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Existence of minimizers of free autonomous variational problems via solvability of constrained ones

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Abstract

We consider the following autonomous variational problem

$$\text{minimize}\left\{\int_a^b f\big(v(x),v'(x)\big)\,\mathrm{d}x\colon\ v\in W^{1,1}(a,b),\ v(a)=\alpha,\ v(b)=\beta\right\}$$

where the Lagrangian f is assumed to be continuous, but not necessarily coercive, nor convex. We show that the existence of the minimum is linked to the solvability of certain constrained variational problems. This allows us to derive existence theorems covering a wide class of nonconvex noncoercive problems.

Résumé

On considère la classe des problèmes variationels autonomes ci-dessous :

$$\text{minimiser}\left\{\int_a^b f\big(v(x),v'(x)\big)\,\mathrm{d}x\colon\ v\in W^{1,1}(a,b),\ v(a)=\alpha,\ v(b)=\beta\right\}$$

où le lagrangien f est une fonction continue sans hypothèse de coercivité ou de convexité. On démontre que l'existence de solutions pour ces problèmes est liée à l'existence de solutions de certains problèmes variationels sous contraintes. Ce résultat permet d'obtenir des théorèmes d'existence pour une classe étendue de problèmes variationels ni coercifs ni convexes.

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1. Introduction

This paper is devoted to the investigation of the solvability of the classical autonomous Lagrange variational problem

$$\text{minimize}\left\{F(v)=\int_a^b f\big(v(x),v'(x)\big)\,\mathrm{d}x\colon\ v\in\Upsilon\right\};\tag{P}$$

where

$$\Upsilon:=\left\{v\in W^{1,1}(a,b)\colon\ v(a)=\alpha,\ v(b)=\beta,\ v(x)\in I\right\},$$

$I\subseteq\mathbb{R}$ is a given interval, bounded or unbounded, with $\alpha,\beta\in I$, and $f:I\times\mathbb{R}\to[0,+\infty)$, $f=f(s,z)$, is a continuous function. We do not assume any convexity or coercivity condition on f, so that the classical direct methods of the Calculus of Variations cannot be applied and the variational problem can have no solution.

This problem has been widely investigated for various types of Lagrangian and the mathematical literature on this context is very wide. We mention, in a not exhaustive way, for the case of f coercive but not convex the papers [12,15–17]. Instead, if f is noncoercive, but convex, the existence of Lipschitz continuous minimizers is established in [6] under different types of assumptions, allowing some cases of linear growth on the integrand f, and, in [8], even for Lagrangians f which do not exhibit any growth. Results involving noncoercive nonautonomous Lagrangians have been provided in [1–3,13]. The solvability of (P) for a class of nonconvex noncoercive integrands, under a growth assumption similar to that employed in [6] and [7], has been considered in [4]; as for relaxation and Lipschitz regularity of the minimizers we refer to [5].

Actually, the study of suitable growth assumptions for noncoercive integrands does not seem sufficiently developed, mainly for those functional admitting non-Lipschitz minimizers. For instance, for $f(s,z)=s|z|^p$, $p>1$, following the same argument proposed in the solution of [11, Ex. 2.2.10], it is easy to show that F admits minimum (attained at a possible non-Lipschitz trajectory), but this situation does not seem to be covered by any general existence theorem.

In the recent paper [14] a detailed analysis about the existence and the Lipschitz regularity of the solutions to autonomous constrained variational problems is accomplished, where f can be nonsmooth, nonconvex, noncoercive, and the competition set is restricted to the monotone functions. In such a paper, a nonsmooth version of the DuBois–Reymond equation is proved, expressed in terms of an inclusion involving the subdifferential of the Convex Analysis. This condition turns out to be both necessary and sufficient for the optimality of a trajectory. Moreover, after a relaxation result, a new necessary and sufficient condition for the existence of the minimum of F is introduced, which is expressed in terms of an upper bound for the assigned slope $\frac{\beta-\alpha}{b-a}$.

On the other hand, a certain monotonicity property of the minimizers of free problems has been recently studied in [9] and [10], where it was proved that, under very mild assumptions, the competition set Υ can be restricted, without loss of generality, to those trajectories admitting at most one change of monotonicity. More precisely, if (P) is solvable, then it admits a minimizer which is increasing in $[a,x_0]$ and decreasing in $[x_0,b]$ (or vice versa) for some $x_0\in[a,b]$. In light of this property, it is natural to investigate the link between the solvability of the free problem and the constrained one, in order to obtain necessary or sufficient conditions for the solvability of the free problem (P).

As for the necessary condition, we have proved in [9], subsequently improving the results in [10], that if (P) is solvable then there exists a minimizer u satisfying the monotonicity property described above, such that $\partial f(u(x),u'(x))\neq\emptyset$ for a.e. $x\in(a,b)$ and the DuBois–Reymond type necessary condition

$$f\big(u(x),u'(x)\big)-c\in u'(x)\partial f\big(u(x),u'(x)\big)\quad\text{a.e. in }(a,b)\tag{1.1}$$

holds, where ∂f is the subdifferential of Convex Analysis and c is a constant subjected to the limitation from above $c\leqslant\min_{s\in u([a,b])}f(s,0)$. But contrary to the constrained case investigated in [14], such a condition is no more sufficient for unconstrained problems. For instance, consider

$$f(z)=|z|,\quad (a,b)=(-1,2),\quad (\alpha,\beta)=(1,2),\quad I=\mathbb{R}.$$

The function $u(x)=|x|$ satisfies (1.1) with constant $c=0=f(0)$ but is not a minimizer, whereas every increasing absolutely continuous function satisfying the boundary conditions solves (P).

Nevertheless, in the present paper we show that condition (1.1) plays a relevant role in the study of the existence of the minimum. Indeed, we prove (see Proposition 4.3) that the existence of a pair of functions in Υ satisfying (1.1) for suitable values of the constant c contributes to further restrict the competition set (see (2.6)); the region of the plane included by the graphs of these "functions-barrier" can be seen as a cage in which the graph of a minimizer, whenever it exists, is trapped. In this compact region we define a suitable lower semicontinuous function mapping each point (x, y) into the sum of the infima of the integral functional evaluated at the monotone trajectories joining (a, α) with (x, y) and (x, y) with (b, β). The minimum point of such a function, say $(\bar{x}, \bar{y})$, plays a key role since we show that one can look for a minimizer for problem (P) among those trajectories passing throughout $(\bar{x}, \bar{y})$ and admitting at most one change of monotonicity exactly in that point. So, the solvability of the free problem (P) is reduced to the solvability of a pair of suitable constrained problems, that is the minimization of the integral functional evaluated among the monotone functions joining (a, α) with $(\bar{x}, \bar{y})$ and the monotone functions joining $(\bar{x}, \bar{y})$ with (b, β). Indeed, a minimizer of (P) can be obtained by matching the monotone minimizers of both the constrained problems.

Therefore, if the constrained problem admits minimum whatever the endpoints may be, then problem (P) is solvable too. When f is convex with respect to the last variable, this can be achieved by using the existence criteria for constrained problems stated in [14]. To deal with the nonconvex case, we combine them with a relaxation result recently proved in [10].

In this way, we obtain sufficient criteria for the solvability of problem (P) under rather mild growth conditions, covering situations which were not previously included in known existence results. In particular, we introduce the following condition

$$\lim_{|z|\to+\infty} \inf\{f^{**}(s,z) - z\partial f^{**}(s,z)\} = -\infty \quad \text{for a.e. } s \in I \tag{1.2}$$

which, jointly to some other slight technical assumptions, guarantees the existence of the minimum (see Theorem 7.1 for the convex case and its combination with the relaxation Theorem 7.8 for the nonconvex case).

The main feature of condition (1.2) is the requirement that it is satisfied just *almost everywhere* in I. This allows to include integrands $f(s, z)$ having superlinear growth with respect to z, but with the exception of straight lines in a set of null measure in the (s, z)-plane, where f vanishes. For instance, we are able to handle integrands of the type $f(s, z) = \phi(s) + \psi(s)h(z)$ with h coercive and ψ vanishing in a set of null measure.

More in detail, we herein present our existence criteria, limiting ourselves to integrands having affine-type structure, for the sake of simplicity, but we refer the reader to Theorems 7.1 and 7.8 for the general case.

Theorem 1.1. *Let $f(s, z) = \phi(s) + \psi(s)h(z)$, with ϕ, h continuous and nonnegative, ψ continuous and almost everywhere positive. Suppose that*

$$\lim_{|z|\to+\infty} \inf\{h^{**}(z) - z\partial h^{**}(z)\} = -\infty, \tag{1.3}$$

and that there exist

$$\min_{s\leqslant\beta, s\in I} \{\phi(s) + \psi(s)h(0)\} \quad \text{and} \quad \min_{s\geqslant\alpha, s\in I} \{\phi(s) + \psi(s)h(0)\}. \tag{1.4}$$

Moreover, assume that there exist two sequences $(z_n)_n$, $(\zeta_n)_n$ such that

$$z_n \uparrow 0, \quad \zeta_n \downarrow 0, \quad \text{and} \quad h(z_n) = h^{**}(z_n), \quad h(\zeta_n) = h^{**}(\zeta_n), \tag{1.5}$$

and that

$$\operatorname{co}\{z\colon\ h(z) = h^{**}(z)\} = \mathbb{R}. \tag{1.6}$$

Then, the functional F admits minimum. Moreover, if $\psi(s) > 0$ for every $s \in I$, then F admits a Lipschitz continuous minimizer.

Of course, when h is convex then conditions (1.5) and (1.6) are trivially satisfied. So the existence of the minimum in the convex case is ensured by conditions (1.3), (1.4). For instance, if $\phi \equiv 0$ and ψ vanishes in a nonempty set of null measure in $[\alpha, \beta]$, then (1.4) is satisfied. Therefore, the functional $F(u) = \int_a^b \psi(u(x))h(u'(x))\,dx$ admits minimum (attained at a possible non-Lipschitz minimizer) provided that condition (1.3) holds true (see also Examples 7.3, 7.4). Notice that in this case the functional F is not coercive and condition (1.2) holds just almost everywhere in I. According to our knowledge, such a situation was not previously covered by any existence result.

Moreover, in some cases the minimum can exist, whatever the boundary data α, β may be, even if condition (1.3) does not hold, as the following criterium states (see Corollary 7.5 and Theorem 7.8).

Theorem 1.2. *Let $f(s,z) = \psi(s)h(z)$, with ψ continuous and positive, h continuous and nonnegative, having h^{**} not affine in any half-line. Let the assumptions of Theorem* 1.1 *be satisfied, but with* (1.3) *replaced by*

$$\lim_{|z|\to+\infty} \inf\{h^{**}(z) - z\partial h^{**}(z)\} = 0.$$

Then the problem (P) is solvable and there exists a Lipschitz continuous minimizer.

For instance, the classical example $f(s,z) = \psi(s)\sqrt{1+z^2}$, with $\psi(s) > 0$ for every s, can be handled by using Theorem 1.2. We recall that in the convex case the existence of the minimum for such a type of integrands have been already proved in [8].

As for assumption (1.4), notice that it is trivially satisfied if $f(\cdot,0)$ is increasing and I has minimum or if $f(\cdot,0)$ is decreasing and I admits maximum. Moreover, it of course holds true even if I is compact. In order to treat situations in which (1.4) is not satisfied, we provide an a priori estimate in L^∞ for the trajectories lying in the level sets of the functional F (see Theorem 2.4 and Corollary 2.6), in such a way that the interval I can assumed to be bounded, without loss of generality. By virtue of Corollary 2.6, assumption (1.4) in Theorems 1.1, 1.2 can be replaced by

$$I \text{ is an unbounded closed interval} \quad \text{and} \quad \liminf_{|s|\to+\infty} \psi(s) > 0. \tag{1.4*}$$

Concerning assumption (1.5), notice that it requires that the origin is a cluster point, both from the right and the left side, for the contact set between h and h^{**}, and hence, by continuity, $h(0) = h^{**}(0)$. Condition (1.5) derives from a relaxation result proved in [10], where it was also discussed the possibility of removing it, keeping the requirement $h(0) = h^{**}(0)$. More precisely (see Theorem 7.8), condition (1.5) in Theorems 1.1, 1.2 can be replaced by the following requirement:

$$\text{the map } s \mapsto \phi(s) + \psi(s)h(0), \quad s \in [\alpha, \beta], \text{ has at most countable many minimizers} \tag{1.5*}$$

provided that $h(0) = h^{**}(0)$.

Finally, notice that condition (1.6) cannot be removed, owing to the possible existence of non-Lipschitz minimizers; in fact, as a consequence of condition (1.1), f and f^{**} should coincide along the minimizer.

The paper is organized as follows: after some notations and preliminary results, in Section 3 we recall the known results about constrained problems and the DuBois–Reymond necessary condition. In Section 4 we show that the trajectories satisfying the DuBois–Reymond necessary condition for suitable values of the constant c can be taken as upper or lower barriers for the competition set; whereas in Section 5 we study the lower-semicontinuity of the infimum of the integral functional with respect to the boundary data, which may have interest in itself. Section 6 contains the results connecting the solvability of certain constrained problems with that of (P) and finally in Section 7 we prove our main existence results.

2. Notations and preliminary results

As mentioned in Introduction, we consider the autonomous variational problem (P), that is

$$\text{minimize} \left\{ F(v) = \int_a^b f\big(v(x), v'(x)\big)\,\mathrm{d}x:\ v \in \Upsilon \right\}; \tag{P}$$

where

$$\Upsilon := \{ v \in W^{1,1}(a,b):\ v(a) = \alpha,\ v(b) = \beta,\ v(x) \in I \},$$

where $I \subseteq \mathbb{R}$ is a generic interval, with $\alpha, \beta \in I$, $f: I \times \mathbb{R} \to [0, +\infty)$, $f = f(s,z)$, is a continuous function. Without loss of generality, from now on we assume that $\alpha \leqslant \beta$.

Our approach in dealing with the optimality of problem (P) is based on the study of suitable variational problems with constraints on the derivatives. To do this, we will adopt the following notations.

Given $a \leqslant x_1 < x_2 \leqslant b$ and $0 \leqslant y_1 \leqslant y_2 \leqslant \beta$ we set

$$\Upsilon^+[(x_1,y_1),(x_2,y_2)] := \{v \in W^{1,1}(x_1,x_2):\ v(x_1)=y_1,\ v(x_2)=y_2,\ v'(x) \geqslant 0 \text{ a.e. in } (x_1,x_2)\}$$

and we consider the constrained variational problem

$$(P^+)[(x_1,y_1),(x_2,y_2)]:\ \text{minimize} \int_{x_1}^{x_2} f(v(x),v'(x))\,\mathrm{d}x \quad \text{for } v \in \Upsilon^+[(x_1,y_1),(x_2,y_2)].$$

Similarly, if $0 \leqslant y_2 \leqslant y_1 \leqslant \beta$ we define

$$\Upsilon^-[(x_1,y_1),(x_2,y_2)] := \{v \in W^{1,1}(x_1,x_2):\ v(x_1)=y_1,\ v(x_2)=y_2,\ v'(x) \leqslant 0 \text{ a.e. in } (x_1,x_2)\},$$

$$(P^-)[(x_1,y_1),(x_2,y_2)]:\ \text{minimize} \int_{x_1}^{x_2} f(v(x),v'(x))\,\mathrm{d}x \quad \text{for } v \in \Upsilon^-[(x_1,y_1),(x_2,y_2)].$$

Moreover, we write $(P)^{**}$, $(P^+)^{**}$ or $(P^-)^{**}$ if the integrand function f is replaced by f^{**}, where, as usual, f^{**} denotes the convex envelope of f with respect to the second variable, i.e., fixed $s \in I$, $f^{**}(s,\cdot)$ is the largest convex function lower than $f(s,\cdot)$.

For the sake of simplicity, if we do not write explicitly the dependence on the initial and the final boundary points, we mean that they are (a,α) and (b,β), respectively.

The link between the free problem (P) and the constrained ones (P^+), (P^-) is based upon the study of the monotonicity properties of the minimizers. More in detail, we will say that a function $u \in \Upsilon$ satisfies the *maximum principle* if the following property holds true

(M) there exists $x_0 \in [a,b]$ such that u is decreasing in $[a,x_0]$ and increasing in $[x_0,b]$

and the *minimum principle* if

(m) there exists $x_0 \in [a,b]$ such that u is increasing in $[a,x_0]$ and decreasing in $[x_0,b]$.

We define

$$\Upsilon_M := \{u \in \Upsilon:\ u \text{ satisfies (M)}\}, \qquad \Upsilon_m := \{u \in \Upsilon:\ u \text{ satisfies (m)}\}, \qquad \Upsilon^* := \Upsilon_M \cup \Upsilon_m. \tag{2.1}$$

The expression *maximum* [*minimum*] *principle* can be justified observing that any function in Υ_M [Υ_m] has the remarkable property that any restriction on a subinterval of $[a,b]$ assumes its maximum [minimum] value in correspondence of one of the endpoints. In particular, notice that $\Upsilon_M \cap \Upsilon_m$ is the set of the increasing functions of Υ; i.e., $\Upsilon_M \cap \Upsilon_m = \Upsilon^+[(a,\alpha),(b,\beta)]$.

Remark 2.1. Observe that if $u_1,u_2 \in \Upsilon_M$, then $\max\{u_1,u_2\}$ belongs to Υ_M too, and if u_2 is increasing then $\max\{u_1,u_2\}$ is increasing. Similarly, if $u_1,u_2 \in \Upsilon_m$, then $\min\{u_1,u_2\}$ belongs to Υ_m. Moreover, if $v_1 \in \Upsilon_M$ and $v_2 \in \Upsilon_m$ then $\min\{v_1,v_2\}$ belongs to Υ_M, $\max\{v_1,v_2\}$ belongs to Υ_m and, if v_1 is increasing, $\min\{v_1,v_2\}$ is increasing. We will use these facts in the proof of Proposition 4.3.

Throughout the paper we make use of the subdifferential in the sense of Convex Analysis, that is

$$\partial f(s,z) := \{\xi \in \mathbb{R}:\ f(s,w) - f(s,z) \geqslant \xi(w-z) \text{ for every } w \in \mathbb{R}\}.$$

We consider very mild assumptions on the continuous Lagrangian $f : I \times \mathbb{R} \to [0,+\infty)$. We list here the main conditions recurrent in our statements.

$$\partial f^{**}(\cdot,0) \text{ has a continuous selection;} \tag{2.2}$$

$$f(s,0) = f^{**}(s,0) \quad \text{for every } s \in I; \tag{2.3}$$

$$\text{there exist } m_1 := \min_{s \leqslant \beta,\, s \in I} f(s,0) \quad \text{and} \quad m_2 := \min_{s \geqslant \alpha,\, s \in I} f(s,0). \tag{2.4}$$

Whenever (2.4) holds true, we put

$$s_1 := \max\{s \leqslant \beta:\ f(s,0) = m_1\}, \qquad s_2 := \min\{s \geqslant \alpha:\ f(s,0) = m_2\}, \tag{2.5}$$

and we define

$$\tilde{\Upsilon} := \{u \in \Upsilon^*:\ \min\{\alpha, s_1\} \leqslant u(x) \leqslant \max\{\beta, s_2\},\ x \in [a,b]\}. \tag{2.6}$$

Moreover, if $w_1, w_2 \in \Upsilon^*$ and $w_1(x) \leqslant w_2(x)$ for every $x \in [a,b]$ then $\Upsilon^*_{w_1,w_2}$ denotes the following set of functions:

$$\Upsilon^*_{w_1,w_2} := \{v \in \tilde{\Upsilon}:\ w_1(x) \leqslant v(x) \leqslant w_2(x) \text{ for every } x \in [a,b]\}. \tag{2.7}$$

Remark 2.2. Condition (2.2) is trivially satisfied if the integrand has the affine-type structure $f(s,z) = \phi(s) + \psi(s)h(z)$, provided that ψ is continuous. Moreover, (2.2) holds even if f^{**} is continuous and $f^{**}(s,\cdot)$ is differentiable at $z = 0$ for every $s \in I$ (see Remark 2.2 in [9]). If the previous conditions are not satisfied, then condition (2.2) does not hold in general. Consider for instance the function $f(s,z) := |z - s|$, for which $\partial f^{**}(s,0)$ does not admit a continuous selection.

Without loss of generality, under conditions (2.2), (2.3), possibly by subtracting an affine function, we can also assume the validity of the following assumption

$$f(s,0) = \min_{z\in\mathbb{R}} f(s,z) \quad \text{for every } s \in I. \tag{2.8}$$

For the sake of completeness we give a precise statement of this observation, involving condition (1.1) too.

Lemma 2.3. *(See [9, Lemma 2.1].) Let $f : I \times \mathbb{R} \to [0,+\infty)$ be a continuous function satisfying* (2.2) *and* (2.3). *Then the function*

$$\tilde{f}(s,z) := f(s,z) - g(s)z,$$

where g is a continuous selection of $\partial f^{**}(\cdot,0)$, satisfies the following properties:

(a) $f(s,0) = \tilde{f}(s,0) = \tilde{f}^{**}(s,0)$ *for every $s \in I$*;
(b) $\tilde{f}(s,0) = \min_{z\in\mathbb{R}} \tilde{f}(s,z)$ *for every $s \in I$*;
(c) *$u \in \Upsilon$ satisfies condition* (1.1) *relatively to function f if and only if it satisfies the same condition relatively to $\tilde{f}$ too, with the same constant c*;
(d) *there exists $k \in \mathbb{R}$ such that $F(u) = \tilde{F}(u) + k$, for every $u \in \Upsilon$, where $\tilde{F}(u)$ stands for $\int_a^b \tilde{f}(u(x), u'(x))\,dx$.*

We now discuss the possibility of assuming, without loss of generality, condition (2.4). To this aim, let us present an a priori estimate in L^∞ for the trajectories lying in the level sets of the functional F, in such a way that I can assumed to be bounded. Therefore, if I is also closed, condition (2.4) trivially holds true. However, we need to combine this with the possibility to assume also condition (2.8); so the following result proved in Appendix A furnishes a sufficient condition for assuming both the conditions (2.4) and (2.8), without loss of generality.

Theorem 2.4. *Let I be a closed, unbounded real interval and let $f : I \times \mathbb{R} \to [0,+\infty)$ be a continuous function satisfying* (2.2) *and* (2.3). *Assume that there exists a continuous selection $g = g(s)$ of $\partial f^{**}(s,0)$ and positive constants ϵ, K, M such that*

$$f^{**}(s,z) \geqslant f^{**}(s,0) + (g(s) + \epsilon)z \quad \text{for every } z \geqslant K,\ |s| \geqslant M \tag{2.9+}$$

or

$$f^{**}(s,z) \geqslant f^{**}(s,0) + (g(s) - \epsilon)z \quad \text{for every } z \leqslant -K,\ |s| \geqslant M. \tag{2.9−}$$

Then there exists $L > \max\{|\alpha|, |\beta|\}$ such that, put $\tilde{f}(s,z) := f(s,z) - g(s)z$, if there exists a solution to

$$\textit{minimize}\left\{\tilde{F}(v) := \int_a^b \tilde{f}(v(x), v'(x))\,dx:\ v \in \Upsilon,\ \|v\|_\infty \leqslant L\right\} \tag{2.10}$$

then (P) is solvable too.

Remark 2.5. In view of the previous result and Lemma 2.3, if (2.2), (2.3) and one of the conditions (2.9+), (2.9−) hold, then both (2.4) and (2.8) can be assumed to hold, without loss of generality.

The following corollary proved in Appendix A is a consequence of the previous theorem, for affine-type integrands.

Corollary 2.6. *Let I be a closed unbounded interval. Let $f : I \times \mathbb{R} \to [0, +\infty)$, be defined by*

$$f(s,z) = \phi(s) + \psi(s)h(z)$$

with ϕ, ψ, h continuous and nonnegative. Suppose that $h^{**}(z)$ is not linear on the whole real line and

$$\liminf_{|s|\to+\infty} \psi(s) > 0.$$

Then conditions (2.4) *and* (2.8) *can be assumed to hold, without restriction.*

The next result proved in [10], concerns the monotonicity properties of minimizers. It states that the class Υ can be restricted to Υ^* or $\tilde{\Upsilon}$ (see (2.1), (2.6)).

Theorem 2.7. *(See [10, Theorem 3.1, Corollary 3.2].) Let $f, f^{**} : I \times \mathbb{R} \to [0, +\infty)$ be Borel-measurable functions, with $s \mapsto f(s, 0)$ lower semicontinuous, satisfying* (2.3)*. Then*

$$\inf_{\Upsilon} F = \inf_{\Upsilon^*} F.$$

Moreover, if $\inf_\Upsilon F$ is attained, then there exists a minimizer in Υ^*.

Finally, if (2.4) *is satisfied too, then the class Υ^* can be replaced by the subclass $\tilde{\Upsilon}$.*

As a consequence of the previous result we get that under suitable assumptions the unconstrained problem (P) reduces to the constrained one (P^+) and all the existence results proved in [14] for constrained problems can be applied in the present setting.

Corollary 2.8. *Let $f, f^{**} : I \times \mathbb{R} \to [0, +\infty)$ be Borel-measurable functions, with $s \mapsto f(s, 0)$ lower semicontinuous, satisfying* (2.3)*. If $\min_{s\in[\alpha,\beta]} f(s, 0) = \inf_{s\in I} f(s, 0)$, then*

$$\inf_{u\in\Upsilon} F(u) = \inf_{u\in\Upsilon^+} F(u).$$

Proof. By assumption, (2.4) is satisfied and both s_1 and s_2 are in $[\alpha, \beta]$. Thus, $\tilde{\Upsilon}$ in Theorem 2.7 coincides with Υ^+. □

3. The DuBois–Reymond condition and the constrained problem

Our main tool for studying the solvability of problem (P) is the DuBois–Reymond condition (1.1). It plays a central role when dealing with autonomous problems constrained to monotone functions, since, in this context, it is not only a necessary condition, but also a sufficient one, provided that the constant c satisfies a suitable upper limitation. Indeed, in [14] it was proved the following result:

Theorem 3.1. *(See [14, Theorem 7 and Remark 3].) Let $f : I \times [0, +\infty) \to [0, +\infty)$ be lower semicontinuous. Then a function $u \in \Upsilon^+[(x_1, y_1), (x_2, y_2)]$ is a minimizer for the constrained problem $(P^+)[(x_1, y_1), (x_2, y_2)]$ if and only if there exists a constant $c \leqslant \min_{s\in[y_1,y_2]} f(s, 0)$, such that*

$$f\big(u(x), u'(x)\big) - c \in u'(x)\partial_+ f\big(u(x), u'(x)\big) \quad \text{a.e. in } (x_1, x_2),$$

(with the position $0 \cdot \emptyset = 0$), where

$$\partial_+ f(s,z) := \big\{\eta \in \mathbb{R}\colon\ f(s,\zeta) - f(s,z) \geqslant \eta(\zeta - z) \text{ for every } \zeta > 0\big\}, \quad \text{for every } z > 0.$$

Remark 3.2. As proved in [9, Lemma 4.3], if $f(s,\cdot)$ satisfies (2.8) and is continuous at 0, then $\partial_+ f(s,z) = \partial f(s,z)$ for every $s \in I$ and $z > 0$.

Thanks to the above remark and Lemma 2.3, the following variant of Theorem 3.1 holds.

Proposition 3.3. *Let $f : I \times \mathbb{R} \to [0,+\infty)$ be continuous, satisfying* (2.2), (2.3). *Let y_1, y_2 be in I, $y_1 \leqslant y_2$. Then $u \in \Upsilon^+[(x_1,y_1),(x_2,y_2)]$ satisfies* (1.1) *with $c \leqslant \min_{s\in[y_1,y_2]} f(s,0)$ if and only if u solves the constrained problem $(P^+)[(x_1,y_1),(x_2,y_2)]$.*

Remark 3.4. Of course, results analogous to Theorem 3.1 and Proposition 3.3 hold for the constrained problem $(P^-)[(x_1,y_2),(x_2,y_1)]$, with $\partial_+ f(s,z)$ replaced by

$$\partial_- f(s,z) := \{\eta \in \mathbb{R}\colon\ f(s,\zeta) - f(s,z) \geqslant \eta(\zeta - z) \text{ for every } \zeta < 0\},$$

and again $c \leqslant \min_{s\in[y_1,y_2]} f(s,0)$. In fact, define the bijective function $T : \Upsilon^-[(x_1,y_2),(x_2,y_1)] \to \Upsilon^+[(x_1,y_1),(x_2,y_2)]$, as $T(v) = v^*$ where $v^*(x) := v(x_1 + x_2 - x)$ and $f_*(s,z) := f(s,-z)$. It is easy to see that $\partial_+ f_*(s,z) = -\partial_- f(s,-z)$, and that the following constrained variational problems are equivalent

$$\inf\left\{\int_{x_1}^{x_2} f\big(v(t), v'(t)\big)\,\mathrm{d}t\colon\ v \in \Upsilon^-\big[(x_1,y_2),(x_2,y_1)\big]\right\}$$

and

$$\inf\left\{\int_{x_1}^{x_2} f_*\big(v^*(t), v^{*\prime}(t)\big)\,\mathrm{d}t\colon\ v^* \in \Upsilon^+\big[(x_1,y_1),(x_2,y_2)\big]\right\}.$$

Unfortunately, a necessary and sufficient condition, analogous to Theorem 3.1, for the solvability of the unconstrained problem (P) cannot be obtained, as mentioned in Introduction. Nevertheless, by Corollary 2.8, when $\min_{s\in[\alpha,\beta]} f(s,0) = \inf_{s\in I} f(s,0)$, then the free problem (P) reduces to the constrained one (P^+), and so Theorem 3.1 holds with the same statement, for problem (P) too.

However, the necessary part holds for free problems, as stated by the following result, proved in [10] (see also [9]).

Theorem 3.5. *(See [10, Theorem 4.1].) Let $f : I \times \mathbb{R} \to [0,+\infty)$ be a continuous function satisfying* (2.2), (2.3).

If (P) is solvable then there exists a solution $u \in \Upsilon^*$ (see (2.1)*) such that $\partial f(u(x), u'(x)) \neq \emptyset$ for a.e. $x \in (a,b)$ and* (1.1) *holds for some constant $c \leqslant \min_{s\in u([a,b])} f(s,0)$. Moreover, if $u'(x) = 0$ in a set having positive measure then $c = \min_{s\in u([a,b])} f(s,0)$.*

4. Upper and lower barriers for the minimizers

The aim of this section is to restrict further the competition set $\tilde{\Upsilon}$ (see (2.6)), by means of "functions-barrier" satisfying condition (1.1).

The first step in this direction is the following result.

Lemma 4.1. *Let $f : I \times \mathbb{R} \to [0,+\infty)$ be continuous, satisfying* (2.2)–(2.4). *Let $v_0 \in \Upsilon_M$ satisfy* (1.1) *with a constant $c \leqslant f(s_1,0)$ (see* (2.5)*) and assume that $v_0'(x) \neq 0$ a.e. in (a,b).*

Then $F(v) \geqslant F(v_0)$ for every $v \in \Upsilon_M$ such that $\min_{x\in[a,b]} v(x) \leqslant \min_{x\in[a,b]} v_0(x)$.

Proof. By Lemma 2.3, without loss of generality, we can assume (2.8) too. Set

$$\Gamma := \{u \in W^{1,1}(\lambda_u, b)\colon\ \lambda_u \in [a,b),\ u(b) = \beta,\ u(x) \in I,\ u'(x) \geqslant 0 \text{ a.e.}\}.$$

For every $u \in \Gamma$, put

$$\chi_u(\xi) := \lambda_u + \int_{u(\lambda_u)}^{\xi} \phi_u'(\tau)\,\mathrm{d}\tau \quad \text{where } \phi_u(\xi) := \min\{x \in [\lambda_u, b] : u(x) = \xi\}.$$

In [15, Lemma 2] it was proved that $\chi_u'(\xi) > 0$ a.e. in $[u(\lambda_u), \beta]$, $\chi_u(\beta) \leqslant b$ and $\chi_u(\beta) = b \Leftrightarrow u'(x) > 0$ a.e. in $[\lambda_u, b]$. Furthermore, in [15, Lemma 3] it was proved that

$$\int_{\lambda_u}^{b} f\big(u(x), u'(x)\big)\,\mathrm{d}x \geqslant \int_{u(\lambda_u)}^{\beta} f\left(\xi, \frac{1}{\chi_u'(\xi)}\right)\chi_u'(\xi)\,\mathrm{d}\xi + f(s_1, 0)\big[b - \chi_u(\beta)\big] \tag{4.1}$$

and

$$\int_{\lambda_u}^{b} f\big(u(x), u'(x)\big)\,\mathrm{d}x = \int_{u(\lambda_u)}^{\beta} f\left(\xi, \frac{1}{\chi_u'(\xi)}\right)\chi_u'(\xi)\,\mathrm{d}\xi \quad \text{if } u'(x) > 0 \text{ a.e. in } (\lambda_u, b). \tag{4.2}$$

Moreover, if $f(u(x), u'(x)) - c \in u'(x)\partial f(u(x), u'(x))$ a.e. in (λ_u, b), then

$$\chi_u'(\xi)\left[f\left(\xi, \frac{1}{\chi_u'(\xi)}\right) - c\right] \in \partial f\left(\xi, \frac{1}{\chi_u'(\xi)}\right) \quad \text{a.e. in } \big(u(\lambda_u), \beta\big). \tag{4.3}$$

Finally, easy calculations show that (4.3) implies

$$f\left(\xi, \frac{1}{z}\right)z - f\left(\xi, \frac{1}{\chi_u'(\xi)}\right)\chi_u'(\xi) \geqslant c\big(z - \chi_u'(\xi)\big) \quad \text{for every } z > 0, \text{ a.e. in } \big(u(\lambda_u), \beta\big). \tag{4.4}$$

Let us now fix $v \in \Upsilon_M$ such that set $m := \min_{x\in[a,b]} v(x)$ and $m_0 := \min_{x\in[a,b]} v_0(x)$, we have $m \leqslant m_0$. Let $\bar{x} \in [a, b]$ be such that $v(\bar{x}) = m$ and let $x_0 \in [a, b]$ be such that $v_0(x_0) = m_0$. By the assumption on function v_0, the point x_0 is univocally determined and $v_0'(x) < 0$ for a.e. $x \in (a, x_0)$, $v_0'(x) > 0$ for a.e. $x \in (x_0, b)$.

Put $w := \chi_{v|[\bar{x},b]}$ and $w_0 := \chi_{v_0|[x_0,b]}$. Observe that by (4.1), (4.2), (4.4), (2.8) and being $v_0'(x) \neq 0$ a.e., we get

$$\begin{aligned}
&\int_{\bar{x}}^{b} f\big(v(x), v'(x)\big)\,\mathrm{d}x - \int_{x_0}^{b} f\big(v_0(x), v_0'(x)\big)\,\mathrm{d}x \\
&\quad\geqslant \int_{m}^{\beta} f\left(\xi, \frac{1}{w'(\xi)}\right)w'(\xi)\,\mathrm{d}\xi + f(s_1, 0)\big(b - w(\beta)\big) - \int_{m_0}^{\beta} f\left(\xi, \frac{1}{w_0'(\xi)}\right)w_0'(\xi)\,\mathrm{d}\xi \\
&\quad= \int_{m_0}^{\beta}\left[f\left(\xi, \frac{1}{w'(\xi)}\right)w'(\xi) - f\left(\xi, \frac{1}{w_0'(\xi)}\right)w_0'(\xi)\right]\mathrm{d}\xi + \int_{m}^{m_0} f\left(\xi, \frac{1}{w'(\xi)}\right)w'(\xi)\,\mathrm{d}\xi + f(s_1, 0)\big(b - w(\beta)\big) \\
&\quad\geqslant c\int_{m_0}^{\beta}\big[w'(\xi) - w_0'(\xi)\big]\,\mathrm{d}\xi + f(s_1, 0)\int_{m}^{m_0} w'(\xi)\,\mathrm{d}\xi + f(s_1, 0)\big(b - w(\beta)\big) \\
&\quad= c\big[w(\beta) - w_0(\beta)\big] - c\big[w(m_0) - w_0(m_0)\big] + f(s_1, 0)\big[w(m_0) - w(m)\big] + f(s_1, 0)\big(b - w(\beta)\big) \\
&\quad= \big[f(s_1, 0) - c\big]\big[b - w(\beta)\big] + \big[f(s_1, 0) - c\big]\big[w(m_0) - w(m)\big] + c(x_0 - \bar{x})
\end{aligned}$$

since $w_0(\beta) = b$, $w_0(m_0) = x_0$ and $w(m) = \bar{x}$.

In order to handle the integrals $\int_a^{\bar{x}} f(v(x), v'(x))\,\mathrm{d}x$ and $\int_a^{x_0} f(v_0(x), v_0'(x))\,\mathrm{d}x$, we reason as in Remark 3.4. Let us make the change of variable $t := a + b - x$, for $x \in [a, \bar{x}]$ and define $v^*(t) := v(a + b - t)$. Of course, v^* is defined in $[a + b - \bar{x}, b]$, satisfies $v^*(b) = v(a) = \alpha$, $v^*(a + b - \bar{x}) = v(\bar{x}) = m$, and $v^{*\prime}(t) = -v'(a + b - t) \geqslant 0$ a.e. in $(a + b - \bar{x}, b)$. Similarly, put $v_0^*(t) := v_0(a + b - t)$ for $t \in [a + b - x_0, b]$, we have $v_0^*(b) = v_0(a) = \alpha$, $v_0^*(a + b - x_0) = v_0(x_0) = m_0$ and $v_0^{*\prime}(t) = -v_0'(a + b - t) > 0$ a.e. in $(a + b - x_0, b)$.

Now, consider $f_*(s, \xi) := f(s, -\xi)$. As it is easy to check, $\partial f_*(s, z_0) = -\partial f(s, -z_0)$, so

$$f_*\big(v_0^*(t), v_0^{*\prime}(t)\big) - c \in v_0^{*\prime}(t)\partial f_*\big(v_0^*(t), v_0^{*\prime}(t)\big) \quad \text{a.e. in } (a + b - x_0, b).$$

Hence, put $w^* := \chi_{v^*}$ and $w_0^* := \chi_{v_0^*}$, by repeating the same argument above developed (replacing β with α), we obtain

$$\int_a^{\bar{x}} f\big(v(x), v'(x)\big)\,\mathrm{d}x - \int_a^{x_0} f\big(v_0(x), v_0'(x)\big)\,\mathrm{d}x$$

$$= \int_{a+b-\bar{x}}^{b} f_*\big(v^*(t), v^{*\prime}(t)\big)\,\mathrm{d}t - \int_{a+b-x_0}^{b} f_*\big(v_0^*(t), v_0^{*\prime}(t)\big)\,\mathrm{d}t$$

$$\geqslant \big(f(s_1,0)-c\big)\big(b-w^*(\alpha)\big) + \big(f(s_1,0)-c\big)\big(w^*(m_0)-w^*(m)\big) + c(\bar{x}-x_0).$$

Therefore,

$$F(v)-F(v_0) \geqslant \big(f(s_1,0)-c\big)\big(b-w(\beta)\big) + \big(f(s_1,0)-c\big)\big(w(m_0)-w(m)\big)$$
$$+ \big(f(s_1,0)-c\big)\big(b-w^*(\alpha)\big) + \big(f(s_1,0)-c\big)\big(w^*(m_0)-w^*(m)\big) \geqslant 0$$

since $c \leqslant f(s_1,0)$, $m \leqslant m_0$ and the functions w, w^* are monotone increasing. □

In an analogous way we can prove the following

Lemma 4.2. *Let $f : I\times\mathbb{R} \to [0,+\infty)$ be continuous, satisfying* (2.2)–(2.4). *Let $v_0 \in \Upsilon_m$ satisfy* (1.1) *with a constant $c \leqslant f(s_2,0)$ (see* (2.5)*) and assume that $v_0'(x) \neq 0$ a.e. in (a,b).*

Then $F(v) \geqslant F(v_0)$ for every $v \in \Upsilon_m$ such that $\max_{x\in[a,b]} v(x) \geqslant \max_{x\in[a,b]} v_0(x)$.

The next proposition is the main result of this section. It establishes sufficient conditions to have a pair of trajectories forming a further barrier for the competition set.

Proposition 4.3. *Let $f : I\times\mathbb{R} \to [0,+\infty)$ be continuous, satisfying* (2.2)–(2.4). *Let v_1, v_2 be functions in Υ_M and Υ_m respectively, such that $\min_{x\in[a,b]} v_1(x) = \min\{\alpha, s_1\}$ and $\max_{x\in[a,b]} v_2(x) = \max\{\beta, s_2\}$. Set*

$$x_1 := \min\big\{x\in[a,b]:\ v_1(x) = \min\{\alpha,s_1\}\big\}, \qquad x_2 := \max\big\{x\in[a,b]:\ v_1(x) = \min\{\alpha,s_1\}\big\},$$
$$x_3 := \min\big\{x\in[a,b]:\ v_2(x) = \max\{\beta,s_2\}\big\}, \qquad x_4 := \max\big\{x\in[a,b]:\ v_2(x) = \max\{\beta,s_2\}\big\}.$$

Suppose that there exist some constants $k_1, k_2 \leqslant f(s_1,0)$ and $k_3, k_4 \leqslant f(s_2,0)$ such that

$$f\big(v_1(x), v_1'(x)\big) - k_1 \in v_1'(x)\partial f\big(v_1(x), v_1'(x)\big) \quad \text{a.e. in } (a,x_1), \tag{4.5}$$
$$f\big(v_1(x), v_1'(x)\big) - k_2 \in v_1'(x)\partial f\big(v_1(x), v_1'(x)\big) \quad \text{a.e. in } (x_2,b),$$
$$f\big(v_2(x), v_2'(x)\big) - k_3 \in v_2'(x)\partial f\big(v_2(x), v_2'(x)\big) \quad \text{a.e. in } (a,x_3), \tag{4.6}$$
$$f\big(v_2(x), v_2'(x)\big) - k_4 \in v_2'(x)\partial f\big(v_2(x), v_2'(x)\big) \quad \text{a.e. in } (x_4,b).$$

Then, set $w_1 = \min\{v_1, v_2\}$ and $w_2 = \max\{v_1, v_2\}$, we have that $w_1 \in \Upsilon_M$, $w_2 \in \Upsilon_m$ and

$$\inf_{v\in\Upsilon} F(v) = \inf_{\Upsilon^*_{w_1,w_2}} F(v) \tag{4.7}$$

where $\Upsilon^*_{w_1,w_2}$ is defined in (2.7).

Proof. We have to prove (4.7) only, since $w_1 \in \Upsilon_M$ and $w_2 \in \Upsilon_m$ by what observed in Remark 2.1. By Lemma 2.3, without loss of generality we assume (2.8).

We claim that $v_1'(x) \neq 0$ for a.e. $x \in (a,x_1)$. By contradiction, suppose that $v_1'(x) = 0$ in a subset H of (a,x_1) having positive measure. Then, in particular, $a < x_1$, hence $s_1 < \alpha$.

Let H^* denote the subset of H, with $\mathrm{meas}(H\setminus H^*) = 0$, where (4.5) holds true. We get

$$f\big(v_1(x), 0\big) = k_1 \leqslant f(s_1,0) \quad \text{for every } x \in H^*.$$

On the other hand, $v_1(x) \leqslant \beta$ and so $f(v_1(x),0) \geqslant f(s_1,0)$ for each $x\in[a,x_1]$. Thus, $f(v_1(x),0) = f(s_1,0)$ for every $x \in H^*$, in contradiction with the definition of s_1 since $v_1(x) > s_1$ for every $x \in [a,x_1)$.

Similarly we can prove that $v_1'(x) \neq 0$ a.e. in (x_2,b), and also that $v_2'(x) \neq 0$ a.e. in $(a,b)\setminus(x_3,x_4)$.

By Theorem 2.7 it suffices to show that for every fixed $u \in \tilde{\Upsilon}$ there exists $w \in \Upsilon^*_{w_1,w_2}$ such that $F(w) \leqslant F(u)$.

Let us first assume that $u \in \Upsilon_M$ and set

$$A := \{x \in [a,b]:\ u(x) < v_1(x)\}.$$

A is an open set hence, if not empty, it is union of no more than countably many pairwise disjoint open intervals $\{(a_n,b_n)\}$, such that $[a_n,b_n] \subseteq [a,b] \setminus (x_1,x_2)$ and $u(a_n) = v_1(a_n)$, $u(b_n) = v_1(b_n)$. If $(a_n,b_n) \subseteq (a,x_1)$ then

$$s_1 \leqslant \min_{x\in[a_n,b_n]} u(x) \leqslant u(b_n) = v_1(b_n) = \min_{x\in[a_n,b_n]} v_1(x)$$

since v_1 is decreasing in $[a_n,b_n]$. Analogously, we get $\min_{x\in[a_n,b_n]} u(x) \leqslant \min_{x\in[a_n,b_n]} v_1(x)$ when $(a_n,b_n) \subseteq (x_2,b)$. By applying Lemma 4.1 in $[a_n,b_n]$, we get

$$\int_{a_n}^{b_n} f\big(u(x),u'(x)\big)\,\mathrm{d}x \geqslant \int_{a_n}^{b_n} f\big(v_1(x),v_1'(x)\big)\,\mathrm{d}x.$$

Thus, set $v := \max\{u, v_1\}$, by what observed in Remark 2.1 we have $v \in \Upsilon_M$ and

$$\begin{aligned} F(u) = \int_a^b f\big(u(x),u'(x)\big)\,\mathrm{d}x &= \sum_n \int_{a_n}^{b_n} f\big(u(x),u'(x)\big)\,\mathrm{d}x + \int_{[a,b]\setminus A} f\big(u(x),u'(x)\big)\,\mathrm{d}x \\ &\geqslant \sum_n \int_{a_n}^{b_n} f\big(v_1(x),v_1'(x)\big)\,\mathrm{d}x + \int_{[a,b]\setminus A} f\big(u(x),u'(x)\big)\,\mathrm{d}x = F(v). \end{aligned}$$

Now set $w = \min\{v, v_2\}$; again by Remark 2.1 we have $w \in \Upsilon_M \cap \Upsilon^*_{w_1,w_2}$. Now, if $w(x) \equiv v(x)$ then $F(w) = F(v) \leqslant F(u)$ and the thesis follows. Otherwise, consider the nonempty open set

$$B := \{x \in [a,b]:\ v(x) > v_2(x)\}$$

which is union of no more than countably many pairwise disjoint open intervals $\{(c_n,d_n)\}$, such that $v(c_n) = v_2(c_n)$, $v(d_n) = v_2(d_n)$. Since $v_2 \in \Upsilon_m$ then necessarily v_2 is increasing on each (c_n,d_n), that is $B \subseteq (a,x_3)$. Therefore, since also v is increasing in (c_n,d_n), by (4.6) and Proposition 3.3 we have that

$$\int_{c_n}^{d_n} f\big(v(x),v'(x)\big)\,\mathrm{d}x \geqslant \int_{c_n}^{d_n} f\big(v_2(x),v_2'(x)\big)\,\mathrm{d}x = \int_{c_n}^{d_n} f\big(w(x),w'(x)\big)\,\mathrm{d}x,$$

hence $F(w) \leqslant F(v) \leqslant F(u)$.

If, instead, $u \in \Upsilon_m$, define $v = \min\{u, v_2\}$; then $v \in \Upsilon_m$ and, reasoning as above, applying Lemma 4.2 in place of Lemma 4.1, we get that $F(v) \leqslant F(u)$. Then, considering $w = \max\{v, v_1\}$, it turns out that $w \in \Upsilon_m \cap \Upsilon^*_{w_1,w_2}$ and satisfies the thesis of the proposition. □

5. Regularity w.r.t. the boundary data

Another step towards our general existence results for free problems is the study of the regularity of the minima of constrained variational problems with varying endpoints.

Fixed $\gamma_1 \leqslant \alpha$ and $\gamma_2 \geqslant \beta$ with $\gamma_1, \gamma_2 \in I$, put $D := [a,b] \times [\gamma_1,\gamma_2]$ and according to the values of $y \in [\gamma_1,\gamma_2]$ define the functions $H^i : D \to \mathbb{R}$, for $i = 1,2$, as follows:

$$H^1(x,y) := \begin{cases} \inf_{v\in\Upsilon^-[(a,\alpha),(x,y)]} \int_a^x f(v(\xi),v'(\xi))\,\mathrm{d}\xi & \text{if } a < x \leqslant b \text{ and } \gamma_1 \leqslant y \leqslant \alpha, \\ \inf_{v\in\Upsilon^+[(a,\alpha),(x,y)]} \int_a^x f(v(\xi),v'(\xi))\,\mathrm{d}\xi & \text{if } a < x \leqslant b \text{ and } \alpha < y \leqslant \gamma_2, \\ 0 & \text{if } x = a \end{cases}$$

and

$$H^2(x,y) := \begin{cases} \inf_{v\in\Upsilon^+[(x,y),(b,\beta)]} \int_x^b f(v(\xi),v'(\xi))\,\mathrm{d}\xi & \text{if } a \leqslant x < b \text{ and } \gamma_1 \leqslant y \leqslant \beta, \\ \inf_{v\in\Upsilon^-[(x,y),(b,\beta)]} \int_x^b f(v(\xi),v'(\xi))\,\mathrm{d}\xi & \text{if } a \leqslant x < b \text{ and } \beta < y \leqslant \gamma_2, \\ 0 & \text{if } x = b. \end{cases}$$

Finally, let $H : D \to [0,+\infty)$ be defined by

$$H(x,y) := H^1(x,y) + H^2(x,y). \tag{5.1}$$

The following result holds.

Theorem 5.1. *Let $f : I \times \mathbb{R} \to [0,+\infty)$ be a continuous function. Then H is lower semicontinuous in D.*

Proof. Let us prove that both functions H^1 and H^2 are lower semicontinuous. More precisely, fixed $(x_0,y_0) \in D$ and $\epsilon > 0$, we prove that the properties below hold true (here $Q((x,y);\rho)$ denotes the square centered at (x,y) with size 2ρ):

(1) if $x_0 < b$ and $y_0 < \beta$ there exists $\rho < \min\{b - x_0, \beta - y_0\}$ such that for all $(x,y) \in Q((x_0,y_0);\rho) \cap D$ and all $v \in \Upsilon^+[(x,y),(b,\beta)]$ we have

$$\int_a^x f\big(v(\xi),v'(\xi)\big)\,\mathrm{d}\xi \geqslant H^2(x_0,y_0) - \epsilon,$$

(2) if $x_0 < b$ and $y_0 > \beta$ there exists $\rho < \min\{b - x_0, y_0 - \beta\}$ such that for all $(x,y) \in Q((x_0,y_0);\rho) \cap D$ and all $v \in \Upsilon^-[(x,y),(b,\beta)]$ we have

$$\int_a^x f\big(v(\xi),v'(\xi)\big)\,\mathrm{d}\xi \geqslant H^2(x_0,y_0) - \epsilon,$$

(3) if $x_0 < b$ and $y_0 = \beta$ there exists $\rho < b - x_0$ such that for all $(x,y) \in Q((x_0,y_0);\rho) \cap D$, for all $v \in \Upsilon^+[(x,y),(b,\beta)]$ (if $y \leqslant \beta$), or for all $v \in \Upsilon^-[(x,y),(b,\beta)]$ (if $y > \beta$) we have

$$\int_a^x f\big(v(\xi),v'(\xi)\big)\,\mathrm{d}\xi \geqslant H^2(x_0,y_0) - \epsilon.$$

Notice that if $x_0 = b$ then $H^2(x,y) \geqslant H^2(x_0,y_0)$ for all $(x,y) \in D$, so the lower semicontinuity of H^2 at (b,y_0) is trivial.

Analogous properties have to be proved for H^1.

The proof is rather long hence we divide it into various steps.

Step 1. Let $x_0 \in [a,b)$, $y_0 \in [\gamma_1,\beta)$ and consider the function H^2. We split the proof of this case into several claims.

Claim 1. *For every $\epsilon > 0$ there exists $\delta_1 = \delta_1(\epsilon,x_0,y_0)$ such that*

$$\int_x^b f\big(v(\xi),v'(\xi)\big)\,\mathrm{d}\xi \geqslant H^2(x_0,y_0) - \frac{\epsilon}{2} \tag{5.2}$$

for every $(x,y) \in D$, such that $x \in (x_0 - \delta_1, x_0]$, $y \in [y_0, y_0 + \delta_1)$, and for every $v \in \Upsilon^+[(x,y),(b,\beta)]$.

To this aim, let

$$M := \max\left\{ f(s,z)\colon\ s \in [\gamma_1,\beta],\ z \in \left[0, \max\left\{1, 4\frac{\beta - y_0}{b - x_0}\right\}\right]\right\}.$$

Let $\delta_1 := \min\{\frac{\epsilon}{8M+1}, \frac{b-x_0}{4}, \beta - y_0\}$. Let us fix (x, y) such that $x \in (x_0 - \delta_1, x_0]$, $y \in [y_0, y_0 + \delta_1)$ and let us fix $v \in \Upsilon^+[(x, y), (b, \beta)]$.

Note that there exists an interval $[c, d] \subset [x_0, b]$ such that

$$d - c = 4\delta_1 \quad \text{and} \quad v(d) - v(c) \leqslant 8\delta_1 \frac{\beta - y_0}{b - x_0}. \tag{5.3}$$

Indeed, otherwise, put $n := [\frac{b-x_0}{4\delta_1}]$, that is such that $n \leqslant \frac{b-x_0}{4\delta_1} < n + 1$, we have

$$\begin{aligned} v(b) - v(x_0) &\geqslant v(x_0 + 4n\delta_1) - v(x_0) = \sum_{i=1}^{n} \bigl\{ v(x_0 + 4i\delta_1) - v(x_0 + 4(i-1)\delta_1) \bigr\} \\ &> 2\frac{\beta - y_0}{b - x_0} 4n\delta_1 > 2(\beta - y_0)\frac{n}{n+1} \geqslant \beta - y_0, \end{aligned}$$

that is $v(x_0) < y_0$, hence $y_0 \leqslant y = v(x) \leqslant v(x_0) < y_0$, a contradiction. Therefore, (5.3) holds.

So, let $r_1(\xi)$ be the equation of the straight line joining the points (x_0, y_0) and $(x + 2\delta_1, y)$, and let $r_2(\xi)$ be the equation of the straight line joining the points $(c + 2\delta_1, v(c))$ and $(d, v(d))$. Let us define $\tilde{v} : [x_0, b] \to \mathbb{R}$, as

$$\tilde{v}(\xi) := \begin{cases} r_1(\xi) & \text{for } x_0 \leqslant \xi \leqslant x + 2\delta_1, \\ v(\xi - 2\delta_1) & \text{for } x + 2\delta_1 \leqslant \xi \leqslant c + 2\delta_1, \\ r_2(\xi) & \text{for } c + 2\delta_1 \leqslant \xi \leqslant d, \\ v(\xi) & \text{for } d \leqslant \xi \leqslant b. \end{cases}$$

Note that $\tilde{v} \in W^{1,1}(x_0, b)$, with $\tilde{v}(x_0) = y_0$, $\tilde{v}(b) = \beta$ and $\tilde{v}'(\xi) \geqslant 0$, hence $\tilde{v} \in \Upsilon^+[(x_0, y_0), (b, \beta)]$. Moreover, observe that

$$r_1'(\xi) = \frac{y - y_0}{x + 2\delta_1 - x_0} \leqslant 1, \qquad r_2'(\xi) = \frac{v(d) - v(c)}{d - c - 2\delta_1} = \frac{v(d) - v(c)}{2\delta_1} \leqslant 4\frac{\beta - y_0}{b - x_0}.$$

Then,

$$\int_{x_0}^{x+2\delta_1} f\bigl(r_1(\xi), r_1'(\xi)\bigr)\, \mathrm{d}\xi \leqslant M(x + 2\delta_1 - x_0) \leqslant 2M\delta_1 \leqslant \frac{\epsilon}{4},$$

$$\int_{c+2\delta_1}^{d} f\bigl(r_2(\xi), r_2'(\xi)\bigr)\, \mathrm{d}\xi \leqslant M(d - c - 2\delta_1) = 2\delta_1 M \leqslant \frac{\epsilon}{4}.$$

Therefore,

$$\begin{aligned} &\int_{x}^{b} f\bigl(v(\xi), v'(\xi)\bigr)\, \mathrm{d}\xi \\ &\quad \geqslant \int_{x}^{c} f\bigl(v(\xi), v'(\xi)\bigr)\, \mathrm{d}\xi + \int_{d}^{b} f\bigl(v(\xi), v'(\xi)\bigr)\, \mathrm{d}\xi \\ &\quad = \int_{x+2\delta_1}^{c+2\delta_1} f\bigl(\tilde{v}(\xi), \tilde{v}'(\xi)\bigr)\, \mathrm{d}\xi + \int_{d}^{b} f\bigl(\tilde{v}(\xi), \tilde{v}'(\xi)\bigr)\, \mathrm{d}\xi \\ &\quad = \int_{x_0}^{b} f\bigl(\tilde{v}(\xi), \tilde{v}'(\xi)\bigr)\, \mathrm{d}\xi - \int_{x_0}^{x+2\delta_1} f\bigl(r_1(\xi), r_1'(\xi)\bigr)\, \mathrm{d}\xi - \int_{c+2\delta_1}^{d} f\bigl(r_2(\xi), r_2'(\xi)\bigr)\, \mathrm{d}\xi \geqslant H^2(x_0, y_0) - \frac{\epsilon}{2} \end{aligned}$$

and (5.2) follows.

Claim 2. *For every $\epsilon > 0$ there exists $\delta_2 = \delta_2(\epsilon, x_0, y_0)$ such that*

$$\int_x^b f\big(v(\xi), v'(\xi)\big)\,\mathrm{d}\xi \geqslant H^2(x_0, y_0) - \epsilon \tag{5.4}$$

for every $(x, y) \in D$, such that $x < b$, $x \in [x_0, x_0 + \delta_2)$, $y \in [y_0, y_0 + \delta_2)$ and every $v \in \Upsilon^+[(x, y), (b, \beta)]$.

To this aim, let $m := \max\{f(s, 0)\colon s \in [\gamma_1, \beta]\}$ and $\delta_2 := \min\{\frac{\epsilon}{2m+1}, \delta_1\}$ (where δ_1 has been defined in Claim 1), and let us fix (x, y) such that $0 \leqslant x - x_0 < \delta_2$, $0 \leqslant y - y_0 < \delta_2$. Fixed $v \in \Upsilon^+[(x, y), (b, \beta)]$, define

$$\tilde{v}(\xi) := \begin{cases} v(\xi + x - x_0) & \text{for } x_0 \leqslant \xi \leqslant b - x + x_0, \\ \beta & \text{for } b - x + x_0 \leqslant \xi \leqslant b. \end{cases}$$

Observe that $\tilde{v} \in \Upsilon^+[(x_0, y), (b, \beta)]$ and

$$\int_x^b f\big(v(\xi), v'(\xi)\big)\,\mathrm{d}\xi \geqslant \int_{x_0}^b f\big(\tilde{v}(\xi), \tilde{v}'(\xi)\big)\,\mathrm{d}\xi - m(x - x_0) > \int_{x_0}^b f\big(\tilde{v}(\xi), \tilde{v}'(\xi)\big)\,\mathrm{d}\xi - \frac{\epsilon}{2}.$$

Moreover, by applying Claim 1 we deduce

$$\int_{x_0}^b f\big(\tilde{v}(\xi), \tilde{v}'(\xi)\big)\,\mathrm{d}\xi > H^2(x_0, y_0) - \frac{\epsilon}{2}$$

and (5.4) follows.

Claim 3. *Let $y_0 > \gamma_1$ (otherwise this claim is meaningless).*
Then for every $\epsilon > 0$ there exists $\delta_3 = \delta_3(\epsilon, x_0, y_0) > 0$ such that

$$\int_{x_0}^b f\big(v(\xi), v'(\xi)\big)\,\mathrm{d}\xi \geqslant H^2(x_0, y_0) - \frac{\epsilon}{4} \tag{5.5}$$

for every $v \in \Upsilon^+[(x_0, y), (b, \beta)]$, with $(x_0, y) \in D$, $y \in (y_0 - \delta_3, y_0)$.

Indeed, since f is uniformly continuous in $[\gamma_1, \beta] \times [0, 1]$, there exists a real $\eta > 0$ such that

$$\big|f(x_1, z) - f(x_2, 0)\big| < \frac{\epsilon}{8(b-a)} \quad \text{if } |x_1 - x_2| < \eta,\ z \in [\gamma_1, \eta]. \tag{5.6}$$

Let $\delta_3 := \min\{\eta, \frac{\eta\epsilon}{8m+1}, y_0 - \gamma_1\}$ (with m as in Claim 2) and let us fix $y \in (y_0 - \delta_3, y_0)$ and $v \in \Upsilon^+[(x_0, y), (b, \beta)]$. Let $\bar{x} > x_0$ be such that $v(\bar{x}) = y_0$. Then

$$H^2(x_0, y_0) \leqslant \int_{x_0}^{\bar{x}} f(y_0, 0)\,\mathrm{d}\xi + \int_{\bar{x}}^b f\big(v(\xi), v'(\xi)\big)\,\mathrm{d}\xi.$$

Hence, to prove (5.5) it suffices to show that

$$\int_{x_0}^{\bar{x}} \big[f\big(v(\xi), v'(\xi)\big) - f(y_0, 0)\big]\,\mathrm{d}\xi \geqslant -\frac{\epsilon}{4}. \tag{5.7}$$

To this purpose, put

$$A := \big\{\xi \in [x_0, \bar{x}]\colon v'(\xi) > \eta\big\}, \qquad B := \big\{\xi \in [x_0, \bar{x}]\colon v'(\xi) \leqslant \eta\big\}.$$

Observe that $\mathrm{meas}(A) \leqslant \frac{\epsilon}{8m+1}$, indeed otherwise we would have

$$v(\bar{x}) - v(x_0) \geqslant \int_A v'(\xi)\,\mathrm{d}\xi > \eta\frac{\epsilon}{8m+1} \geqslant \delta_3,$$

a contradiction, since $v(\bar{x}) = y_0$ and $v(x) = y$ with $|y - y_0| < \delta_3$. Therefore, since f is nonnegative and (5.6) holds, we get

$$\int_{x_0}^{\bar{x}} \big[f\big(v(\xi), v'(\xi)\big) - f(y_0, 0)\big]\,\mathrm{d}\xi \geqslant \int_B \big[f\big(v(\xi), v'(\xi)\big) - f(y_0, 0)\big]\,\mathrm{d}\xi - \int_A f(y_0, 0)\,\mathrm{d}\xi$$
$$\geqslant -\frac{\epsilon}{8(b-a)}\,\mathrm{meas}(B) - f(y_0, 0)\,\mathrm{meas}(A) \geqslant -\frac{\epsilon}{8} - m\,\mathrm{meas}(A) \geqslant -\frac{\epsilon}{4}$$

that is (5.7).

Claim 4. *Let $y_0 > \gamma_1$. Then for every $\epsilon > 0$ there exists $\delta_4 = \delta_4(\epsilon, x_0, y_0) > 0$ such that*

$$\int_x^b f\big(v(\xi), v'(\xi)\big)\,\mathrm{d}\xi \geqslant H^2(x_0, y_0) - \frac{\epsilon}{2} \tag{5.8}$$

for every $(x, y) \in D$, with $x < b$, $x \in (x_0, x_0 + \delta_4)$, $y \in (y_0 - \delta_4, y_0)$ and every $v \in \Upsilon^+[(x, y), (b, \beta)]$.

Let $\delta_4 := \min\{\delta_3, \frac{\epsilon}{4m+1}, b - x_0\}$ (with m as in Claim 2 and δ_3 as in Claim 3).

Fix $x \in (x_0, x_0 + \delta_4)$, $y \in (y_0 - \delta_4, y_0)$ and $v \in \Upsilon^+[(x, y), (b, \beta)]$. Let us consider the function

$$\tilde{v}(\xi) := \begin{cases} v(\xi + x - x_0) & \text{for } x_0 \leqslant \xi \leqslant b - x + x_0, \\ \beta & \text{for } b - x + x_0 \leqslant \xi \leqslant b. \end{cases}$$

We have $\tilde{v}(x_0) = v(x) = y$ and then $\tilde{v} \in \Upsilon^+[(x_0, y), (b, \beta)]$. Moreover,

$$\int_x^b f\big(v(\xi), v'(\xi)\big)\,\mathrm{d}\xi = \int_{x_0}^{b-x+x_0} f\big(\tilde{v}(\xi), \tilde{v}'(\xi)\big)\,\mathrm{d}\xi$$
$$= \int_{x_0}^{b} f\big(\tilde{v}(\xi), \tilde{v}'(\xi)\big)\,\mathrm{d}\xi - m(x - x_0)$$
$$\geqslant \int_{x_0}^{b} f\big(\tilde{v}(\xi), \tilde{v}'(\xi)\big)\,\mathrm{d}\xi - \frac{\epsilon}{4}.$$

Then, by applying what proved in Claim 3, we deduce (5.8).

Claim 5. *Let $y_0 > \gamma_1$. Then for every $\epsilon > 0$ there exists $\delta_5 = \delta_5(\epsilon, x_0, y_0) > 0$ such that*

$$\int_x^b f\big(v(\xi), v'(\xi)\big)\,\mathrm{d}\xi \geqslant H^2(x_0, y_0) - \epsilon \tag{5.9}$$

for every $(x, y) \in D$, with $x \in (x_0 - \delta_5, x_0)$, $y \in (y_0 - \delta_5, y_0)$ and every $v \in \Upsilon^+[(x, y), (b, \beta)]$.

Similarly to what done in Claim 1, let

$$K := \max\left\{f(s, z)\colon\ s \in [\gamma_1, \beta],\ z \in \left[0, \max\left\{1, 4\frac{\beta - y_0 + 1}{b - x_0}\right\}\right]\right\}.$$

Let $\delta_5 := \min\{1, \delta_4, \frac{\epsilon}{2K+1}, \frac{b-x_0}{2}\}$, with δ_4 as in Claim 4.

Let us fix (x, y) such that $x \in (x_0 - \delta_5, x_0]$, $y \in (y_0 - \delta_5, y_0]$ and let us fix $v \in \Upsilon^+[(x, y), (b, \beta)]$.

Note that there exists an interval $[c, d] \subset [x_0, b]$ such that

$$d - c = 2\delta_5 \quad \text{and} \quad v(d) - v(c) \leqslant 4\delta_5 \frac{\beta - y_0 + 1}{b - x_0}. \tag{5.10}$$

Indeed, otherwise, put $n := [\frac{b-x_0}{2\delta_5}]$, that is such that $n \leqslant \frac{b-x_0}{2\delta_5} < n + 1$, we have

$$\begin{aligned} v(b) - v(x) &\geqslant v(x_0 + 2n\delta_5) - v(x_0) = \sum_{i=1}^{n} \bigl\{ v(x_0 + 2i\delta_5) - v\bigl(x_0 + 2(i-1)\delta_5\bigr) \bigr\} \\ &> 4\frac{\beta - y_0 + 1}{b - x_0} n\delta_5 > 2(\beta - y_0 + 1)\frac{n}{n+1} \geqslant \beta - y_0 + 1 > \beta - y, \end{aligned}$$

that is $v(x) < y$, a contradiction. Therefore, (5.10) holds.

So, let $r(\xi)$ be the equation of the straightline joining the points $(c + \delta_5, v(c))$ and $(d, v(d))$. Let us define $\tilde{v} : [x + \delta_5, b] \to \mathbb{R}$, as

$$\tilde{v}(\xi) := \begin{cases} v(\xi - \delta_5) & \text{for } x + \delta_5 \leqslant \xi \leqslant c + \delta_5, \\ r(\xi) & \text{for } c + \delta_5 \leqslant \xi \leqslant d, \\ v(\xi) & \text{for } d \leqslant \xi \leqslant b. \end{cases}$$

Note that $\tilde{v} \in W^{1,1}(x + \delta_5, b)$, with $\tilde{v}(x + \delta_5) = y$, $\tilde{v}(b) = \beta$ and $\tilde{v}'(\xi) \geqslant 0$, hence $\tilde{v} \in \Upsilon^+[(x + \delta_5, y), (b, \beta)]$. Moreover, by (5.10) observe that $r'(\xi) = \frac{v(d)-v(c)}{d-c-\delta_5} = \frac{v(d)-v(c)}{\delta_5} \leqslant 4\frac{\beta-y_0+1}{b-x_0}$. Then, by definition of K and δ_5,

$$\int_{c+\delta_5}^{d} f\bigl(r(\xi), r'(\xi)\bigr)\,\mathrm{d}\xi \leqslant K(d - c - \delta_5) = \delta_5 K \leqslant \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned} \int_x^b f\bigl(v(\xi), v'(\xi)\bigr)\,\mathrm{d}\xi &\geqslant \int_x^c f\bigl(v(\xi), v'(\xi)\bigr)\,\mathrm{d}\xi + \int_d^b f\bigl(v(\xi), v'(\xi)\bigr)\,\mathrm{d}\xi \\ &= \int_{x+\delta_5}^{c+\delta_5} f\bigl(\tilde{v}(\xi), \tilde{v}'(\xi)\bigr)\,\mathrm{d}\xi + \int_d^b f\bigl(\tilde{v}(\xi), \tilde{v}'(\xi)\bigr)\,\mathrm{d}\xi \\ &= \int_{x+\delta_5}^{b} f\bigl(\tilde{v}(\xi), \tilde{v}'(\xi)\bigr)\,\mathrm{d}\xi - \int_{c+\delta_5}^{d} f\bigl(r(\xi), r'(\xi)\bigr)\,\mathrm{d}\xi \\ &\geqslant \int_{x+\delta_5}^{b} f\bigl(\tilde{v}(\xi), \tilde{v}'(\xi)\bigr)\,\mathrm{d}\xi - \frac{\epsilon}{2}. \end{aligned}$$

Since $x + \delta_5 \in (x_0, x_0 + \delta_4)$ and $y \in (y_0 - \delta_5, y_0)$, we can apply what proved in Claim 4 to conclude the validity of (5.9).

Summarizing, put $\rho := \min_{i=1,\dots,5} \delta_i$, by virtue of Claims 1–5, the thesis follows for H^2 when $y_0 \in [\gamma_1, \beta)$.

Notice that Claims 3–5 work even when $y_0 = \beta$.

Step 2. In a similar way we can treat the lower semicontinuity of H^1 when $\alpha < y_0 \leqslant \gamma_2$; we sketch here only one claim, analogous to Claim 1 of Step 1.

Fixed $x_0 > a$ and $\alpha < y_0 \leqslant \gamma_2$, consider $x \in [x_0, x_0 + \delta_1)$ and $y \in (y_0 - \delta_1, y_0]$, where

$$\delta_1 := \min\left\{ \frac{\epsilon}{8M+1}, \frac{x_0 - a}{4}, y_0 - \alpha \right\}$$

and

$$M := \max\left\{ f(s,z)\colon\ s \in [\alpha,\gamma_2]\ z \in \left[0, \max\left\{1, 4\frac{y_0-\alpha}{x_0-a}\right\}\right]\right\}.$$

Given $v \in \Upsilon^+[(a,\alpha),(x,y)]$, arguing as in Claim 1, there exists an interval $[c,d] \subset [a,x_0]$ such that

$$d-c = 4\delta_1 \quad \text{and} \quad v(d)-v(c) \leqslant 8\delta_1\frac{y_0-\alpha}{x_0-a}.$$

Let $r_1(\xi)$ be the equation of the straight line joining the points $(c,v(c))$ and $(d-2\delta_1, v(d))$ and $r_2(\xi)$ be the equation of the straight line joining the points $(x-2\delta_1, y)$ and (x_0,y_0). Define $\tilde{v}:[a,x_0]\to\mathbb{R}$, as

$$\tilde{v}(\xi) := \begin{cases} v(\xi) & \text{for } a \leqslant \xi \leqslant c,\\ r_1(\xi) & \text{for } c \leqslant \xi \leqslant d-2\delta_1,\\ v(\xi+2\delta_1) & \text{for } d-2\delta_1 \leqslant \xi \leqslant x-2\delta_1,\\ r_2(\xi) & \text{for } x-2\delta_1 \leqslant \xi \leqslant x_0. \end{cases}$$

Note that $\tilde{v} \in W^{1,1}(a,x_0)$, with $\tilde{v}(a)=\alpha$, $\tilde{v}(x_0)=y_0$, and $\tilde{v}'(\xi)\geqslant 0$, hence $\tilde{v}\in\Upsilon^+[(a,\alpha),(x_0,y_0)]$ and following what done in Claim 1 we can prove that

$$\int_a^x f\big(v(\xi),v'(\xi)\big)\,\mathrm{d}\xi \geqslant H^1(x_0,y_0)-\frac{\epsilon}{2}.$$

Observe that we applied the same argument developed in Claim 1 of Step 1 in a sector which is the symmetric of the one considered in such a claim, with respect to the point (x_0,y_0). Similarly, using the same argument of Claim 2 in Step 1 we achieve the assertion for (x,y) close to (x_0,y_0), with $x<x_0$, $y<y_0$. Analogous considerations hold for Claims 3–5 which now work for $y>y_0\geqslant\alpha$.

Step 3. In order to prove the result for $H^2(x_0,y_0)$ when $\beta<y_0\leqslant\gamma_2$ and for $H^1(x_0,y_0)$ when $\gamma_1\leqslant y_0<\alpha$, we can reduce ourselves to the above cases by a change of variable. If $(x,y)\in[a,b)\times(\beta,\gamma_2]$, let us define $T:\Upsilon^-[(x,y),(b,\beta)]\to\Upsilon^+[(a,\beta),(a+b-x,y)]$

$$T(v)(s) = v^*(s) = v(a+b-s). \tag{5.11}$$

Such an operator T is bijective. Moreover, define $f_*(s,z) := f(s,-z)$ and

$$H^1_*(\xi,y) := \inf\left\{\int_a^\xi f_*\big(v^*(t),{v^*}'(t)\big)\,\mathrm{d}t\colon\ v^*\in\Upsilon^+\big[(a,\beta),(\xi,y)\big]\right\} \tag{5.12}$$

for $(\xi,y)\in(a,b]\times(\beta,\gamma_2]$. It is easy to see that $H^1_*(a+b-x,y)=H^2(x,y)$.

From what proved in Step 2, for every $\epsilon>0$ there exists a constant $r=r(\epsilon,x_0,y_0)>0$, such that

$$\int_a^{a+b-x} f_*\big(v^*(t),{v^*}'(t)\big)\,\mathrm{d}t \geqslant H^1_*(a+b-x_0,y_0)-\epsilon$$

for every $(x,y)\in[a,b)\times(\beta,\gamma_2]$, such that $|x-x_0|<r$, $|y-y_0|<r$, and every $v^*\in\Upsilon^+[(a,\beta),(a+b-x,y)]$. Therefore, the change of variable $\xi=a+b-t$ implies

$$\int_x^b f\big(v(\xi),v'(\xi)\big)\,\mathrm{d}\xi \geqslant H^1_*(a+b-x_0,y_0)-\epsilon = H^2(x_0,y_0)-\epsilon.$$

Similarly, when $(x,y)\in(a,b]\times[\gamma_1,\alpha)$ consider the change of variable in (5.11). This time

$$T:\Upsilon^-\big[(a,\alpha),(x,y)\big]\to\Upsilon^+\big[(a+b-x,y),(b,\alpha)\big]$$

and we can reason as above applying what proved in Step 1.

Step 4. When $y_0 = \beta$ and $a \leqslant x_0 < b$ we need to compare $H^2(x_0, y_0)$ with $\int_x^b f(v(t), v'(t))\,dt$, where $v \in \Upsilon^+[(x, y), (b, \beta)]$ if $y < \beta$ and $v \in \Upsilon^-[(x, y), (b, \beta)]$ if $y \geqslant \beta$, with (x, y) close to (x_0, y_0). In the first case we can apply Claims 3–5 of Step 1, that still work if $y_0 = \beta$, while when $y \geqslant \beta$ we consider the change of variable defined in Step 3, see (5.11). Defining $H^1_*(\xi, y)$ as in (5.12) for $(\xi, y) \in (a, b] \times [\beta, \gamma_2]$, we apply Step 2 to get an inequality for H^1_* which reduces to the inequality for H^2 we were looking for, by inverting the change variable.

In a similar way we can prove the lower semicontinuity of $H^1(x, y)$ at (x_0, α), $x_0 > a$. □

6. Link between the solvability of free and constrained problems

In this section we prove that the solvability of suitable constrained problems $(P^{\pm})$ with varying endpoints implies the solvability of (P). Notice that in view of Theorem 2.4 all the results stated in this section hold even if condition (2.4) is replaced by one of the conditions (2.9+), (2.9−), provided that I is closed.

Theorem 6.1. *Let $f : I \times \mathbb{R} \to [0, +\infty)$ be a continuous function, satisfying* (2.2)–(2.4). *Let v_1 and v_2 be as in Proposition* 4.3 *and set $w_1 := \min\{v_1, v_2\}$ and $w_2 := \max\{v_1, v_2\}$.*

Assume that the following constrained problems, whenever they are well defined, are solvable:

$$(P^-)\big[(a,\alpha),(x,y)\big] \text{ and } \big(P^+\big)\big[(a,\alpha),(x,y)\big] \quad \text{for all } (x,y) \text{ such that } x \in (a,b],\ w_1(x) \leqslant y \leqslant w_2(x); \tag{6.1}$$

$$(P^+)\big[(x,y),(b,\beta)\big] \text{ and } (P^-)\big[(x,y),(b,\beta)\big] \quad \text{for all } (x,y) \text{ such that } x \in [a,b),\ w_1(x) \leqslant y \leqslant w_2(x). \tag{6.2}$$

Then (P) is solvable.

Proof. Since Proposition 4.3 holds true, we have that $\inf_{v \in \Upsilon} F(v) = \inf_{v \in \Upsilon^*_{w_1,w_2}} F(v)$. Thus, it is sufficient to prove that the last infimum is attained.

Define

$$D^* = \big\{(x,y)\colon\ x \in [a,b],\ w_1(x) \leqslant y \leqslant w_2(x)\big\}$$

and let H be defined as in (5.1), with $\gamma_1 = \min\{\alpha, s_1\}$ and $\gamma_2 = \max\{\beta, s_2\}$. Since D^* is compact, by Theorem 5.1 we get that the function H has a minimum in $(x^*, y^*) \in D^*$. By assumptions (6.1) and (6.2) we deduce that if $x^* > a$ then the infimum $H^1(x^*, y^*)$ is attained at a certain function $u^*_1 \in \Upsilon[(a, \alpha), (x^*, y^*)]$ decreasing if $y^* \leqslant \alpha$, increasing if $y^* > \alpha$. If moreover $x^* < b$ then the infimum $H^2(x^*, y^*)$ is attained at a certain function $u^*_2 \in \Upsilon[(x^*, y^*), (b, \beta)]$, increasing if $y^* \leqslant \beta$ decreasing if $y^* > \beta$. Reasoning as in the proof of Proposition 4.3 it is easy to see that, without loss of generality, we may assume that $w_1(x) \leqslant u^*_1(x) \leqslant w_2(x)$ for every $x \in [a, x^*]$ and $w_1(x) \leqslant u^*_2(x) \leqslant w_2(x)$ for every $x \in [x^*, b]$.

Now, we set

$$u^*(x) := \begin{cases} u^*_1(x) & \text{if } x \in [a, x_*], \\ u^*_2(x) & \text{if } x \in [x_*, b]; \end{cases} \tag{6.3}$$

then $u^* \in \Upsilon^*_{w_1,w_2}$ and $F(u^*) = \min_{(x,y) \in D^*} H(x, y)$.

Let us fix $v \in \Upsilon^*_{w_1,w_2}$ and let $\hat{x}$ satisfy $v(\hat{x}) = \min_{x \in [a,b]} v(x)$ if $v \in \Upsilon_M$, or $v(\hat{x}) = \max_{x \in [a,b]} v(x)$ if $v \in \Upsilon_m$. Then $(\hat{x}, v(\hat{x})) \in D^*$, so

$$F(v) \geqslant H\big(\hat{x}, v(\hat{x})\big) \geqslant H(x^*, y^*) = F(u^*).$$

Thus, u^* is a minimizer for problem (P). □

Thanks to the link between the solvability of constrained problems with that of the unconstrained one (P), expressed by Theorem 6.1, we can immediately deduce the existence of the minimum for problem (P) when (P^+) and (P^-) admit minimum whatever the boundary conditions may be. In fact, the following result hold.

Corollary 6.2. *Let $f : I \times \mathbb{R} \to [0, +\infty)$ be a continuous function, satisfying* (2.2)–(2.4). *If the following constrained variational problems* (*see* (2.5))

- $(P^+)[(a,\alpha),(x,y)]$ *for all* $(x,y) \in (a,b] \times [\alpha, \max\{\beta, s_2\}]$,
- $(P^-)[(a,\alpha),(x,y)]$ *for all* $(x,y) \in (a,b] \times [\min\{\alpha, s_1\}, \alpha]$,
- $(P^+)[(x,y),(b,\beta)]$ *for all* $(x,y) \in [a,b) \times [\min\{\alpha, s_1\}, \beta]$,
- $(P^-)[(x,y),(b,\beta)]$ *for all* $(x,y) \in [a,b) \times [\beta, \max\{\beta, s_2\}]$,

are solvable, then (P) *admits minimum.*

Moreover, if the above constrained problems have a Lipschitz solution, also (P) *does.*

Proof. Let u_1^-, u_1^+ be minimizers of $(P^-)[(a,\alpha),(\frac{a+b}{2},\min\{\alpha,s_1\})]$ and of $(P^+)[(\frac{a+b}{2},\min\{\alpha,s_1\}),(b,\beta)]$, respectively. Analogously, let u_2^-, u_2^+ be, respectively, minimizers of $(P^+)[(a,\alpha),(\frac{a+b}{2},\max\{\beta,s_2\})]$ and of $(P^-)[(\frac{a+b}{2},\max\{\beta,s_2\}),(b,\beta)]$ respectively. Then by Proposition 3.1 the "glued" functions

$$v_1(x) := \begin{cases} u_1^-(x) & \text{if } a \leqslant x \leqslant \frac{a+b}{2}, \\ u_1^+(x) & \text{if } \frac{a+b}{2} \leqslant x \leqslant b, \end{cases} \qquad v_2(x) := \begin{cases} u_2^-(x) & \text{if } a \leqslant x \leqslant \frac{a+b}{2}, \\ u_2^+(x) & \text{if } \frac{a+b}{2} \leqslant x \leqslant b \end{cases}$$

satisfy all the assumptions of Proposition 4.3, so the assertion follows from Theorem 6.1.

As regards the Lipschitz regularity, observe that the minimizer of (P) just found is obtained by gluing two minimizers of the constrained problems (see (6.3)). So, if they are Lipschitz, also the minimizer of (P) enjoys the same regularity property. □

7. Existence results

As a consequence of Corollary 6.2, when the constrained problems (P^+), (P^-) admits minimum whatever the endpoints may be, then problem (P) is solvable. Therefore, using the criteria for the existence of the minimum for constrained problems stated in [14, Corollaries 16, 17], we can deduce results concerning the free problem (P).

• *Convex case*

We first consider the case of noncoercive but convex integrands. In the sequel, whenever (2.4) holds true, we will adopt the following notation

$$\sigma_1 := \min\{\alpha, s_1\}, \qquad \sigma_2 := \max\{\beta, s_2\}$$

where s_1, s_2 were defined in (2.5).

Theorem 7.1. *Let* $f : I \times \mathbb{R} \to [0,+\infty)$ *be a continuous function satisfying* (2.2)–(2.4). *Suppose that for every* $s \in [\sigma_1, \sigma_2]$ *the function* $f(s,\cdot)$ *is convex. Assume that*

$$\lim_{|z|\to+\infty} \inf\bigl\{f(s,z) - z\partial f(s,z)\bigr\} = -\infty, \quad \textit{for a.e. } s \in [\sigma_1, \sigma_2]. \tag{7.1}$$

Then problem (P) *admits minimum. Moreover, if* (7.1) *holds for every* $s \in [\sigma_1, \sigma_2]$ *then* (P) *admits a Lipschitz minimizer.*

Proof. In order to apply Corollary 6.2, we have to show that the constrained problems (P^+) and (P^-) admit minimum whatever the endpoints may be. This is a consequence of the existence result proved in [14, Theorem 15]. Indeed, following the notations in [14, Section 5] for the constrained problem (P^+), the growth condition (7.1) means that $\lambda(s) = -\infty$ for a.e. $s \in [s_1, s_2]$, hence $c_0 := \operatorname{ess\,sup} \lambda(s) = -\infty$. Then, in order to apply [14, Theorem 15(ii)] we only need to verify the validity of condition (17) in [14, Theorem 12], that is put

$$g^-(s,z) := \sup\bigl\{f(s,z) - z\partial f(s,z)\bigr\}, \qquad \gamma(s,y) := \sup\bigl\{z > 0\colon\ g^-(s,z) \geqslant y\bigr\}$$

we have to show that

$$1/\gamma(s,c) \in L^1(\sigma_1,\sigma_2) \quad \text{for every } c < \min_{s\in[\sigma_1,\sigma_2]} f(s,0). \tag{7.2}$$

Indeed, we will show now that

$$\inf_{s\in[\sigma_1,\sigma_2]} \gamma(s,c) > 0 \quad \text{for every } c < \min_{s\in[\sigma_1,\sigma_2]} f(s,0)$$

implying (7.2). In fact, assume by contradiction that $\inf_{s\in[\sigma_1,\sigma_2]}\gamma(s,c)=0$ for some $c<\min_{s\in[\sigma_1,\sigma_2]} f(s,0)$. Then there exists a sequence $(s_h)_h$ in $[\sigma_1,\sigma_2]$ such that

$$\sup\left\{f\left(s_h,\frac{1}{h}\right)-\frac{1}{h}\partial f\left(s_h,\frac{1}{h}\right)\right\}<c. \tag{7.3}$$

Let us prove that

$$0\leqslant f_-\left(s_h,\frac{1}{h}\right)\leqslant \max_{s\in[\sigma_1,\sigma_2]} f(s,2), \tag{7.4}$$

where $f_-(s,z)$ denotes the left derivative of $f(s,\cdot)$ at the point $z>0$. Since we can assume (2.8) without loss of generality, the first inequality is trivial. By the convexity of $f(s_h,\cdot)$ and being $f(s,z)\geqslant 0$ we have

$$f(s_h,2)\geqslant f\left(s_h,\frac{1}{h}\right)+f_-\left(s_h,\frac{1}{h}\right)\left(2-\frac{1}{h}\right)\geqslant f_-\left(s_h,\frac{1}{h}\right)$$

and (7.4) follows. By (7.3) and (2.8) we have

$$c>f\left(s_h,\frac{1}{h}\right)-\frac{1}{h}f_-\left(s_h,\frac{1}{h}\right)\geqslant f(s_h,0)-\frac{1}{h}\max_{s\in[\sigma_1,\sigma_2]} f(s,2).$$

So, taking a converging subsequence $s_{h_k}\to s_0$ and passing to the limit we obtain

$$c\geqslant f(s_0,0)\geqslant \min_{s\in[\sigma_1,\sigma_2]} f(s,0),$$

in contradiction with the choice of c.

Therefore, all the conditions of [14, Theorem 15(ii)] are satisfied and the constrained problem (P^+) admits minimum whatever the endpoints may be. A similar argument works for (P^-) too and this concludes the proof of the existence of the minimum. As for the Lipschitz regularity, it follows from [14, Theorem 12]. □

Condition (7.1) is weaker than similar ones considered in [4–6], by various points of view, but mainly owing to the requirement that it is satisfied just almost everywhere. In this way we can handle integrands $f(s,z)$ possibly vanishing in straight lines $s=s_0$ with s_0 lying in a given set of null measure. For instance, when the Lagrangian has an affine-type structure, the statement of the previous theorem becomes the following.

Corollary 7.2. *Let $f(s,z):=\phi(s)+\psi(s)h(z)$ with ϕ continuous and nonnegative, ψ continuous and almost everywhere positive, h nonnegative and convex. Assume that condition* (2.4) *is satisfied and that*

$$\lim_{|z|\to+\infty}\inf\left\{h(z)-z\partial h(z)\right\}=-\infty. \tag{7.5}$$

Then problem (P) is solvable. Moreover, if $\psi(s)>0$ for every $s\in I$, then (P) admits a Lipschitz minimizer.

Example 7.3. Let

$$f(s,z)=\phi(s)+\psi(s)\sqrt{1+|z|^q},\quad q>2$$

with ϕ,ψ as in Corollary 7.2. Clearly the associated functional is not coercive since $\psi(s)$ may vanish. Nevertheless (P) admits minimizers for every boundary data.

Example 7.4. Let

$$f(s,z)=\psi(s)\max\left\{1,|z|-\log|z|\right\}\quad\text{or}\quad f(s,z)=\psi(s)\left(|z|-\sqrt{|z|}\right).$$

These functions f are both noncoercive. The former has been considered by Cellina–Ferriero [6] proving the existence of a Lipschitz solution, and the second one, has been treated by Celada and Perrotta [4]. Notice that in these two cases the authors assume that $\inf_s\psi(s)>0$. We are now able to remove this last requirement, indeed the integrands satisfy condition (7.5) and the existence of the minimum is guaranteed even if $\inf_s\psi(s)=0$, provided that ψ is as in Corollary 7.2.

There are also situations in which condition (7.1) does not hold, but nevertheless problem (P) admits minimum. The following existence result, immediate consequence of [14, Corollary 16], holds for integrands having a product structure.

Corollary 7.5. *Let $f(s,z) := \psi(s)h(z)$ with ψ continuous and positive, h nonnegative, convex, but not affine in any half-line. Let condition* (2.4) *be satisfied and assume*

$$\lim_{|z|\to+\infty} \inf\{h(z) - z\partial h(z)\} = 0. \tag{7.6}$$

Then, problem (P) admits a Lipschitz continuous minimizer.

Example 7.6. Let

$$f(s,z) = \psi(s)\sqrt{1+z^2}$$

with ψ as in Corollary 7.5. Condition (7.6) holds and (P) admits a Lipschitz minimizer. Note that this result is also consequence of the existence result obtained by Clarke [8].

Thanks to Corollary 2.6, the above results admit the following variant.

Corollary 7.7. *If in Corollaries* 7.2 *and* 7.5 *condition* (2.4) *is replaced by*

$$I \text{ is closed and unbounded and } \liminf_{|s|\to+\infty} \psi(s) > 0, \tag{7.7}$$

then the conclusions still hold.

• *Nonconvex case*

Let us consider the relaxed problem

$$\text{minimize} \left\{ F^{**}(v) = \int_a^b f^{**}\big(v(x), v'(x)\big)\,\mathrm{d}x\colon\ v \in \Upsilon \right\}. \tag{P^{**}}$$

When the solvability of (P^{**}) implies that of (P), then all the previous existence results hold, provided that the assumptions are referred to f^{**} instead of f. We mention now a recent relaxation theorem proved in [10, Theorem 4.4], which can be usefully applied in this context.

Theorem 7.8. *Let $f, f^{**} : I \times \mathbb{R} \to [0,+\infty)$ be Borel-measurable functions. Suppose that $s \mapsto f(s,0)$ is lower semicontinuous, satisfying* (2.3)*. Assume also that*

$$\mathrm{co}\{z\colon\ f^{**}(s,z) = f(s,z)\} = \mathbb{R} \quad \text{for all } s \in I \tag{7.8}$$

and that at least one of the following conditions is satisfied

- *the restriction of $f^{**}(\cdot,0)$ to $[\alpha,\beta]$ has at most countable many minimizers,*
- *for every $s \in [\alpha,\beta]$ the detachment set $\{z \in \mathbb{R}\colon\ f^{**}(s,z) < f(s,z)\}$ does not contain any interval $(-\delta,0)$ or $(0,\delta)$.*

Then, if (P^{**}) is solvable, also (P) is solvable.

Therefore, if the Lagrangian f satisfies the assumptions of such a result and f^{**} satisfies the assumptions of Theorem 7.1 (or the subsequent corollaries), then (P) admits minimum.

Appendix A

In this section we provide the proofs of Theorem 2.4 and Corollary 2.6. We begin by proving a preliminary result about condition (2.4), stating that the interval I can be assumed to be bounded, without loss of generality, if one of conditions (A.2+), (A.2−) below holds true, so that condition (2.4) is trivially satisfied if I is closed.

Lemma A.1. *Let $f : I \times \mathbb{R} \to [0, +\infty)$ be a Borel-measurable function. Let there exists measurable functions $c_1, c_2 : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\liminf_{|s|\to+\infty} c_1(s) > -\infty, \qquad \liminf_{|s|\to+\infty} c_2(s) > 0, \tag{A.1}$$

such that

$$f(s,z) \geqslant c_1(s) + c_2(s)|z| \quad \textit{for every } s \in I,\ z \geqslant 0 \tag{A.2+}$$

or

$$f(s,z) \geqslant c_1(s) + c_2(s)|z| \quad \textit{for every } s \in I,\ z \leqslant 0. \tag{A.2−}$$

Then there exists $L > 0$ such that

$$\inf_{\Upsilon} F = \inf_{\Upsilon^{(L)}} F$$

where $\Upsilon^{(L)} := \{u \in \Upsilon\colon\ |u(x)| \leqslant L$ for every $x \in [a,b]\}$.

Proof. Let us assume (A.2+) (the proof under condition (A.2−) is analogous). Moreover, assume that $\lambda := \inf_{u\in\Upsilon} F(u) < +\infty$ (otherwise there is nothing to prove).

By (A.1) there exist constants $\epsilon > 0$, $K > 0$, $M > \max\{|\alpha|, |\beta|\}$, such that for every $|s| > M$ we have $c_1(s) > -K$ and $c_2(s) > \epsilon$. Therefore,

$$f(s,z) \geqslant -K + \epsilon z \quad \text{for every } s \in I \text{ with } |s| \geqslant M,\ \text{and every } z > 0. \tag{A.3}$$

Fixed a positive real constant $H > \lambda$, set $L := \frac{1}{\epsilon}(H + K(b-a)) + M$.

Let us fix $u \in \Upsilon$ such that $F(u) \leqslant H$, and let $\bar{x} \in [a,b]$ be such that $|u(\bar{x})| > M$ (if such a point does not exist then $\|u\|_{L^\infty} \leqslant M < L$). Suppose that $u(\bar{x}) > M$ (the proof is analogous if $u(\bar{x}) < -M$). Let $x_0 := \inf\{x \in [a,b]\colon\ u(\xi) > M$ for every $\xi \in (x, \bar{x})\}$. Since $M > \alpha$ then $x_0 > a$ and $u(x_0) = M$. So, by (A.3) we get

$$u(\bar{x}) \leqslant u(x_0) + \int_{x_0}^{\bar{x}} \big[u'(x)\big]^+ \,\mathrm{d}x \leqslant M + \frac{1}{\epsilon}\left(\int_{x_0}^{\bar{x}} f\big(u(x), u'(x)\big)\,\mathrm{d}x + K(b-a)\right) \leqslant L.$$

Hence, $\|u\|_{L^\infty} \leqslant L$ for every $u \in \Upsilon$ such that $F(u) < H$ and the assertion follows. □

Proof of Theorem 2.4. Let us assume (2.9+). Let us put

$$c_1(s) := f^{**}(s,0) - \epsilon K, \qquad c_2(s) := \begin{cases} 0 & \text{if } |s| < M, \\ \epsilon & \text{if } |s| \geqslant M. \end{cases}$$

Hence assumption (A.1) of Proposition A.2 is satisfied and we have

$$\begin{aligned}
&f^{**}(s,z) - g(s)z \geqslant f^{**}(s,0) + \epsilon z \geqslant f^{**}(s,z) - \epsilon K + \epsilon z && \text{if } z \geqslant K,\ |s| \geqslant M,\\
&f^{**}(s,z) - g(s)z \geqslant f^{**}(s,0) \geqslant f^{**}(s,0) - \epsilon(K - z) && \text{if } 0 \leqslant z \leqslant K,\ |s| \geqslant M,\\
&f^{**}(s,z) - g(s)z \geqslant f^{**}(s,0) > f^{**}(s,0) - \epsilon K = c_1(s) + c_2(s)z && \text{if } z \geqslant 0,\ |s| \leqslant M
\end{aligned}$$

so that function $\tilde{f}$ satisfies condition (A.2+).

If (2.9−) holds, the proof is analogous. □

Proof of Corollary 2.6. Chosen a constant $\xi_0 \in \partial h^{**}(0)$, in this case conditions (2.9+) or (2.9−) are satisfied provided that:

$$\lim_{z\to+\infty} \frac{h^{**}(z)}{z} - \xi_0 > 0 \quad \text{or} \quad \xi_0 - \lim_{z\to+\infty} \frac{h^{**}(z)}{z} > 0,$$

respectively. If h^{**} is not linear in the whole real line, then at least one of the limits $\lim_{z\to\pm\infty} \frac{h^{**}(z)}{z}$ is different from ξ_0. □

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