsilon z)$, $\overline{V}_\varepsilon(z) = v_\varepsilon(Q_\varepsilon + \varepsilon z)$. Then $(\overline{U}_\varepsilon, \overline{V}_\varepsilon) \to (\overline{U}_0, \overline{V}_0)$.

If $V_0 \equiv \overline{U}_0 \equiv 0$, then we obtain

$$\begin{aligned}
E_\Omega[u_\varepsilon, v_\varepsilon] &= \frac{\varepsilon^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{\lambda_1}{2}\int_\Omega u_\varepsilon^2 - \frac{\mu_1}{4}\int_\Omega u_\varepsilon^4 - \frac{\beta}{2}\int_\Omega u_\varepsilon^2 v_\varepsilon^2 + \frac{\varepsilon^2}{2}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{\lambda_2}{2}\int_\Omega v_\varepsilon^2 - \frac{\mu_2}{4}\int_\Omega v_\varepsilon^4 \\
&= \frac{\mu_1}{4}\int_\Omega u_\varepsilon^4 + \frac{\mu_2}{4}\int_\Omega v_\varepsilon^4 + \frac{\beta}{2}\int_\Omega u_\varepsilon^2 v_\varepsilon^2 \geqslant \frac{\mu_1}{4}\int_\Omega u_\varepsilon^4 + \frac{\mu_2}{4}\int_\Omega v_\varepsilon^4 \\
&\geqslant \varepsilon^N\big(\big[\lambda_1^{(4-N)/2}\mu_1^{-1} + \lambda_2^{(4-N)/2}\mu_2^{-1}\big]I[w] + o(1)\big).
\end{aligned}$$

We may assume that $V_0 \not\equiv 0$. Then we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} E_\Omega[u_\varepsilon, v_\varepsilon] \geqslant \inf_{\substack{U,V\geqslant 0,\ U,V\not\equiv 0,\\ (U,V)\in M_1}} I[U,V] = I[U_0, V_0].$$

Here we have used the fact that $0 < \beta < \sqrt{\mu_1\mu_2}$. On the other hand, the upper bound in Lemma 4.2 gives

$$\lim_{\varepsilon\to 0} \varepsilon^{-N} E_\Omega[u_\varepsilon, v_\varepsilon] \leqslant \inf_{\substack{U,V\geqslant 0,\ U,V\not\equiv 0,\\ (U,V)\in M_1}} I[U,V] = I[U_0, V_0].$$

So $(U_\varepsilon, V_\varepsilon) \to (U_0, V_0)$ which is a ground-state solution. Since U_0, V_0 attains its maximum at zero, we see that v_ε must also attain a local maximum near P_ε. This contradicts $|P_\varepsilon - Q_\varepsilon|/\varepsilon \to +\infty$. In fact, $|P_\varepsilon - Q_\varepsilon|/\varepsilon \to 0$. □

Claim 8 (assuming $\beta > 0$). *$P_\varepsilon, Q_\varepsilon$ are unique.*

Proof. We first claim that

$$\Delta U_0(0) < 0, \qquad \Delta V_0(0) < 0. \tag{6.9}$$

Suppose not , we may assume that $\Delta U_0(0) = 0$. Let $\varphi(r) = U_0(r) - U_0(0)$, $\psi(r) = V_0(r) - V_0(0)$. Then we compute

$$\begin{aligned}
&\Delta\varphi = \Delta U_0(r) = \lambda_1\big(U_0(0)+\varphi\big) - \mu_1\big(U_0(0)+\varphi\big)^3 - \beta\big(U_0(0)+\varphi\big)\big(V_0(0)+\psi\big)^2,\\
&\Delta\varphi + \big[-\lambda_1 + 3\mu_1\big(U_0^2(0) + U_0(0)\varphi\big) + \beta\big(V_0(0)+\psi\big)^2\big]\varphi + \beta U_0(0)\big(2V_0(0)+\psi\big)\psi + \mu_1\varphi^3\\
&\quad = \lambda_1 U_0(0) - \mu_1 U_0^3(0) - \beta U_0(0)V_0^2(0) = \Delta U_0(0) = 0.
\end{aligned}$$

Since $\beta > 0, \varphi \leqslant 0, \psi \leqslant 0$, we have

$$\Delta\varphi + C(r)\varphi \geqslant -\beta U_0(0)\big(2V_0(0)+\psi\big)\psi \geqslant 0,$$

where $C(r) \leqslant 0$. By the strong maximum principle, either $\varphi \equiv 0$ or $\varphi < 0$, which is impossible. Hence $\Delta U_0(0) < 0$ which yields that $NU_0''(0) = \Delta U_0(0) < 0$.

Suppose now u_ε has two maximum points $P_{\varepsilon,1}, P_{\varepsilon,2}$. Then $|P_{\varepsilon,1} - P_{\varepsilon,2}|/\varepsilon \to 0$. Since $U_\varepsilon \to U_0$ in C^2_{loc}, and $U_0''(0) < 0$, it is easy to see that $P_{\varepsilon,1} = P_{\varepsilon,2}$. □

Claim 9 (assuming $\beta > 0$).

$$\begin{aligned}
&\varepsilon|\nabla u_\varepsilon|, |u_\varepsilon| \leqslant C\,\mathrm{e}^{-\sqrt{\lambda_1}(1-\sigma)|x-P_\varepsilon|/\varepsilon},\\
&\varepsilon|\nabla v_\varepsilon|, |v_\varepsilon| \leqslant C\,\mathrm{e}^{-\sqrt{\lambda_2}(1-\sigma)|x-Q_\varepsilon|/\varepsilon}.
\end{aligned}$$

Proof. Since $u_\varepsilon, v_\varepsilon$ are uniformly small outside $B_{\varepsilon R}(P_\varepsilon) \cup B_{\varepsilon R}(Q_\varepsilon)$, we see that u_ε satisfies

$$\varepsilon^2\Delta u_\varepsilon - \lambda_1 u_\varepsilon \geqslant -\lambda_1\frac{\sigma}{2}u_\varepsilon \quad \text{on } \Omega \setminus \big[B_{\varepsilon R}(P_\varepsilon) \cup B_{\varepsilon R}(Q_\varepsilon)\big].$$

Then Claim 8 follows from standard maximum principle and Harnack inequality. □

7. Lower bound for c_ε in the case of $\beta < 0$ (repulsive case)

In this section, we establish the following lower bound for c_ϵ in the case of $\beta < 0$.

Theorem 7.1. *Assume that $\beta < 0$ (repulsive case). Then we have*

$$\begin{aligned}
c_\varepsilon \geqslant \varepsilon^N\big[&(\lambda_1^{(4-N)/2}\mu_1^{-1} + \lambda_2^{(4-N)/2}\mu_2^{-1})I[w] + \big(a_4 + o(1)\big)\varphi_{1,\varepsilon,P_\varepsilon}(P_\varepsilon)\\
&+ \big(a_5 + o(1)\big)\varphi_{2,\varepsilon,Q_\varepsilon}(Q_\varepsilon) + \big(a_6 + o(1)\big)I_\varepsilon^{1+\sigma}(P_\varepsilon, Q_\varepsilon)\big]
\end{aligned} \tag{7.1}$$

where $P_\varepsilon, Q_\varepsilon$ are the two (unique) maximum points of u_ε, v_ε respectively, and a_4, a_5, a_6 are three positive constants, and $\sigma \in (0, \frac{1}{100})$ is any fixed constant.

Assuming Theorem 7.1, we can finish the proof of (3) Theorem 1.1.

Completion of the proof of (3) of Theorem 1.1. Comparing the upper and lower bound of c_ε, we obtain

$$\begin{aligned}&\big(a_4+o(1)\big)\varphi_{1,\varepsilon,P_\varepsilon}(P_\varepsilon)+\big(a_5+o(1)\big)\varphi_{2,\varepsilon,Q_\varepsilon}(Q_\varepsilon)+\big(a_6+o(1)\big)I_\varepsilon^{1+\sigma}(P_\varepsilon,Q_\varepsilon)\\&\quad\leqslant\big(a_1+o(1)\big)\varphi_{1,\varepsilon,P}(P)+\big(a_2+o(1)\big)\varphi_{2,\varepsilon,Q}(Q)+\big(a_3+o(1)\big)I_\varepsilon(P,Q),\quad\text{for all } P,Q,\end{aligned}$$

which then implies that

$$\delta_\epsilon^{1+\sigma}(P_\varepsilon,Q_\varepsilon)\leqslant\min_{(P,Q)\in\Omega^2}\delta_\epsilon(P,Q)\tag{7.2}$$

and hence (using (5.12)) $\lim_{\varepsilon\to0}\varphi^{1+\sigma}(P_\varepsilon,Q_\varepsilon)\geqslant\max_{(P,Q)\in\Omega^2}\varphi(P,Q)$. Now letting $\sigma\to0$, we obtain that

$$\lim_{\varepsilon\to0}\varphi(P_\varepsilon,Q_\varepsilon)\geqslant\max_{(P,Q)\in\Omega^2}\varphi(P,Q).\qquad\Box$$

From Theorem 6.1 (1), we have a first order approximation for $u_\varepsilon, v_\varepsilon$. However, this is not good enough to obtain the second order expansion of c_ε. We follow (and simplify) the method in [28] by expanding u_ϵ, v_ϵ. The main difficulty and the main difference lies in the computation of interactions between u_ϵ and v_ϵ. See the definition of $\phi_{1,\epsilon,P}$ and $\phi_{2,\epsilon,Q}$.

Without loss of generality, we may assume that

$$\lambda_1\leqslant\lambda_2.\tag{7.3}$$

Fix $P\in\Omega$, we define $w_{1,\varepsilon,P}$ to be the unique solution of (5.1) with $\lambda_i=\lambda_1$, $\mu_i=\mu_1$. We also define

$$w_{1,P}:=w_1\left(\frac{x-P}{\varepsilon}\right).$$

Similarly, we can define $w_{2,\varepsilon,Q}, w_{2,Q}$.

Next we set $\phi_{1,\varepsilon,P}$ to be the unique solution of

$$\begin{cases}\varepsilon^2\Delta\phi_{1,\varepsilon,P}-\lambda_1\phi_{1,\varepsilon,P}+\beta w_{2,\varepsilon,Q}^2\phi_{1,\varepsilon,P}+\beta w_{1,\varepsilon,P}w_{2,\varepsilon,Q}^2=0 & \text{in } \Omega,\\ \phi_{1,\varepsilon,P}=0 & \text{on } \partial\Omega,\end{cases}\tag{7.4}$$

and $\phi_{2,\varepsilon,Q}$ to be the unique solution of

$$\begin{cases}\varepsilon^2\Delta\phi_{2,\varepsilon,Q}-\lambda_2\phi_{2,\varepsilon,Q}+\beta w_{1,\varepsilon,P}^2\phi_{2,\varepsilon,Q}+\beta w_{2,\varepsilon,Q}w_{1,\varepsilon,P}^2=0 & \text{in } \Omega,\\ \phi_{2,\varepsilon,Q}=0 & \text{on } \partial\Omega.\end{cases}\tag{7.5}$$

Remark 7.1. Since $\beta<0$, both (7.4) and (7.5) have a unique solution. The function $\phi_{1,\epsilon,P}$ measures the interactions from spike at Q to spike at P. Similarly, the function $\phi_{2,\epsilon,Q}$ measures the interactions from spike at P to spike at Q.

Finally we set

$$\widetilde{w}_{1,\varepsilon,P}=w_{1,\varepsilon,P}+\phi_{1,\varepsilon,P},\qquad\widetilde{w}_{2,\varepsilon,Q}=w_{2,\varepsilon,Q}+\phi_{2,\varepsilon,Q},\tag{7.6}$$

where $w_{1,\varepsilon,P}$, $w_{2,\varepsilon,Q}$ have been studied in Section 5.

We now study $\phi_{1,\varepsilon,P}$. To this end, we need to introduce two functions: let $b\in S^{N-1}$ and we define $\Psi_{1,b}(y)$ to be the unique solution of

$$\Delta\Psi_{1,b}-\lambda_1\Psi_{1,b}+\beta w_2^2(y)\Psi_{1,b}+\beta\,\mathrm{e}^{-\langle b,y\rangle}w_2^2(y)=0,\quad\Psi_{1,b}\in H^1(\mathbb{R}^N),\tag{7.7}$$

and $\Psi_{2,b}(y)$ to be the unique solution of

$$\Delta\Psi_{2,b}-\lambda_2\Psi_{2,b}+\beta w_1^2(y)\Psi_{2,b}+\beta\,\mathrm{e}^{-\langle b,y\rangle}w_1^2(y)=0,\quad\Psi_{2,b}\in H^1(\mathbb{R}^N).\tag{7.8}$$

Then we have

Lemma 7.2. (1) *Suppose* $\lambda_1 \leqslant \lambda_2$. *Then*

$$\left\| e^{\sqrt{\lambda_1}(1-\sigma)|y|}\left(\phi_{1,\varepsilon,P}(Q+\varepsilon y) - w_1\left(\frac{|Q-P|}{\varepsilon}\right)\Psi_{1,\frac{Q-P}{|Q-P|}}(y)\right)\right\|_{L^\infty(\Omega_{\varepsilon,Q})} \to 0 \tag{7.9}$$

as $\varepsilon \to 0$, *where* $y = \frac{x-Q}{\varepsilon}$, $\Omega_{\varepsilon,Q} = \{y \mid \varepsilon y + Q \in \Omega\}$.

(2) *If* $\lambda_1 < \lambda_2$, *then we have*

$$|\phi_{2,\varepsilon,Q}| \leqslant o\left(w_1\left(\frac{|P-Q|}{\varepsilon}\right)\right). \tag{7.10}$$

If $\lambda_1 = \lambda_2$, *then we have*

$$\left\| e^{\sqrt{\lambda_2}(1-\sigma)|y|}\left(\phi_{2,\varepsilon,Q}(P+\varepsilon y) - w_2\left(\frac{|Q-P|}{\varepsilon}\right)\Psi_{2,\frac{P-Q}{|P-Q|}}(y)\right)\right\|_{L^\infty(\Omega_{\varepsilon,P})} \to 0 \tag{7.11}$$

as $\varepsilon \to 0$, *where* $y = \frac{x-P}{\varepsilon}$, $\Omega_{\varepsilon,P} = \{y \mid \varepsilon y + P \in \Omega\}$.

Proof. (1) We note that for $y = \frac{x-Q}{\varepsilon}$

$$w_{1,\varepsilon,P}w_{2,\varepsilon,Q}^2 \approx w_{1,P}w_{2,Q}^2 \approx w_1\left(\frac{|P-Q|}{\varepsilon}\right) \cdot e^{-\sqrt{\lambda_1}\langle \frac{Q-P}{|Q-P|}, y\rangle} w_2^2(y). \tag{7.12}$$

Hence

$$w_1^{-1}\left(\frac{|Q-P|}{\varepsilon}\right)\phi_{1,\varepsilon,P}(x) \to \Psi_{1,b}(y) \quad \text{as } \varepsilon \to 0.$$

The decay estimate follows from the equation since $\lambda_1 \leqslant \lambda_2$.

(2) Similarly,

$$w_{2,\varepsilon,Q}w_{1,\varepsilon,P}^2 \leqslant w_{2,Q}w_{1,P}^2 = o\left(w_1\left(\frac{|P-Q|}{\varepsilon}\right)\right) e^{-\langle \frac{P-Q}{|P-Q|}, y\rangle} w_1^2(y).$$

This proves (7.10).

(3) The proof of (7.11) is similar to that of (7.9). □

Set now

$$s_1[u,v] = \varepsilon^2\Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2, \tag{7.13}$$

$$s_2[u,v] = \varepsilon^2\Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v. \tag{7.14}$$

Then we have

$$\begin{aligned} s_1[\widetilde{w}_{1,\varepsilon,P}, \widetilde{w}_{2,\varepsilon,Q}] &= \varepsilon^2\Delta(w_{1,\varepsilon,P} + \phi_{1,\varepsilon,P}) - \lambda_1(w_{1,\varepsilon,P} + \phi_{1,\varepsilon,P}) \\ &\quad + \mu_1(w_{1,\varepsilon,P} + \phi_{1,\varepsilon,P})^3 + \beta(w_{1,\varepsilon,P} + \phi_{1,\varepsilon,P})(w_{2,\varepsilon,Q} + \phi_{2,\varepsilon,Q})^2 \\ &= \mu_1\left[(w_{1,\varepsilon,P} + \phi_{1,\varepsilon,P})^3 - w_{1,P}^3\right] + 2\beta(w_{1,\varepsilon,P} + \phi_{1,\varepsilon,P})w_{2,\varepsilon,Q}\phi_{2,\varepsilon,Q} \\ &\quad + \beta(w_{1,\varepsilon,P} + \phi_{1,\varepsilon,P})\phi_{2,\varepsilon,Q}^2. \end{aligned} \tag{7.15}$$

Similarly

$$\begin{aligned} s_2[\widetilde{w}_{1,\varepsilon,P}, \widetilde{w}_{2,\varepsilon,Q}] &= \mu_2\left[(w_{2,\varepsilon,Q} + \phi_{2,\varepsilon,Q})^3 - w_{2,Q}^3\right] + 2\beta w_{1,\varepsilon,P}(w_{2,\varepsilon,Q} + \phi_{2,\varepsilon,Q})\phi_{1,\varepsilon,P} \\ &\quad + \beta(w_{2,\varepsilon,Q} + \phi_{2,\varepsilon,Q})\phi_{1,\varepsilon,P}^2. \end{aligned} \tag{7.16}$$

Recall that

$$\delta_\epsilon(P,Q)=\varphi_{1,\epsilon,P}(P)+\varphi_{2,\epsilon,Q}(Q)+I_\epsilon(P,Q).$$

We then have

Lemma 7.3.

$$\|s_1\|_{L^\infty(\Omega)}+\|s_2\|_{L^\infty(\Omega)}\leqslant C\delta_\varepsilon^{1-\sigma}(P,Q), \tag{7.17}$$

$$\|s_1\|_{L^1(\Omega)}+\|s_2\|_{L^1(\Omega)}\leqslant C\varepsilon^N\delta_\varepsilon^{1-\sigma}(P,Q), \tag{7.18}$$

where $\sigma\in(0,\frac{1}{100})$ is a fixed number.

Proof. We consider s_1 only. Actually, s_1 contains three terms. The first term can be bounded by $c|w_{1,\varepsilon,P}^3-w_{1,P}^3|+cw_{1,P}^2|\phi_{1,\varepsilon,P}|$.

Note that by Lemma 5.1 (3),

$$|w_{1,\varepsilon,P}^3-w_{1,P}^3|\leqslant C\varphi_{1,\epsilon,P}(P). \tag{7.19}$$

By Lemma 7.2, we see that

$$|\phi_{1,\varepsilon,P}|\leqslant Cw_1\left(\frac{P-Q}{\varepsilon}\right)\cdot\left|\Psi_{1,\frac{Q-P}{|Q-P|}}\left(\frac{|x-Q|}{\varepsilon}\right)\right|\mathrm{e}^{-\sqrt{\lambda_1}(1-\sigma)|\frac{x-Q}{\epsilon}|}$$

and hence

$$\begin{aligned}|w_{1,P}|^2|\phi_{1,\varepsilon,P}|&\leqslant Cw_1\left(\frac{P-Q}{\varepsilon}\right)\left|w_1\left(\frac{x-P}{\varepsilon}\right)\right|^2\left|\Psi_{1,\frac{Q-P}{|Q-P|}}\left(\frac{|x-Q|}{\varepsilon}\right)\right|\\&\leqslant Cw_1^{2(1-\sigma)}\left(\frac{|P-Q|}{\varepsilon}\right).\end{aligned} \tag{7.20}$$

Combining (7.19) and (7.20), we obtain that

$$\left\|\mu_1\left[(w_{1,\epsilon,P}+\phi_{1,\epsilon,P})^3-w_{1,P}^3\right]\right\|_{L^\infty(\Omega)}\leqslant C\delta_\varepsilon^{1-\sigma}(P,Q). \tag{7.21}$$

The second term can be estimated as follows

$$\begin{aligned}\left|2\beta(w_{1,\varepsilon,P}+\phi_{1,\varepsilon,P})w_{2,\varepsilon,Q}\phi_{2,\varepsilon,Q}\right|\leqslant|w_{1,P}||w_{2,Q}||\phi_{2,\varepsilon,Q}|&\leqslant w_1\left(\frac{|P-Q|}{\varepsilon}\right)|\phi_{2,\varepsilon,Q}|\\&\leqslant Cw_1^{2(1-\sigma)}\left(\frac{P-Q}{\varepsilon}\right).\end{aligned}$$

The third term

$$\left|(w_{1,\varepsilon,P}+\phi_{1,\varepsilon,P})\phi_{2,\varepsilon,Q}^2\right|\leqslant C|\phi_{2,\varepsilon,Q}|^2\leqslant Cw_1^{2(1-\sigma)}\left(\frac{|P-Q|}{\varepsilon}\right).\qquad\square$$

We now decompose

$$\begin{aligned}u_\varepsilon(x)&=\widetilde{w}_{1,\varepsilon,P_\varepsilon}+f_\varepsilon,\\v_\varepsilon(x)&=\widetilde{w}_{2,\varepsilon,Q_\varepsilon}+g_\varepsilon.\end{aligned} \tag{7.22}$$

Substituting the decomposition of u_ε and v_ε into the equations for u_ε and v_ε, respectively, we obtain that f_ε satisfies

$$\begin{aligned}&\varepsilon^2\Delta f_\varepsilon-\lambda_1 f_\varepsilon+3\mu_1\widetilde{w}_{1,\varepsilon,P_\varepsilon}^2 f_\varepsilon+\beta\widetilde{w}_{2,\varepsilon,Q_\varepsilon}^2 f_\varepsilon+2\beta\tilde{w}_{1,\varepsilon,P_\varepsilon}\widetilde{w}_{2,\varepsilon,Q_\varepsilon}g_\varepsilon\\&\quad+o(1)f_\varepsilon+o(1)g_\varepsilon+s_1[\widetilde{w}_{1,\varepsilon,P_\varepsilon},\tilde{w}_{2,\varepsilon,Q_\varepsilon}]=0.\end{aligned} \tag{7.23}$$

Similarly, g_ε satisfies

$$\varepsilon^2\Delta g_\varepsilon - \lambda_2 g_\varepsilon + 3\mu_2 \widetilde{w}_{2,\varepsilon,Q_\varepsilon}^2 g_\varepsilon + \beta\widetilde{w}_{1,\varepsilon,P_\varepsilon}^2 g_\varepsilon + 2\beta\tilde{w}_{1,\varepsilon,P_\varepsilon}\widetilde{w}_{2,\varepsilon,Q_\varepsilon} f_\varepsilon + o(1)g_\varepsilon + o(1)f_\varepsilon + s_2[\widetilde{w}_{1,\varepsilon,P_\varepsilon}, \tilde{w}_{2,\varepsilon,Q_\varepsilon}] = 0. \tag{7.24}$$

Next we claim that

Lemma 7.4.

(1) $$\|f_\varepsilon\|_{L^\infty(\Omega)} + \|g_\varepsilon\|_{L^\infty(\Omega)} \leqslant C\delta_\varepsilon^{1-\sigma}(P_\varepsilon, Q_\varepsilon), \tag{7.25}$$

(2) $$\varepsilon^2\int_\Omega |\nabla f_\varepsilon|^2 + \lambda_1\int_\Omega f_\varepsilon^2 + \varepsilon^2\int_\Omega |\nabla g_\varepsilon|^2 + \lambda_2\int_\Omega g_\varepsilon^2 \leqslant C\varepsilon^N\delta_\varepsilon^{2(1-\sigma)}(P_\varepsilon, Q_\varepsilon). \tag{7.26}$$

Proof. Without loss of generality, we may assume that

$$\|f_\varepsilon\|_{L^\infty(\Omega)} \geqslant \|g_\varepsilon\|_{L^\infty(\Omega)}, \qquad \|f_\varepsilon\|_{L^\infty(\Omega)} = f_\varepsilon(x_\varepsilon).$$

We first prove a weaker result:

$$\|f_\varepsilon\|_{L^\infty(\Omega)} \leqslant C\delta_\varepsilon^{(1-\sigma/2)/2}(P_\varepsilon, Q_\varepsilon). \tag{7.27}$$

In fact suppose not. That is $\|f_\varepsilon\|_{L^\infty(\Omega)} \gg \delta_\varepsilon^{(1-\sigma/2)/2}(P_\varepsilon, Q_\varepsilon)$. We let

$$\tilde{f}_\varepsilon(y) := \frac{f_\varepsilon(P_\varepsilon + \varepsilon y)}{\|f_\varepsilon\|_{L^\infty(\Omega)}}.$$

Then $\tilde{f}_\varepsilon(y)$ satisfies

$$\Delta_y\tilde{f}_\varepsilon(y) - \lambda_1\tilde{f}_\varepsilon(y) + 3\mu_1\tilde{w}_{1,\varepsilon,P_\varepsilon}^2\tilde{f}_\varepsilon + \beta\widetilde{w}_{2,\varepsilon,Q_\varepsilon}^2\tilde{f}_\varepsilon + 2\beta\frac{\widetilde{w}_{1,\varepsilon,P_\varepsilon}\widetilde{w}_{2,\varepsilon,Q_\varepsilon}}{\|f_\varepsilon\|_{L^\infty(\Omega)}}g_\varepsilon + o(1)\tilde{f}_\varepsilon + o(1) = 0 \quad \text{in } \Omega.$$

Since $\widetilde{w}_{1,\varepsilon,P_\varepsilon}\widetilde{w}_{2,\varepsilon,Q_\varepsilon} = O(w_1((P_\varepsilon - Q_\varepsilon)/\varepsilon)) = O(\delta_\varepsilon^{(1-\sigma/2)/2}(P_\varepsilon, Q_\varepsilon))$, then $\tilde{f}_\varepsilon(y)$ satisfies

$$\Delta_y\tilde{f}_\varepsilon - \lambda_1\tilde{f}_\varepsilon + 3\mu_1\tilde{w}_{1,\varepsilon,P_\varepsilon}^2\tilde{f}_\varepsilon + \beta\widetilde{w}_{2,\varepsilon,Q_\varepsilon}^2\tilde{f}_\varepsilon + o(1) = 0 \quad \text{in } \Omega_{\varepsilon,P_\varepsilon} \tag{7.28}$$

where

$$\Omega_{\varepsilon,P_\varepsilon} = \{y \mid \varepsilon y + P_\varepsilon \in \Omega\}.$$

Note that at $y_\varepsilon = (x_\varepsilon - P_\varepsilon)/\varepsilon$, $\tilde{f}_\varepsilon(y_\varepsilon) = \max_{y\in\Omega_{\varepsilon,P_\varepsilon}}\tilde{f}_\varepsilon(y) = 1$. We claim that $x_\varepsilon \in B_{\varepsilon R}(P_\varepsilon)$, i.e. $|y_\varepsilon| \leqslant R$, for some $R > 0$. Suppose $|y_\varepsilon| \to +\infty$. Then $\widetilde{w}_{1,\varepsilon,P_\varepsilon} \to 0$. Moreover, at y_ε, by (7.28), we have

$$\Delta_y\tilde{f}_\varepsilon(y_\varepsilon) = \lambda_1\tilde{f}_\varepsilon(y_\varepsilon) + o(1)\tilde{f}_\varepsilon(y_\varepsilon) - \beta\widetilde{w}_{2,\varepsilon,Q_\varepsilon}(y_\varepsilon)\tilde{f}_\varepsilon(y_\varepsilon) + o(1) \leqslant 0.$$

Hence $\tilde{f}_\varepsilon(y_\varepsilon) = o(1)$, and we get a contradiction. So $|y_\varepsilon| \leqslant R$.

On the other hand, as $\varepsilon \to 0$, $\tilde{f}_\varepsilon(y) \to f_0(y)$, in $C^1_{\text{loc}}(\mathbb{R}^N)$, where f_0 satisfies

$$\Delta f_0 - \lambda_1 f_0 + 3\mu_1 w_1^2 f_0 = 0, \ |f_0| \leqslant 1.$$

By the nondegeneracy of w_1, $f_0(y) = \sum_{j=1}^N a_j\frac{\partial w_1}{\partial y_j}$ for some constants a_j, $j = 1, \ldots, N$. Since $f_\varepsilon(x) = u_\varepsilon(x) - \widetilde{w}_{1,\varepsilon,P_\varepsilon}$, then

$$\nabla_y f_\varepsilon(0) = -\nabla_y[w_{1,\varepsilon,P_\varepsilon} + \phi_{1,\varepsilon,P_\varepsilon}],$$
$$\nabla_y \tilde f_\varepsilon(0) = -\frac{1}{\|f_\varepsilon\|_{L^\infty}}\nabla_y[w_{1,\varepsilon,P_\varepsilon} + \phi_{1,\varepsilon,P_\varepsilon}] = -\frac{1}{\|f_\varepsilon\|_{L^\infty}}\nabla_y\big[w(y) - \varphi_{1,\varepsilon,P_\varepsilon}(P_\varepsilon)V_{1,\varepsilon,P_\varepsilon}(y) + \phi_{1,\varepsilon,P_\varepsilon}\big]$$
$$= O\left(\frac{1}{\|f_\varepsilon\|_{L^\infty}}\left[\varphi_{1,\varepsilon,P_\varepsilon}(P_\varepsilon) + w_1\left(\frac{P_\varepsilon - Q_\varepsilon}{\varepsilon}\right)\right]\right) \leqslant O\left(\frac{O(\delta_\varepsilon^{1-\sigma/4}(P_\varepsilon, Q_\varepsilon))}{\delta_\varepsilon^{(1-\sigma/2)/2}(P_\varepsilon, Q_\varepsilon)}\right).$$

Here we have used Lemmas 5.1 and 7.2. Hence $\nabla_y \tilde f_\varepsilon(0) \to 0$ which implies that $\nabla f_0(0) = 0$, and $\sum_{j=1}^N a_j \cdot \nabla \frac{\partial w_1}{\partial y_j}(0) = 0$. So $a_j \equiv 0$, $f_0(y) \equiv 0$ and $\tilde f_\varepsilon(y) \to 0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$. However, $\tilde f_\varepsilon(y_\varepsilon) = 1$, $|y_\varepsilon| \leqslant R$. A contradiction!

So we have proved $\|f_\varepsilon\|_{L^\infty(\Omega)} = O(\delta_\varepsilon^{(1-\sigma/2)/2}(P_\varepsilon, Q_\varepsilon))$. Thus $\|f_\varepsilon\|_{L^\infty(\Omega)} + \|g_\varepsilon\|_{L^\infty(\Omega)} \leqslant C\delta_\varepsilon^{(1-\sigma/2)/2}(P_\varepsilon, Q_\varepsilon)$. This implies

$$\widetilde w_{1,\varepsilon,P_\varepsilon}\widetilde w_{2,\varepsilon,Q_\varepsilon} g_\varepsilon = O\big(\delta_\varepsilon^{1-\sigma}(P_\varepsilon, Q_\varepsilon)\big), \tag{7.29}$$
$$\widetilde w_{1,\varepsilon,P_\varepsilon}\widetilde w_{2,\varepsilon,Q_\varepsilon} f_\varepsilon = O\big(\delta_\varepsilon^{1-\sigma}(P_\varepsilon, Q_\varepsilon)\big). \tag{7.30}$$

Substituting the estimates (7.29) and (7.30) into (7.23) and (7.24), we can use similar argument to prove (1).

To prove (2), we multiply (7.23) by f_ε, and integrate by parts. Then we may obtain

$$\int_\Omega \big(\varepsilon^2|\nabla f_\varepsilon|^2 + \lambda_1 f_\varepsilon^2\big) \leqslant \int_\Omega w_{1,P_\varepsilon}^2|f_\varepsilon|^2 + |\beta|\int_\Omega w_{2,Q_\varepsilon}^2|f_\varepsilon|^2 + \int_\Omega |\widetilde w_{1,\varepsilon,P_\varepsilon}||\widetilde w_{2,\varepsilon,Q_\varepsilon}||g_\varepsilon||f_\varepsilon| + o(1)\int_\Omega f_\varepsilon^2$$
$$+ o(1)\int_\Omega |f_\varepsilon g_\varepsilon| + \int_\Omega |s_1[\widetilde w_{1,\varepsilon,P_\varepsilon}, \widetilde w_{2,\varepsilon,Q_\varepsilon}]||f_\varepsilon|$$
$$\leqslant C\varepsilon^N\delta_\varepsilon^{2(1-\sigma)}(P_\varepsilon, Q_\varepsilon) + o(1)\int_\Omega |f_\varepsilon g_\varepsilon|. \tag{7.31}$$

Here we have used Lemma 7.3. Similarly multiplying (7.24) by g_ε and integrating by parts, we obtain

$$\int_\Omega \big(\varepsilon^2|\nabla g_\varepsilon|^2 + \lambda_2 g_\varepsilon^2\big) \leqslant C\varepsilon^N\delta_\varepsilon^{2(1-\sigma)}(P_\varepsilon, Q_\varepsilon) + o(1)\int_\Omega |f_\varepsilon g_\varepsilon|. \tag{7.32}$$

Adding (7.31) and (7.32) proves (2). □

Now we may compute c_ε and complete the proof of Theorem 7.1: using Lemma 7.3,

$$c_\varepsilon = E_\Omega[u_\varepsilon, v_\varepsilon]$$
$$= \frac{\varepsilon^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 + \frac{\lambda_1}{2}\int_\Omega u_\varepsilon^2 - \frac{\mu_1}{4}\int_\Omega u_\varepsilon^4 + \frac{\varepsilon^2}{2}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{\lambda_1}{2}\int_\Omega - \frac{\mu_2}{4}\int_\Omega v_\varepsilon^4 - \frac{\beta}{2}\int_\Omega u_\varepsilon^2 v_\varepsilon^2$$
$$= E_\Omega[\widetilde w_{1,\varepsilon,P_\varepsilon}, \widetilde w_{2,\varepsilon,Q_\varepsilon}] + \varepsilon^2\int_\Omega \nabla\widetilde w_{1,\varepsilon,P_\varepsilon}\nabla f_\varepsilon + \lambda_1\int_\Omega \widetilde w_{1,\varepsilon,P_\varepsilon} f_\varepsilon - \mu_1\int_\Omega \widetilde w_{1,\varepsilon,P_\varepsilon}^3 f_\varepsilon$$
$$+ \varepsilon^2\int_\Omega \nabla\widetilde w_{2,\varepsilon,Q_\varepsilon}\nabla g_\varepsilon + \lambda_2\int_\Omega \widetilde w_{2,\varepsilon,Q_\varepsilon} g_\varepsilon - \mu_2\int_\Omega \widetilde w_{2,\varepsilon,Q_\varepsilon}^3 g_\varepsilon$$
$$- \frac{\beta}{2}\int_\Omega [2\widetilde w_{1,\varepsilon,P_\varepsilon}\widetilde w_{2,\varepsilon,Q_\varepsilon}^2 f_\varepsilon + 2\widetilde w_{1,\varepsilon,P_\varepsilon}^2\widetilde w_{2,\varepsilon,Q_\varepsilon} g_\varepsilon] + o\big(\varepsilon^N\,\delta_\varepsilon(P_\varepsilon, Q_\varepsilon)\big)$$
$$= E_\Omega[\widetilde w_{1,\varepsilon,P_\varepsilon}, \widetilde w_{2,\varepsilon,Q_\varepsilon}]$$

$$
\begin{aligned}
&+\mu_1\int_\Omega (w^3_{1,P_\varepsilon}-\widetilde{w}^3_{1,\varepsilon,P_\varepsilon})f_\varepsilon+\mu_2\int_\Omega (w^3_{2,Q_\varepsilon}-\widetilde{w}^3_{2,\varepsilon,Q_\varepsilon})g_\varepsilon\\
&+\beta\Big[\int_\Omega (w_{1,\varepsilon,P_\varepsilon}w^2_{2,\varepsilon,Q_\varepsilon}-\widetilde{w}_{1,\varepsilon,P_\varepsilon}\widetilde{w}^2_{2,\varepsilon,Q_\varepsilon})f_\varepsilon+(w_{2,\varepsilon,Q_\varepsilon}w^2_{1,\varepsilon,P_\varepsilon}-\widetilde{w}_{2,\varepsilon,Q_\varepsilon}\widetilde{w}^2_{1,\varepsilon,P_\varepsilon})g_\varepsilon\Big]\\
&+o\big(\varepsilon^N\delta_\varepsilon(P_\varepsilon,Q_\varepsilon)\big)\\
=\,&E_\Omega[\widetilde{w}_{1,\varepsilon,P_\varepsilon},\widetilde{w}_{2,\varepsilon,Q_\varepsilon}]+o\big(\varepsilon^N\delta_\varepsilon\big(P_\varepsilon,Q_\varepsilon)\big).
\end{aligned}
$$

It remains to compute $E_\Omega[\widetilde{w}_{1,\varepsilon,P_\varepsilon},\widetilde{w}_{2,\varepsilon,Q_\varepsilon}]$. Note that

$$
\begin{aligned}
E_\Omega[\widetilde{w}_{1,\varepsilon,P_\varepsilon},\widetilde{w}_{2,\varepsilon,Q_\varepsilon}]&=\frac{\varepsilon^2}{2}\int_\Omega|\nabla w_{1,\varepsilon,P_\varepsilon}|^2+\frac{\lambda_1}{2}\int_\Omega w^2_{1,\varepsilon,P_\varepsilon}-\frac{\mu_1}{4}\int_\Omega w^4_{1,\varepsilon,P_\varepsilon}\\
&\quad+\frac{\varepsilon^2}{2}\int_\Omega|\nabla w_{2,\varepsilon,Q_\varepsilon}|^2+\frac{\lambda_2}{2}\int_\Omega w^2_{2,\varepsilon,Q_\varepsilon}-\frac{\mu_2}{4}\int_\Omega w^4_{2,\varepsilon,Q_\varepsilon}\\
&\quad+\varepsilon^2\int_\Omega\nabla w_{1,\varepsilon,P_\varepsilon}\nabla\phi_{1,\varepsilon,P_\varepsilon}+\lambda_1\int_\Omega w_{1,\varepsilon,P_\varepsilon}\phi_{1,\varepsilon,P_\varepsilon}-\mu_1\int_\Omega w^3_{1,\varepsilon,P_\varepsilon}\phi_{1,\varepsilon,P_\varepsilon}\\
&\quad+\frac{\varepsilon^2}{2}\int_\Omega|\nabla\phi_{1,\varepsilon,P_\varepsilon}|^2+\frac{\lambda_1}{2}\int_\Omega|\phi_{1,\varepsilon,P_\varepsilon}|^2\\
&\quad+\varepsilon^2\int_\Omega\nabla w_{2,\varepsilon,Q_\varepsilon}\nabla\phi_{2,\varepsilon,Q_\varepsilon}+\lambda_2\int_\Omega w_{2,\varepsilon,Q_\varepsilon}\phi_{2,\varepsilon,Q_\varepsilon}-\mu_2\int_\Omega w^3_{2,\varepsilon,Q_\varepsilon}\phi_{2,\varepsilon,Q_\varepsilon}\\
&\quad+\frac{\varepsilon^2}{2}\int_\Omega|\nabla\phi_{2,\varepsilon,Q_\varepsilon}|^2+\frac{\lambda_2}{2}\int_\Omega|\phi_{2,\varepsilon,Q_\varepsilon}|^2\\
&\quad-\frac{\beta}{2}\int_\Omega(w_{1,\varepsilon,P_\varepsilon}+\phi_{1,\varepsilon,P_\varepsilon})^2(w_{2,\varepsilon,Q_\varepsilon}+\phi_{2,\varepsilon,Q_\varepsilon})^2+o\big(\varepsilon^N\delta_\varepsilon(P_\varepsilon,Q_\varepsilon)\big)\\
&\geqslant\frac{\varepsilon^2}{2}\int_\Omega|\nabla w_{1,\varepsilon,P_\varepsilon}|^2+\frac{\lambda_1}{2}\int_\Omega w^2_{1,\varepsilon,P_\varepsilon}-\frac{\mu_1}{4}\int_\Omega w^4_{1,\varepsilon,P_\varepsilon}\\
&\quad+\frac{\varepsilon^2}{2}\int_\Omega|\nabla w_{2,\varepsilon,Q_\varepsilon}|^2+\frac{\lambda_2}{2}\int_\Omega w^2_{2,\varepsilon,Q_\varepsilon}-\frac{\mu_2}{4}\int_\Omega w^4_{2,\varepsilon,Q_\varepsilon}\\
&\quad-\frac{\beta}{2}\int_\Omega(w_{1,\varepsilon,P_\varepsilon}+\phi_{1,\varepsilon,P_\varepsilon})^2(w_{2,\varepsilon,Q_\varepsilon}+\phi_{2,\varepsilon,Q_\varepsilon})^2+o\big(\varepsilon^N\delta_\varepsilon(P_\varepsilon,Q_\varepsilon)\big)\\
&=\varepsilon^N\big[\lambda_1^{(4-N)/2}\mu_1^{-1}I[w]+\lambda_2^{(4-N)/2}\mu_2^{-1}I[w]+\big(a_1+o(1)\big)\varphi_{1,\varepsilon,P_\varepsilon}(P_\varepsilon)\\
&\quad+\big(a_2+o(1)\big)\varphi_{2,\varepsilon,Q_\varepsilon}(Q_\varepsilon)\big]-\frac{\beta}{2}\int_\Omega(w_{1,\varepsilon,P_\varepsilon}+\phi_{1,\varepsilon,P_\varepsilon})^2(w_{2,\varepsilon,Q_\varepsilon}+\phi_{2,\varepsilon,Q_\varepsilon})^2\\
&\quad+o\big(\varepsilon^N\delta_\varepsilon\big(P_\varepsilon,Q_\varepsilon)\big).
\end{aligned}
$$

Suppose $\lambda_1<\lambda_2$ first, then by Lemma 7.2,

$$\phi_{1,\varepsilon,P_\varepsilon}(x) \sim w_{1,\varepsilon,P_\varepsilon}(Q_\varepsilon)\cdot \Psi_{1,b_\epsilon}\left(\frac{|x-Q_\varepsilon|}{\varepsilon}\right),$$
$$w_{1,\varepsilon,Q_\varepsilon}(x) \sim w_{1,\varepsilon,P_\varepsilon}(P_\varepsilon)\,\mathrm{e}^{-\langle b_\epsilon, y\rangle},$$
$$\phi_{2,\varepsilon,Q_\varepsilon} \sim o\big(w_{1,\varepsilon,P_\varepsilon}(Q_\varepsilon)\big),$$

where $b_\epsilon = (Q_\epsilon - P_\epsilon)/|Q_\epsilon - P_\epsilon|$.

So we have

$$\begin{aligned}
&-\frac{\beta}{2}\int_\Omega (w_{1,\varepsilon,P_\varepsilon}+\phi_{1,\varepsilon,P_\varepsilon})^2(w_{2,\varepsilon,Q_\varepsilon}+\phi_{2,\varepsilon,Q_\varepsilon})^2\\
&\quad\geqslant -\frac{\beta}{2}\int_{B_{\varepsilon R}(Q_\varepsilon)} (w_{1,\varepsilon,P_\varepsilon}+\phi_{1,\varepsilon,P_\varepsilon})^2(w_{2,\varepsilon,Q_\varepsilon}+\phi_{2,\varepsilon,Q_\varepsilon})^2\\
&\quad\geqslant \left(-\frac{\beta}{4}\right)\varepsilon^N\int_{\mathbb{R}^N}\big(\mathrm{e}^{-\langle b_\epsilon,y\rangle}+\Psi_{1,b_\epsilon}(y)\big)^2 w_2^2(y)\,\mathrm{d}y\cdot\big(w_{1,\varepsilon,P_\varepsilon}(Q_\varepsilon)\big)^2\\
&\quad\geqslant a_6 I_\varepsilon(P_\varepsilon,Q_\varepsilon)\geqslant a_6 I_\varepsilon^{1+\sigma}(P_\varepsilon,Q_\varepsilon)
\end{aligned}$$

where $a_6>0$ is a positive number, using Lemma 3.5.

Similarly, for $\lambda_1=\lambda_2$,

$$\begin{aligned}
&-\frac{\beta}{2}\int_\Omega (w_{1,\varepsilon,P_\varepsilon}+\phi_{1,\varepsilon,P_\varepsilon})^2(w_{2,\varepsilon,Q_\varepsilon}+\phi_{2,\varepsilon,Q_\varepsilon})^2\\
&\quad\geqslant -\frac{\beta}{2}\int_{B_{\epsilon R}(P_\varepsilon)} (w_{1,\varepsilon,P_\varepsilon}+\phi_{1,\varepsilon,P_\varepsilon})^2(w_{2,\varepsilon,Q_\varepsilon}+\phi_{2,\varepsilon,Q_\varepsilon})^2\\
&\qquad+\left(-\frac{\beta}{2}\right)\int_{B_{\epsilon R}(Q_\varepsilon)} (w_{1,\varepsilon,P_\varepsilon}+\phi_{1,\varepsilon,P_\varepsilon})^2(w_{2,\varepsilon,Q_\varepsilon}+\phi_{2,\varepsilon,Q_\varepsilon})^2\\
&\quad\geqslant \left(-\frac{\beta}{2}\right)\varepsilon^N\int_{B_R}\big(\mathrm{e}^{-\langle b_\epsilon,y\rangle}+\Psi_{1,b_\epsilon}(y)\big)^2 w_2^2(y)\,\mathrm{d}y\big(w_{1,\varepsilon,P_\varepsilon}(Q_\varepsilon)\big)^2\\
&\qquad+\left(-\frac{\beta}{2}\right)\varepsilon^N\int_{B_R}\big(\mathrm{e}^{\langle b_\epsilon,y\rangle}+\Psi_{2,-b_\epsilon}(y)\big)^2 w_1^2(y)\,\mathrm{d}y\big(w_{2,\varepsilon,Q_\varepsilon}(P_\varepsilon)\big)^2\\
&\quad\geqslant a_6 I_\varepsilon^{1+\sigma}(P_\varepsilon,Q_\varepsilon).
\end{aligned}$$

Thus we have

$$\begin{aligned}
c_\varepsilon \geqslant \varepsilon^N\big[&(\lambda_1^{(4-N)/2}\mu_1^{-1}+\lambda_2^{(4-N)/2}\mu_2^{-1})I[w]+\big(a_4+o(1)\big)\varphi_{1,\varepsilon,P_\varepsilon}(P_\varepsilon)\\
&+\big(a_5+o(1)\big)\varphi_{2,\varepsilon,Q_\varepsilon}(Q_\varepsilon)+\big(a_6+o(1)\big)I_\varepsilon^{1+\sigma}(P_\varepsilon,Q_\varepsilon)\big],
\end{aligned}$$

where $a_4, a_5, a_6 > 0$ are three positive numbers.

This proves Theorem 7.1.

8. A lower bound for c_ε in the case of $\beta > 0$ (attractive case)

We obtain a lower bound for c_ε, assuming that $\beta>0$. Let P_ε, Q_ε be the unique local maximum points of u_ε, v_ε, respectively. Recall that by Theorem 6.1, $|P_\varepsilon - Q_\varepsilon|/\varepsilon \to 0$.

We may assume that, passing to a subsequence, that P_ε (or Q_ε) $\to x_0 \in \overline{\Omega}$. Thus

$$d_\varepsilon = d(P_\varepsilon, \partial\Omega) \to d_0 = d(x_0, \partial\Omega), \quad \text{as } \varepsilon \to 0.$$

(Note that $d(x_0, \partial\Omega)$ may be 0.)

Given $\sigma > 0$, we choose a number $d_0' > 0$ so that

$$\mathrm{vol}\big(B(x_0, d_0')\big) = \mathrm{vol}\big(\Omega \cap B(x_0, d_0 + \sigma)\big)$$

choose $\sigma' > 0$ slightly smaller than σ with $d_0' < d_0 + \sigma'$.

Now consider a C^∞ cut-off function η_ε such that

$$\eta_\varepsilon(s) = 1 \quad \text{for } 0 \leqslant s \leqslant d_\varepsilon + \sigma', \quad \eta_\varepsilon(s) = 0 \quad \text{for } s > d_\varepsilon + \sigma, \quad 0 \leqslant \eta_\varepsilon \leqslant 1, \quad |\eta_\varepsilon'| \leqslant C.$$

Let $\tilde{u}_\varepsilon = u_\varepsilon \eta_\varepsilon(|P_\varepsilon - x|)$, $\tilde{v}_\varepsilon = v_\varepsilon \eta_\varepsilon(|Q_\varepsilon - x|)$. Then using estimates (6.3) and (6.4) of Theorem 6.1, we obtain that

$$c_\varepsilon \geqslant E_\Omega[tu_\varepsilon, sv_\varepsilon] \geqslant E_{\widetilde{\Omega}}[t\tilde{u}_\varepsilon, s\tilde{v}_\varepsilon] - \varepsilon^N \exp\left[-\frac{2\sqrt{\lambda_1}}{\varepsilon}(d_\varepsilon + \sigma')\right] - \varepsilon^N \exp\left[-\frac{2\sqrt{\lambda_2}}{\varepsilon}(d_\varepsilon + \sigma')\right]$$

for all $t, s \in [0, 2]$, where

$$\widetilde{\Omega} = \Omega \cap B(x_\varepsilon, d_\varepsilon + \sigma).$$

Let $R_\varepsilon = d_\varepsilon'/\varepsilon$, where d_ε' is chosen such that

$$\mathrm{vol}\big(B(0, d_\varepsilon')\big) = \mathrm{vol}\big(\Omega \cap B(x_\varepsilon, d_\varepsilon + \sigma)\big).$$

Using Schwartz's symmetrization, we have

$$\int_{B(0,d_\varepsilon')} (\tilde{u}_\varepsilon^*)^2 (\tilde{v}_\varepsilon^*)^2 \geqslant \int_{\widetilde{\Omega}} \tilde{u}_\varepsilon^2 \tilde{v}_\varepsilon^2$$

and hence (since $\beta > 0$)

$$E_{B(0,d_\varepsilon')}[t\tilde{u}_\varepsilon^*, s\tilde{v}_\varepsilon^*] \leqslant E_{\widetilde{\Omega}}[t\tilde{u}_\varepsilon, s\tilde{v}_\varepsilon].$$

Now we choose, t^*, s^* such that

$$E_{B(0,d_\varepsilon')}[t^*\tilde{u}_\varepsilon^*, s^*\tilde{v}_\varepsilon^*] \geqslant E_{B(0,d_\varepsilon')}[t\tilde{u}_\varepsilon^*, s\tilde{v}_\varepsilon^*], \quad \forall t, s \geqslant 0.$$

Then

$$\begin{aligned} E_{B(0,d_\varepsilon')}[t^*\tilde{u}_\varepsilon^*, s^*\tilde{v}_\varepsilon^*] &\leqslant E_{\widetilde{\Omega}}[t^*\tilde{u}_\varepsilon, s^*\tilde{v}_\varepsilon] \\ &\leqslant c_\varepsilon + \varepsilon^N \exp\left[-\frac{2\sqrt{\lambda_1}}{\varepsilon}(d_\varepsilon + \sigma')\right] + \varepsilon^N \exp\left[-\frac{2\sqrt{\lambda_2}}{\varepsilon}(d_\varepsilon + \sigma')\right], \end{aligned}$$

$$\begin{aligned} E_{B(0,d_\varepsilon')}[t^*\tilde{u}_\varepsilon^*, s^*\tilde{v}_\varepsilon^*] &= \sup_{t,s} E_{B(0,d_\varepsilon')}[t\tilde{u}_\varepsilon^*, s\tilde{v}_\varepsilon^*] \geqslant \epsilon^N \inf_{\substack{u,v\geqslant 0,\ u\not\equiv 0,\ v\not\equiv 0,\\ (u,v)\in M_{R_\varepsilon}}} I_{R_\varepsilon}[u,v] \\ &\geqslant \varepsilon^N \left\{ I_0 + c_3 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_1}}{\varepsilon}\big(d_\varepsilon + o(1)\big)\right] + c_4 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_2}}{\varepsilon}\big(d_\varepsilon + o(1)\big)\right]\right\} \end{aligned}$$

where the last inequality follows from Theorem 4.1.

Thus

$$c_\varepsilon \geqslant \varepsilon^N \left\{ I_0 + c_3 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_1}}{\varepsilon}\big(d_\varepsilon + o(1)\big)\right] + c_4 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_2}}{\varepsilon}\big(d_\varepsilon + o(1)\big)\right]\right\}. \tag{8.1}$$

Completion of proof of (2) of Theorem 1.1. If $\beta > 0$, then combining the lower and upper bound of c_ε, we obtain

$$c_3 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_1}}{\varepsilon}\big(d_\varepsilon + o(1)\big)\right] + c_4 \exp\left[-\frac{2(1+\sigma)\sqrt{\lambda_2}}{\varepsilon}\big(d_\varepsilon + o(1)\big)\right]$$
$$\leqslant c_1 \exp\left[-\frac{2(1-\sigma)\sqrt{\lambda_1}}{\varepsilon}\big(d_0 + o(1)\big)\right] + c_2 \exp\left[-\frac{2(1-\sigma)\sqrt{\lambda_2}}{\varepsilon}\big(d_0 + o(1)\big)\right].$$

This then shows that $d(P_\varepsilon, \partial\Omega), d(Q_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega)$ since $|P_\varepsilon - Q_\varepsilon| \to 0$. $\square$

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