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A vanishing viscosity approach to a quasistatic evolution problem with nonconvex energy

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Abstract

We study a quasistatic evolution problem for a nonconvex elastic energy functional. Due to lack of convexity, the natural energetic formulation can be obtained only in the framework of Young measures. Since the energy functional may present multiple wells, an evolution driven by global minimizers may exhibit unnatural jumps from one well to another one, which overcome large potential barriers. To avoid this phenomenon, we study a notion of solution based on a viscous regularization. Finally we compare this solution with the one obtained with global minimization.

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1. Introduction

In this paper we consider a quasistatic evolution problem for an elastic material described by the deformation variable and by a further parameter which plays the role of internal variable connected to energy dissipation. The energetic formulation of this problem is expressed in terms of the elastic energy functional $\mathcal{W}$, depending both on the deformation and on the internal variable, and of the dissipation functional $\mathcal{H}$, depending just on the internal variable (see, e.g. [10,12,13]).

As in [8] we assume that $\mathcal{W}$ and $\mathcal{H}$ have the following integral form

$$\mathcal{W}(z,v) := \int_D W\big(z(x), \nabla v(x)\big)\, dx,$$

$$\mathcal{H}(z) := \int_D H\big(z(x)\big)\, dx$$

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where $D \subseteq \mathbb{R}^d$ is the reference configuration, $v : D \to \mathbb{R}^N$ the deformation, and $z : D \to \mathbb{R}^m$ the internal variable. We also assume that the body is subjected to volume forces $\boldsymbol{f}(t)$ depending on time, and to a time-dependent prescribed boundary condition (which in this introduction is imposed on the whole boundary for simplicity).

The standard method to attack the problem of the quasistatic evolution is via time-discretization and resolution of incremental minimum problems, which in our case would take the form

$$\min\bigl\{\mathcal{W}(z, v) - \langle \boldsymbol{f}(t), v\rangle + \mathcal{H}(z - z_0)\bigr\} \tag{1.1}$$

among all functions (z, v) which satisfy the prescribed boundary condition at time t. Here z_0 is the known value of z at the previous instant.

For the mechanical applications considered in [8], the convexity of $\mathcal{W}$ with respect to the internal variable z is not a natural assumption. Thus, as observed in [7], the minimum problem (1.1) may have no solutions; moreover, since the lack of convexity allows the functional to have multiple wells, a quasistatic evolution driven by global minimizers, if they would exist, could prescribe abrupt jumps from one well to another one; therefore it is preferable to follow a path composed by local minimizers rather than global minimizers.

In this spirit, the properties of global minimality (stability) and energy balance characterizing the usual energetic solution (see e.g. [11, Section 3] and references therein) are weakened: they are replaced by a stationarity condition and an energy inequality, i.e.

(1) for every $t \in [0, T]$

$$-\operatorname{div} \boldsymbol{\sigma}(t) = \boldsymbol{f}(t),$$
$$\boldsymbol{\zeta}(t) \in \partial\mathcal{H}(0)$$

where $\boldsymbol{\sigma}(t) := \frac{\partial W}{\partial F}(\boldsymbol{z}(t), \nabla \boldsymbol{v}(t))$ and $\boldsymbol{\zeta}(t) := -\frac{\partial W}{\partial \theta}(\boldsymbol{z}(t), \nabla \boldsymbol{v}(t))$;

(2) for every $t \in [0, T]$

$$\mathcal{W}\bigl(\boldsymbol{z}(t), \boldsymbol{v}(t)\bigr) - \bigl\langle \boldsymbol{f}(t), \boldsymbol{v}(t)\bigr\rangle + \operatorname{Var}_H(\boldsymbol{z}; 0, t)$$
$$\leqslant \mathcal{W}(z_0, v_0) - \bigl\langle \boldsymbol{f}(0), v_0\bigr\rangle + \int_0^t \bigl\langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\bigr\rangle\, ds - \int_0^t \bigl[\bigl\langle \dot{\boldsymbol{f}}(s), \boldsymbol{v}(s)\bigr\rangle + \bigl\langle \boldsymbol{f}(s), \dot{\boldsymbol{\varphi}}(s)\bigr\rangle\bigr]\, ds,$$

where Var_H is a suitably defined notion of variation (see (3.1)).

Moreover in the spirit of [5,3,15,1,9], we propose a selection criterion among the solutions of (1) and (2), based on a sort of viscous approximation. The underlying idea is that an evolution obtained in this way does not jump over potential barriers. Given a regularizing parameter $\varepsilon > 0$, we first consider the ε-regularized problem:

(a) *equilibrium condition*: for a.e. $t \in [0, T]$

$$-\operatorname{div} \boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\Delta\dot{\boldsymbol{v}}_\varepsilon(t) = \boldsymbol{f}(t); \tag{1.2}$$

(b) *regularized flow rule*: for a.e. $t \in [0, T]$

$$\dot{\boldsymbol{z}}_\varepsilon(t) = N_K^\varepsilon\bigl(\boldsymbol{\zeta}_\varepsilon(t)\bigr) \quad \text{a.e. in } D,$$

where we look for a solution $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$ in $H^1([0, T]; L^2(D; \mathbb{R}^m) \times H^1(D; \mathbb{R}^N))$ satisfying the boundary condition, $\boldsymbol{\sigma}_\varepsilon(t) := \frac{\partial W}{\partial F}(z_\varepsilon(t), \nabla \boldsymbol{v}_\varepsilon(t))$, $\boldsymbol{\zeta}_\varepsilon(t) := -\frac{\partial W}{\partial \theta}(z_\varepsilon(t), \nabla \boldsymbol{v}_\varepsilon(t))$, and $N_K^\varepsilon(\boldsymbol{\zeta}) := \frac{1}{\varepsilon}(\boldsymbol{\zeta} - P_K(\boldsymbol{\zeta}))$, P_K being the projection onto $K := \partial H(0)$.

In Theorem 4.4 we prove that every ε-regularized problem is equivalent to the following one:

$(1)_\varepsilon$ *equilibrium condition*: for a.e. $t \in [0, T]$

$$-\operatorname{div} \boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\Delta\dot{\boldsymbol{v}}_\varepsilon(t) = \boldsymbol{f}(t) \tag{1.3}$$

relaxed dual constraint: for a.e. $t \in [0, T]$

$$\boldsymbol{\zeta}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{z}}_\varepsilon(t) \in \partial\mathcal{H}(0);$$

$(2)_\varepsilon$ *energy equality*: for every $t \in [0, T]$,

$$\mathcal{W}\big(\boldsymbol{z}_\varepsilon(t), \boldsymbol{v}_\varepsilon(t)\big) - \big\langle \boldsymbol{f}(t), \boldsymbol{v}_\varepsilon(t)\big\rangle + \int_0^t \mathcal{H}\big(\dot{\boldsymbol{z}}_\varepsilon(s)\big)\,ds + \varepsilon\int_0^t \big\|\dot{\boldsymbol{z}}_\varepsilon(s)\big\|_2^2\,ds + \varepsilon\int_0^t \big\langle \nabla\dot{\boldsymbol{v}}_\varepsilon(s), \nabla\dot{\boldsymbol{v}}_\varepsilon(s) - \nabla\dot{\boldsymbol{\varphi}}(s)\big\rangle\,ds$$
$$= \mathcal{W}(z_0, v_0) - \big\langle \boldsymbol{f}(0), v_0\big\rangle + \int_0^t \big\langle \boldsymbol{\sigma}_\varepsilon(s), \nabla\dot{\boldsymbol{\varphi}}(s)\big\rangle\,ds - \int_0^t \big[\big\langle \dot{\boldsymbol{f}}(s), \boldsymbol{v}_\varepsilon(s)\big\rangle + \big\langle \boldsymbol{f}(s), \dot{\boldsymbol{\varphi}}(s)\big\rangle\big]\,ds$$

where we look for a solution $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$ in $H^1([0,T]; L^2(D;\mathbb{R}^m) \times H^1(D;\mathbb{R}^N))$ satisfying the boundary condition.

To prove the existence and uniqueness of the solutions to the ε-regularized problems we use the analysis of discretized minimum problems. We fix a time discretization of the interval $[0, T]$

$$0 = t_0 < t_1 < \cdots < t_n = T,$$

and we set $\tau_i := t_i - t_{i-1}$, for $i > 0$. Given $\varepsilon > 0$ and an initial value (z_0, v_0), we define inductively the value of an approximate solution of the ε-regularized problem at time t_i, for $i > 0$, as a minimizer of the functional

$$\mathcal{W}(z, v) - \big\langle \boldsymbol{f}(t_i), v\big\rangle + \mathcal{H}\big(z - z(t_{i-1})\big) + \frac{\varepsilon}{2\tau_i}\big\|z - z(t_{i-1})\big\|_2^2 + \frac{\varepsilon}{2\tau_i}\big\|\nabla v - \nabla v(t_{i-1})\big\|_2^2, \tag{1.4}$$

among all (z, v) which satisfy the boundary condition at time t_i. Under suitable regularity assumptions on W, the functional (1.4) is convex for τ_i sufficiently small, thanks to the presence of the regularizing terms $\frac{\varepsilon}{2\tau_i}\|z - z(t_{i-1})\|_2^2$ and $\frac{\varepsilon}{2\tau_i}\|\nabla v - \nabla v(t_{i-1})\|_2^2$; hence the minimizer exists and is unique. The study of the limit of the approximate solutions, when the discretization step tends to 0 and the parameter ε is kept fixed, proves the existence (see Theorem 4.6) of a unique solution of the ε-regularized problem.

We come back now to the original problem: as in [3], we accept only solutions to (1) and (2) which can be approximated by solutions of ε-regularized problems; nevertheless, due to the nonconvexity of the problem, the ε-regularized solutions may develop stronger and stronger oscillations in space as ε tends to 0, and this prevents the passage to the limit of $(1)_\varepsilon$ and $(2)_\varepsilon$ in the usual function spaces.

Therefore (1) and (2) need a weaker formulation in terms of Young measures, or equivalently (see [7, Section 3]) in terms of stochastic processes. In the latter formulation the function $(\boldsymbol{z}(t,x), \boldsymbol{v}(t,x))$ is substituted by a stochastic process $(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega))$, defined on a product probability space $(D \times \Omega, P)$, with

$$\pi_D(P) = \mathcal{L}^d, \tag{1.5}$$

and (1), (2) become

$(1')$ *equilibrium condition*: for every $t \in [0, T]$

$$-\operatorname{div}\boldsymbol{\sigma}(t) = \boldsymbol{f}(t); \tag{1.6}$$

dual constraint: for every $t \in [0, T]$

$$\boldsymbol{\zeta}(t) \in \partial\mathcal{H}(0); \tag{1.7}$$

$(2')$ *energy inequality*: for every $t \in [0, T]$ we have

$$\int_{D\times\Omega} W\big(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega)\big)\,dP(x,\omega) - \big\langle \boldsymbol{f}(t), \boldsymbol{v}(t)\big\rangle + \mathrm{Var}_H(\boldsymbol{Z}, P; 0, t)$$
$$\leqslant \mathcal{W}(z_0, v_0) - \big\langle \boldsymbol{f}(0), v_0\big\rangle + \int_0^t \big\langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\big\rangle\,ds - \int_0^t \big[\big\langle \boldsymbol{f}(s), \dot{\boldsymbol{\varphi}}(s)\big\rangle + \big\langle \dot{\boldsymbol{f}}(s), \boldsymbol{v}(s)\big\rangle\big]\,ds, \tag{1.8}$$

where $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_t$ is the stochastic process representing the evolution, $\boldsymbol{v}(t)$ is the unique function satisfying the boundary condition and $\nabla \boldsymbol{v}(t) = \operatorname{bar}((\pi_D, \boldsymbol{Y}_t)(P))$, $\boldsymbol{\sigma}(t)$, $\boldsymbol{\zeta}(t)$ are the unique elements of $L^2(D; \mathbb{R}^{N\times d})$ and $L^2(D; \mathbb{R}^m)$, such that

$$\int_D \boldsymbol{\sigma}(t,x) g(x)\,dx = \int_{D\times\Omega} \frac{\partial W}{\partial F}\big(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega)\big) g(x)\,dP(x,\omega),$$

$$\int_D \boldsymbol{\zeta}(t,x) h(x)\,dx = \int_{D\times\Omega} -\frac{\partial W}{\partial \theta}\big(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega)\big) h(x)\,dP(x,\omega),$$

for every $g \in L^2(D; \mathbb{R}^{N\times d})$, $h \in L^2(D; \mathbb{R}^m)$, and

$$\operatorname{Var}_H(\boldsymbol{Z}, P; 0, t) := \sup \sum_{i=1}^{k} \int_{D\times\Omega} H\big(\boldsymbol{Z}_{t_i}(x,\omega) - \boldsymbol{Z}_{t_{i-1}}(x,\omega)\big)\,dP(x,\omega),$$

where the supremum is taken over all finite partitions $0 = t_0 < \cdots < t_k = t$.

In this formalism ordinary functions can be seen as stochastic processes independent of ω; in this case conditions (1′) and (2′) are exactly (1) and (2), thanks to (1.5).

In Theorem 7.16 we study the passage to the limit in the space of Young measures of the solutions to the ε-regularized problems, as ε tends to 0, while in Theorem 7.13 we show that the stochastic process corresponding to this limit is a solution of the generalized problem (1′) and (2′).

In the last section, we compare the notion of approximable quasistatic evolution considered in the present paper with the quasistatic evolution based on global minimization defined in [7, Definition 6.12]. More precisely we study an example in which the approximable quasistatic evolution is unique and is a classical function which can be described explicitly. Theorem 8.2 proves that this evolution cannot be a globally stable quasistatic evolution, since it does not satisfy the stability condition.

2. Mathematical preliminaries and notations

For the mathematical preliminaries about functions, measures, and Young measures we refer to [7, Section 2].

The symbol $\langle\cdot,\cdot\rangle$ will denote a duality pairing depending on the context.

We refer to [7, Section 3] for a presentation of compatible systems of Young measures with probabilistic language; here we just recall the statement of the main theorem about the correspondence between compatible systems of Young measures and stochastic processes.

Theorem 2.1. *Let T be a set of indices, and let, for every $t \in T$, V_t and W_t be finite dimensional Hilbert spaces. Given two compatible systems $\boldsymbol{\mu} \in SY^2(D; (V_t)_{t\in T})$ and $\boldsymbol{\nu} \in SY^2(D; (V_t \times W_t)_{t\in T})$, satisfying*

$$\pi_{D\times V_t}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t, \quad \text{for every } t \in T, \tag{2.1}$$

there exist a probability space of the form $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$, where $\mathcal{B}(D)$ is the Borel σ-algebra on D, $(\Omega, \mathcal{F})$ is a measurable space, and P a probability measure with $\pi_D(P) = \mathcal{L}^d$, and a stochastic process $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in T}$ with

$$\boldsymbol{Z}_t \in L^2(D \times \Omega; V_t),$$
$$\boldsymbol{Y}_t \in L^2(D \times \Omega; W_t),$$

for every $t \in T$, such that

$$\big(\pi_D, (\boldsymbol{Z}_t)_{t\in F}\big)(P) = \boldsymbol{\mu}_F, \tag{2.2}$$

for every nonempty finite subset F of T, and

$$(\pi_D, \boldsymbol{Z}_t, \boldsymbol{Y}_t)(P) = \boldsymbol{\nu}_t, \tag{2.3}$$

for every $t \in T$.

3. Mechanical model

The *reference configuration* D is a bounded connected open subset of $\mathbb{R}^d$ with Lipschitz boundary $\partial D = \Gamma_0 \cup \Gamma_1$, where Γ_0 is assumed to be a nonempty closed subset of ∂D with $\mathcal{H}^{d-1}(\Gamma_0) \neq 0$, and $\Gamma_1 = \partial D \setminus \Gamma_0$. Without loss of generality we also assume for simplicity that $\mathcal{L}^d(D) = 1$.

We indicate the deformation by v and the internal variable by z.

We denote *the stored energy density* by $W : \mathbb{R}^m \times \mathbb{R}^{N\times d} \to [0, +\infty)$ and the *dissipation rate density* by $H : \mathbb{R}^m \to [0, +\infty)$. For every $\theta \in \mathbb{R}^m$ and every $F \in \mathbb{R}^{N\times d}$, we make the following assumptions:

(W.1) there exist positive constants c, C such that

$$c\big(|\theta|^2 + |F|^2\big) - C \leqslant W(\theta, F) \leqslant C\big(1 + |\theta|^2 + |F|^2\big);$$

(W.2) W is of class $\mathcal{C}^2$ and there exists a positive constant M such that

$$\left|\frac{\partial^2 W}{\partial(\theta, F)^2}(\theta, F)\right| \leqslant M,$$

where $\frac{\partial^2 W}{\partial(\theta,F)^2}$ denotes the matrix of all second derivatives with respect to (the components of) θ and F;

(H.1) H is positively homogeneous of degree one and convex;

(H.2) there exists a positive constant λ, such that $\frac{1}{\lambda}|\theta| \leqslant H(\theta) \leqslant \lambda|\theta|$.

Let $\mathcal{W}$ be the functional $\mathcal{W}(z, v) := \int_D W(z(x), \nabla v(x))\,dx$, for every $z \in L^2(D; \mathbb{R}^m)$ and every $v \in H^1(D; \mathbb{R}^N)$, and $\mathcal{H}$ the functional $\mathcal{H}(z) := \int_D H(z(x))\,dx$, for every $z \in L^1(D; \mathbb{R}^m)$.

Given two distinct times $s < t$, the *global dissipation* of a possibly discontinuous function $\boldsymbol{z} : [0, T] \to L^2(D; \mathbb{R}^m)$ in the interval $[s, t]$ is

$$\mathrm{Var}_H(\boldsymbol{z}; s, t) := \sup\left\{\sum_{i=1}^{k} \mathcal{H}\big(\boldsymbol{z}(\tau_i) - \boldsymbol{z}(\tau_{i-1})\big)\right\}, \tag{3.1}$$

where the supremum is taken among all finite partitions $s = \tau_0 < \tau_1 < \cdots < \tau_k = t$.

Note that for $\boldsymbol{z} \in H^1([0, T]; L^2(D; \mathbb{R}^m))$ we have

$$\mathrm{Var}_H(\boldsymbol{z}; s, t) = \int_s^t \mathcal{H}\big(\dot{\boldsymbol{z}}(\tau)\big)\,d\tau \tag{3.2}$$

(see, e.g., [2, Theorem 7.1]).

The external load at time t and the prescribed boundary datum on Γ_0 at time t are denoted by $\boldsymbol{l}(t)$ and $\boldsymbol{\varphi}(t)$, respectively; we assume that $\boldsymbol{l} \in H^1([0, T]; H^1(D; \mathbb{R}^N)^*)$, where $H^1(D; \mathbb{R}^N)^*$ is the dual of $H^1(D; \mathbb{R}^N)$, and $\boldsymbol{\varphi} \in H^1([0, T]; H^1(D; \mathbb{R}^N))$.

The kinematically admissible values at time t for z and v are those who make the total energy finite and satisfy the boundary condition, i.e. $v = \boldsymbol{\varphi}(t)$ on Γ_0 $\mathcal{H}^{d-1}$-a.e. (in the sense of traces). From previous assumptions it follows that the kinematically admissible values at time t are contained in $L^2(D; \mathbb{R}^m) \times \mathcal{A}(t)$, where

$$\mathcal{A}(t) = H^1_{\Gamma_0}\big(\boldsymbol{\varphi}(t)\big) := \big\{v \in H^1\big(D; \mathbb{R}^N\big) \text{ such that } v = \boldsymbol{\varphi}(t)\mathcal{H}^{d-1}\text{-a.e. on } \Gamma_0\big\}.$$

Before concluding this section, we want to point out some properties of the functional $\mathcal{H}$ and to introduce some new notation.

From the hypotheses on H, it follows that $\mathcal{H}$ is l.s.c. with respect to the weak topology of $L^2(D; \mathbb{R}^m)$ and satisfies the triangle inequality, i.e.

$$\mathcal{H}(z_1 + z_2) \leqslant \mathcal{H}(z_1) + \mathcal{H}(z_2).$$

For every $\varepsilon > 0$ we define the function $H_\varepsilon : \mathbb{R}^m \to \mathbb{R}$ as

$$H_\varepsilon(\theta) := H(\theta) + \frac{\varepsilon}{2}|\theta|^2, \tag{3.3}$$

and the corresponding integral functional $\mathcal{H}_\varepsilon : L^2(D;\mathbb{R}^m) \to \mathbb{R}$ as

$$\mathcal{H}_\varepsilon(z) := \int_D H_\varepsilon\big(z(x)\big)\,dx. \tag{3.4}$$

The convex conjugate $H_\varepsilon^* : \mathbb{R}^m \to \mathbb{R}$ of H_ε is

$$H_\varepsilon^*(\zeta) := \sup_{\theta\in\mathbb{R}^m} \big\{\zeta\theta - H_\varepsilon(\theta)\big\}.$$

Since the convex conjugate H^* of H is the indicator function of the convex set $K := \partial H(0)$ (see [14, Theorem 13.2]), using [14, Theorem 16.4], it can be proved that

$$H_\varepsilon^*(\zeta) = \frac{1}{2\varepsilon}\big|\zeta - P_K(\zeta)\big|^2, \tag{3.5}$$

where $P_K : \mathbb{R}^m \to K$ is the projection onto K. Therefore H_ε^* is differentiable with gradient

$$N_K^\varepsilon(\zeta) := \frac{1}{\varepsilon}\big(\zeta - P_K(\zeta)\big). \tag{3.6}$$

In particular N_K^ε is Lipschitz continuous.

Let $\mathcal{H}_\varepsilon^* : L^2(D;\mathbb{R}^m) \to \mathbb{R}$ be the convex conjugate of $\mathcal{H}_\varepsilon$. It can be easily shown (using a general property of integral functionals, see e.g., [6, Proposition IX.2.1]) that

$$\mathcal{H}_\varepsilon^*(\zeta) = \int_D H_\varepsilon^*\big(\zeta(x)\big)\,dx,$$

so that the gradient $\partial\mathcal{H}_\varepsilon^*$ is given by

$$\partial\mathcal{H}_\varepsilon^*(\zeta)(x) = N_K^\varepsilon\big(\zeta(x)\big), \quad \text{for a.e. } x \in D. \tag{3.7}$$

Therefore $\partial\mathcal{H}_\varepsilon^*$ is Lipschitz continuous.

4. Regularized evolution

In this section we give the definition and an existence result for the solution of the ε-regularized evolution problem. We will assume that the initial condition $(z_0, v_0) \in L^2(D;\mathbb{R}^m) \times \mathcal{A}(0)$ satisfies the following condition

$$\langle\sigma_0, \nabla\tilde{u}\rangle = \big\langle \boldsymbol{l}(0), \tilde{u}\big\rangle \quad \text{for every } \tilde{u} \in H^1_{\Gamma_0}(0), \tag{4.1}$$

$$\zeta_0 \in \partial\mathcal{H}(0), \tag{4.2}$$

where $\sigma_0(x) := \frac{\partial W}{\partial F}(z_0(x), \nabla v_0(x))$, $\zeta_0(x) := -\frac{\partial W}{\partial\theta}(z_0(x), \nabla v_0(x))$, for a.e. $x \in D$.

Definition 4.1. Let $\varepsilon > 0$, $\boldsymbol{l} \in H^1([0,T]; H^1(D;\mathbb{R}^N)^*)$, $\boldsymbol{\varphi} \in H^1([0,T]; H^1(D;\mathbb{R}^N))$, $(z_0, v_0) \in L^2(D;\mathbb{R}^m) \times \mathcal{A}(0)$, and $T > 0$. Assume that (z_0, v_0) satisfies (4.1) and (4.2). A *solution of the ε-regularized problem* in the time interval $[0,T]$, with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0, v_0) is a pair $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon) \in H^1([0,T]; L^2(D;\mathbb{R}^m) \times H^1(D;\mathbb{R}^N))$, satisfying the following conditions:

$(\text{ev0})_\varepsilon$ *initial condition*: $(\boldsymbol{z}_\varepsilon(0), \boldsymbol{v}_\varepsilon(0)) = (z_0, v_0)$;
$(\text{ev1})_\varepsilon$ *kinematic admissibility*: $\boldsymbol{v}_\varepsilon(t) \in \mathcal{A}(t)$, for every $t \in [0,T]$;
$(\text{ev2})_\varepsilon$ *equilibrium condition*: for a.e. $t \in [0,T]$ and for every $\tilde{u} \in H^1_{\Gamma_0}(0)$,

$$\big\langle \boldsymbol{\sigma}_\varepsilon(t) + \varepsilon\nabla\dot{\boldsymbol{v}}_\varepsilon(t), \nabla\tilde{u}\big\rangle = \big\langle \boldsymbol{l}(t), \tilde{u}\big\rangle, \tag{4.3}$$

where $\boldsymbol{\sigma}_\varepsilon(t) := \frac{\partial W}{\partial F}(\boldsymbol{z}_\varepsilon(t), \nabla\boldsymbol{v}_\varepsilon(t))$;
$(\hat{\text{ev3}})_\varepsilon$ *regularized flow rule*: for a.e. $t \in [0,T]$,

$$\dot{\boldsymbol{z}}_\varepsilon(t) = N_K^\varepsilon\big(\boldsymbol{\zeta}_\varepsilon(t)\big) \quad \text{a.e. in } D, \tag{4.4}$$

where $\boldsymbol{\zeta}_\varepsilon(t) := -\frac{\partial W}{\partial\theta}(\boldsymbol{z}_\varepsilon(t), \nabla\boldsymbol{v}_\varepsilon(t))$.

Remark 4.2. If $\boldsymbol{l}$ can be written in the form

$$\langle \boldsymbol{l}(t), \tilde{v}\rangle = \int_D f(t,x)\cdot \tilde{v}(x)\,dx + \int_{\partial D} g(t,x)\cdot \tilde{v}(x)\,d\mathcal{H}^{d-1}(x),$$

with $f(t) \in L^2(D;\mathbb{R}^N)$ and $g(t) \in L^2(\partial D;\mathbb{R}^N)$, for every $\tilde{v} \in H^1(D;\mathbb{R}^N)$, then $(\mathrm{ev2})_\varepsilon$ takes the form

$$-\operatorname{div}\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\Delta\dot{\boldsymbol{v}}_\varepsilon(t) = f(t), \tag{4.5}$$

$$\big[\boldsymbol{\sigma}_\varepsilon(t) + \varepsilon\nabla\dot{\boldsymbol{v}}_\varepsilon(t)\big]\cdot\nu = g(t), \quad \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_1, \tag{4.6}$$

where ν is the outer unit normal to ∂D.

Indeed, choosing first $\tilde{u} \in H^1_0(D;\mathbb{R}^N)$ in $(\mathrm{ev2})_\varepsilon$, we obtain (4.5) and this ensures that $-\operatorname{div}\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\Delta\dot{\boldsymbol{v}}_\varepsilon(t) \in L^2(D;\mathbb{R}^N)$; hence we can apply integration by parts to get (4.6).

Remark 4.3. Let us fix $t \in [0,T]$ such that $\dot{\boldsymbol{z}}_\varepsilon(t)$ and $\dot{\boldsymbol{v}}_\varepsilon(t)$ exist. Then the following conditions are equivalent

$$\dot{\boldsymbol{z}}_\varepsilon(t) = N_K^\varepsilon\big(\boldsymbol{\zeta}_\varepsilon(t)\big) \quad \text{a.e. in } D, \tag{4.7}$$

$$\boldsymbol{\zeta}_\varepsilon(t) \in \partial\mathcal{H}_\varepsilon\big(\dot{\boldsymbol{z}}_\varepsilon(t)\big), \tag{4.8}$$

$$\boldsymbol{\zeta}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{z}}_\varepsilon(t) \in \partial\mathcal{H}\big(\dot{\boldsymbol{z}}_\varepsilon(t)\big). \tag{4.9}$$

Indeed, by (3.7), $\partial\mathcal{H}_\varepsilon^*(\boldsymbol{\zeta}_\varepsilon(t)) = N_K^\varepsilon(\boldsymbol{\zeta}_\varepsilon(t))$, so that (4.7) and (4.8) are equivalent by standard property of conjugate functions (see, e.g., [6, Corollary I.5.2]). The equivalence of (4.8) and (4.9) comes from the definition of $\mathcal{H}_\varepsilon$.

In the following theorem we prove that the modified flow rule $(\mathrm{ev}\hat{3})_\varepsilon$ can be replaced by a suitable constraint on $\boldsymbol{\zeta}_\varepsilon$ and an energy equality.

Theorem 4.4. *Let $\boldsymbol{l}$, $\boldsymbol{\varphi}$, z_0, v_0, ε, and T be as in Definition* 4.1.

Then $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon) \in H^1([0,T]; L^2(D;\mathbb{R}^m)\times H^1(D;\mathbb{R}^N))$ is a solution of the ε-regularized problem in the time interval $[0,T]$, with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0, v_0) if and only if it satisfies the initial condition $(\mathrm{ev0})_\varepsilon$, *the kinematic admissibility* $(\mathrm{ev1})_\varepsilon$, *the equilibrium condition* $(\mathrm{ev2})_\varepsilon$, *and the following conditions*:

$(\mathrm{ev3})_\varepsilon$ *relaxed dual constraint*: $\boldsymbol{\zeta}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{z}}_\varepsilon(t) \in \partial\mathcal{H}(0)$, *for a.e.* $t \in [0,T]$;
$(\mathrm{ev4})_\varepsilon$ *energy equality*: *for every* $t \in [0,T]$,

$$\mathcal{W}\big(\boldsymbol{z}_\varepsilon(t), \boldsymbol{v}_\varepsilon(t)\big) - \big\langle \boldsymbol{l}(t), \boldsymbol{v}_\varepsilon(t)\big\rangle + \int_0^t \mathcal{H}\big(\dot{\boldsymbol{z}}_\varepsilon(s)\big)\,ds + \varepsilon\int_0^t \big\|\dot{\boldsymbol{z}}_\varepsilon(s)\big\|_2^2\,ds + \varepsilon\int_0^t \big\langle \nabla\dot{\boldsymbol{v}}_\varepsilon(s), \nabla\dot{\boldsymbol{v}}_\varepsilon(s) - \nabla\dot{\boldsymbol{\varphi}}(s)\big\rangle\,ds$$
$$= \mathcal{W}(z_0, v_0) - \big\langle \boldsymbol{l}(0), v_0\big\rangle + \int_0^t \big\langle \boldsymbol{\sigma}_\varepsilon(s), \nabla\dot{\boldsymbol{\varphi}}(s)\big\rangle\,ds - \int_0^t \big[\big\langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}_\varepsilon(s)\big\rangle + \big\langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s)\big\rangle\big]\,ds.$$

Remark 4.5. Since we are dealing with the ε-regularized problem, in the *energy equality* there are two extra terms proportional to ε and depending on the time derivatives of $\boldsymbol{z}_\varepsilon$ and $\nabla\boldsymbol{v}_\varepsilon$; they represent a sort of viscous dissipation.

Proof of Theorem 4.4. Suppose that $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$ satisfies $(\mathrm{ev0})_\varepsilon$, $(\mathrm{ev1})_\varepsilon$, $(\mathrm{ev2})_\varepsilon$ and $(\mathrm{ev}\hat{3})_\varepsilon$. As $\mathcal{H}$ is positively homogeneous of degree one, by a general property of integral functionals (see, e.g., [6, Proposition IX.2.1])

$$\partial\mathcal{H}\big(\dot{\boldsymbol{z}}_\varepsilon(t)\big) \subset \partial\mathcal{H}(0) = \big\{\zeta \in L^2\big(D;\mathbb{R}^m\big)\colon\ \zeta(x) \in \partial H(0) \text{ for a.e. } x \in D\big\}, \tag{4.10}$$

for a.e. $t \in [0,T]$. Hence from (4.9) we derive $(\mathrm{ev3})_\varepsilon$.

Since $\mathcal{H}$ is positively homogeneous of degree one, we have $\langle\zeta, z\rangle = \mathcal{H}(z)$, for every $z \in L^2(D;\mathbb{R}^m)$ and $\zeta \in \partial\mathcal{H}(z)$. Therefore, by (4.9),

$$\mathcal{H}\big(\dot{\boldsymbol{z}}_\varepsilon(t)\big) = \big\langle \boldsymbol{\zeta}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{z}}_\varepsilon(t), \dot{\boldsymbol{z}}_\varepsilon(t)\big\rangle, \tag{4.11}$$

for a.e. $t \in [0, T]$.

Choosing $\tilde{u} = \dot{\boldsymbol{v}}_\varepsilon(t) - \dot{\boldsymbol{\varphi}}(t)$ in $(\text{ev2})_\varepsilon$ and using (4.11) we obtain

$$\left\langle \frac{\partial W}{\partial \theta}\big(\boldsymbol{z}_\varepsilon(t), \nabla \boldsymbol{v}_\varepsilon(t)\big), \dot{\boldsymbol{z}}_\varepsilon(t)\right\rangle + \left\langle \frac{\partial W}{\partial F}\big(\boldsymbol{z}_\varepsilon(t), \nabla \boldsymbol{v}_\varepsilon(t)\big), \nabla \dot{\boldsymbol{v}}_\varepsilon(t) - \nabla \dot{\boldsymbol{\varphi}}(t)\right\rangle - \big\langle \boldsymbol{l}(t), \dot{\boldsymbol{v}}_\varepsilon(t) - \dot{\boldsymbol{\varphi}}(t)\big\rangle + \mathcal{H}\big(\dot{\boldsymbol{z}}_\varepsilon(t)\big)$$
$$+ \varepsilon \big\|\dot{\boldsymbol{z}}_\varepsilon(t)\big\|_2^2 + \varepsilon\big\langle \nabla \dot{\boldsymbol{v}}_\varepsilon(t), \nabla \dot{\boldsymbol{v}}_\varepsilon(t) - \nabla \dot{\boldsymbol{\varphi}}(t)\big\rangle = 0, \tag{4.12}$$

for a.e. $t \in [0, T]$. By integration of (4.12) from 0 to t, we obtain $(\text{ev4})_\varepsilon$.

Conversely, suppose that $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$ satisfies $(\text{ev0})_\varepsilon$, $(\text{ev1})_\varepsilon$, $(\text{ev2})_\varepsilon$, $(\text{ev3})_\varepsilon$ and $(\text{ev4})_\varepsilon$. Then, by derivation with respect to t of $(\text{ev4})_\varepsilon$, we obtain (4.12), which gives (4.11), thanks to $(\text{ev2})_\varepsilon$. Combining (4.11) with $(\text{ev3})_\varepsilon$, it immediately follows (4.9) for a.e. $t \in [0, T]$, and hence $(\text{ev}\hat{3})_\varepsilon$. □

We conclude by stating the existence theorem for the solutions of the ε-regularized problems, which will be proved in the next section.

Theorem 4.6. *Let ε, $\boldsymbol{l}$, $\boldsymbol{\varphi}$, z_0, v_0, and T as in Definition* 4.1. *Then there exists a unique solution of the ε-regularized problem in the time interval* $[0, T]$ *with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$ and initial condition* (z_0, v_0).

5. Proof of Theorem 4.6

The proof is obtained via time-discretization, resolution of incremental minimum problems, and passing to the limit as the time step tends to 0.

5.1. The incremental minimum problem

In this section we study the incremental minimum problem used in the discrete-time formulation of the evolution problem.

Let us fix a sequence of subdivisions of $[0, T]$, $0 = t_n^0 < t_n^1 < \cdots < t_n^{k(n)} = T$, such that $\tau_n := \sup_{i=1,\dots,k(n)} \tau_n^i \to 0$, as $n \to \infty$, where $\tau_n^i := t_n^i - t_n^{i-1}$, for every $i = 1, \dots, k(n)$. We assume that $\tau_n < \frac{\varepsilon}{M}$ for every n, where M is the constant appearing in (W.2).

For every $i = 0, 1, \dots, k(n)$ we set $l_n^i := \boldsymbol{l}(t_n^i)$ and $\varphi_n^i := \boldsymbol{\varphi}(t_n^i)$.

For every n, we define $(z_n^i, v_n^i) \in L^2(D; \mathbb{R}^m) \times \mathcal{A}(t_n^i)$ by induction on i: set $(z_n^0, v_n^0) := (z_0, v_0)$, and for $i > 0$ we define (z_n^i, v_n^i) as the solution (see Lemma 5.1 below) to the incremental minimum problem

$$\min\left\{\mathcal{W}(z, v) - \big\langle l_n^i, v\big\rangle + \mathcal{H}\big(z - z_n^{i-1}\big) + \frac{\varepsilon}{2\tau_n^i}\big\|z - z_n^{i-1}\big\|_2^2 + \frac{\varepsilon}{2\tau_n^i}\big\|\nabla v - \nabla v_n^{i-1}\big\|_2^2\right\}, \tag{5.1}$$

among all $z \in L^2(D; \mathbb{R}^m)$ and all $v \in \mathcal{A}(t_n^i)$.

Lemma 5.1. *Let $\varepsilon > 0$, then for every n and every $i > 0$ there exists a unique solution to* (5.1) *in* $L^2(D; \mathbb{R}^m) \times \mathcal{A}(t_n^i)$.

Proof. Let $(z_k, v_k) \in L^2(D; \mathbb{R}^m) \times \mathcal{A}(t_n^i)$ be a minimizing sequence. By the bounds on W and the assumption on $\boldsymbol{l}$, we have

$$c\big(\|z_k\|_2^2 + \|\nabla v_k\|_2^2\big) - C'\big(1 + \|v_k\|_{H^1}\big) \leqslant \mathcal{W}(z_k, v_k) - \big\langle l_n^i, v_k\big\rangle \leqslant C', \tag{5.2}$$

for suitable positive constants c, C'. Since $v_k \in \mathcal{A}(t_n^i)$, using Poincaré inequality we can deduce from (5.2) that $(z_k, v_k)_k$ is a bounded sequence in $L^2(D; \mathbb{R}^m) \times H^1(D; \mathbb{R}^N)$.

From the boundedness of the second derivative of W, guaranteed by (W.2), and since $\tau_n \in (0, \varepsilon/M)$ by our assumption on τ_n, it easily follows that the functional in (5.1) is strictly convex, and hence weakly lower semicontinuous on $L^2(D; \mathbb{R}^m) \times H^1(D; \mathbb{R}^N)$. Now the existence of minimizers comes from the direct methods of the Calculus of Variations; the uniqueness is a consequence of the strict convexity of the functional. □

Now we derive Euler conditions for the minimizers.

Theorem 5.2. *Let $\varepsilon > 0$, then for every n and every $i > 0$ we have*

$$\begin{aligned}&\mathcal{H}(\tilde z + z_n^i - z_n^{i-1}) - \mathcal{H}(z_n^i - z_n^{i-1}) - \langle l_n^i, \tilde u\rangle\\ &\quad\geqslant \left\langle -\frac{\partial W}{\partial\theta}(z_n^i, \nabla v_n^i), \tilde z\right\rangle - \left\langle \frac{\partial W}{\partial F}(z_n^i, \nabla v_n^i), \nabla\tilde u\right\rangle - \frac{\varepsilon}{\tau_n^i}\langle z_n^i - z_n^{i-1}, \tilde z\rangle - \frac{\varepsilon}{\tau_n^i}\langle \nabla v_n^i - \nabla v_n^{i-1}, \nabla\tilde u\rangle,\end{aligned} \tag{5.3}$$

for every $\tilde z \in L^2(D;\mathbb{R}^N)$ and $\tilde u \in H^1_{\Gamma_0}(0)$.

Hence we can deduce the following Euler conditions:

$$-\frac{\partial W}{\partial\theta}(z_n^i, \nabla v_n^i) - \frac{\varepsilon}{\tau_n^i}(z_n^i - z_n^{i-1}) \in \partial\mathcal{H}(z_n^i - z_n^{i-1}), \tag{5.4}$$

$$\left\langle \frac{\partial W}{\partial F}(z_n^i, \nabla v_n^i) + \frac{\varepsilon}{\tau_n^i}(\nabla v_n^i - \nabla v_n^{i-1}), \nabla\tilde u\right\rangle = \langle l_n^i, \tilde u\rangle \quad \textit{for every } \tilde u \in H^1_{\Gamma_0}(0). \tag{5.5}$$

Conversely, conditions (5.4) *and* (5.5) *imply that (z_n^i, v_n^i) is a solution of* (5.1).

Proof. Since for every $\tilde z \in L^2(D;\mathbb{R}^m)$, $\tilde u \in H^1_{\Gamma_0}(0)$ and $s \geqslant 0$, $z_n^i + s\tilde z \in L^2(D;\mathbb{R}^m)$ and $v_n^i + s\tilde u \in \mathcal{A}(t_n^i)$, the minimality property of (z_n^i, v_n^i) leads to

$$\begin{aligned}&\mathcal{W}(z_n^i, v_n^i) - \langle l_n^i, v_n^i\rangle + \mathcal{H}(z_n^i - z_n^{i-1}) + \frac{\varepsilon}{2\tau_n^i}\|z_n^i - z_n^{i-1}\|_2^2 + \frac{\varepsilon}{2\tau_n^i}\|\nabla v_n^i - \nabla v_n^{i-1}\|_2^2\\ &\quad\leqslant \mathcal{W}(z_n^i + s\tilde z, v_n^i + s\tilde u) - \langle l_n^i, v_n^i + s\tilde u\rangle + \mathcal{H}(z_n^i + s\tilde z - z_n^{i-1})\\ &\qquad + \frac{\varepsilon}{2\tau_n^i}\|z_n^i + s\tilde z - z_n^{i-1}\|_2^2 + \frac{\varepsilon}{2\tau_n^i}\|\nabla v_n^i + s\nabla\tilde u - \nabla v_n^{i-1}\|_2^2.\end{aligned} \tag{5.6}$$

Hence from the convexity of $\mathcal{H}$ we can deduce for every $s \in [0,1]$

$$\begin{aligned}s\big[\mathcal{H}(z_n^i + \tilde z - z_n^{i-1}) - \mathcal{H}(z_n^i - z_n^{i-1})\big] &\geqslant \mathcal{W}(z_n^i, v_n^i) - \mathcal{W}(z_n^i + s\tilde z, v_n^i + s\tilde u) - \langle l_n^i, v_n^i\rangle + \langle l_n^i, v_n^i + s\tilde u\rangle\\ &\quad + \frac{\varepsilon}{2\tau_n^i}\|z_n^i - z_n^{i-1}\|_2^2 - \frac{\varepsilon}{2\tau_n^i}\|z_n^i + s\tilde z - z_n^{i-1}\|_2^2\\ &\quad + \frac{\varepsilon}{2\tau_n^i}\|\nabla v_n^i - \nabla v_n^{i-1}\|_2^2 - \frac{\varepsilon}{2\tau_n^i}\|\nabla v_n^i + s\nabla\tilde u - \nabla v_n^{i-1}\|_2^2.\end{aligned} \tag{5.7}$$

Taking the derivative of (5.7) with respect to s at $s = 0$ we obtain (5.3). For $\tilde u = 0$, (5.3) is

$$\mathcal{H}(z_n^i + \tilde z - z_n^{i-1}) - \mathcal{H}(z_n^i - z_n^{i-1}) \geqslant \left\langle -\frac{\partial W}{\partial\theta}(z_n^i, \nabla v_n^i) - \frac{\varepsilon}{\tau_n^i}(z_n^i - z_n^{i-1}), \tilde z\right\rangle,$$

for every $\tilde z \in L^2(D;\mathbb{R}^m)$, i.e. (5.4). Taking $\tilde z = 0$ in (5.3) we obtain (5.5).

Conversely, (5.4) and (5.5) imply the minimality of (z_n^i, v_n^i), thanks to the strict convexity of the functional. □

Remark 5.3. (5.4) is equivalent to

$$\frac{1}{\tau_n^i}(z_n^i - z_n^{i-1}) = \partial\mathcal{H}_\varepsilon^*\left(-\frac{\partial W}{\partial\theta}(z_n^i, \nabla v_n^i)\right). \tag{5.8}$$

Indeed, since $\partial\mathcal{H}$ is positively homogeneous of degree 0, (5.4) can be rewritten as

$$-\frac{\partial W}{\partial\theta}(z_n^i, \nabla v_n^i) - \frac{\varepsilon}{\tau_n^i}(z_n^i - z_n^{i-1}) \in \partial\mathcal{H}\left(\frac{1}{\tau_n^i}(z_n^i - z_n^{i-1})\right),$$

which is equivalent to

$$-\frac{\partial W}{\partial\theta}(z_n^i, \nabla v_n^i) \in \partial\mathcal{H}_\varepsilon\left(\frac{1}{\tau_n^i}(z_n^i - z_n^{i-1})\right).$$

This is equivalent to (5.8), thanks to a general duality formula (see, e.g., [6, Corollary I.5.2]).

Since $\mathcal{A}(t_n^i) = \varphi_n^i + H^1_{\Gamma_0}(0)$, for every n and every $i = 0, 1, \dots, k(n)$, there exists $u_n^i \in H^1_{\Gamma_0}(0)$ such that $v_n^i = \varphi_n^i + u_n^i$.

Set, for every n and every $i = 0, 1, \dots, k(n)$,

$$\zeta_n^i := -\frac{\partial W}{\partial \theta}\big(z_n^i, \nabla u_n^i + \nabla \varphi_n^i\big),$$
$$\sigma_n^i := \frac{\partial W}{\partial F}\big(z_n^i, \nabla u_n^i + \nabla \varphi_n^i\big).$$

Let define the piecewise constant interpolations $(\boldsymbol{z}_n, \boldsymbol{u}_n) : [0, T] \to L^2(D; \mathbb{R}^m) \times H^1_{\Gamma_0}(0)$ and $(\boldsymbol{\zeta}_n, \boldsymbol{\sigma}_n) : [0, T] \to L^2(D; \mathbb{R}^m) \times L^2(D; \mathbb{R}^{N\times d})$ as

$$\boldsymbol{z}_n(t) := z_n^i, \quad \boldsymbol{u}_n(t) := u_n^i, \quad \boldsymbol{\zeta}_n(t) := \zeta_n^i, \quad \boldsymbol{\sigma}_n(t) := \sigma_n^i, \quad \text{for } t_n^i \leqslant t < t_n^{i+1},$$

where we set $t_n^{k(n)+1} = T + \frac{1}{n}$. Set $\tau_n(t) := t_n^i$ whenever $t_n^i \leqslant t < t_n^{i+1}$. We introduce also the piecewise affine interpolations $\boldsymbol{z}_n^{\triangle} : [0, T] \to L^2(D; \mathbb{R}^m)$, $\boldsymbol{u}_n^{\triangle} : [0, T] \to H^1_{\Gamma_0}(0)$, $\boldsymbol{\varphi}_n^{\triangle} : [0, T] \to H^1(D; \mathbb{R}^N)$, $\boldsymbol{l}_n^{\triangle} : [0, T] \to H^1(D; \mathbb{R}^N)^*$, defined by

$$\begin{aligned}
\boldsymbol{z}_n^{\triangle}(t) &:= z_n^i + \big(t - t_n^i\big)\frac{z_n^{i+1} - z_n^i}{t_n^{i+1} - t_n^i},\\
\boldsymbol{u}_n^{\triangle}(t) &:= u_n^i + \big(t - t_n^i\big)\frac{u_n^{i+1} - u_n^i}{t_n^{i+1} - t_n^i},\\
\boldsymbol{\varphi}_n^{\triangle}(t) &:= \varphi_n^i + \big(t - t_n^i\big)\frac{\varphi_n^{i+1} - \varphi_n^i}{t_n^{i+1} - t_n^i},\\
\boldsymbol{l}_n^{\triangle}(t) &:= l_n^i + \big(t - t_n^i\big)\frac{l_n^{i+1} - l_n^i}{t_n^{i+1} - t_n^i}
\end{aligned} \tag{5.9}$$

for $t_n^i \leqslant t \leqslant t_n^{i+1}$.

Observe that, thanks to Remark 5.3 and to (3.7), we can obtain from (5.4) that

$$\frac{z_n^i - z_n^{i-1}}{t_n^i - t_n^{i-1}} = N_K^{\varepsilon}\big(\zeta_n^i\big). \tag{5.10}$$

Analogously we can deduce from (5.5) that

$$\left\langle \sigma_n^i + \varepsilon\left(\frac{\nabla u_n^i - \nabla u_n^{i-1}}{t_n^i - t_n^{i+1}} + \frac{\nabla \varphi_n^i - \varphi_n^{i-1}}{t_n^i - t_n^{i-1}}\right), \nabla \tilde{u}\right\rangle = \big\langle l_n^{i-1}, \tilde{u}\big\rangle, \tag{5.11}$$

for every $\tilde{u} \in H^1_{\Gamma_0}(0)$.

5.2. A priori estimates

Now we obtain an a priori bound on the piecewise constant interpolations, from an energy estimate for the solutions of the incremental problems.

Since $u_n^{i-1} + \varphi_n^i \in \mathcal{A}(t_n^i)$, from the minimum property of $(z_n^i, u_n^i + \varphi_n^i)$ we deduce the following inequality:

$$\begin{aligned}
&\mathcal{W}\big(z_n^i, u_n^i + \varphi_n^i\big) - \big\langle l_n^i, u_n^i + \varphi_n^i\big\rangle + \mathcal{H}\big(z_n^i - z_n^{i-1}\big) + \frac{\varepsilon}{2\tau_n^i}\big\|z_n^i - z_n^{i-1}\big\|_2^2 + \frac{\varepsilon}{2\tau_n^i}\big\|\nabla u_n^i - \nabla u_n^{i-1} + \nabla\varphi_n^i - \nabla\varphi_n^{i-1}\big\|_2^2\\
&\quad\leqslant \mathcal{W}\big(z_n^{i-1}, u_n^{i-1} + \varphi_n^{i-1}\big) - \big\langle l_n^{i-1}, u_n^{i-1} + \varphi_n^{i-1}\big\rangle + \frac{\varepsilon}{2\tau_n^i}\big\|\nabla\varphi_n^i - \nabla\varphi_n^{i-1}\big\|_2^2 - \int_{t_n^{i-1}}^{t_n^i}\big\langle \dot{\boldsymbol{l}}(s), u_n^{i-1} + \boldsymbol{\varphi}(s)\big\rangle\, ds
\end{aligned}$$

$$- \int_{t_n^{i-1}}^{t_n^i} \langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s) \rangle \, ds + \int_{t_n^{i-1}}^{t_n^i} \left\langle \frac{\partial W}{\partial F}\big(z_n^{i-1}, \nabla u_n^{i-1} + \nabla \boldsymbol{\varphi}(s)\big), \nabla \dot{\boldsymbol{\varphi}}(s) \right\rangle ds. \tag{5.12}$$

Fixed $t \in [0, T]$, iterating (5.12) we obtain

$$\begin{aligned}
&\mathcal{W}\big(z_n(t), \boldsymbol{u}_n(t) + \boldsymbol{\varphi}\big(\tau_n(t)\big)\big) - \big\langle \boldsymbol{l}\big(\tau_n(t)\big), \boldsymbol{u}_n(t) + \boldsymbol{\varphi}\big(\tau_n(t)\big)\big\rangle + \mathrm{Var}_H(z_n; 0, t) \\
&\quad + \frac{\varepsilon}{2} \int_0^{\tau_n(t)} \big\| \dot{z}_n^{\triangle}(s) \big\|_2^2 \, ds + \frac{\varepsilon}{4} \int_0^{\tau_n(t)} \big\| \nabla \dot{\boldsymbol{u}}_n^{\triangle}(s) \big\|_2^2 \, ds \\
&\leqslant \mathcal{W}(z_0, v_0) - \big\langle \boldsymbol{l}(0), v_0 \big\rangle + \varepsilon \int_0^{\tau_n(t)} \big\| \nabla \dot{\boldsymbol{\varphi}}(s) \big\|_2^2 \, ds + \int_0^{\tau_n(t)} \left\langle \frac{\partial W}{\partial F}\big(z_n(s), \nabla \boldsymbol{u}_n(s) + \nabla \boldsymbol{\varphi}(s)\big), \nabla \dot{\boldsymbol{\varphi}}(s) \right\rangle ds \\
&\quad - \int_0^{\tau_n(t)} \big[\big\langle \dot{\boldsymbol{l}}(s), \boldsymbol{u}_n(s) + \boldsymbol{\varphi}(s) \big\rangle + \big\langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s) \big\rangle\big] \, ds,
\end{aligned} \tag{5.13}$$

where we have used the identity

$$\frac{\varepsilon}{\tau_n^i} \big\| \nabla \varphi_n^i - \nabla \varphi_n^{i-1} \big\|_2^2 = \frac{\varepsilon}{\tau_n^i} \left\| \int_{t_n^{i-1}}^{t_n^i} \nabla \dot{\boldsymbol{\varphi}}(t) \, dt \right\|_2^2, \tag{5.14}$$

for every $i = 1, \dots, k(n)$.

Using the fact that $\sup_{t\in[0,T]} \|\boldsymbol{l}(t)\|_{(H^1)^*}$, $\sup_{t\in[0,T]} \|\nabla \boldsymbol{\varphi}(t)\|_2$, $\int_0^T \|\dot{\boldsymbol{l}}(t)\|_{(H^1)^*} \, dt$ and $\int_0^T \|\nabla \dot{\boldsymbol{\varphi}}(t)\|_2 \, dt$ are bounded, the growth hypothesis on W, (W.1), and the fact that $z_n \in L^\infty([0, T]; L^2(D; \mathbb{R}^m))$, $\boldsymbol{u}_n \in L^\infty([0, T]; H^1(D; \mathbb{R}^N))$ (since they are piecewise constant functions), (5.13) leads to

$$\tilde{c}\big(\big\|z_n(t)\big\|_2 + \big\|\nabla \boldsymbol{u}_n(t)\big\|_2\big)^2 \leqslant \tilde{C} \sup_{s\in[0,T]} \big(1 + \big\|z_n(s)\big\|_2 + \big\|\nabla \boldsymbol{u}_n(s)\big\|_2\big),$$

for suitable positive constants $\tilde{c}$, $\tilde{C}$. Since this can be repeated for every $t \in [0, T]$, we can conclude that there exists a positive constant C_ε, depending on ε but independent of n, such that

$$\sup_{t\in[0,T]} \big\|z_n(t)\big\|_2 \leqslant C_\varepsilon, \qquad \mathrm{Var}_H(z_n; 0, T) \leqslant C_\varepsilon, \qquad \int_0^T \big\|\dot{z}_n^{\triangle}(t)\big\|_2^2 \, dt \leqslant C_\varepsilon; \tag{5.15}$$

$$\sup_{t\in[0,T]} \big\|\nabla \boldsymbol{u}_n(t)\big\|_2 \leqslant C_\varepsilon, \qquad \int_0^T \big\|\nabla \dot{\boldsymbol{u}}_n^{\triangle}(t)\big\|_2^2 \, dt \leqslant C_\varepsilon, \tag{5.16}$$

for every $t \in [0, T]$.

5.3. Passage to the limit

To establish the convergence of the interpolations we need the following lemma, based on Gronwall's inequality.

Lemma 5.4. *The sequences $(z_n)_n$, $(\boldsymbol{u}_n)_n$ satisfy*

$$\sup_{t\in[0,T]} \big\|z_n(t) - z_m(t)\big\|_2 \to 0,$$
$$\sup_{t\in[0,T]} \big\|\boldsymbol{u}_n(t) - \boldsymbol{u}_m(t)\big\|_{H^1} \to 0,$$

as $n, m \to \infty$.

Proof. From the construction of z_n^i, u_n^i, and using (5.10), (5.11), we can deduce that, for every $\tilde z \in L^2(D;\mathbb{R}^N)$, $\tilde u \in H^1_{\Gamma_0}(0)$,

$$\langle \nabla u_n^j + \nabla\varphi_n^j, \nabla\tilde u\rangle - \langle \nabla u_n^{j-1} + \nabla\varphi_n^{j-1}, \nabla\tilde u\rangle + \langle z_n^j, \tilde z\rangle - \langle z_n^{j-1}, \tilde z\rangle = -\frac{\tau_n^j}{\varepsilon}\big[\langle\sigma_n^j, \nabla\tilde u\rangle - \langle l_n^j, \tilde u\rangle\big] + \tau_n^j\langle N_K^\varepsilon(\zeta_n^j), \tilde z\rangle.$$

Fixed $t \in [0,T]$, for every n, there exists i such that $t_n^i \leqslant t < t_n^{i+1}$; summing for j from 1 to i we obtain

$$\begin{aligned}
&\langle z_n(t), \tilde z\rangle - \langle z_0, \tilde z\rangle + \langle \nabla \boldsymbol{u}_n(t) + \nabla\boldsymbol{\varphi}(\tau_n(t)), \nabla\tilde u\rangle - \langle \nabla u_0 + \nabla\boldsymbol{\varphi}(0), \nabla\tilde u\rangle \\
&\quad= \frac{1}{\varepsilon}\sum_{j=1}^{i}\tau_n^j\big[\varepsilon\langle N_K^\varepsilon(\zeta_n^{j-1}), \tilde z\rangle - \langle\sigma_n^{j-1}, \tilde u\rangle + \langle l_n^{j-1}, \tilde u\rangle + \varepsilon\langle N_K^\varepsilon(\zeta_n^j) - N_K^\varepsilon(\zeta_n^{j-1}), \tilde z\rangle \\
&\qquad - \langle \sigma_n^j - \sigma_n^{j-1}, \nabla\tilde u\rangle + \langle l_n^j - l_n^{j-1}, \tilde u\rangle\big] \\
&\quad= \frac{1}{\varepsilon}\int_0^t \big[\varepsilon\langle N_K^\varepsilon(\boldsymbol{\zeta}_n(s)), \tilde z\rangle - \langle\boldsymbol{\sigma}_n(s), \nabla\tilde u\rangle + \langle \boldsymbol{l}(\tau_n(s)), \tilde u\rangle\big]\,ds + R_n(t),
\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}
R_n(t) := &-\frac{1}{\varepsilon}\int_{\tau_n(t)}^{t} \big[\varepsilon\langle N_K^\varepsilon(\boldsymbol{\zeta}_n(s)), \tilde z\rangle - \langle\boldsymbol{\sigma}_n(s), \nabla\tilde u\rangle + \langle \boldsymbol{l}(\tau_n(s)), \tilde u\rangle\big]\,ds \\
&+ \frac{1}{\varepsilon}\sum_{j=1}^{i}\tau_n^j\big[\varepsilon\langle N_K^\varepsilon(\zeta_n^j) - N_K^\varepsilon(\zeta_n^{j-1}), \tilde z\rangle - \langle\sigma_n^j - \sigma_n^{j-1}, \nabla\tilde u\rangle + \langle l_n^j - l_n^{j-1}, \tilde u\rangle\big].
\end{aligned}$$

Observe that, since $\frac{\partial W}{\partial\theta}$, $\frac{\partial W}{\partial F}$ are M-Lipschitz thanks to (W.2),

$$\|\boldsymbol{\zeta}_n(s)\|_2 \leqslant \tilde M\big(1 + \|z_n(s)\|_2 + \|\nabla\boldsymbol{u}_n(s) + \nabla\boldsymbol{\varphi}(\tau_n(s))\|_2\big), \tag{5.18}$$
$$\|\boldsymbol{\sigma}_n(s)\|_2 \leqslant \tilde M\big(1 + \|z_n(s)\|_2 + \|\nabla\boldsymbol{u}_n(s) + \nabla\boldsymbol{\varphi}(\tau_n(s))\|_2\big), \tag{5.19}$$

for a suitable positive constant $\tilde M$ and for every $s \in [0,T]$. Hence from (5.15) and (5.16), we can deduce that

$$\int_{\tau_n(t)}^{t} \|\boldsymbol{\zeta}_n(s)\|_2\,ds \leqslant \tilde M\tilde C_\varepsilon\tau_n, \tag{5.20}$$
$$\int_{\tau_n(t)}^{t} \|\boldsymbol{\sigma}_n(s)\|_2\,ds \leqslant \tilde M\tilde C_\varepsilon\tau_n. \tag{5.21}$$

Since N_K^ε is $1/\varepsilon$-Lipschitz, and $\partial W/\partial\theta$, $\partial W/\partial F$ are M-Lipschitz, thanks to (5.20) and (5.21) we can estimate $R_n(t)$ in the following way

$$|R_n(t)| \leqslant \frac{1}{\varepsilon}\beta_\varepsilon\tau_n\big(\|\tilde z\|_2 + \|\nabla\tilde u\|_2\big), \tag{5.22}$$

for a suitable positive constant β_ε, depending on ε but independent of t and n.

Let n, m be two different indexes. Subtracting term by term the equations corresponding to (5.17), we obtain, for every $\tilde z \in L^2(D;\mathbb{R}^m)$ and $\tilde u \in H^1_{\Gamma_0}(0)$,

$$\begin{aligned}
&\langle z_n(t) - z_m(t), \tilde z\rangle + \langle \nabla\boldsymbol{u}_n(t) - \nabla\boldsymbol{u}_m(t) + \nabla\boldsymbol{\varphi}(\tau_n(t)) - \nabla\boldsymbol{\varphi}(\tau_m(t)), \nabla\tilde u\rangle \\
&\quad= \frac{1}{\varepsilon}\int_0^t \big[\varepsilon\langle N_K^\varepsilon(\boldsymbol{\zeta}_n(s)) - N_K^\varepsilon(\boldsymbol{\zeta}_m(s)), \tilde z\rangle - \langle\boldsymbol{\sigma}_n(s) - \boldsymbol{\sigma}_m(s), \nabla\tilde u\rangle\big]\,ds
\end{aligned}$$

$$+\int_0^t \langle \boldsymbol{l}(\tau_n(s)) - \boldsymbol{l}(\tau_m(s)), \tilde{u}\rangle \, ds + R_n(t) - R_m(t). \tag{5.23}$$

Now using again the fact that N_K^ε, $\partial W/\partial\theta$ and $\partial W/\partial F$ are Lipschitzian, and the estimate (5.22), we can deduce that

$$\begin{aligned}
&\langle z_n(t) - z_m(t), \tilde{z}\rangle + \langle \nabla \boldsymbol{u}_n(t) - \nabla \boldsymbol{u}_m(t) + \nabla\boldsymbol{\varphi}(\tau_n(t)) - \nabla\boldsymbol{\varphi}(\tau_m(t)), \nabla\tilde{u}\rangle \\
&\quad \leqslant \frac{\gamma_\varepsilon}{\varepsilon}\Bigg\{\int_0^t \big[\|z_n(s) - z_m(s)\|_2 - \|\nabla \boldsymbol{u}_n(s) - \nabla \boldsymbol{u}_m(s)\|_2 + \|\nabla\boldsymbol{\varphi}(\tau_n(s)) - \nabla\boldsymbol{\varphi}(\tau_m(s))\|_2 \\
&\qquad + \|\boldsymbol{l}_n(s) - \boldsymbol{l}_m(s)\|_{(H^1)^*}\big]\, ds + \beta_\varepsilon(\tau_n + \tau_m)\Bigg\}\big(\|\tilde{z}\|_2 + \|\nabla\tilde{u}\|_2\big),
\end{aligned} \tag{5.24}$$

for a suitable positive constant γ_ε.

Since $\boldsymbol{\varphi} \in H^1([0,T]; H^1(D;\mathbb{R}^N))$ and $\boldsymbol{l} \in H^1([0,T]; H^1(D;\mathbb{R}^N)^*)$, there exists a positive constant α, such that

$$\|\nabla\boldsymbol{\varphi}(t_1) - \nabla\boldsymbol{\varphi}(t_2)\|_2 \leqslant \alpha|t_1 - t_2|^{1/2}, \tag{5.25}$$

$$\|\boldsymbol{l}(t_1) - \boldsymbol{l}(t_2)\|_{(H^1)^*} \leqslant \alpha|t_1 - t_2|^{1/2}, \tag{5.26}$$

for every $t_1, t_2 \in [0,T]$.

It follows that

$$\|\nabla\boldsymbol{\varphi}(\tau_n(t)) - \nabla\boldsymbol{\varphi}(\tau_m(t))\|_2 \leqslant \alpha\big((\tau_n)^{1/2} + (\tau_m)^{1/2}\big), \tag{5.27}$$

$$\|\boldsymbol{l}(\tau_n(t)) - \boldsymbol{l}(\tau_m(t))\|_{(H^1)^*} \leqslant \alpha\big((\tau_n)^{1/2} + (\tau_m)^{1/2}\big), \tag{5.28}$$

for every $t \in [0,T]$ and every n, m.

If we choose $\tilde{z} = z_n(t) - z_m(t)$ and $\tilde{u} = \boldsymbol{u}_n(t) - \boldsymbol{u}_m(t)$, taking into account (5.27) and (5.28), (5.24) gives

$$\begin{aligned}
&\|z_n(t) - z_m(t)\|_2 + \|\nabla \boldsymbol{u}_n(t) - \nabla \boldsymbol{u}_m(t)\|_2 \\
&\quad \leqslant \frac{\gamma_\varepsilon}{\varepsilon}\Bigg\{\int_0^t \big[\|z_n(s) - z_m(s)\|_2 + \|\nabla \boldsymbol{u}_n(s) - \nabla \boldsymbol{u}_m(s)\|_2\big]\, ds + \tilde{\alpha}\big((\tau_n)^{1/2} + (\tau_m)^{1/2}\big)\Bigg\},
\end{aligned}$$

for a suitable positive constant $\tilde{\alpha}$ independent of t, m and n.

Applying Gronwall's inequality we conclude that

$$\sup_{t\in[0,T]} \|z_n(t) - z_m(t)\|_2 \to 0,$$

$$\sup_{t\in[0,T]} \|\nabla \boldsymbol{u}_n(t) - \nabla \boldsymbol{u}_m(t)\|_2 \to 0,$$

for n, m tending to ∞. Since $\boldsymbol{u}_n(t) - \boldsymbol{u}_m(t) \in H^1_{\Gamma_0}(0)$, applying Poincaré inequality we obtain

$$\sup_{t\in[0,T]} \|\boldsymbol{u}_n(t) - \boldsymbol{u}_m(t)\|_{H^1} \to 0 \tag{5.29}$$

as n, m tend to ∞. □

From Lemma 5.4, we can deduce that there exist

$$z : [0,T] \to L^2(D;\mathbb{R}^m),$$

$$\boldsymbol{u} : [0,T] \to H^1(D;\mathbb{R}^N),$$

bounded, such that

$$\sup_{t\in[0,T]} \|z_n(t) - z(t)\|_2 \to 0, \tag{5.30}$$

$$\sup_{t\in[0,T]} \|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_{H^1} \to 0. \tag{5.31}$$

Moreover $\boldsymbol{u}(t) \in H^1_{\Gamma_0}(0)$, for every $t \in [0, T]$.

Set

$$\boldsymbol{\zeta}(t) := -\frac{\partial W}{\partial \theta}\big(z(t), \nabla\boldsymbol{u}(t) + \nabla\boldsymbol{\varphi}(t)\big), \tag{5.32}$$

$$\boldsymbol{\sigma}(t) := \frac{\partial W}{\partial F}\big(z(t), \nabla\boldsymbol{u}(t) + \nabla\boldsymbol{\varphi}(t)\big). \tag{5.33}$$

Thanks to (5.25) and to the convergence of z_n, $\boldsymbol{u}_n$, we have

$$\sup_{t\in[0,T]} \|\boldsymbol{\zeta}_n(t) - \boldsymbol{\zeta}(t)\|_2 \to 0, \tag{5.34}$$

$$\sup_{t\in[0,T]} \|\boldsymbol{\sigma}_n(t) - \boldsymbol{\sigma}(t)\|_2 \to 0, \tag{5.35}$$

as n tends to ∞.

Thanks to (5.15) and (5.16), we have that $(z_n^\triangle)_n$ and $(\boldsymbol{u}_n^\triangle)_n$ are bounded sequences in $H^1([0,T]; L^2(D;\mathbb{R}^m))$ and $H^1([0,T]; H^1(D;\mathbb{R}^N))$, respectively; hence there exist $\hat{z}$, $\hat{\boldsymbol{u}}$ such that, up to subsequences, $z_n^\triangle \rightharpoonup \hat{z}$ and $\boldsymbol{u}_n^\triangle \rightharpoonup \hat{\boldsymbol{u}}$ weakly in $H^1([0,T]; L^2(D;\mathbb{R}^m))$ and $H^1([0,T]; H^1(D;\mathbb{R}^N))$, respectively.

Moreover using the identities

$$z_n^\triangle(t) = z_n(t) + \int_{\tau_n(t)}^{t} \dot{z}_n^\triangle(s)\,ds,$$

$$\nabla\boldsymbol{u}_n^\triangle(t) = \nabla\boldsymbol{u}_n(t) + \int_{\tau_n(t))}^{t} \nabla\dot{\boldsymbol{u}}_n^\triangle(s)\,ds,$$

for every $t \in [0, T]$, we deduce that

$$\sup_{t\in[0,T]} \|z_n^\triangle(t) - z_n(t)\|_2 \to 0,$$

$$\sup_{t\in[0,T]} \|\nabla\boldsymbol{u}_n^\triangle(t) - \nabla\boldsymbol{u}_n(t)\|_2 \to 0.$$

Hence we can conclude that $\hat{z} = z$, $\nabla\hat{\boldsymbol{u}} = \nabla\boldsymbol{u}$ and the whole sequences $z_n^\triangle$ and $\boldsymbol{u}_n^\triangle$ satisfy

$$z_n^\triangle \rightharpoonup z \quad \text{weakly in } H^1\big([0,T]; L^2\big(D;\mathbb{R}^m\big)\big), \tag{5.36}$$

$$\boldsymbol{u}_n^\triangle \rightharpoonup \boldsymbol{u} \quad \text{weakly in } H^1\big([0,T]; H^1\big(D;\mathbb{R}^N\big)\big). \tag{5.37}$$

It is immediate to see that $(\mathrm{ev0})_\varepsilon$ follows from the construction of $(z_n, \boldsymbol{v}_n)$ and from (5.30), (5.31).

Since $\boldsymbol{u}(t) \in H^1_{\Gamma_0}(0)$ for every $t \in [0, T]$ also $(\mathrm{ev1})_\varepsilon$ is immediate.

We prove now $(\mathrm{ev2})_\varepsilon$. From the construction of $\boldsymbol{u}_n^\triangle$ and (5.11), it follows that

$$\big\langle \boldsymbol{\sigma}_n(t) + R_n^u(t) + \varepsilon\big(\nabla\dot{\boldsymbol{u}}_n^\triangle(t) + \nabla\dot{\boldsymbol{\varphi}}_n^\triangle(t)\big), \nabla\tilde{u}\big\rangle = \big\langle \boldsymbol{l}\big(\tau_n(t)\big) + R_n^l(t), \tilde{u}\big\rangle, \tag{5.38}$$

for every $\tilde{u} \in H^1_{\Gamma_0}(0)$, where $R_n^u(t) := \sigma_n^{i+1} - \sigma_n^i$ and $R_n^l(t) := l_n^{i+1} - l_n^i$, for $t_n^i < t < t_n^{i+1}$.

Thanks to (5.26), $\sup_{t\in[0,T]} \|R_n^l(t)\|_{(H^1)^*} \to 0$. Using the fact that N_K^ε is $1/\varepsilon$-Lipschitz, (W.2), the hypothesis on $\boldsymbol{\varphi}$, and (5.15), (5.16), we deduce that $\sup_{t\in[0,T]} \|R_n^u(t)\|_2 \to 0$.

From (5.14) we deduce that $\int_0^T \|\nabla\dot{\boldsymbol{\varphi}}_n^\triangle(t)\|_2^2\,dt$ is uniformly bounded with respect to n and then $\nabla\dot{\boldsymbol{\varphi}}_n^\triangle \rightharpoonup \nabla\dot{\boldsymbol{\varphi}}$ weakly in $L^2([0,T]; L^2(D;\mathbb{R}^{N\times d}))$.

Thus $t \mapsto \boldsymbol{\sigma}_n(t) + R_n^u(t) + \varepsilon(\nabla \dot{\boldsymbol{u}}_n^{\triangle}(t) + \nabla \dot{\boldsymbol{\varphi}}_n^{\triangle}(t))$ weakly converges in $L^2([0,T]; L^2(D; \mathbb{R}^{N\times d}))$ to $t \mapsto \boldsymbol{\sigma}(t) + \varepsilon(\nabla \dot{\boldsymbol{u}}(t) + \nabla \dot{\boldsymbol{\varphi}}(t))$; $t \mapsto \boldsymbol{l}(\tau_n(t)) + R_n^l(t)$ strongly converges in $L^2([0,T]; H^1(D; \mathbb{R}^N)^*)$ to $\boldsymbol{l}(t)$ as $n \to +\infty$.

Therefore from (5.38) we can obtain $(\mathrm{ev2})_\varepsilon$.

Finally we prove $(\mathrm{ev3})_\varepsilon$. From the construction of $\boldsymbol{z}_n^{\triangle}$ and (5.10), it follows that

$$\dot{\boldsymbol{z}}_n^{\triangle}(t) = N_K^\varepsilon\big(\boldsymbol{\zeta}_n(t)\big) + R_n^z(t) \quad \text{a.e. in } D,$$

where $R_n^z(t) := N_K^\varepsilon(\zeta_n^{i+1}) - N_K^\varepsilon(\zeta_n^i)$, for $t_n^i < t < t_n^{i+1}$.

Repeating the previous argument we deduce that $\sup_{t\in[0,T]} \|R_n^z(t)\|_2 \to 0$, as $n \to +\infty$, so that, taking into account (5.34), we conclude that

$$\sup_{t\in[0,T]} \big\|\dot{\boldsymbol{z}}_n^{\triangle}(t) - N_K^\varepsilon\big(\boldsymbol{\zeta}(t)\big)\big\|_2 \to 0. \tag{5.39}$$

In particular this implies that $\dot{\boldsymbol{z}}_n^{\triangle}$ converges strongly in $L^\infty([0,T]; L^2(D; \mathbb{R}^m))$ and the limit must coincide with $\dot{\boldsymbol{z}}$, thanks to (5.36); hence from (5.39) we obtain

$$\dot{\boldsymbol{z}}(t) = N_K^\varepsilon\big(\boldsymbol{\zeta}(t)\big), \quad \text{a.e. in } D,$$

for a.e. $t \in [0,T]$.

5.4. Uniqueness

It remains to show that the solution of the ε-regularized problem is unique.

Let $(\boldsymbol{z}_1, \boldsymbol{v}_1)$, $(\boldsymbol{z}_2, \boldsymbol{v}_2)$ be two solutions of the ε-regularized problem in the time interval $[0,T]$ with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0, v_0), and set

$$\boldsymbol{\zeta}_i(t) := -\frac{\partial W}{\partial \theta}\big(\boldsymbol{z}_i(t), \nabla \boldsymbol{v}_i(t)\big),$$
$$\boldsymbol{\sigma}_i(t) := \frac{\partial W}{\partial F}\big(\boldsymbol{z}_i(t), \nabla \boldsymbol{v}_i(t)\big),$$

for $i = 1, 2$.

In particular the following equations hold for a.e. $t \in [0,T]$:

$$\dot{\boldsymbol{z}}_i(t) = N_K^\varepsilon\big(\boldsymbol{\zeta}_i(t)\big) \quad \text{a.e. in } D,$$
$$\big\langle \boldsymbol{\sigma}_i(t) + \varepsilon \nabla \dot{\boldsymbol{v}}_i(t), \nabla \tilde{u}\big\rangle = \big\langle \boldsymbol{l}(t), \tilde{u}\big\rangle \quad \text{for every } \tilde{u} \in H^1_{\Gamma_0}(0),$$

for $i = 1, 2$.

Hence, for a.e. $t \in [0,T]$, for every $\tilde{z} \in L^2(D; \mathbb{R}^m)$, and every $\tilde{u} \in H^1_{\Gamma_0}(0)$, we have

$$\big\langle \nabla \dot{\boldsymbol{v}}_1(t) - \nabla \dot{\boldsymbol{v}}_2(t), \nabla \tilde{u}\big\rangle + \big\langle \dot{\boldsymbol{z}}_1(t) - \dot{\boldsymbol{z}}_2(t), \tilde{z}\big\rangle = -\frac{1}{\varepsilon}\big\langle \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t), \nabla \tilde{u}\big\rangle + \big\langle N_K^\varepsilon\big(\boldsymbol{\zeta}_1(t)\big) - N_K^\varepsilon\big(\boldsymbol{\zeta}_2(t)\big), \tilde{z}\big\rangle.$$

Therefore, by integration and $(\mathrm{ev0})_\varepsilon$, we obtain

$$\begin{aligned}&\big\langle \nabla \boldsymbol{v}_1(t) - \nabla \boldsymbol{v}_2(t), \nabla \tilde{u}\big\rangle + \big\langle \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t), \tilde{z}\big\rangle \\ &\quad = \int_0^t \Big[-\frac{1}{\varepsilon}\big\langle \boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s), \nabla \tilde{u}\big\rangle + \big\langle N_K^\varepsilon\big(\boldsymbol{\zeta}_1(s)\big) - N_K^\varepsilon\big(\boldsymbol{\zeta}_2(s)\big), \tilde{z}\big\rangle\Big]\,ds.\end{aligned} \tag{5.40}$$

We observe that $\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t) \in H^1_{\Gamma_0}(0)$, for every $t \in [0,T]$. Hence we can take $\tilde{z} = \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)$, $\tilde{u} = \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)$, and we derive from (5.40) the following estimate:

$$\big\|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\big\|_2 + \big\|\nabla \boldsymbol{v}_1(t) - \nabla \boldsymbol{v}_2(t)\big\|_2 \leqslant \frac{M'}{\varepsilon}\int_0^t \big[\big(\big\|\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)\big\|_2 + \big\|\nabla \boldsymbol{v}_1(s) - \nabla \boldsymbol{v}_2(s)\big\|_2\big)\big]\,ds,$$

for a suitable positive constant M' and for a.e. $t \in [0, T]$.

Hence Gronwall's inequality guarantees that $\boldsymbol{z}_1(t) = \boldsymbol{z}_2(t)$ and $\boldsymbol{v}_1(t) = \boldsymbol{v}_2(t)$, for a.e. $t \in [0, T]$; since, for $i = 1, 2$, $\boldsymbol{z}_i$ and $\boldsymbol{v}_i$ are absolutely continuous functions from $[0, T]$ into $L^2(D; \mathbb{R}^m)$ and $H^1(D; \mathbb{R}^N)$, respectively, we have the thesis.

6. Some properties of the solutions of the regularized problems

In this section we want to point out some useful properties satisfied by the solutions of the ε-regularized problems.

Remark 6.1. In the special case of $\boldsymbol{l} \equiv 0$, $\Gamma_0 = \partial\Omega$, $\boldsymbol{\varphi}(t, x) = F(t)x$, for $F(t) \in H^1([0, T]; \mathbb{R}^{N\times d})$, for every $t \in [0, T]$ and a.e. $x \in D$, and $v_0(x) = F(0)x$, $z_0 \equiv \theta_0 \in \mathbb{R}^m$, the solution $(\boldsymbol{v}_\varepsilon, \boldsymbol{z}_\varepsilon)$ of the ε-regularized problems satisfies the following properties:

(1) $\boldsymbol{v}_\varepsilon = \boldsymbol{\varphi}$,
(2) $x \mapsto \boldsymbol{z}_\varepsilon(t, x)$ is a.e. constant on D, for a.e. $t \in [0, T]$.

Indeed the Cauchy problem

$$\begin{cases} \dot{\theta}_\varepsilon(t) = N_K^\varepsilon\left(-\dfrac{\partial W}{\partial \theta}\big(\theta_\varepsilon(t), F(t)\big)\right), \\ \theta_\varepsilon(0) = \theta_0 \end{cases}$$

has a unique solution $\theta_\varepsilon : [0, T] \to \mathbb{R}^m$, since the right-hand side is Lipschitz.

The function $(\boldsymbol{z}_\varepsilon(t), \boldsymbol{v}_\varepsilon(t)) = (\theta_\varepsilon(t), \boldsymbol{\varphi}(t))$ satisfies conditions $(\mathrm{ev0})_\varepsilon$, $(\mathrm{ev1})_\varepsilon$, $(\mathrm{ev2})_\varepsilon$, $(\mathrm{ev3})_\varepsilon$, and $(\mathrm{ev4})_\varepsilon$, hence by uniqueness it is the solution of the ε-regularized problem.

Using the energy equality, we can prove the following bounds on the solution of the ε-regularized problems.

Lemma 6.2. *Let $\boldsymbol{\varphi}$, $\boldsymbol{l}$, z_0, v_0, and $T > 0$ be as in Definition* 4.1. *Then there exists a positive constant C', independent of ε, such that*

$$\sup_{t\in[0,T]} \|\boldsymbol{z}_\varepsilon(t)\|_2 \leqslant C', \qquad \mathrm{Var}_H(\boldsymbol{z}_\varepsilon; 0, T) \leqslant C', \qquad \varepsilon \int_0^T \|\dot{\boldsymbol{z}}_\varepsilon(s)\|_2^2\, ds \leqslant C', \tag{6.1}$$

$$\sup_{t\in[0,T]} \|\nabla \boldsymbol{v}_\varepsilon(t)\|_2 \leqslant C', \qquad \varepsilon \int_0^T \|\nabla \dot{\boldsymbol{v}}_\varepsilon(s)\|_2^2\, ds \leqslant C'. \tag{6.2}$$

Proof. The proof can be obtained from the energy equality for $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$ reasoning as in the second step of the proof of Theorem 4.6. □

Remark 6.3. From (6.1) and (6.2) we can deduce that, for every sequence $\varepsilon_k \to 0$, we have $\varepsilon_k \dot{\boldsymbol{z}}_{\varepsilon_k} \to 0$ and $\varepsilon_k \nabla \dot{\boldsymbol{v}}_{\varepsilon_k} \to 0$ strongly in $L^2([0, T]; L^2(D; \mathbb{R}^m))$ and $L^2([0, T]; L^2(D; \mathbb{R}^{N\times d}))$, respectively. In particular

$$\varepsilon_k \dot{\boldsymbol{z}}_{\varepsilon_k}(t) \to 0 \quad \text{strongly in } L^2\big(D; \mathbb{R}^m\big), \tag{6.3}$$

$$\varepsilon_k \nabla \dot{\boldsymbol{v}}_{\varepsilon_k}(t) \to 0 \quad \text{strongly in } L^2\big(D; \mathbb{R}^{N\times d}\big), \tag{6.4}$$

for a.e. $t \in [0, T]$.

7. Approximable quasistatic evolution

In this section we give the definition of approximable quasistatic evolution in terms both of stochastic processes and of compatible systems of Young measures. We prove an existence result and that this evolution satisfies suitable properties of equilibrium, dual constraint and an energy inequality, so that it can be considered as a solution of our evolution problem.

7.1. Approximable quasistatic evolution in terms of stochastic processes

Here we give the definition using a probabilistic language.

A probability space of the form $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$, where $\mathcal{B}(D)$ is the Borel σ-algebra on D, $(\Omega, \mathcal{F})$ is a measurable space, and P a probability measure on $\mathcal{B}(D) \otimes \mathcal{F}$ satisfying $\pi_D(P) = \mathcal{L}^d$, will be called $(D, \mathcal{L}^d)$-*probability space*.

Definition 7.1. A stochastic process $(\boldsymbol{X}_t)_{t\in[0,T]}$ defined on a $(D, \mathcal{L}^d)$-probability space $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$ is said to be 2-*weakly* left continuous* if for every finite sequence $t_1 < \cdots < t_n$ in $[0, T]$ we have

$$(\pi_D, \boldsymbol{X}_{s_1^j}, \ldots, \boldsymbol{X}_{s_n^j})(P) \rightharpoonup (\pi_D, \boldsymbol{X}_{t_1}, \ldots, \boldsymbol{X}_{t_n})(P)$$

as $j \to \infty$, whenever $s_i^j \to t_i$ and $s_i^j \leqslant t_i$ for $i = 1, \ldots, n$.

Definition 7.2. Given a subset Θ of $[0, T]$ satisfying $\mathcal{L}^1([0, T] \setminus \Theta) = 0$, a stochastic process $(\boldsymbol{X}_t)_{t\in[0,T]}$ defined on a $(D, \mathcal{L}^d)$-probability space $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$ is said to be Θ-2-*weakly* approximable from the left* if for every $t \in [0, T] \setminus \Theta$ there exists a sequence s^j in Θ converging to t, with $s^j \leqslant t$ and

$$(\pi_D, \boldsymbol{X}_{s^j})(P) \rightharpoonup (\pi_D, \boldsymbol{X}_t)(P) \quad \text{2-weakly*} \tag{7.1}$$

as $j \to \infty$.

Remark 7.3. Note that the notion of 2-weakly* left continuity is much stronger than Θ-2-weakly* approximability from the left: indeed the first one requires that the convergence condition is satisfied not only for a single time but for every finite sequence of times, and does not depend on the choice of the sequence s_i^j approximating t_i.

Definition 7.4. Given a boundary datum $\boldsymbol{\varphi} \in H^1([0, T]; H^1(D; \mathbb{R}^N))$, an external load $\boldsymbol{l} \in H^1([0, T]; H^1(D; \mathbb{R}^N)^*)$, an initial condition $(z_0, v_0) \in L^2(D; \mathbb{R}^m) \times \mathcal{A}(0)$ satisfying (4.1) and (4.2), and $T > 0$, an *approximable quasistatic evolution* of stochastic processes in the time interval $[0, T]$ is a pair of stochastic processes $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ on a $(D, \mathcal{L}^d)$-probability space $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$, with $\boldsymbol{Z}_t \in L^2(D \times \Omega; \mathbb{R}^m)$ and 2-weakly* left continuous and $\boldsymbol{Y}_t \in L^2(D \times \Omega; \mathbb{R}^{N\times d})$, for which there exist a positive sequence $\varepsilon_k \to 0$ and a subset Θ of $[0, T]$ with $0 \in \Theta$ and $\mathcal{L}^1([0, T] \setminus \Theta) = 0$, such that the solutions $(\boldsymbol{z}_{\varepsilon_k}, \boldsymbol{v}_{\varepsilon_k})$ of the ε_k-regularized problems satisfy the following conditions:

(a) for every finite sequence $t_1 < \cdots < t_n$ in Θ, we have

$$\big(\pi_D, \boldsymbol{z}_{\varepsilon_k}(t_1), \ldots, \boldsymbol{z}_{\varepsilon_k}(t_n)\big)(P) \rightharpoonup (\pi_D, \boldsymbol{Z}_{t_1}, \ldots, \boldsymbol{Z}_{t_n})(P) \quad \text{2-weakly*}$$

as $k \to \infty$;

(b) for every $t \in \Theta$, there exists a subsequence $(\varepsilon_{k_j^t})_j$ of $(\varepsilon_k)_k$, possibly depending on t, with

$$\big(\pi_D, \boldsymbol{z}_{\varepsilon_{k_j^t}}(t), \nabla \boldsymbol{v}_{\varepsilon_{k_j^t}}(t)\big)(P) \rightharpoonup (\pi_D, \boldsymbol{Z}_t, \boldsymbol{Y}_t)(P) \quad \text{2-weakly*},$$

as $j \to \infty$ and

$$\limsup_{\varepsilon_k}\big[\langle \boldsymbol{\sigma}_{\varepsilon_k}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \boldsymbol{l}(t), \boldsymbol{v}_{\varepsilon_k}(t)\rangle\big] = \lim_{\varepsilon_{k_j^t}}\big[\langle \boldsymbol{\sigma}_{\varepsilon_{k_j^t}}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \boldsymbol{l}(t), \boldsymbol{v}_{\varepsilon_{k_j^t}}(t)\rangle\big];$$

(c) $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is Θ-2-weakly* approximable from the left, for every $t \in \Theta$ (6.3) and (6.4) hold, and for every $t \in \Theta \setminus 0$ (4.3) and (4.4) hold for every ε_k.

In Theorem 7.13 we will prove that the evolution defined in this way satisfies properties (1.6), (1.7), and (1.8). Since the proof will be given using the language of Young measures, we translate the previous definition in terms of Young measures.

7.2. Approximable quasistatic evolution in terms of Young measures

The definition of approximable quasistatic evolution is now presented in terms of Young measures.

We recall that a compatible system $\boldsymbol{\mu} \in SY^2([0,T], D; \mathbb{R}^M)$ is said to be *left continuous* if for every finite sequence $t_1 < \cdots < t_n$ in $[0,T]$

$$\boldsymbol{\mu}_{s_1^j \dots s_n^j} \rightharpoonup \boldsymbol{\mu}_{t_1 \dots t_n} \quad \text{2-weakly*}$$

as $j \to \infty$, whenever $s_i^j \to t_i$ and $s_i^j \leqslant t_i$ for $i = 1, \dots, n$. We denote the set of such compatible systems by $SY^2_-([0,T], D; \mathbb{R}^M)$.

Definition 7.5. Given a subset Θ of $[0,T]$ with $\mathcal{L}^1([0,T] \setminus \Theta) = 0$, a family of Young measures $\boldsymbol{\nu} \in Y^2(D; \mathbb{R}^M)^{[0,T]}$ is said to be Θ-2-*weakly* approximable from the left* if for every $t \in [0,T] \setminus \Theta$ there exists a sequence s^j in Θ converging to t, with $s^j \leqslant t$, such that

$$\boldsymbol{\nu}_{s^j} \rightharpoonup \boldsymbol{\nu}_t \quad \text{2-weakly*} \tag{7.2}$$

as $j \to \infty$.

Definition 7.6. Given a boundary datum $\boldsymbol{\varphi} \in H^1([0,T]; H^1(D; \mathbb{R}^N))$, an external load $\boldsymbol{l} \in H^1([0,T]; H^1(D; \mathbb{R}^N)^*)$, an initial condition $(z_0, v_0) \in L^2(D; \mathbb{R}^m) \times \mathcal{A}(0)$ satisfying (4.1) and (4.2), and $T > 0$, an *approximable quasistatic evolution* of Young measures in the time interval $[0,T]$ is a pair $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N \times d})^{[0,T]} \times SY^2_-([0,T], D; \mathbb{R}^m)$, for which there exist a positive sequence $\varepsilon_k \to 0$ and a subset Θ of $[0,T]$ with $0 \in \Theta$ and $\mathcal{L}^1([0,T] \setminus \Theta) = 0$, such that the solutions $(\boldsymbol{z}_{\varepsilon_k}, \boldsymbol{v}_{\varepsilon_k})$ of the ε_k-regularized problems satisfy the following conditions:

(a) for every finite sequence $t_1 < \cdots < t_n$ in Θ we have

$$\delta_{(z_{\varepsilon_k}(t_1), \dots, z_{\varepsilon_k}(t_n))} \rightharpoonup \boldsymbol{\mu}_{t_1 \dots t_n} \quad \text{2-weakly*},$$

as $k \to \infty$;

(b) for every $t \in \Theta$, there exists a subsequence $(\varepsilon_{k_j^t})_j$ of $(\varepsilon_k)_k$, possibly depending on t, with

$$\delta_{(z_{\varepsilon_{k_j^t}}(t), \nabla \boldsymbol{v}_{\varepsilon_{k_j^t}}(t))} \rightharpoonup \boldsymbol{\nu}_t \quad \text{2-weakly*}, \tag{7.3}$$

as $j \to \infty$ and

$$\limsup_{\varepsilon_k} \big[\langle \boldsymbol{\sigma}_{\varepsilon_k}(t), \nabla \dot{\boldsymbol{\varphi}}(t) \rangle - \langle \boldsymbol{l}(t), \boldsymbol{v}_{\varepsilon_k}(t) \rangle \big] = \lim_{\varepsilon_{k_j^t}} \big[\langle \boldsymbol{\sigma}_{\varepsilon_{k_j^t}}(t), \nabla \dot{\boldsymbol{\varphi}}(t) \rangle - \langle \boldsymbol{l}(t), \boldsymbol{v}_{\varepsilon_{k_j^t}}(t) \rangle \big]; \tag{7.4}$$

(c) $\boldsymbol{\nu}$ is Θ-2-approximable from the left, for every $t \in \Theta$ (6.3) and (6.4) hold, and for every $t \in \Theta \setminus 0$ (4.3) and (4.4) hold for every ε_k.

In the next subsection we will show that an evolution defined in this way, besides fulfilling the selection criterion mentioned in the Introduction, satisfies also conditions (1′) and (2′) suitably reformulated in terms of Young measures. In particular the technical condition (7.4) will be crucial to apply the argument in [4, Section 7].

Before stating this result, we clarify in which sense the notions of evolution given in terms of stochastic processes and in terms of Young measures are equivalent, and we make some technical remarks which will be useful in the proof of the main theorem.

Remark 7.7. If $(\boldsymbol{\nu}, \boldsymbol{\mu})$ is an approximable quasistatic evolution, then $\pi_{D \times \mathbb{R}^m}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t$, for every $t \in [0,T]$. Indeed if $t \in \Theta$, we have $\delta_{z_{\varepsilon_{k_j^t}}} \rightharpoonup \boldsymbol{\mu}_t$ and $\delta_{(z_{\varepsilon_{k_j^t}}, \nabla \boldsymbol{v}_{\varepsilon_{k_j^t}})} \rightharpoonup \boldsymbol{\nu}_t$ 2-weakly*; in particular $\pi_{D \times \mathbb{R}^m}(\delta_{(z_{\varepsilon_{k_j^t}}, \nabla \boldsymbol{v}_{\varepsilon_{k_j^t}})}) = \delta_{z_{\varepsilon_{k_j^t}}} \rightharpoonup \pi_{D \times \mathbb{R}^m}(\boldsymbol{\nu}_t)$ 2-weakly* and this prove the claim for $t \in \Theta$. Let now $t \in [0,T] \setminus \Theta$, and let $s^j \to t$ a sequence satisfying (7.2); we have $\boldsymbol{\nu}_{s^j} \rightharpoonup \boldsymbol{\nu}_t$ 2-weakly* and $\boldsymbol{\mu}_{s^j} \rightharpoonup \boldsymbol{\mu}_t$ 2-weakly* by left continuity of $\boldsymbol{\mu}$, but $\pi_{D \times \mathbb{R}^m}(\boldsymbol{\nu}_{s^j}) = \boldsymbol{\mu}_{s^j}$ for every j, hence we have the thesis for every $t \in [0,T]$.

Remark 7.8. If $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is an approximable quasistatic evolution of stochastic processes, the pair $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]} \times SY^2_-([0,T], D; \mathbb{R}^m)$ defined by

$$\boldsymbol{\nu}_t := (\pi_D, \boldsymbol{Z}_t, \boldsymbol{Y}_t)(P) \quad \text{for every } t \in [0,T],$$
$$\boldsymbol{\mu}_{t_1\dots t_n} := (\pi_D, \boldsymbol{Z}_{t_1}, \dots, \boldsymbol{Z}_{t_n})(P) \quad \text{for every finite sequence } t_1 < \cdots < t_n \text{ in } [0,T],$$

is an approximable quasistatic evolution of Young measures.

On the other side, thanks to Remark 7.7 and Theorem 2.1, given an approximable quasistatic evolution of Young measures $(\boldsymbol{\nu}, \boldsymbol{\mu})$ there exists a stochastic process $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ such that

$$(\pi_D, \boldsymbol{Z}_t, \boldsymbol{Y}_t)(P) = \boldsymbol{\nu}_t \quad \text{for every } t \in [0,T],$$
$$(\pi_D, \boldsymbol{Z}_{t_1}, \dots, \boldsymbol{Z}_{t_n})(P) = \boldsymbol{\mu}_{t_1\dots t_n} \quad \text{for every finite sequence } t_1 < \cdots < t_n \text{ in } [0,T];$$

in particular $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is an approximable quasistatic evolution of stochastic processes.

Remark 7.9. If $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]} \times SY^2_-([0,T], D; \mathbb{R}^m)$ is an approximable quasistatic evolution of Young measures, for every $t \in [0,T]$ there exists a unique function $\boldsymbol{v}(t) \in \mathcal{A}(t)$ such that $\nabla \boldsymbol{v}(t) = \mathrm{bar}(\pi_{D\times\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t))$, where $\mathrm{bar}(\pi_{D\times\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t))$ denotes the barycentre of the Young measure $\pi_{D\times\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t)$. Indeed, if $t \in \Theta$ this follows from condition (b) of Definition 7.6 and [7, Lemma 4.9]; if $t \in [0,T] \setminus \Theta$ we observe that if s^j is a sequence satisfying (7.2), then

$$\tilde{\mathcal{T}}^2_{\nabla\boldsymbol{\varphi}(t)-\nabla\boldsymbol{\varphi}(s^j)}(\boldsymbol{\nu}_{s^j}) \rightharpoonup \boldsymbol{\nu}_t \quad \text{2-weakly*};$$

hence we can use again [7, Lemma 4.9] to obtain the thesis.

Translating the previous remark in terms of stochastic processes we obtain the following

Remark 7.10. If $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is an approximable quasistatic evolution of stochastic processes, for every $t \in [0,T]$ there exists a unique function $\boldsymbol{v}(t) \in \mathcal{A}(t)$ such that $\nabla \boldsymbol{v}(t) = \mathrm{bar}((\pi_D, \boldsymbol{Y}_t)(P))$.

Remark 7.11. If $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]} \times SY^2_-([0,T], D; \mathbb{R}^m)$ is an approximable quasistatic evolution of Young measures, for every $t \in [0,T]$ we define

$$\boldsymbol{\sigma}(t,x) := \int_{\mathbb{R}^m\times\mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\theta, F)\, d\boldsymbol{\nu}_t^x(\theta, F), \tag{7.5}$$

$$\boldsymbol{\zeta}(t,x) := \int_{\mathbb{R}^m\times\mathbb{R}^{N\times d}} -\frac{\partial W}{\partial \theta}(\theta, F)\, d\boldsymbol{\nu}_t^x(\theta, F), \tag{7.6}$$

for a.e. $x \in D$. For every $t \in [0,T]$ we have that $\boldsymbol{\sigma}(t) \in L^2(D; \mathbb{R}^{N\times d})$ and $\boldsymbol{\zeta}(t) \in L^2(D; \mathbb{R}^m)$: this comes immediately from (W.2), from $\pi_D(\boldsymbol{\nu}_t) = \mathcal{L}^d$, and from the fact that $\boldsymbol{\nu}_t \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})$. In the language of stochastic processes $\boldsymbol{\sigma}(t)$ and $\boldsymbol{\zeta}(t)$ can be characterized as the unique elements of $L^2(D; \mathbb{R}^{N\times d})$ and $L^2(D; \mathbb{R}^m)$, respectively, such that

$$\int_D \boldsymbol{\sigma}(t,x) g(x)\, dx = \int_{D\times\Omega} \frac{\partial W}{\partial F}\big(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega)\big) g(x)\, dP(x,\omega), \tag{7.7}$$

$$\int_D \boldsymbol{\zeta}(t,x) h(x)\, dx = \int_{D\times\Omega} -\frac{\partial W}{\partial \theta}\big(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega)\big) h(x)\, dP(x,\omega), \tag{7.8}$$

for every $g \in L^2(D; \mathbb{R}^{N\times d})$, $h \in L^2(D; \mathbb{R}^m)$, where $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is the stochastic process corresponding to $(\boldsymbol{\nu}, \boldsymbol{\mu})$.

7.3. Properties of an approximable quasistatic evolution

Definition 7.12. Given a stochastic process $(\boldsymbol{X}_t)_{t\in[0,T]}$ on a $(D,\mathcal{L}^d)$-probability space $(D\times\Omega,\mathcal{B}(D)\otimes\mathcal{F},P)$, with $\boldsymbol{X}_t\in L^2(D\times\Omega;\mathbb{R}^m)$, we define the dissipation associated to $(\boldsymbol{X}_t)_{t\in[0,T]}$ as

$$\mathrm{Var}_H(\boldsymbol{X},P;0,t):=\sup\sum_{i=1}^{k}\int_{D\times\Omega}H\big(\boldsymbol{X}_{t_i}(x,\omega)-\boldsymbol{X}_{t_{i-1}}(x,\omega)\big)\,dP(x,\omega)<\infty,$$

where the supremum is taken over all finite partitions $0=t_0<\cdots<t_k=t$.

The next theorem shows that an approximable quasistatic evolution of stochastic processes satisfies suitable properties of equilibrium, dual constraint, and energy inequality.

Theorem 7.13. *Let $\boldsymbol{\varphi}$, $\boldsymbol{l}$, (z_0,v_0), ε_k, and $T>0$ be as in Definition* 7.4. *If $(\boldsymbol{Z}_t,\boldsymbol{Y}_t)_{t\in[0,T]}$ is an approximable quasistatic evolution of stochastic processes, then the following conditions are satisfied*:

(ev0) *initial condition*: $(\boldsymbol{Z}_0,\boldsymbol{Y}_0)=(z_0,v_0)$;
(ev1) *kinematic admissibility*: *for every $t\in[0,T]$, there exists a unique function $\boldsymbol{v}(t)\in\mathcal{A}(t)$ such that* $\nabla\boldsymbol{v}(t)=\mathrm{bar}((\pi_D,\boldsymbol{Y}_t)(P))$;
(ev2) *equilibrium condition*: *for every $t\in[0,T]$ and every $\tilde{u}\in H^1_{\Gamma_0}(0)$*,

$$\langle\boldsymbol{\sigma}(t),\nabla\tilde{u}\rangle=\langle\boldsymbol{l}(t),\tilde{u}\rangle;$$

(ev3) *dual constraint*: $\boldsymbol{\zeta}(t)\in\partial\mathcal{H}(0)$, *for every $t\in[0,T]$*;
(ev4) *energy inequality*: *for every $t\in[0,T]$ the map*

$$t\mapsto\big[\langle\boldsymbol{\sigma}(t),\nabla\dot{\boldsymbol{\varphi}}(t)\rangle-\langle\dot{\boldsymbol{l}}(t),\boldsymbol{v}(t)\rangle\big],$$

where $\boldsymbol{v}(t)$ is the function appearing in (ev1)*, is integrable on* $[0,T]$*, and we have*

$$\int_{D\times\Omega}W\big(\boldsymbol{Z}_t(x,\omega),\boldsymbol{Y}_t(x,\omega)\big)\,dP(x,\omega)-\langle\boldsymbol{l}(t),\boldsymbol{v}(t)\rangle+\mathrm{Var}_H(\boldsymbol{Z},P;0,t)$$

$$\leqslant\mathcal{W}(z_0,v_0)-\langle\boldsymbol{l}(0),v_0\rangle+\int_0^t\langle\boldsymbol{\sigma}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle\,ds-\int_0^t\big[\langle\boldsymbol{l}(s),\dot{\boldsymbol{\varphi}}(s)\rangle+\langle\dot{\boldsymbol{l}}(s),\boldsymbol{v}(s)\rangle\big]\,ds.$$

Thanks to Remark 7.8, to prove the previous theorem it is enough to prove the equivalent version for Young measures.

Definition 7.14. Given a compatible system $\boldsymbol{\mu}\in SY^2([0,T],D;\mathbb{R}^m)$, we define the dissipation associated to $\boldsymbol{\mu}$ as

$$\mathrm{Var}_H(\boldsymbol{\mu};0,t):=\sup\sum_{i=1}^{k}\int_{D\times(\mathbb{R}^m)^{k+1}}H(\theta_i-\theta_{i-1})\,d\boldsymbol{\mu}_{t_0\ldots t_k}(x,\theta_0,\ldots,\theta_k)<\infty,$$

where the supremum is taken over all finite partitions $0=t_0<\cdots<t_k=t$.

Theorem 7.15. *Let $\boldsymbol{\varphi}$, $\boldsymbol{l}$, (z_0,v_0), ε_k, and $T>0$ be as in Definition* 7.6. *If $(\boldsymbol{\nu},\boldsymbol{\mu})\in Y^2(D;\mathbb{R}^m\times\mathbb{R}^{N\times d})^{[0,T]}\times SY^2_{-}([0,T],D;\mathbb{R}^m)$ is an approximable quasistatic evolution of Young measures, then the following conditions are satisfied*:

(ev0) *initial condition*: $\boldsymbol{\nu}_0=\delta_{(z_0,v_0)}$;
(ev1) *kinematic admissibility*: *for every $t\in[0,T]$, there exists a unique function $\boldsymbol{v}(t)\in\mathcal{A}(t)$ such that*

$$\nabla\boldsymbol{v}(t)=\mathrm{bar}\big(\pi_{D\times\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t)\big);\tag{7.9}$$

(ev2) *equilibrium condition*: *for every* $t \in [0,T]$ *and every* $\tilde{u} \in H^1_{\Gamma_0}(0)$,

$$\langle \boldsymbol{\sigma}(t), \nabla\tilde{u}\rangle = \langle \boldsymbol{l}(t), \tilde{u}\rangle; \tag{7.10}$$

(ev3) *dual constraint*: $\boldsymbol{\zeta}(t) \in \partial\mathcal{H}(0)$, *for every* $t \in [0,T]$;

(ev4) *energy inequality*: *for every* $t \in [0,T]$ *the map*

$$t \mapsto \big[\langle \boldsymbol{\sigma}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}(t)\rangle\big], \tag{7.11}$$

where $\boldsymbol{v}(t)$ *is the function appearing in* (ev1), *is integrable on* $[0,T]$, *and we have*

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\,d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t), \boldsymbol{v}(t)\rangle + \mathrm{Var}_H(\boldsymbol{\mu};0,t)$$
$$\leqslant \mathcal{W}(z_0,v_0) - \langle \boldsymbol{l}(0), v_0\rangle + \int_0^t \langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\,ds - \int_0^t \big[\langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s)\rangle + \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}(s)\rangle\big]\,ds.$$

Proof. Let $(\boldsymbol{\nu},\boldsymbol{\mu})$ be an approximable quasistatic evolution of Young measures.

Condition (ev0) follows immediately from condition (b) of Definition (7.6) and $(\mathrm{ev0})_{\varepsilon_{k^0_j}}$.

Condition (ev1) has been proved in Remark 7.9.

We now prove (ev2); we observe that condition (b) and (W.2) imply that

$$\boldsymbol{\sigma}_{\varepsilon_{k^t_j}}(t) \rightharpoonup \boldsymbol{\sigma}(t) \quad \text{weakly in } L^2\big(D;\mathbb{R}^{N\times d}\big), \tag{7.12}$$

for every $t \in \Theta$, where $(\varepsilon_{k^t_j})_j$ is the sequence appearing in (b). Hence, for every $t \in \Theta\setminus 0$ (7.10) follows from $(\mathrm{ev2})_{\varepsilon_{k^t_j}}$, (7.12), and condition (c) of Definition 7.6, while for $t=0$ is a direct consequence of (4.1), $(\mathrm{ev0})_{\varepsilon_{k^0_j}}$, and (7.12). If $t\in[0,T]\setminus\Theta$, let $s^j \leqslant t$ be a sequence satisfying (7.2); from (7.10) for s^j, we can obtain (7.12) for t, using the continuity of the map $\boldsymbol{l}:[0,T]\to H^1(D;\mathbb{R}^N)^*$.

We show now (ev3). As for $\boldsymbol{\sigma}$ it is easy to see that

$$\boldsymbol{\zeta}_{\varepsilon_{k^t_j}}(t) \rightharpoonup \boldsymbol{\zeta}(t) \quad \text{weakly in } L^2\big(D;\mathbb{R}^m\big), \tag{7.13}$$

for every $t\in\Theta$, where $(\varepsilon_{k^t_j})_j$ is the sequence in (b). Thanks to (c), (7.13) implies that $\boldsymbol{\zeta}_{\varepsilon_{k^t_j}}(t) - \varepsilon_{k^t_j}\dot{\boldsymbol{z}}_{\varepsilon_{k^t_j}}(t) \rightharpoonup \boldsymbol{\zeta}(t)$ weakly in $L^2(D;\mathbb{R}^m)$, for every $t\in\Theta$, and thus, since $\partial\mathcal{H}(0)$ is sequentially weakly closed in $L^2(D;\mathbb{R}^m)$, we obtain from (c) and $(\mathrm{ev3})_{\varepsilon_{k^t_j}}$ that

$$\boldsymbol{\zeta}(t) \in \partial\mathcal{H}(0), \tag{7.14}$$

for every $t\in\Theta\setminus 0$, while for $t=0$ it comes immediately from (4.2), $(\mathrm{ev0})_{\varepsilon_{k^0_j}}$, and (7.13). For $t\in[0,T]\setminus\Theta$, (7.14) follows now easily from (c).

Finally we want to prove (ev4). First of all we observe that if $(\varepsilon_{k^t_j})_j$ is the sequence appearing in (b), we have

$$\boldsymbol{v}_{\varepsilon_{k^t_j}}(t) \rightharpoonup \boldsymbol{v}(t) \quad \text{weakly in } H^1\big(D;\mathbb{R}^N\big); \tag{7.15}$$

hence

$$\langle \boldsymbol{\sigma}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}(t)\rangle = \limsup_k\big[\langle \boldsymbol{\sigma}_{\varepsilon_k}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}_{\varepsilon_k}(t)\rangle\big] \tag{7.16}$$

for every $t\in\Theta$, thanks to (7.12), (7.15), and (7.4). Therefore the map (7.11) is measurable on $[0,T]$. Moreover, from Lemma 6.2 we deduce that

$$\big|\langle \boldsymbol{\sigma}_\varepsilon(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}_\varepsilon(t)\rangle\big| \leqslant C'\big[\|\nabla\dot{\boldsymbol{\varphi}}(t)\|_2 + \|\dot{\boldsymbol{l}}(t)\|_{(H^1)^*}\big]; \tag{7.17}$$

hence, thanks to the hypotheses on $\boldsymbol{\varphi}$ and $\boldsymbol{l}$ and to (7.16), the map (7.11) is integrable on $[0,T]$.

Fix $t \in \Theta$ and let $(\varepsilon_{k_j^t})_j$ be the sequence appearing in (b); since the term containing W is weakly lower semicontinuous and the variation is weakly lower semicontinuous too, thanks to condition (a) of Definition 7.6, we have

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\,d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t),\boldsymbol{v}(t)\rangle + \mathrm{Var}_H(\boldsymbol{\mu};0,t)$$
$$\leqslant \liminf_j \big[\mathcal{W}\big(z_{\varepsilon_{k_j^t}}(t),\boldsymbol{v}_{\varepsilon_{k_j^t}}(t)\big) - \langle \boldsymbol{l}(t),\boldsymbol{v}_{\varepsilon_{k_j^t}}(t)\rangle + \mathrm{Var}_H(z_{\varepsilon_{k_j^t}};0,t)\big].$$

Using $(\mathrm{ev4})_{\varepsilon_{k_j^t}}$ and (3.2), we deduce that

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\,d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t),\boldsymbol{v}(t)\rangle + \mathrm{Var}_H(\boldsymbol{\mu};0,t)$$
$$\leqslant \mathcal{W}(z_0,v_0) - \langle \boldsymbol{l}(0),v_0\rangle - \int_0^t \langle \boldsymbol{l}(s),\dot{\boldsymbol{\varphi}}(s)\rangle\,ds + \limsup_j \int_0^t \big[\langle \boldsymbol{\sigma}_{\varepsilon_{k_j^t}}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}_{\varepsilon_{k_j^t}}(s)\rangle\big]\,ds.$$

We can deduce, using Fatou Lemma thanks to (7.17), that

$$\limsup_j \int_0^t \big[\langle \boldsymbol{\sigma}_{\varepsilon_{k_j^t}}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}_{\varepsilon_{k_j^t}}(s)\rangle\big]\,ds$$
$$\leqslant \limsup_k \int_0^t \big[\langle \boldsymbol{\sigma}_{\varepsilon_k}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}_{\varepsilon_k}(s)\rangle\big]\,ds \leqslant \int_0^t \limsup_k \big[\langle \boldsymbol{\sigma}_{\varepsilon_k}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}_{\varepsilon_k}(s)\rangle\big]\,ds.$$

Thanks to (7.4) this implies that

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\,d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t),\boldsymbol{v}(t)\rangle + \mathrm{Var}_H(\boldsymbol{\mu};0,t)$$
$$\leqslant \mathcal{W}(z_0,v_0) - \langle \boldsymbol{l}(0),v_0\rangle - \int_0^t \langle \boldsymbol{l}(s),\dot{\boldsymbol{\varphi}}(s)\rangle\,ds + \int_0^t \big[\langle \boldsymbol{\sigma}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}(s)\rangle\big]\,ds. \tag{7.18}$$

Let now $t \in [0,T]\setminus\Theta$ and let $s^j \to t$ be a sequence satisfying (7.2); it is easy to verify that

$$\mathrm{Var}_H(\boldsymbol{\mu};0,t) \leqslant \liminf_j \mathrm{Var}_H\big(\boldsymbol{\mu};0,s^j\big),$$

hence (ev2) for t can be deduced from (7.18) for s^j. □

The following result is an existence theorem for approximable quasistatic evolution of stochastic processes.

Theorem 7.16. *Given an external load $\boldsymbol{l} \in H^1([0,T];H^1(D;\mathbb{R}^N)^*)$, a boundary datum $\boldsymbol{\varphi} \in H^1([0,T];H^1(D;\mathbb{R}^N))$, an initial condition $(z_0,v_0) \in L^2(D;\mathbb{R}^m)\times\mathcal{A}(0)$ satisfying* (4.1) *and* (4.2)*, and $T>0$, there exists an approximable quasistatic evolution of stochastic processes* (*or of Young measures*) *in the time interval* $[0,T]$.

Proof. Thanks to Remark 7.8, it is enough to prove that there exists an approximable quasistatic evolution of Young measures. Fixed a positive sequence $\varepsilon_k \to 0$, let $(z_{\varepsilon_k},v_{\varepsilon_k})$ be the solution of the ε_k-regularized problem in the time interval $[0,T]$, with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$ and initial condition (z_0,v_0). Thanks to (5.15) and (H.2) we are in the hypothesis of Helly's Theorem for compatible systems of Young measures (see [7, Theorem 4.10]). Therefore, by passing to a subsequence still denoted by $(\varepsilon_k)_k$, we can conclude that there exist $\Theta \subset [0,T]$, with $0 \in \Theta$ and $\mathcal{L}^1([0,T]\setminus\Theta)=0$, and $\boldsymbol{\mu} \in SY^2_-([0,T],D;\mathbb{R}^m)$, which satisfy condition (a) of Definition 7.6.

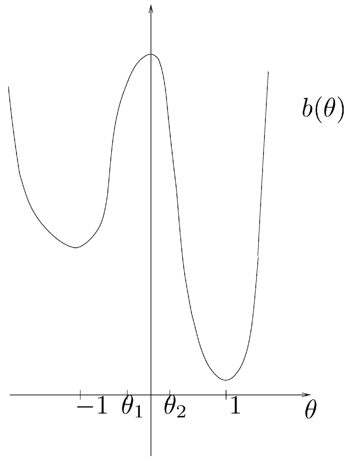

Fig. 1. The function b.

Thanks to Remark 6.3, we can assume that (6.3), (6.4) hold for every $t \in \Theta$, by choosing a subset of Θ if necessary; analogously we can assume that (4.3) and (4.4) hold for every $t \in \Theta \setminus 0$ and every ε_k. For every $t \in \Theta$ select a subsequence $(\varepsilon_{k_j^t})_j$ of $(\varepsilon_k)_k$ which satisfies (7.4); thanks to (5.15) and (5.16), we can apply [7, Lemma 4.13] to the sequence of compatible systems $(\delta_{(z_{\varepsilon_k}, \nabla \boldsymbol{v}_{\varepsilon_k})})_k$ and we obtain a family of Young measures $\boldsymbol{\nu} \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]}$, which is Θ-2-weakly* approximable from the left and satisfies (7.3), for a suitable subsequence of $(\varepsilon_{k_j^t})_j$. This proves (b) and (c). □

8. A finite dimensional example

In this section we will propose the complete analysis of the approximable quasistatic evolution for a concrete case, in which the hypotheses of Remark 6.1 are fulfilled and hence the internal variable and the gradient of the deformation are functions from $[0, T]$ into a finite dimensional space.

Let consider the case $d = N = m = 1$, $D = (0, 1)$, and $\Gamma_0 = \{0, 1\}$. We assume $H = |\cdot|$, $\boldsymbol{l} \equiv 0$, and

$$W(\theta, y) := \frac{1}{10}\big[\eta(y)\big(y - a(\theta)\big)^2 + \big(1 - \eta(y)\big)y^2\big] + b(\theta) \quad \text{for every } \theta, y \in \mathbb{R}, \tag{8.1}$$

where $a \in \mathcal{C}^2(\mathbb{R})$ is bounded with its first and second derivative and $a(\theta) = \theta$ if $|\theta| \leqslant 2$, $\eta \in \mathcal{C}^2_c(\mathbb{R})$ is a cut off-function with $\eta(y) = y$ if $|y| \leqslant 7 + 5\,b'(2)$, and b is a $\mathcal{C}^2$ function satisfying the following properties, for every $\theta \in \mathbb{R}$ (see Fig. 1):

(b.1) $b(\theta) \geqslant c\theta^2 + d$, for suitable positive constant c, d;
(b.2) $b(\theta) + |\theta + 1| > 2$, for every $\theta \neq -1$;
(b.3) b has a local minimum at -1, with $b(-1) = 2$, a global minimum at 1, with $b(1) < 1$, and a local maximum in 0, and there are no other local extrema;
(b.4) $5b'' + 1$ is bounded and has exactly two zeros, $-1 < \theta_1 < 0 < \theta_2 < 1$, with $b'(\theta_1) < b'(2)$.

It is immediate to verify that such a W satisfies hypotheses (W.1) and (W.2).

Let us fix T such that $6 + \theta_1 + 5\,b'(\theta_1) < T < 8 + 5\,b'(2)$; we will study the approximable quasistatic evolution in the time interval $[0, T]$ with $\boldsymbol{\varphi}(t, x) := (t - 1)x$ for every $t \in [0, T]$ and every $x \in [0, 1]$ (which corresponds to the boundary condition $\boldsymbol{v}(t, 0) = 0$ and $\boldsymbol{v}(t, 1) = t - 1$), and initial condition $(z_0, v_0) = (-1, \boldsymbol{\varphi}(0))$.

Theorem 8.1. *Let W, $\boldsymbol{l}$, H, $\boldsymbol{\varphi}$, T, and (z_0, v_0) satisfy the assumptions at the beginning of this section. Then the unique approximable quasistatic evolution corresponding to this data is given by*

$$\boldsymbol{v}(t, x) = \boldsymbol{\varphi}(t, x) \quad \textit{for every } t \in [0, T];$$

$$z(t,x) = z(t) := \begin{cases} -1 & \textit{for } 0 \leqslant t \leqslant 5, \\ z_1(t) & \textit{for } 5 < t \leqslant t_1, \\ z_2(t) & \textit{for } t_1 < t \leqslant T \end{cases} \tag{8.2}$$

where $t_1 := 6 + \theta_1 + 5b'(\theta_1)$, *for every* $t \in [5, t_1]$ $z_1(t)$ *is the unique solution in the interval* $[-1, \theta_1]$ *of the equation*

$$\frac{1}{5}\big(t - 1 - \theta(t)\big) - b'\big(\theta(t)\big) = 1, \tag{8.3}$$

and for $t \in (t_1, T]$ z_2 *is the unique solution of* (8.3).

Proof. We are in the case of Remark 6.1, hence the solution $(\boldsymbol{v}_\varepsilon, \boldsymbol{z}_\varepsilon)$ of the ε-regularized problem is

$$\boldsymbol{v}_\varepsilon(t,x) := \boldsymbol{\varphi}(t,x), \tag{8.4}$$

$$\boldsymbol{z}_\varepsilon(t,x) = z_\varepsilon(t), \tag{8.5}$$

where z_ε is the solution of the Cauchy problem

$$\begin{cases} \dot{z_\varepsilon}(t) = \dfrac{1}{\varepsilon}\big[-W_\theta\big(z_\varepsilon(t), t-1\big) - P_{[-1,1]}\big(-W_\theta\big(z_\varepsilon(t), t-1\big)\big)\big], \\ z_\varepsilon(0) = -1, \end{cases} \tag{8.6}$$

where $P_{[-1,1]}$ is the projection on the interval $[-1, 1]$. By the upper bound on T, $-W_\theta(\theta, t-1)$ takes the form

$$g(t,\theta) := \frac{1}{5}(t - 1 - \theta) - b'(\theta),$$

for every $t \in [0, T]$ and for every $|\theta| \leqslant 2$. Hence the equation in (8.6) becomes

$$\varepsilon \dot{z}_\varepsilon(t) = \begin{cases} g(t, z_\varepsilon(t)) - 1 & \text{if } g(t, z_\varepsilon(t)) > 1, \\ 0 & \text{if } |g(t, z_\varepsilon(t))| \leqslant 1, \\ g(t, z_\varepsilon(t)) + 1 & \text{if } g(t, z_\varepsilon(t)) < -1 \end{cases} \tag{8.7}$$

until $|z_\varepsilon(t)| \leqslant 2$. For every ε let t_ε be the greatest time in $[0, T]$ such that $|z_\varepsilon(t)| \leqslant 2$ for every $t \in [0, t_\varepsilon]$. In particular $z_\varepsilon(t_\varepsilon) = 2$. Since $0 \leqslant g(t, -1) = \frac{1}{5}t \leqslant 1$ for $t \in [0, 5]$, we have $z_\varepsilon(t) = -1$, for every $t \leqslant 5$ and every ε. In particular we have $g(5, z_\varepsilon(5)) = 1$ and $t_\varepsilon > 5$. It is easy to see that $g(t, z_\varepsilon(t)) \geqslant 1$ for $t > 5$. Indeed let U_ε be the open set $\{t \in (5, t_\varepsilon)\colon\ g(t, z_\varepsilon(t)) < 1\}$ and let (α, β) be any connected component of U_ε. Since $g(\alpha, z_\varepsilon(\alpha)) = 1$, there exists $0 < \delta_\varepsilon < \beta - \alpha$ such that $0 < g(t, z_\varepsilon(t)) < 1$ for every $t \in (\alpha, \alpha + \delta_\varepsilon)$; therefore $\dot{z}_\varepsilon(t) = 0$ for every $t \in (\alpha, \alpha + \delta_\varepsilon)$, in particular $z_\varepsilon(t) = z_\varepsilon(\alpha)$. Since $g(\cdot, z_\varepsilon(\alpha))$ is strictly increasing (indeed $\frac{\partial g}{\partial t}(t,\theta) = \frac{1}{5}$), we have $1 = g(\alpha, z_\varepsilon(\alpha)) < g(t, z_\varepsilon(\alpha)) = g(t, z_\varepsilon(t))$ for every $t \in (\alpha, \alpha + \delta_\varepsilon)$, which contradicts $g(t, z_\varepsilon(t)) < 1$. Hence $U_\varepsilon = \emptyset$ and $g(t, z_\varepsilon(t)) \geqslant 1$ for every $t \in [5, t_\varepsilon]$. Thanks to the upper bound on T we have $g(T, 2) < 1$, but, if $t_\varepsilon < T$, we have $1 \leqslant g(t_\varepsilon, z_\varepsilon(t_\varepsilon)) = g(t_\varepsilon, 2) < g(T, 2)$ which contradicts $g(T, 2) < 1$. Therefore $t_\varepsilon = T$ for every ε, and $g(t, z_\varepsilon(t)) \geqslant 1$ for every $t \in [5, T]$.

Hence we can conclude that, for $t \in (5, T]$, z_ε is the unique solution of the equation

$$\varepsilon \dot{z}_\varepsilon(t) = g\big(t, z_\varepsilon(t)\big) - 1. \tag{8.8}$$

Note that

$$\frac{\partial g}{\partial \theta}(t,\theta) = -\frac{1}{5}\big(1 + 5b''(\theta)\big),$$

for every t and θ. Therefore from (b.4) we know that $\partial g/\partial\theta$ has exactly two zeros θ_1 and θ_2 with $-1 < \theta_1 < 0 < \theta_2 < 1$.

First of all we want to show that there exists a unique solution $z_1(t) \in (-1, \theta_1)$ to the equation

$$g\big(t, z(t)\big) = 1, \tag{8.9}$$

for $t \in (5, t_1)$ where $t_1 = 6 + \theta_1 + 5b'(\theta_1)$. Note that $g(t_1, \theta_1) = 1$; since $g(\cdot, \theta)$ is strictly increasing for every θ, we have $g(t, -1) > g(5, -1) = 1 = g(t_1, \theta_1) > g(t, \theta_1)$ for every $t \in (5, t_1)$; hence for every $t \in (5, t_1)$ there exists a unique $z_1(t) \in (-1, \theta)$ solving (8.9) (because $\frac{\partial g}{\partial \theta}(t, \cdot)$ never vanishes on $(-1, \theta_1)$). By the Inverse Function Theorem the map $t \mapsto z_1(t)$ is $\mathcal{C}^1$ and we can deduce that $\lim_{t \to t_1^-} z_1(t) = \theta_1$ (indeed if not, let θ^* be this limit; we have $-1 \leqslant \theta^* < \theta_1$ and $g(t_1, \theta^*) = 1$, which contradicts the fact that $g(t_1, \cdot)$ is strictly decreasing on $(-1, \theta_1)$).

It is easy to see that for $t > t_1$ Eq. (8.9) has a unique solution: indeed we can write

$$g(t,\theta) - 1 = \frac{1}{5}(t-1) - \psi(\theta),$$

where $\psi(\theta) := \frac{1}{5}\theta + b'(\theta) + 1$; since, for every $t > t_1$, $\psi(\theta) \leqslant \psi(\theta_1) = \frac{1}{5}(t_1 - 1) < \frac{1}{5}(t-1)$, for every $\theta \leqslant \theta_2$, and $\lim_{\theta\to+\infty}\psi(\theta) = +\infty$, we deduce that the zeros of $\frac{1}{5}(t-1) - \psi(\theta)$ exist and are contained in $(\theta_2, +\infty)$; since in this interval $\frac{\partial g}{\partial \theta}$ never vanishes, we can apply again the Inverse Function Theorem to obtain the existence of a unique continuous function $z_2 : (t_1, T] \to \mathbb{R}$ solving (8.9).

We want now to show that z_1 is the unique approximable quasistatic evolution in $[5, t_1]$. First of all we observe that, since $z_\varepsilon(5) = z_1(5) = -1$ and $\varepsilon\dot z_1(t) > 0 = g(t, z_1(t)) - 1$ while $\varepsilon\dot z_\varepsilon(t) = g(t, z_\varepsilon(t)) - 1$, by the comparison principle $z_1(t) \geqslant z_\varepsilon(t)$ for every $t \in [5, t_1)$. Let now fix $\eta > 0$ and $\bar t \in (5, t_1)$; if we show that there exists ε_0 such that for every $\varepsilon \leqslant \varepsilon_0$ we have $z_\varepsilon(t) \in [z_1(t) - \eta, z_1(t)]$ for every $t \in (5, \bar t)$, we can conclude that $z(t) = z_1(t)$ on $(5, \bar t)$. Let

$$c_\eta := \min_{t\in[5,\bar t]} g\big(t, z_1(t) - \eta\big) - 1,$$
$$m := \max_{t\in[5,\bar t]} \dot z_1(t);$$

we have $m < +\infty$ by continuity of $\dot z_1$, and $c_\eta > 0$ because $g(t, z_1(t) - \eta) > g(t, z_1(t)) = 1$. Therefore we can find $\varepsilon_0 > 0$ such that $\varepsilon_0 m < c_\eta$, and for every $\varepsilon \leqslant \varepsilon_0$ we have $z_1(5) - \eta < z_\varepsilon(5)$, $\varepsilon\dot z_1(t) < g(t, z_1(t) - \eta) - 1$ while $\varepsilon\dot z_\varepsilon(t) = g(t, z_\varepsilon(t)) - 1$, hence $z_\varepsilon(t) \geqslant z_1(t) - \eta$ for every $t \in (5, \bar t)$.

Since z is left continuous by definition we can conclude that $z(t) = z_1(t)$ for every $t \in [5, t_1]$.

Finally we show that z must coincide with z_2 on $(t_1, T]$.

As $|z_\varepsilon(t)| \leqslant 2$ for every $t \in [0, T]$ and every ε, condition (ev3) satisfied by z can be written as

$$g\big(t, z(t)\big) \in [-1, 1]. \tag{8.10}$$

Since we have proved that $g(t, z_\varepsilon(t)) \geqslant 1$, for every $t > t_1$, it follows that z satisfies (8.9), for $t > t_1$. As this equation has a unique solution z_2 defined on (t_1, T), we can conclude that $z(t) = z_2(t)$ for every $t \in (t_1, T]$. □

We prove now that the approximable quasistatic evolution described in Theorem 8.1 does not fulfill the requirements of the definition of globally quasistatic evolution given in [7, Definition 6.12].

Theorem 8.2. *The datum $(-1, \boldsymbol{\varphi}(0))$ is stable for the considered problem, but the approximable quasistatic evolution described in Theorem* 8.1 *does not satisfies global stability, i.e. there exists $t \in [0, T]$, $\tilde z \in L^2(0, 1)$, and $\tilde u \in H^1_0(0, 1)$ with*

$$W\big(z(t), t-1\big) > \int_0^1 W\big(z(t) + \tilde z(x), t - 1 + \tilde u'(x)\big)\,dx + \|\tilde z\|_1. \tag{8.11}$$

Proof. First of all we have to verify that the initial condition satisfies the minimality condition requested in [7, Definition 6.12]. To this aim we have to check that

$$W(-1, -1) \leqslant \int_0^1 W\big(-1 + \tilde z(x), -1 + \tilde u'(x)\big)\,dx + \|\tilde z\|_1,$$

for every $\tilde z \in L^2((0, 1))$ and $\tilde u \in H^1_0(0, 1)$; this is immediate because $W(-1, -1) = 2$, while

$$\int_0^1 W\big(-1 + \tilde z(x), -1 + \tilde u'(x)\big)\,dx + \|\tilde z\|_1 \geqslant \int_0^1 \big[b\big(-1 + \tilde z(x)\big) + \big|\tilde z(x)\big|\big]\,dx \geqslant 2,$$

thanks to assumption (b.2).

Let now consider $t \in (4, 5]$, $\tilde z = 2$, and $\tilde u = 0$. We have $W(z(t), t - 1) = \frac{1}{10}(t^2 + 20)$, while thanks to (b.3)

$$\int_0^1 W\big(z(t)+\tilde z(x), t-1+\tilde u'(x)\big)\,dx + \|\tilde z\|_1$$
$$= W(1, t-1) + 2 = \frac{1}{10}(t-2)^2 + b(1) + 2 < \frac{1}{10}(t-2)^2 + 3 = \frac{1}{10}\big(t^2-4t+34\big) < \frac{1}{10}(t^2+18). \qquad \square$$

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