Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 26 (2009) 1099–1119
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

# Liouville results for $m$-Laplace equations of Lane–Emden–Fowler type ☆

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Received 14 November 2007; accepted 30 June 2008

Available online 25 July 2008

**Abstract**

We consider sign changing solutions of the equation $-\Delta_m(u) = |u|^{p-1}u$ in possibly unbounded domains or in $\mathbb{R}^N$. We prove Liouville type theorems for stable solutions or for solutions which are stable outside a compact set. The results hold true for $m > 2$ and $m - 1 < p < p_c(N, m)$. Here $p_c(N, m)$ is a new critical exponent, which is infinity in low dimension and is always larger than the classical critical one.


*MSC:* 35B05; 35B65; 35J70

*Keywords:* Degenerate PDEs; Stable solutions; Critical exponents; Rigidity results

## 1. Introduction and statement of the main results

We consider $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ weak solution of

$$-\Delta_m(u) = |u|^{p-1}u \quad \text{in } \Omega, \tag{1}$$

where $\Delta_m$ denotes the $m$-Laplacean operator, $m > 1$, $\Omega \subseteq \mathbb{R}^N$ is any domain (bounded or not) and $u$ is a possibly unbounded function which may change sign.

The main results of this paper will be collected in Theorems 1.5, 1.7–1.10 and 1.11 below and they will be concerned with Liouville type rigidity results for suitable solutions of (1).

☆ Partially supported by *MIUR Metodi variazionali ed equazioni differenziali nonlineari*.
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0294-1449/$ – see front matter 
doi:10.1016/j.anihpc.2008.06.001

We recall that $u$ is said to be a weak solution of (1) if

$$\int_{\Omega} |\nabla u|^{m-2}(\nabla u, \nabla\varphi)\,dx = \int_{\Omega} |u|^{p-1}u\varphi\,dx \quad \forall\varphi \in C_c^{\infty}(\Omega). \tag{2}$$

In some cases we will also consider the problem

$$\begin{cases} -\Delta_m(u) = |u|^{p-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

We also recall that the $C^{1,\alpha}$ regularity assumption is natural in this setting due to the results in [11,21,27]. See also [20] for related results. The aim of this paper is to prove various non-linear Liouville type theorems for the above considered problems (2) and (3).

Liouville type theorems for the semilinear case $m = 2$ have received much attention in the last decades. We refer to the papers [17,18] for the case of non-negative solutions and subcritical values of the exponent $p$ (see also [2,4]) and to the works [14,15] for the case of changing-sign solutions belonging to one of the following families: stable solutions, finite Morse index solutions or (more generally) solutions stable outside a compact set of $\Omega$ (see Definition 1.2 below). The results proved in [14,15] also cover the case of supercritical values of the exponent $p$. More precisely, it is proved that they hold true for $1 < p < p_c(N)$, where $p_c(N)$ is a second critical exponent, which is always larger than the classical critical one.

For the quasilinear case $m \neq 2$, Liouville type theorems for non-negative weak solutions have been proved in [26]. These results hold true for the corresponding subcritical range of values of the exponent $p$. To the best of our knowledge no Liouville type result is known for changing-sign solutions in the case $m \neq 2$. The main purpose of the present work is to prove such a kind of results for solutions which are either stable, with finite Morse index or stable outside a compact set.

The proofs of our results are different from those of [17,18,26,2,4]. Of course, as in the above papers, an important tool is the choice of suitable test functions and our techniques are inspired by the methods developed by the second author in [14,15].

We start with the following

**Definition 1.1.** We recall (see [7,8] and [3]) that, given a bounded domain $\omega$, if $\rho \in L^1(\omega)$, $\rho \geqslant 0$, $1 \leqslant p < \infty$, the space $H_\rho^{1,p}(\omega)$ is defined as the completion of $C^1(\overline{\omega})$ (or, equivalently, $C^\infty(\overline{\omega})$) under the norm

$$\|v\|_{H_\rho^{1,p}} = \|v\|_{L^p(\omega)} + \|Dv\|_{L^p(\omega,\rho)} \tag{4}$$

where $\|Dv\|_{L^p(\omega,\rho)}^p = \int_\omega |Dv|^p \rho\,dx$.

In this way $H_\rho^{1,p}(\omega)$ is a Banach space and $H_\rho^{1,2}(\omega)$ is a Hilbert space. We also recall that $H_\rho^{1,p}$ may be defined as the space of functions having distributional derivatives represented by a function for which the quantity in (4) is finite. These two definitions are equivalent if the domain has piecewise regular boundary.

Moreover, we define $H_{0,\rho}^{1,p}(\omega)$ as the closure of $C_c^1(\omega)$ (or $C_c^\infty(\omega)$) in $H_\rho^{1,p}(\omega)$.

The linearized operator of (2) at $u$ is given by

$$L_u(v,\varphi) = \int_{\Omega} |\nabla u|^{m-2}(\nabla v, \nabla\varphi)\,dx + \int_{\Omega} (m-2)|\nabla u|^{m-4}(\nabla u, \nabla v)(\nabla u, \nabla\varphi) - p|u|^{p-1}v\varphi\,dx.$$

In this paper we will be mostly concerned with the case $m > 2$. In this case, if we consider

$$\rho = |\nabla u|^{(m-2)} \tag{5}$$

then $\rho \in L^1_{\mathrm{loc}}(\Omega)$ by the $C^{1,\alpha}$ regularity of $u$. Therefore we can consider $v \in H^{1,2}_{\rho,\mathrm{loc}}(\Omega)$ and $\varphi \in C_c^1(\Omega)$ and the linearized operator is well defined for such $(v,\varphi)$.

Also, by density arguments, we can consider the case $v \in H^{1,2}_{\rho,\mathrm{loc}}(\Omega)$ and $\varphi \in H^{1,2}_{0,\rho}(\mathcal{K})$, for some compact set $\mathcal{K} \subset \Omega$, and the linearized operator is well defined for such $(v,\varphi)$.

Observe that, besides the fact that many of our estimates work only in the case $m > 2$, in the case $1 < m < 2$ the weight $|\nabla u|^{m-2}$ might not belong to the space $L^1_{\mathrm{loc}}(\Omega)$. This prevents the use of classical estimates in weighted Sobolev spaces and in this case it is not even clear which definition of stability would be the natural one.

**Definition 1.2.** We say that a weak solution $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$

- is *stable* if
$$\begin{cases} L_u(\varphi,\varphi) \geqslant 0, \\ \forall \varphi \in C^1_c(\Omega), \end{cases}$$
- *has Morse index* equal to $K \geqslant 1$ if $K$ is the maximal dimension of a subspace $\mathcal{Z} \subset C^1_c(\Omega)$ such that $L_u(\varphi,\varphi) < 0$ for any $\varphi \in \mathcal{Z}$, $\varphi \not\equiv 0$,
- is *stable outside a compact set* $\mathcal{K} \subset \Omega$ if $L_u(\varphi,\varphi) \geqslant 0$ for every $\varphi \in C^1_c(\Omega \setminus \mathcal{K})$.

We recall that the stability condition translates into the fact that the second variation of the energy functional is non-negative. Therefore all the minima of the functional are stable weak solutions of the equation $-\Delta_m(u) = |u|^{p-1}u$.

**Remark 1.3.** It is well known that, if $u$ has finite Morse index, then $u$ is stable outside a compact set. Also, for future use, we point out that, if $u$ is stable, for any $\varphi \in H^{1,2}_{0,\rho}(\mathcal{K})$, for any compact set $\mathcal{K} \subset \Omega$, we have

$$p\int_\Omega |u|^{p-1}\varphi^2 \leqslant (m-1)\int_\Omega |\nabla u|^{m-2}|\nabla\varphi|^2. \tag{6}$$

Exploiting the technique introduced in [14,15], we prove the following

**Proposition 1.4.** *Let $\Omega$ be a smooth domain bounded or not of $\mathbb{R}^N$. Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ be a* stable *solution of* (2) *with $p > (m-1)$ and $m > 2$. Then, for every*

$$\gamma \in \left[1; \frac{2p-(m-1)+2\sqrt{p(p-(m-1))}}{(m-1)}\right)$$

*and for any integer $k$ with*

$$k \geqslant \max\left\{\frac{p+\gamma}{p-(m-1)}; 2\right\}$$

*there exists a positive constant $C = C(p,m,k,\gamma)$ such that*

$$\int_\Omega \left(|\nabla u|^m|u|^{\gamma-1}\psi^{mk} + |u|^{p+\gamma}\psi^{mk}\right) \leqslant C\int_\Omega |\nabla\psi|^{m\frac{p+\gamma}{p-(m-1)}} \tag{7}$$

*for all test functions $\psi \in C^1_c(\Omega)$ with $0 \leqslant \psi \leqslant 1$.*

Proposition 1.4 provides an important estimate on the integrability of $u$ and $\nabla u$. As we will see, our non-existence results will follow by showing that the right-hand side of (7) vanishes under the right assumptions on $p$ and $\Omega$. More precisely, as a corollary of Proposition 1.4, we can state our first Liouville type theorem.

**Theorem 1.5.** *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ be a* stable *solution of* (2) *with $m > 2$. Assume that*

$$\begin{cases} (m-1) < p < \infty, & \text{if } N \leqslant \dfrac{m(m+3)}{m-1}, \\ (m-1) < p < p_c(N,m), & \text{if } N > \dfrac{m(m+3)}{m-1}, \end{cases}$$

*with*

$$p_c(N,m) = \frac{[(m-1)N-m]^2 + m^2(m-2) - m^2(m-1)N + 2m^2\sqrt{(m-1)(N-1)}}{(N-m)[(m-1)N-m(m+3)]}. \tag{8}$$

*Then, $u \equiv 0$.*

We observe that the critical exponent $p_c(N,m)$ is always greater than the classic critical exponent $\frac{N(m-1)+m}{N-m}$. When $m=2$ it reduces to the one found in [15], namely

$$p_c(N,2) = p_c(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}$$

if $N > 10$, which appears also in the study of the bifurcation diagram for positive radial solutions of the equation (see [19]).

In our setting the operator is non-linear and the equation has to be understood in the weak sense. Also the solutions are not $C^2$, this causing some technical difficulties that we overcome by an iterate use of Young's inequality and Hölder's inequality, that allows us to carry on the proofs considering the equation only in the weak sense.

We now point out a useful variation of Proposition 1.4:

**Proposition 1.6.** *Let $\Omega$ be a domain bounded or not of $\mathbb{R}^N$. Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ be a* stable *solution of* (2). *Assume $m > 2$ and $p > m-1$. Then, for any*

$$\gamma \in \left[1; \frac{2p-(m-1)+2\sqrt{p(p-(m-1))}}{(m-1)}\right)$$

*and for any integer $k$*

$$k \geqslant \max\left\{\frac{p+\gamma}{p-1}; 2\right\}$$

*there exists a positive constant $C = C(p,m,k,\gamma)$ such that*

$$\int_\Omega \left(|\nabla u|^{m-2}\left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2 + |u|^{p+\gamma}\right)\psi^{2k} \leqslant C\int_\Omega \left(|\nabla u|^{m-2}|\nabla\psi|^2\right)^{\frac{p+\gamma}{p-1}} \tag{9}$$

*for all $\psi \in C^1_c(\Omega)$ with $0 \leqslant \psi \leqslant 1$.*

We observe that Proposition 1.6 is a weighted version of Proposition 1.4, because of the presence of the weight $|\nabla u|^{m-2}$ in (9). We note that this weight is 1 when $m=2$, and that, in this case, Proposition 1.6 reduces to the above Proposition 1.4. As a consequence of Proposition 1.6 we obtain a bound on the possible decay of the gradient of stable solutions of (2), according to the following

**Theorem 1.7.** *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ be a* stable *solution of* (2) *with $m > 2$. Assume that there exists a suitable constant $C > 0$ such that*

$$\left|\nabla u(x)\right| \leqslant C|x|^\beta$$

*for $x \in \mathbb{R}^N$ and large $|x|$.*

*If*

$$N + \left(\beta(m-2)-2\right)\left(\frac{p+\gamma}{p-1}\right) < 0$$

*for some*

$$\gamma \in \left[1; \frac{2p-(m-1)+2\sqrt{p(p-(m-1))}}{(m-1)}\right),$$

*then $u \equiv 0$.*

We recall that all the minima of the energy functional are stable solutions. Also it is possible to show that a solution which is monotone in some direction, is stable (see Lemma 6.2 here or Lemma 7.1 in [16]). This allows us to prove that, if $\Omega$ is a smooth coercive epigraph then any positive solution is monotone thanks to the moving plane technique, and therefore it is stable.

Consequently, a classification result for non-negative solutions follows when the assumptions of Theorem 1.5 are fulfilled and $\Omega$ is a smooth coercive epigraph, without assuming that the solution is stable. More precisely we have

**Theorem 1.8.** *Let $\Omega$ be a smooth domain in $\mathbb{R}^N$. Assume that $\Omega$ is a $C^{2,\alpha}_{\mathrm{loc}}$-smooth* coercive epigraph, *meaning that*

$$\Omega = \{(x', x_N) \in \mathbb{R}^N \colon \varphi(x') < x_N\}$$

*where $\varphi \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^{N-1}, \mathbb{R})$ and*

$$\lim_{|x'| \to +\infty} \varphi(x') = +\infty.$$

*Suppose that $m > 2$. Let $u$ be a non-negative solution of* (3) *with*

$$\begin{cases} (m-1) < p < \infty, & \text{if } N \leqslant \dfrac{m(m+3)}{m-1}, \\ (m-1) < p < p_c(N,m), & \text{if } N > \dfrac{m(m+3)}{m-1}, \end{cases}$$

*where $p_c(N,m)$ is given in* (8)*. Then $u \equiv 0$.*

We remark that the assumption that the coercive epigraph is $C^{2,\alpha}_{\mathrm{loc}}$-smooth in Theorem 1.8 is needed for the moving plane technique.

A Liouville type result for solutions stable outside a compact set is somewhat more complicated, since suitable integrability estimates need to interplay with an appropriate Pohozaev type identity. The Pohozaev identity [22] has been extended to quasilinear degenerate operators by P. Pucci and J. Serrin [23], and by M. Degiovanni, A. Musesti and M. Squassina [10]. We give here (see Corollary 8.3 below) a self-contained proof which can be carried out in our case with simple arguments.

These tools will allow us to obtain the following

**Theorem 1.9.** *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution of* (2) *which is stable outside the compact set $\mathcal{K}$. Assume $m > 2$ and*

$$\begin{cases} (m-1) < p < \infty, & \text{if } N \leqslant m, \\ (m-1) < p < \dfrac{N(m-1)+m}{N-m}, & \text{if } N > m. \end{cases} \tag{10}$$

*Then $u \equiv 0$.*

The result of Theorem 1.9 is sharp: if $p = \frac{N(m-1)+m}{N-m}$, we will prove in Proposition 10.1 that there exist positive radial solutions of the form

$$u_\lambda(|x|) = u_\lambda(r) = \lambda\left(\frac{\lambda^{\frac{1}{m-1}}(N^{\frac{1}{m}}(\frac{N-m}{m-1})^{\frac{m-1}{m}})}{\lambda^{\frac{m}{m-1}} + r^{\frac{m}{m-1}}}\right)^{\frac{N-m}{m}}, \quad \lambda > 0, \tag{11}$$

which are stable outside a sufficiently large ball. The proof of the stability of such solutions depend on Hardy's inequality.

If in Theorem 1.9 we assume that $m < N$ and that $p$ is supercritical with respect to the classic critical exponent, while it is subcritical with respect to the new critical exponent $p_c(N,m)$, i.e.,

$$\begin{cases} \dfrac{N(m-1)+m}{N-m} < p < \infty, & \text{if } N \leqslant \dfrac{m(m+3)}{m-1}, \\ \dfrac{N(m-1)+m}{N-m} < p < p_c(N,m), & \text{if } N > \dfrac{m(m+3)}{m-1}, \end{cases} \tag{12}$$

we are not able to conclude that any solution stable outside a compact set is the trivial one (we recall that this is true when $m = 2$, as proved in [15]). In any case, we are able to prove that any such solution decay at infinity faster than radial solutions (which are classified in [1]):

**Theorem 1.10.** *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution of* (2) *which is stable outside the compact set $\mathcal{K}$. Assume that $m > 2$ and assume that* (12) *holds. Then*

$$\lim_{|x| \to \infty} |x|^{\frac{m}{p-m+1}} u(x) = 0. \tag{13}$$

As a consequence, we will obtain:

**Theorem 1.11.** *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ be a radial solution of* (2) *which is stable outside a compact set $\mathcal{K}$ and $m > 2$. Assume that*

$$\begin{cases} \dfrac{N(m-1)+m}{N-m} < p < \infty, & \text{if } N \leqslant \dfrac{m(m+3)}{m-1}, \\ \dfrac{N(m-1)+m}{N-m} < p < p_c(N,m), & \text{if } N > \dfrac{m(m+3)}{m-1}, \end{cases}$$

*with $p_c(N,m)$ as in Theorem* 1.5. *Then $u \equiv 0$.*

Finally, we note that for $p \geqslant p_c(N,m)$, positive bounded radial solutions always exist (see [1,13]). For $m = 2$ all these solutions are stable as it was shown in [15].

The rest of the paper is organized as follows. Sections 2 and 3 contain the proofs of the auxiliary integrability estimates of Propositions 1.4 and 1.6. Sections 4, 5, 7, 9, 11 and 12 are devoted to the proofs of the main results. Section 6 discusses the relation between monotone and stable solutions. The Pohozaev type Identity needed for our purposes is contained in Section 8. Section 10 contains the example that shows the sharpness of Theorem 1.9.

## 2. Proof of Proposition 1.4

The proof is inspired by the techniques developed in [14,15] for the semilinear case $m = 2$.

**Step 1.** We claim that for any $\gamma \geqslant 1$ and for any $0 < \epsilon < \sqrt{\gamma}$, there exists a constant $C_\epsilon$ possibly depending on $\epsilon$ and $m$, such that

$$\left(\gamma - \epsilon^2\right) \int_\Omega |\nabla u|^m |u|^{\gamma-1} \varphi^m \leqslant \int_\Omega |u|^{p+\gamma} \varphi^m + C_\epsilon \int_\Omega |\nabla \varphi|^m |u|^{\gamma+(m-1)} \tag{14}$$

for any non-negative $\varphi \in C^\infty_c(\Omega)$.

To prove this, let us consider $\Phi = |u|^{\gamma-1} u \varphi^m$. We use $\Phi$ as test function in (2). Since

$$\nabla\Phi = m|u|^{\gamma-1} u \varphi^{m-1} \nabla\varphi + \gamma |u|^{\gamma-1} (\nabla u) \varphi^m$$

we get

$$\gamma \int_\Omega |\nabla u|^m |u|^{\gamma-1} \varphi^m = -m \int_\Omega |\nabla u|^{m-2} (\nabla u, \nabla\varphi) |u|^{\gamma-1} u \varphi^{m-1} + \int_\Omega |u|^{p+\gamma} \varphi^m$$

and therefore

$$\gamma \int_\Omega |\nabla u|^m |u|^{\gamma-1} \varphi^m \leqslant m \int_\Omega |\nabla u|^{m-1} |\nabla\varphi| |u|^\gamma \varphi^{m-1} + \int_\Omega |u|^{p+\gamma} \varphi^m.$$

Writing $|u|^\gamma = |u|^{[(\gamma-1)\frac{m-1}{m} + \frac{\gamma+(m-1)}{m}]}$ and exploiting Young's inequality with exponents $m$ and $\frac{m}{m-1}$ we have

$$\gamma \int_\Omega |\nabla u|^m |u|^{\gamma-1} \varphi^m \leqslant \epsilon^2 \int_\Omega |\nabla u|^m |u|^{\gamma-1} \varphi^m + C_\epsilon \int_\Omega |\nabla\varphi|^m |u|^{\gamma+(m-1)} + \int_\Omega |u|^{p+\gamma} \varphi^m$$

that is (14).

**Step 2.** We set

$$\alpha = \alpha_\epsilon = \frac{p}{m-1} - \left(\frac{(\gamma+1)^2}{4} + \epsilon^2\right) \frac{1}{\gamma - \epsilon^2} \tag{15}$$

and we claim that there exists a positive constant $\beta = \beta(p, m, \gamma, \epsilon)$ such that

$$\alpha \int_\Omega |u|^{p+\gamma}\varphi^m \leqslant \beta \int_\Omega |\nabla\varphi|^m |u|^{\gamma+(m-1)} \tag{16}$$

for any non-negative $\varphi \in C_c^\infty(\Omega)$.

To prove this, we use the stability assumption with $\tilde\Phi = |u|^{\frac{\gamma-1}{2}} u\varphi^{\frac{m}{2}}$. Since

$$\nabla\tilde\Phi = \left(\frac{\gamma+1}{2}\right)|u|^{\frac{\gamma-1}{2}}\varphi^{\frac{m}{2}}\nabla u + \frac{m}{2}|u|^{\frac{\gamma-1}{2}} u\varphi^{\frac{m}{2}-1}\nabla\varphi$$

recalling (6), we get

$$\frac{p}{m-1}\int_\Omega |u|^{p+\gamma}\varphi^m \leqslant \left(\frac{1+\gamma}{2}\right)^2 \int_\Omega |\nabla u|^m |u|^{\gamma-1}\varphi^m + \frac{m^2}{4}\int_\Omega |\nabla u|^{m-2}|u|^{\gamma+1}|\nabla\varphi|^2\varphi^{m-2}$$
$$+ m\frac{\gamma+1}{2}\int_\Omega |\nabla u|^{m-1}|u|^\gamma\varphi^{m-1}|\nabla\varphi|. \tag{17}$$

By writing $|u|^{\gamma+1} = |u|^{[(\gamma-1)\frac{(m-2)}{m} + 2\frac{(\gamma+(m-1))}{m}]}$ and using Young's inequality with exponents $\frac{m}{m-2}$ and $\frac{m}{2}$ (recall that $m > 2$), we get

$$\frac{m^2}{4}\int_\Omega |u|^{\gamma+1}\varphi^{m-2}|\nabla u|^{m-2}|\nabla\varphi|^2 \leqslant \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^m |u|^{\gamma-1}\varphi^m + C'_\epsilon \int_\Omega |\nabla\varphi|^m |u|^{\gamma+(m-1)}.$$

Also, by Young's inequality with exponents $\frac{m}{m-1}$ and $m$, we obtain

$$m\frac{\gamma+1}{2}\int_\Omega |\nabla u|^{m-1}|u|^\gamma\varphi^{m-1}|\nabla\varphi| \leqslant \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^m\varphi^m|u|^{\gamma-1} + C''_\epsilon\int_\Omega |\nabla\varphi|^m|u|^{\gamma+(m-1)}.$$

As a consequence of (17), we get

$$\frac{p}{m-1}\int_\Omega |u|^{p+\gamma}\varphi^m \leqslant \left[\left(\frac{\gamma+1}{2}\right)^2 + \epsilon^2\right]\int_\Omega |\nabla u|^m|u|^{\gamma-1}\varphi^m + (C'_\epsilon + C''_\epsilon)\int_\Omega |\nabla\varphi|^m|u|^{\gamma+(m-1)}. \tag{18}$$

By (14) and (18),

$$\frac{p}{m-1}\int_\Omega |u|^{p+\gamma}\varphi^m \leqslant \left[\left(\frac{\gamma+1}{2}\right)^2 + \epsilon^2\right]\frac{1}{\gamma-\epsilon^2}\left(\int_\Omega |u|^{p+\gamma}\varphi^m + C_\epsilon\int_\Omega |\nabla\varphi|^m|u|^{\gamma+(m-1)}\right)$$
$$+ (C'_\epsilon + C''_\epsilon)\int_\Omega |\nabla\varphi|^m|u|^{\gamma+(m-1)}$$
$$\leqslant \left[\left(\frac{\gamma+1}{2}\right)^2 + \epsilon^2\right]\frac{1}{\gamma-\epsilon^2}\int_\Omega |u|^{p+\gamma}\varphi^m + C'''_\epsilon\int_\Omega |\nabla\varphi|^m|u|^{\gamma+(m-1)}$$

where we take $C'''_\epsilon = (C'_\epsilon + C''_\epsilon) + [(\frac{\gamma+1}{2})^2 + \epsilon^2]\frac{1}{(\gamma-\epsilon^2)}C_\epsilon$, so that

$$\left[\frac{p}{m-1} - \left(\frac{(\gamma+1)^2}{4} + \epsilon^2\right)\frac{1}{\gamma-\epsilon^2}\right]\int_\Omega |u|^{p+\gamma}\varphi^m \leqslant C'''_\epsilon\int_\Omega |\nabla\varphi|^m|u|^{\gamma+(m-1)}.$$

By setting $C'''_\epsilon = \beta$ and recalling (15), we get (16).

**Remark 2.1.** We have that

$$\lim_{\epsilon\to 0}\left[\frac{p}{m-1}-\left(\frac{(\gamma+1)^2}{4}+\epsilon^2\right)\frac{1}{\gamma-\epsilon^2}\right]=\frac{p}{m-1}-\frac{(\gamma+1)^2}{4\gamma}$$

and, as follows by elementary calculus $\frac{p}{m-1}-\frac{(\gamma+1)^2}{4\gamma}>0$ for

$$\gamma\in\left[1;\frac{2p-(m-1)+2\sqrt{p(p-(m-1))}}{(m-1)}\right)$$

Therefore under our assumption, we can assume that $\alpha>0$ for $\epsilon$ small.

**Remark 2.2.** Combining (14) and (16), we get that

$$\begin{aligned}\int_\Omega |\nabla u|^m|u|^{\gamma-1}|\varphi|^m &\leqslant \frac{1}{\gamma-\epsilon^2}\int_\Omega |u|^{p+\gamma}|\varphi|^m+\frac{C_\epsilon}{\gamma-\epsilon^2}\int_\Omega |\nabla\varphi|^m|u|^{\gamma+(m-1)}\\ &\leqslant \tilde C_\epsilon\int_\Omega |\nabla\varphi|^m|u|^{\gamma+(m-1)}\end{aligned}\tag{19}$$

for example with $\tilde C_\epsilon=\frac{C_\epsilon}{\gamma-\epsilon^2}+\frac{1}{\gamma-\epsilon^2}\frac{\beta}{\alpha}$.

**Step 3.** We claim that there exists a constant $C=C(p,m,k,\gamma)>0$ such that

$$\int_\Omega |u|^{p+\gamma}\psi^{mk}\leqslant C\int_\Omega |\nabla\psi|^{m\frac{p+\gamma}{p-(m-1)}}.\tag{20}$$

To prove this, let $\psi\in C_c^\infty(\Omega)$ with $0\leqslant\psi\leqslant 1$ and take $\varphi=\psi^k$. Note that $\nabla\varphi=k\psi^{k-1}\nabla\psi$ and so, by (16),

$$\begin{aligned}\int_\Omega |u|^{p+\gamma}\psi^{mk}&\leqslant C\int_\Omega |u|^{\gamma+(m-1)}\psi^{m(k-1)}|\nabla\psi|^m\\ &\leqslant C\left[\int_\Omega\left(|u|^{\gamma+(m-1)}\psi^{m(k-1)}\right)^{\frac{p+\gamma}{\gamma+(m-1)}}\right]^{\frac{\gamma+(m-1)}{p+\gamma}}\left[\int_\Omega\left(|\nabla\psi|^m\right)^{\frac{p+\gamma}{p-(m-1)}}\right]^{\frac{p-(m-1)}{p+\gamma}}\end{aligned}\tag{21}$$

where this makes sense since $p>m-1$. Also

$$m(k-1)\left(\frac{p+\gamma}{\gamma+(m-1)}\right)\geqslant mk\tag{22}$$

since we assumed $k\geqslant\frac{p+\gamma}{p-(m-1)}$. Consequently, recalling that $0\leqslant\psi\leqslant 1$, we have

$$\int_\Omega |u|^{p+\gamma}\psi^{mk}\leqslant C\left[\int_\Omega |u|^{p+\gamma}\psi^{mk}\right]^{\frac{\gamma+(m-1)}{p+\gamma}}\left[\int_\Omega |\nabla\psi|^{m\frac{p+\gamma}{p-(m-1)}}\right]^{\frac{p-(m-1)}{p+\gamma}}$$

and consequently

$$\int_\Omega |u|^{p+\gamma}\psi^{mk}\leqslant C\int_\Omega |\nabla\psi|^{m\frac{p+\gamma}{p-(m-1)}}$$

that is (20).

**Step 4.** There exists a constant $C=C(p,m,k,\gamma)>0$ such that

$$\int_\Omega |\nabla u|^m|u|^{\gamma-1}\psi^{mk}\leqslant C\int_\Omega |\nabla\psi|^{m\frac{p+\gamma}{p-(m-1)}}\tag{23}$$

for any $\psi \in C_c^\infty(\Omega)$ such that $0 \leqslant \psi \leqslant 1$. Indeed, we consider (19) for $\varphi = \psi^k$ (thence $\nabla\varphi = k\psi^{k-1}\nabla\psi$), obtaining

$$\int_\Omega |\nabla u|^m |u|^{\gamma-1}\psi^{mk} \leqslant C\int_\Omega |u|^{\gamma+(m-1)}\psi^{m(k-1)}|\nabla\psi|^m$$
$$\leqslant C\bigg[\int_\Omega \big(|u|^{\gamma+(m-1)}\psi^{m(k-1)}\big)^{\frac{p+\gamma}{\gamma+(m-1)}}\bigg]^{\frac{\gamma+(m-1)}{p+\gamma}}\bigg[\int_\Omega |\nabla\psi|^{m\frac{p+\gamma}{p-(m-1)}}\bigg]^{\frac{p-(m-1)}{p+\gamma}}$$

and so, by taking into account (20), we get (23).

**Conclusion.** The thesis of Proposition 1.4 follows by collecting the estimates in (20) and (23).

Arguing exactly as above (see [14,15] for related results when $m = 2$), we can also state and prove the following

**Proposition 2.3.** *Let $\Omega$ be a smooth domain bounded or not of $\mathbb{R}^N$. Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ be a solution of* (3) *which is* stable outside a compact set $\mathcal{K} \subset \Omega$ *with $p > (m-1)$ and $m > 2$. Then, for every*

$$\gamma \in \bigg[1; \frac{2p-(m-1)+2\sqrt{p(p-(m-1))}}{(m-1)}\bigg)$$

*and for any integer k with*

$$k \geqslant \max\bigg\{\frac{p+\gamma}{p-(m-1)}; 2\bigg\}.$$

*Then there exists a positive constant $C = C(p,m,k,\gamma)$ such that*

$$\int_\Omega |\nabla u|^m|u|^{\gamma-1}\psi^{mk} + |u|^{p+\gamma}\psi^{mk} \leqslant C\int_\Omega |\nabla\psi|^{m\frac{p+\gamma}{p-(m-1)}} \tag{24}$$

*for all test functions $\psi \in C_c^1(\mathbb{R}^N \setminus \mathcal{K})$ with $0 \leqslant \psi \leqslant 1$.*

**Proof.** The proof is exactly the same as in Proposition 1.4. We only have to note that here, even if $\psi \in C_c^1(\mathbb{R}^N \setminus \mathcal{K})$ does not have compact support in $\Omega$, all the test-functions used in the proof of Proposition 1.4 are good test-functions since $u$ fulfills the zero Dirichlet boundary condition. □

## 3. Proof of Proposition 1.6

We will follow very closely the proof of Proposition 1.4. For shortness, we will assume that $C_\epsilon = C(p,m,k,\gamma,\epsilon) > 0$ is a generic constant (which, as usual, may take different values on different occurrences).

**Step 1.** We have that for any $\gamma \geqslant 1$ and for any $0 < \epsilon < \sqrt{\gamma}$, there exists a constant $C_\epsilon$ such that

$$\frac{4}{(\gamma+1)^2}\big(\gamma-\epsilon^2\big)\int_\Omega |\nabla u|^{m-2}\big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\varphi^2 \leqslant \int_\Omega |u|^{p+\gamma}\varphi^2 + C_\epsilon\int_\Omega |\nabla u|^{m-2}|u|^{\gamma+1}|\nabla\varphi|^2 \tag{25}$$

for any $\varphi \in C_c^\infty(\Omega)$.

To prove this, let us consider $\Phi = |u|^{\gamma-1}u\varphi^2$, so that

$$\nabla\Phi = 2|u|^{\gamma-1}u\varphi\nabla\varphi + \gamma|u|^{\gamma-1}(\nabla u)\varphi^2.$$

Taking $\Phi$ as test function in (2), we have

$$\gamma\int_\Omega |\nabla u|^{m-2}|\nabla u|^2|u|^{\gamma-1}\varphi^2 = -2\int_\Omega |\nabla u|^{m-2}(\nabla u,\nabla\varphi)|u|^{\gamma-1}u\varphi + \int_\Omega |u|^{p+\gamma}\varphi^2.$$

Thence

$$\gamma \int_\Omega |\nabla u|^{m-2}|u|^{\gamma-1}|\nabla u|^2\varphi^2 \leqslant \int_\Omega |u|^{p+\gamma}\varphi^2 + \epsilon^2 \int_\Omega |\nabla u|^{m-2}|u|^{\gamma-1}|\nabla u|^2\varphi^2 + C_\epsilon \int_\Omega |\nabla u|^{m-2}|\nabla\varphi|^2|u|^{\gamma+1}$$

and therefore we get (25).

**Step 2.** Now we set $\frac{4}{(\gamma+1)^2}(\gamma-\epsilon^2) = A_\epsilon = A$,

$$\frac{p}{m-1} - \frac{1+\epsilon^2}{A} = \alpha_\epsilon = \alpha$$

and we prove that for some positive constant, say $\beta$, we have

$$\alpha \int_\Omega |u|^{p+\gamma}\varphi^2\,dx \leqslant \beta \int_\Omega |\nabla u|^{m-2}|u|^{\gamma+1}|\nabla\varphi|^2\,dx. \tag{26}$$

Indeed, exploiting (6) with $\Phi = |u|^{\frac{\gamma-1}{2}}u\varphi$, we get

$$\frac{p}{m-1}\int_\Omega |u|^{p+\gamma}\varphi^2 \leqslant \left(1+\epsilon^2\right)\int_\Omega |\nabla u|^{m-2}\left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\varphi^2 + C_\epsilon\int_\Omega |\nabla u|^{m-2}|u|^{\gamma+1}|\nabla\varphi|^2.$$

By (25)

$$\frac{p}{m-1}\int_\Omega |u|^{p+\gamma}\varphi^2 \leqslant \frac{1+\epsilon^2}{A}\int_\Omega |u|^{p+\gamma}\varphi^2 + C_\epsilon\int_\Omega |\nabla u|^{m-2}|u|^{\gamma+1}|\nabla\varphi|^2$$

hence (26) follows. Note that, as in Remark 2.1, we can take $\epsilon$ so small that $\alpha_\epsilon > 0$.

**Step 3.** Let now $\psi \in C_c^\infty(\Omega)$ such that $0 \leqslant \psi \leqslant 1$. We show that

$$\int_\Omega |u|^{p+\gamma}\psi^{2k} \leqslant C_\epsilon \int_\Omega \left(|\nabla u|^{m-2}|\nabla\psi|^2\right)^{\frac{p+\gamma}{p-1}}. \tag{27}$$

To prove this, we use (26) with $\varphi = \psi^k$, so that

$$\alpha \int_\Omega |u|^{p+\gamma}\psi^{2k} \leqslant \beta k^2 \int_\Omega |\nabla u|^{m-2}\psi^{2k-2}|\nabla\psi|^2|u|^{\gamma+1}$$

with $\alpha,\ \beta > 0$ as above, and, as a consequence, by Hölder's inequality,

$$\int_\Omega |u|^{p+\gamma}\psi^{2k} \leqslant C_\epsilon\left[\int_\Omega \left(|u|^{\gamma+1}\psi^{2k-2}\right)^{\frac{p+\gamma}{1+\gamma}}\right]^{\frac{\gamma+1}{p+\gamma}}\left[\int_\Omega \left(|\nabla u|^{m-2}|\nabla\psi|^2\right)^{\frac{p+\gamma}{p-1}}\right]^{\frac{p-1}{p+\gamma}}.$$

Now we recall that $k \geqslant \max\{\frac{p+\gamma}{p-1}; 2\}$ and so

$$\psi^{\frac{(2k-2)(p+\gamma)}{\gamma+1}} \leqslant \psi^{2k} \tag{28}$$

since $0 \leqslant \psi \leqslant 1$.

Therefore,

$$\int_\Omega |u|^{p+\gamma}\psi^{2k} \leqslant C_\epsilon\left[\int_\Omega |u|^{p+\gamma}\psi^{2k}\right]^{\frac{\gamma+1}{p+\gamma}}\left[\int_\Omega \left(|\nabla u|^{m-2}|\nabla\psi|^2\right)^{\frac{p+\gamma}{p-1}}\right]^{\frac{p-1}{p+\gamma}}.$$

Hence, (27) follows at once.

**Step 4.** We have

$$\int_\Omega |\nabla u|^{m-2}\big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\psi^{2k}\leqslant C_\epsilon\int_\Omega\big(|\nabla u|^{m-2}|\nabla\psi|^2\big)^{\frac{p+\gamma}{p-1}}. \tag{29}$$

To prove this, note that by (25) and (26)

$$\int_\Omega |\nabla u|^{m-2}\big(\big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\big)\varphi^2\leqslant C_\epsilon\int_\Omega |\nabla u|^{m-2}|u|^{\gamma+1}|\nabla\varphi|^2$$

and so, for $\varphi=\psi^k$,

$$\int_\Omega |\nabla u|^{m-2}\big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\psi^{2k}\leqslant C_\epsilon\int_\Omega |\nabla u|^{m-2}|u|^{\gamma+1}\psi^{2k-2}|\nabla\psi|^2.$$

Therefore, using again (28), we have

$$\begin{aligned}\int_\Omega |\nabla u|^{m-2}\big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\psi^{2k}&\leqslant C_\epsilon\bigg[\int_\Omega\big(|u|^{\gamma+1}\psi^{2k-2}\big)^{\frac{p+\gamma}{\gamma+1}}\bigg]^{\frac{\gamma+1}{p+\gamma}}\bigg[\int_\Omega\big(|\nabla u|^{m-2}|\nabla\psi|^2\big)^{\frac{p+\gamma}{p-1}}\bigg]^{\frac{p-1}{p+\gamma}}\\&\leqslant C_\epsilon\bigg[\int_\Omega |u|^{p+\gamma}\psi^{2k}\bigg]^{\frac{\gamma+1}{p+\gamma}}\bigg[\int_\Omega\big(|\nabla u|^{m-2}|\nabla\psi|^2\big)^{\frac{p+\gamma}{p-1}}\bigg]^{\frac{p-1}{p+\gamma}}.\end{aligned}$$

In this way, (29) follows from (27).

**Conclusion.** By collecting the estimates in (27) and (29), we get (9).

## 4. Proof of Theorem 1.5

Let $\psi\in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that $0\leqslant\psi\leqslant 1$ everywhere and

$$\begin{aligned}&\psi\equiv 1,\quad \text{in } B(0,R),\\&\psi\equiv 0,\quad \text{outside } B(0,2R),\\&|\nabla\psi|\leqslant\frac{C}{R}.\end{aligned}$$

Consequently, Proposition 1.4 gives

$$\int_\Omega\big(|\nabla u|^m|u|^{\gamma-1}+|u|^{p+\gamma}\big)\psi^{mk}\leqslant C R^{N-m\frac{p+\gamma}{p-(m-1)}}. \tag{30}$$

When $N-m\frac{p+\gamma}{p-(m-1)}<0$, the desired claim follows by letting $R\to\infty$.

Let us now consider the case in which $N-m\frac{p+\gamma}{p-(m-1)}<0$.

Define, for $t>m-1$,

$$\bar\gamma(t)=\frac{2t-(m-1)+2\sqrt{t(t-(m-1))}}{m-1}$$

and set

$$g(t)=\frac{m(t+\bar\gamma(t))}{t-(m-1)}.$$

We have that $g(t)$ is decreasing. Also,

$$\lim_{t\to\infty}g(t)=\frac{(m+3)m}{m-1}\quad\text{and}\quad\lim_{t\to(m-1)^+}g(t)=+\infty.$$

Therefore, if

$$N \leqslant \frac{(m+3)m}{m-1}$$

then $N < g(t)$ for any $t > m-1$, hence if we fix $\gamma \in [1, \bar{\gamma}(p))$, suitably near $\bar{\gamma}(p)$, we obtain

$$N - \frac{m(p+\gamma)}{p-(m-1)} < 0.$$

For this reason, the desired result follows by letting $R \to \infty$ in (30).

Assume now $N > \frac{(m+3)m}{m-1}$. Since $g$ is decreasing, we get in this case a critical value $p_c(N,m)$ such that

$$N - \frac{m(p+\bar{\gamma}(p))}{p-(m-1)} < 0$$

for $p < p_c(N,m)$.

From this, the desired result follows again by letting $R \to \infty$ in (30).

Of course, $p_c(N,m)$ may be deduced from the equation $N - \frac{m(p+\bar{\gamma})}{p-(m-1)} = 0$, giving the value in (8).

## 5. Proof of Theorem 1.7

The proof is an application of Proposition 1.6.

Let $\psi \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that $0 \leqslant \psi \leqslant 1$ everywhere and

$$\psi \equiv 1, \quad \text{in } B(0,R),$$
$$\psi \equiv 0, \quad \text{outside } B(0,2R),$$
$$|\nabla \psi| \leqslant \frac{C}{R}.$$

Then, Proposition 1.6 gives

$$\int_\Omega \left(|\nabla u|^{m-2}\left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2 + |u|^{p+\gamma}\right)\psi^{2k} \leqslant C R^{N+(\beta(m-2)-2)(\frac{p+\gamma}{p-1})}.$$

The desired claim follows by letting $R \to \infty$.

## 6. Monotonicity and stability

When $m = 2$ it is a well-known fact that monotone solutions are stable. The same holds in our contest as shown by the following simple considerations.

**Definition 6.1.** We say that the Weak Maximum Principle (WMP) holds for the linearized operator $L_u$ in $\Omega$, if, given $v \in H_\rho^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} L_u(v,\varphi) \geqslant 0 & \text{for } \varphi \in C_c^1(\Omega) \\ v \geqslant 0 & \text{on } \partial\Omega \end{cases} \tag{31}$$

it follows that $v \geqslant 0$ in $\Omega$.

**Lemma 6.2.** *Let $u$ be a solution of* (2) *in $\Omega$* (*possibly unbounded*). *Assume that the derivative of $u$ in some direction is positive, say for example $\partial_{x_N} u > 0$. Then,*

*$u$ is* stable.

**Proof.** Let $\varphi \in C_c^1(\Omega)$. Since $\partial_{x_N} u > 0$, we have that $\nabla u \neq 0$ everywhere. Therefore the linearized operator is non-degenerate. Since also $\partial_{x_N} u > 0$ is a solution of the linearized equation, it follows that the WMP holds for $L_u$ in

$B(0,R)$ for any $R>0$. Let us now take $R>0$ such that $\mathrm{supp}(\varphi)\subset B(0,R)$. We have that the first eigenvalue of $L_u$ in $B(0,R)$ is positive and therefore $L_u(\varphi,\varphi)\geqslant 0$. □

Further details about monotonicity and stabilities are given in Section 7 of [16].

## 7. Proof of Theorem 1.8

Since $u\geqslant 0$, by the Strong Maximum Principle (see [28,24]) we have that either $u>0$ or $u=0$. If $u=0$ we are done. If else $u>0$ we are in the position of using the moving plane method (see [7–9,12]) and prove that $u$ has positive derivative in some direction. We recall that, for uniformly elliptic operators, the moving plane technique has been extended to the case of domains which are coercive epigraphs in [12]. We refer to [7,8] for the case of equations involving the $m$-Laplace operator in bounded domains. Combining the arguments in [12,7] it is easily seen that the moving plane technique applies in our case. Therefore the monotonicity of $u$ shows that $u$ is stable by Lemma 6.2, and the result follows by repeating the proof of Theorem 1.5. The only difference is that the use of Proposition 1.4 in Theorem 1.5 is replaced by the use of Proposition 2.3 here.

## 8. Pohozaev type identity

We consider $u\in C^{1,\alpha}$ to be weak solution of

$$-\Delta_m(u)=g(u)\quad \text{in } \Omega$$

where $\Omega\subseteq\mathbb{R}^N$ is bounded and smooth. Therefore we have

$$\int_\Omega |\nabla u|^{m-2}(\nabla u,\nabla\varphi)\,dx=\int_\Omega g(u)\varphi\,dx\quad \forall\varphi\in C_c^\infty(\Omega). \tag{32}$$

Assume that $g(\cdot)$ is locally Lipschitz continuous. We will use $\eta(x)$ to indicate the unit outward normal vector to $\partial\Omega$ at a point $x\in\partial\Omega$.

Also, we will use the notation $u_\eta(x)=\frac{\partial u}{\partial\eta}(x)$ and we define

$$G(t)=\int_0^t g(s)\,ds.$$

**Lemma 8.1.** *Let $u\in C^{1,\alpha}(\Omega)\cap C^0(\overline{\Omega})$ be a weak solution of*

$$-\Delta_m(u)=g(u)\quad \textit{in } \Omega$$

*where $\Omega\subseteq\mathbb{R}^N$ is bounded and smooth and $g$ is locally Lipschitz continuous. Then we have*

$$-\Delta_m(u)=g(u),\quad \textit{a.e. in } \Omega.$$

*More precisely the equation is fulfilled in the classic sense for almost every $x\in\Omega$.*

**Proof.** We recall (see for example [7]) that

$$|\nabla u|^{m-2}\nabla u\in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N).$$

Thus, by (32), integrating by parts, we get

$$\int_\Omega -\Delta_m(u)\varphi\,dx=\int_\Omega g(u)\varphi\,dx\quad \forall\varphi\in C_c^\infty(\Omega)$$

and the thesis. □

**Proposition 8.2.** *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a weak solution of*

$$-\Delta_m(u) = g(u) \quad \text{in } \Omega$$

*where $\Omega \subseteq \mathbb{R}^N$ is bounded and smooth and $g$ is locally Lipschitz continuous. Then we have*

$$N\int_{\Omega} G(u)\,dx - \frac{N-m}{m}\int_{\Omega} |\nabla u|^m\,dx = \int_{\partial\Omega}\left[G(u)(x\cdot\eta) + |\nabla u|^{m-2}(x\cdot\nabla u)u_\eta - \frac{|\nabla u|^m}{m}(x\cdot\eta)\right]dS. \tag{33}$$

**Proof.** Since by Lemma 8.1 we have $-\Delta_m(u) = g(u)$ a.e. in $\Omega$, then

$$\int_{\Omega} \operatorname{div}\big(|\nabla u|^{m-2}\nabla u\big)(x\cdot\nabla u)\,dx = \int_{\Omega} \Delta_m(u)(x\cdot\nabla u)\,dx = \int_{\Omega} -g(u)(x\cdot\nabla u)\,dx. \tag{34}$$

We note now that

$$\int_{\Omega} -g(u)(x\cdot\nabla u)\,dx = \sum_{i=1}^{N}\int_{\Omega} -x_i\big(G(u)\big)_i\,dx = N\int_{\Omega} G(u)\,dx - \int_{\partial\Omega} G(u)(x\cdot\eta)\,dS. \tag{35}$$

Also, since $\operatorname{div}(|\nabla u|^{m-2}\nabla u)(u_i\cdot x_i) \in L^1(\Omega)$ by the regularity results in [7], and $(|\nabla u|^{m-2}\nabla u)u_i\cdot x_i \in C^0(\overline{\Omega})$ by assumption, then we can exploit the divergence theorem (see [5]) obtaining

$$\begin{aligned}
&\int_{\Omega} \operatorname{div}\big(|\nabla u|^{m-2}\nabla u\big)(x\cdot\nabla u)\,dx \\
&\quad = -\int_{\Omega} |\nabla u|^m\,dx + \int_{\partial\Omega} |\nabla u|^{m-2}(x\cdot\nabla u)u_\eta\,dS - \sum_{i=1}^{N}\int_{\Omega} |\nabla u|^{m-2}(\nabla u,\nabla u_i)x_i\,dx \\
&\quad = -\int_{\Omega} |\nabla u|^m\,dx + \int_{\partial\Omega} |\nabla u|^{m-2}(x\cdot\nabla u)u_\eta\,dS - \sum_{i=1}^{N}\int_{\Omega} \frac{1}{m}\left(\frac{\partial}{\partial x_i}|\nabla u|^m\right)\cdot x_i\,dx \\
&\quad = -\int_{\Omega} |\nabla u|^m\,dx + \frac{N}{m}\int_{\Omega} |\nabla u|^m\,dx + \int_{\partial\Omega} |\nabla u|^{m-2}(x\cdot\nabla u)u_\eta\,dS - \int_{\partial\Omega} \frac{|\nabla u|^m}{m}(x\cdot\eta)\,dS
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{\Omega} \operatorname{div}\big(|\nabla u|^{m-2}\nabla u\big)(x\cdot\nabla u)\,dx - \frac{N-m}{m}\int_{\Omega} |\nabla u|^m\,dx \\
&\quad = \int_{\partial\Omega} |\nabla u|^{m-2}(x\cdot\nabla u)u_\eta\,dS - \int_{\partial\Omega} \frac{|\nabla u|^m}{m}(x\cdot\eta)\,dS.
\end{aligned} \tag{36}$$

Exploiting (34), (35) and (36), we get the thesis. □

**Corollary 8.3.** *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ be a weak solution of the equation $-\Delta_m(u) = g(u)$ where $g$ is locally Lipschitz continuous. Assume that*

$$|\nabla u| \in L^m\big(\mathbb{R}^N\big) \quad \text{and} \quad G(u) \in L^1\big(\mathbb{R}^N\big).$$

*Then,*

$$N\int_{\mathbb{R}^N} G(u)\,dx = \frac{N-m}{m}\int_{\mathbb{R}^N} |\nabla u|^m\,dx.$$

**Proof.** By assumption,

$$\int_{\mathbb{R}^N} \left(\left|G(u)\right| + |\nabla u|^m\right) dx < \infty$$

and so

$$\int_0^\infty \int_{\partial B(0,r)} \left(\left|G(u)\right| + |\nabla u|^m\right) dS\, dr < \infty.$$

For this reason,

$$\liminf_{r\to\infty} r \int_{\partial B(0,r)} \left(\left|G(u)\right| + |\nabla u|^m\right) dS = 0. \tag{37}$$

If now we exploit Proposition 8.2 with $\Omega = B(0,r)$ we get the thesis noticing that the boundary term in (33) vanishes at infinity thanks to (37). □

**Remark 8.4.** If, in Corollary 8.3, we consider the case

$$g(u) = |u|^{p-1}u$$

and we assume $u \in L^{p+1}(\mathbb{R}^N)$, we get

$$\frac{N}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\, dx = \frac{N-m}{m} \int_{\mathbb{R}^N} |\nabla u|^m\, dx.$$

## 9. Proof of Theorem 1.9

We claim that there exist positive constants $A$, $B$ and $R_0$, possibly depending on $u$, such that, for $r > R_0 + 3$ and

$$\gamma \in \left[1; \frac{2p-(m-1)+2\sqrt{p(p-(m-1))}}{(m-1)}\right)$$

we have

$$\int_{\{R_0+2<|x|<r\}} \left(|\nabla u|^m |u|^{\gamma-1} + |u|^{p+\gamma}\right) dx \leqslant A + Br^{N-\frac{m(p+\gamma)}{p-(m-1)}}. \tag{38}$$

To prove this, let us first fix $R_0$ so that $B(0,R_0) \supset \mathcal{K}$. Given $r > R_0 + 3$ we consider

$$\phi_r(x) \equiv \begin{cases} 0 & \text{if } |x| < R_0 + 1, \\ 1 & \text{if } R_0 + 2 < |x| < r, \\ 0 & \text{if } |x| > 2r \end{cases}$$

and we may and do assume that $0 \leqslant \phi_r(x) \leqslant 1$, and $|\nabla \phi_r(x)| \leqslant C$ in $B(0,R_0+2) \setminus B(0,R_0+1)$, and $|\nabla \phi_r(x)| \leqslant \frac{C}{r}$ in $B(0,2r) \setminus B(0,r)$. Therefore, exploiting Proposition 1.4, we get

$$\int_{\{R_0+2<|x|<r\}} \left(|\nabla u|^m |u|^{\gamma-1} + |u|^{p+\gamma}\right) dx \leqslant C \int_{\{R_0+1<|x|<R_0+2\}} \left|\nabla \phi_r(x)\right|^{\frac{m(p+\gamma)}{p-(m-1)}} dx + C \int_{\{r<|x|<2r\}} \left|\nabla \phi_r(x)\right|^{\frac{m(p+\gamma)}{p-(m-1)}} dx$$

from which (38) plainly follows.

Taking $\gamma = 1$ in (38), recalling assumption (10), it follows that

$$N - \frac{m(p+\gamma)}{p-(m-1)} < 0.$$

Therefore, letting $r \to \infty$, we conclude that

$$|\nabla u| \in L^m(\mathbb{R}^N) \quad \text{and} \quad u \in L^{p+1}(\mathbb{R}^N).$$

This gives the right summability needed to exploit Pohozaev identity (see Corollary 8.3 and Remark 8.4), that gives

$$\frac{N}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\, dx = \frac{N-m}{m} \int_{\mathbb{R}^N} |\nabla u|^m\, dx. \tag{39}$$

Let us now consider

$$\varphi_R(x) \equiv \begin{cases} 1 & \text{if } |x| < R, \\ 0 & \text{if } |x| > 2R \end{cases}$$

with $0 \leqslant \varphi_R(x) \leqslant 1$, and $|\nabla \varphi_R(x)| \leqslant \frac{C}{R}$ for $R < |x| < 2R$. Using $u \cdot \varphi_R$ as test function in (2), we get

$$\int_{B(0,2R)} |\nabla u|^m \varphi_R\, dx + \int_{B(0,2R)} |\nabla u|^{m-2}(\nabla u, \nabla \varphi_R) u\, dx = \int_{B(0,2R)} |u|^{p+1} \varphi_R\, dx. \tag{40}$$

Note now that

$$\begin{aligned}\left| \int_{B(0,2R)} |\nabla u|^{m-2}(\nabla u, \nabla \varphi_R) u\, dx \right| &\leqslant \int_{B(0,2R)} |\nabla u|^{m-1} |\nabla \varphi_R| |u|\, dx \\ &\leqslant \text{const} \left( \int_{\mathbb{R}^N} |\nabla u|^m\, dx \right)^{\frac{m-1}{m}} \left( \int_{\mathbb{R}^N} |u|^{p+1}\, dx \right)^{\frac{1}{p+1}} R^{\frac{N(p+1-m)}{m(p+1)} - 1}\end{aligned}$$

and so, letting $R \to \infty$ and noticing that $\frac{N(p+1-m)}{m(p+1)} - 1 < 0$, since $N - \frac{m(p+1)}{p-(m-1)} < 0$ under our assumptions, we gather that

$$\lim_{R \to \infty} \left| \int_{B(0,2R)} |\nabla u|^{m-2}(\nabla u, \nabla \varphi_R) u\, dx \right| = 0.$$

As a consequence, (40) becomes

$$\int_{\mathbb{R}^N} |\nabla u|^m\, dx = \int_{\mathbb{R}^N} |u|^{p+1}\, dx. \tag{41}$$

Combining (39) and (41) we get

$$\left( \frac{N-m}{m} - \frac{N}{p+1} \right) \int_{\mathbb{R}^N} |u|^{p+1}\, dx = 0.$$

Since under our assumptions $(\frac{N-m}{m} - \frac{N}{p+1}) \neq 0$, we obtain the desired thesis.

## 10. A counterexample for the critical case

In this section we show the existence of positive radial solutions for (2), which are stable outside a compact set, for the limit case

$$p = \frac{N(m-1)+m}{N-m}$$

assuming $2 < m < N$ (the case $m = 2$ was treated in [14,15]). Recalling the classification results in [1] (see Theorem 6.1 therein), we know that the non-constant global radial solutions of

$$-\Delta_m(u) = |u|^{p-1} u \quad \text{in } \mathbb{R}^N$$

are positive (or negative), and they are given by

$$u_\lambda\big(|x|\big) = u_c(r) = c\left(r^{\frac{m}{m-1}} + \frac{1}{N}\left(\frac{m-1}{N-m}\right)^{(m-1)} c^{(\frac{m^2}{N-m})}\right)^{\frac{m-N}{m}}, \quad c > 0.$$

Equivalently,

$$u_\lambda\big(|x|\big) = u_\lambda(r) = \lambda\left(\frac{\lambda^{\frac{1}{m-1}}(N^{\frac{1}{m}}(\frac{N-m}{m-1})^{\frac{m-1}{m}})}{\lambda^{\frac{m}{m-1}} + r^{\frac{m}{m-1}}}\right)^{\frac{N-m}{m}}, \quad \lambda > 0. \tag{42}$$

In particular the exact behavior of $u_\lambda$ and $\nabla u_\lambda$ at infinity is known and is given by

$$u_\lambda(r) \approx \mathrm{const}\left(\frac{1}{r}\right)^{\frac{N-m}{m-1}}$$

and

$$\big|\nabla u_\lambda(r)\big| \approx \mathrm{const}\left(\frac{1}{r}\right)^{\frac{N-1}{m-1}}.$$

**Proposition 10.1.** *Let $u = u_\lambda$ be a positive radial solution of* (2)*, given by* (42)*. Then there exists $R_0 > 0$ such that $u$ is stable outside $B(0, R_0)$.*

**Proof.** Given $R_0 > 0$, let us consider $\Psi \in C_c^1(\mathbb{R}^N \setminus B(0, R_0))$. We will show that, if $R_0$ is sufficiently large, then

$$\int\limits_{\mathbb{R}^N\setminus B(0,R_0)} |\nabla u|^{m-2}|\nabla\Psi|^2 + (m-2)|\nabla u|^{m-4}(\nabla u, \nabla\Psi)^2 - pu^{p-1}\Psi^2\,dx \geqslant 0$$

recalling that $u = |u|$ since we assumed that $u$ is positive.

We note now that

$$u^{p-1} = \mathrm{O}\left(\frac{1}{r}\right)^{\frac{N(m-2)+2m}{m-1}} = \mathrm{o}\left(\frac{1}{r}\right)^{\frac{N(m-2)+m}{m-1}}.$$

Therefore for $R_0$ sufficiently large we may assume that

$$pu^{p-1}\Psi^2 \leqslant \tilde{C}\cdot\frac{\Psi^2}{|x|^\tau}$$

where $\tilde{C}$ may be taken as small as we like (the value of $\tilde{C}$ will be fixed in (44) below).

Moreover $\tau = \tau(N, m) = \frac{N(m-2)+m}{m-1}$. It follows now that

$$\int\limits_{\mathbb{R}^N\setminus B(0,R_0)} |\nabla u|^{m-2}|\nabla\Psi|^2 + (m-2)|\nabla u|^{m-4}(\nabla u, \nabla\Psi)^2 - pu^{p-1}\Psi^2\,dx$$

$$\geqslant \int\limits_{\mathbb{R}^N\setminus B(0,R_0)} |\nabla u|^{m-2}|\nabla\Psi|^2 - \tilde{C}\cdot\frac{\Psi^2}{|x|^\tau}\,dx \tag{43}$$

where we also used that $m > 2$. Exploiting the weighted Hardy's inequality (see for example Lemma 2.3 in [6]), we get that

$$\int\limits_{\mathbb{R}^N\setminus B(0,R_0)} \frac{\Psi^2}{|x|^\tau}\,dx \leqslant \left(\frac{2}{N-\tau}\right)^2 \int\limits_{\mathbb{R}^N\setminus B(0,R_0)} \frac{|\nabla\Psi|^2}{|x|^{\tau-2}}\,dx$$

where $\tau < N$ since $m < N$. Moreover, recalling that

$$|\nabla u|^{m-2} \approx \left(\frac{1}{|x|}\right)^{(N-1)\frac{m-2}{m-1}} = \left(\frac{1}{|x|}\right)^{\tau-2}$$

we have

$$\frac{|\nabla u|^{2-m}}{|x|^{\tau-2}} \leqslant K(N,m,u)$$

for $R_0$ large and a suitable constant $K(N,m,u)$. Therefore

$$\int\limits_{\mathbb{R}^N\setminus B(0,R_0)} \tilde{C}\cdot\frac{\Psi^2}{|x|^\tau}\,dx \leqslant \tilde{C}\cdot K(N,m,u)\int\limits_{\mathbb{R}^N\setminus B(0,R_0)} |\nabla u|^{m-2}|\nabla\Psi|^2\,dx.$$

Hence, taking $\tilde{C}$ so that

$$\tilde{C}\cdot K(N,m,u) < 1 \tag{44}$$

we gather that

$$\int\limits_{\mathbb{R}^N\setminus B(0,R_0)} |\nabla u|^{m-2}|\nabla\Psi|^2 - \tilde{C}\cdot\frac{\Psi^2}{|x|^\tau}\,dx \geqslant 0.$$

Then, the result follows from (43). □

## 11. The supercritical case: Proof of Theorem 1.10

In this section we will assume that $m < N$ and $p$ is supercritical with respect to the classic critical exponent, while it is subcritical with respect to the new critical exponent. More precisely we assume here to be in the range given by (12), with $p_c(N,m)$ as in (8).

Let us prove some preliminary results:

**Lemma 11.1.** *Let $m > 2$ and $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution of* (2) *which is stable outside the compact set $\mathcal{K}$. Assume that* (12) *holds.*

*Then, there exists a constant C such that, if*

$$\gamma \in \left[1; \frac{2p-(m-1)+2\sqrt{p(p-(m-1))}}{(m-1)}\right)$$

*and $B(y,2R) \subset (B(0,R_0))^c \subset \mathcal{K}^c$, then we have*

$$\int\limits_{B(y,R)} \left(|\nabla u|^m|u|^{\gamma-1} + |u|^{p+\gamma}\right)dx \leqslant CR^{N-\frac{m(p+\gamma)}{p-(m-1)}}. \tag{45}$$

**Proof.** Let us consider

$$\Psi_{R,y}(x) \equiv \begin{cases} 1 & \text{if } |x-y| < R, \\ 0 & \text{if } |x-y| > 2R \end{cases}$$

assuming also that $0 \leqslant \Psi_{R,y}(x) \leqslant 1$, and $|\nabla\Psi_{R,y}(x)| \leqslant \frac{C}{R}$ for $R < |x-y| < 2R$. Exploiting Proposition 1.4 and following the proof in Step 1 of Theorem 1.9, we get (45). □

**Lemma 11.2.** *Let $m > 2$ and $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution of* (2) *which is stable outside the compact set $\mathcal{K}$. Assume that $\mathcal{K} \subset B(0,R_0)$ and that* (12) *holds.*

*We can always set*

$$(p-m+1)\frac{N}{m} = p + \gamma_1 \tag{46}$$

*for some $\gamma_1$ with $1 \leqslant \gamma_1 < \gamma_M(m,p) = \frac{2p-(m-1)+2\sqrt{p(p-(m-1))}}{(m-1)}$, and*

$$1 \leqslant (p-m+1)\frac{N}{m-\varepsilon} - p < \gamma_M(m,p) \tag{47}$$

*for a small $\varepsilon \in (0,1]$.*

*Then there exists $R_1 = R_1(p, m, N) > R_0$ such that $u \in L^{[(p-m+1)\frac{N}{m}]}(\mathbb{R}^N \setminus B(0, R_1))$. In particular, for every $\eta > 0$, eventually taking a larger $R_1$, we can always assume that*

$$\int_{|x|\geqslant R_1} |u|^{(p-m+1)\frac{N}{m}}\, dx < \eta. \tag{48}$$

**Proof.** Eq. (47) can be easily deduced by (46) for $\varepsilon$ sufficiently small. Let us consider (46).

Note that the assumption $p \geqslant \frac{N(m-1)+m}{N-m}$ implies that $(p-m+1)\frac{N}{m} - p \geqslant 1$. Also, arguing as in the proof of Theorem 1.5, we can check that $(p-m+1)\frac{N}{m} - p < \gamma_M(m, p)$.

By Step 1 in Theorem 1.9 we know that there exist constants $A$, $B$, possibly depending on $u$, such that, for $r > R_0 + 3$ we have

$$\int_{\{R_0+2<|x|<r\}} \left(|u|^{(p-m+1)\frac{N}{m}}\right) dx = \int_{\{R_0+2<|x|<r\}} \left(|u|^{p+\gamma_1}\right) dx \leqslant A + Br^{N-\frac{m(p+\gamma_1)}{p-(m-1)}} = A + Br^0 \tag{49}$$

so that $u \in L^{[(p-m+1)\frac{N}{m}]}(\mathbb{R}^N \setminus B(0, R_1))$ follows. □

We are now in the position of ending the proof of Theorem 1.10. For this, we strongly use the results by J. Serrin in [25] to deduce a result on the decay of solutions which are stable outside compact sets and we follow some ideas used in [15] for the semilinear case $m = 2$.

Let $R_0$ as above, with $\mathcal{K} \subset B(0, R_0)$ and define

$$p + \gamma_2 = (p-m+1)\frac{N}{m-\varepsilon}$$

with $1 \leqslant \gamma_2 < \gamma_M(m, p) = \frac{2p-(m-1)+2\sqrt{p(p-(m-1))}}{(m-1)}$. This is possible in view of (47). We look at our equation in the following form

$$-\Delta_m(u) = d(x)|u|^{m-2}u$$

with $d(x) = |u|^{p-m+1}$.

We now consider $y \in \mathbb{R}^n$ with $|y| > 10R_1 > 10R_0$ and set $R = \frac{|y|}{4}$. Note that

$$B(y, 2R) \subset \left\{x \in \mathbb{R}^N \colon |x| > R_1\right\} \subset \left\{x \in \mathbb{R}^N \colon |x| > R_0\right\}.$$

Also, by (48),

$$\int_{|x|\geqslant R_1} |u|^{(p-m+1)\frac{N}{m}}\, dx \leqslant \eta$$

and $d(x) \in L^{\frac{N}{m-\varepsilon}}(B(y, 2R))$.

Therefore, we can exploit Theorem 1 in [25] to get that

$$\|u\|_{L^\infty(B(y,R))} \leqslant C_S \frac{1}{R^{\frac{N}{m}}} \|u\|_{L^m(B(y,2R))}, \tag{50}$$

where the constant $C_S$ may depend on $p, m, N$ and $R^\varepsilon \|d\|_{L^{\frac{N}{m-\varepsilon}}(B(y,2R))}$.

Here below we show that it is possible to control the dependence of $C_S$ on $R^\varepsilon \|d\|_{L^{\frac{N}{m-\varepsilon}}(B(y,2R))}$. Indeed, we have

$$R^\varepsilon \|d\|_{L^{\frac{N}{m-\varepsilon}}(B(y,2R))} = R^\varepsilon \left( \int_{B(y,2R)} \left(|u|^{(p-m+1)\frac{N}{m-\varepsilon}}\right) dx \right)^{\frac{m-\varepsilon}{N}} = R^\varepsilon \left( \int_{B(y,2R)} \left(|u|^{p+\gamma_2}\right) dx \right)^{\frac{m-\varepsilon}{N}} \leqslant \mathrm{const}\, R^\varepsilon \left(R^{N-m\frac{p+\gamma_2}{p-m+1}}\right)^{\frac{m-\varepsilon}{N}},$$

due to Lemma 11.1.

Since $(N - m\frac{p+\gamma_2}{p-m+1}) \cdot \frac{m-\varepsilon}{N} = -\varepsilon$, we get

$$R^{\varepsilon}\|d\|_{L^{\frac{N}{m-\varepsilon}}(B(y,2R))} \leqslant \text{const}$$

where the above constant does not depend on $R$.

As a consequence, $C_S$ depends only on $p$, $m$ and $N$.

We now exploit (50) to get that

$$\begin{aligned}\|u\|_{L^\infty(B(y,R))} &\leqslant C_S \frac{1}{R^{\frac{N}{m}}}\|u\|_{L^m(B(y,2R))} \leqslant C_S\, C(N,p,m)\frac{1}{R^{\frac{m}{p-m+1}}}\|u\|_{L^{[(p-m+1)\frac{N}{m}]}(B(y,2R))}\\ &\leqslant \text{const}\,\frac{1}{R^{\frac{m}{p-m+1}}}\|u\|_{L^{[(p-m+1)\frac{N}{m}]}(B(y,2R))},\end{aligned}$$

where the constant does not depend on $R$ because of (48) and the fact that we proved that $C_S$ does not depends on $R$.

We now recall that $|y| = 4R$ and therefore

$$\begin{aligned}|y|^{\frac{m}{p-m+1}}\left|u(y)\right| &\leqslant |y|^{\frac{m}{p-m+1}}\|u\|_{L^\infty(B(y,R))} \leqslant \text{const}\,\frac{|y|^{\frac{m}{p-m+1}}}{R^{\frac{m}{p-m+1}}}\|u\|_{L^{[(p-m+1)\frac{N}{m}]}(B(y,2R))}\\ &\leqslant \text{const}\,\|u\|_{L^{[(p-m+1)\frac{N}{m}]}(\mathbb{R}^N\setminus B(0,R))}.\end{aligned}$$

Accordingly, recalling (48), we obtain

$$\lim_{|y|\to\infty}|y|^{\frac{m}{p-m+1}}\left|u(y)\right| \leqslant \lim_{R\to\infty}\text{const}\,\|u\|_{L^{[(p-m+1)\frac{N}{m}]}(\mathbb{R}^N\setminus B(0,R))} = 0,$$

as desired. This ends the proof of Theorem 1.10.

## 12. Proof of Theorem 1.11

The proof follows directly from the classifications results in [1] where it is shown that radial solutions decay exactly as $\frac{1}{|x|^{\frac{m}{p-m+1}}}$, which is a contradiction with (13) (see Theorem 1.10) unless $u \equiv 0$.

## Acknowledgement

We thank an anonymous referee for her or his interesting and useful comments.

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