Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 26 (2009) 1223–1236

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

# Vortex analysis of the periodic Ginzburg–Landau model ☆

Hassen Aydi [a], Etienne Sandier [b,c,*]

[a] *Institut supérieur d'informatique de Mahdia, route de Réjiche Km 4, BP 53, Mahdia 5121, Tunisie*
[b] *Université Paris-Est, LAMA – CNRS UMR 8050, 61, Avenue du Général de Gaulle, 94010 Créteil, France*
[c] *Institut Universitaire de France, France*

Received 8 January 2008; received in revised form 19 September 2008; accepted 19 September 2008

Available online 29 October 2008

**Abstract**

We study the vortices of energy minimizers in the London limit for the Ginzburg–Landau model with periodic boundary conditions. For applied fields well below the second critical field we are able to describe the location and number of vortices. Many of the results presented appeared in [H. Aydi, Doctoral Dissertation, Université Paris-XII, 2004], others are new.


**Résumé**

Nous étudions les tourbillons des minimiseurs de l'énergie de Ginzburg–Landau en supraconductivité dans la limite de London, et pour des conditions aux limites périodiques. Lorsque le champ magnétique appliqué est petit devant le second champ critique, nous décrivons le nombre et la localisation de ces tourbillons. Certains de ces résultats étaient présents dans [H. Aydi, Doctoral Dissertation, Université Paris-XII, 2004], d'autres sont nouveaux.


*Keywords:* Superconductivity; Ginzburg–Landau equations; Vortices; Periodic boundary conditions

*Mots-clés :* Supraconductivité ; Équations de Ginzburg–Landau ; Tourbillons ; Conditions aux limites périodiques

## 1. Introduction

Periodic solutions of the Ginzburg–Landau equations of superconductivity with vortices arranged in a lattice were first introduced in the famous work of A. Abrikosov [1], based on an analysis of the linearized equations about the normal solution, where the order parameter is 0. Since then many contributions to the study of this type of solutions have appeared both from physicists and mathematicians, establishing rigorously the existence of Abrikosov type solutions ([20], more recently [5] established the existence of many other families of periodic solutions) or investigating the energy or the minimality of these solutions [18,17] or their numeric analysis [12]. We may refer to the review paper [10] for a broad overview of the subject from a physicist's point of view.

☆ This research was supported by the ANR, project ANR-05-JCJC-0213.
* Corresponding author at: Université Paris-Est, LAMA – CNRS UMR 8050, 61, Avenue du Général de Gaulle, 94010 Créteil, France.
*E-mail address:* sandier@univ-paris12.fr (E. Sandier).

0294-1449/$ – see front matter 
doi:10.1016/j.anihpc.2008.09.004

There is a convenient variational setting for the periodic Ginzburg–Landau equations, we may refer to [20,13] for a mathematically oriented presentation of this setting, the existence of minimizers is proved in [20], the regularity of solutions is established [13] together with other properties of minimizers or critical points of the Ginzburg–Landau functional. Let us also cite the recent works [6] and [14], without going into further detail.

Opposite to the case of solutions close to 0, the so-called London limit or London approximation deals with solutions where the order parameter has modulus close to 1, except in small areas (the vortex cores). This limit was investigated by A. Abrikosov from the beginning, see also [10] for the use of this approximation in numerous situations, or the classical textbook [25]. The mathematical justification of this approach following the methods introduced in [8] may be found in [22], where it is in addition applied to describe the vortices of minimizers of the Ginzburg–Landau energy in a fixed domain $\Omega$ (in units of the penetration depth), when the Ginzburg–Landau parameter $\kappa$ is large and the applied field is small compared to the $\mathbb{R}^2$-upper critical field, i.e. in the parameter region where minimizers may indeed be analyzed using the London approximation.

In [7], the first author carried out for the case of periodic boundary conditions the analysis carried out in [22] in the case of natural boundary conditions in a bounded domain. It was established among other things that in the periodic case, the regime of the lower critical field is $H_{c_1}(\kappa) = \log\kappa/2$ to leading order as $\kappa \to +\infty$, see below for a precise statement. Note that in the units used in [7], and in most regimes where vortices are present, there is a divergent number of vortices as $\kappa \to \infty$ in each periodicity cell. The periodicity cell is large compared to the intervortex distance, and thus the periodicity constraint is not a strong one.

The present paper contains both results present in [7] and new results which complete them, in order to give a fairly unified description of the vortices of minimizers of the Ginzburg–Landau functional in the limit $\kappa \to \infty$, for applied fields well below the upper critical field.

## 2. Statement of the results

**Notation.** Throughout, $K$ will denote a parallelogram with area 1 generated by two vectors $(\vec{u}, \vec{v})$. We will denote by $\mathcal{L}$ the group of translations generated by $(\vec{u}, \vec{v})$.

The parameter of our asymptotic analysis will be the inverse of the Ginzburg–Landau parameter $\kappa$, denoted by $\varepsilon$.

The applied field $h_{\mathrm{ex}}$ is a positive function of $\varepsilon \in \mathbb{R}_+^*$, and we define

$$\Delta_{\mathrm{ex}} = h_{\mathrm{ex}} - \frac{1}{2}|\log\varepsilon|.$$

As noted in the introduction, $\frac{1}{2}|\log\varepsilon|$ is the regime of the so-called lower critical field $H_{c_1}(\varepsilon)$, to be defined below, this motivates the notation $\Delta_{\mathrm{ex}}$.

**Definition 1.** We define $H^1_{\mathrm{per}}$ to be the set of $(u, A)$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that for any integers $k, \ell \in \mathbb{Z}$ the configuration $(u(\cdot + k\vec{u} + \ell\vec{v}), A(\cdot + k\vec{u} + \ell\vec{v}))$ is gauge-equivalent to $(u, A)$.

In a more geometrical language, $u$ is a section of a complex line bundle over the torus $\mathbb{R}^2/\mathcal{L}$, and $A$ is a connection. Then $\operatorname{curl} A$ is the curvature of the connection and

$$\frac{1}{2\pi}\int_K \operatorname{curl} A$$

is an integer, the first chern class of the line bundle.

Given $(u, A)$ in $H^1_{\mathrm{per}}$, and an for each $\varepsilon > 0$ an applied magnetic field $h_{\mathrm{ex}}(\varepsilon)$, we define for any $\varepsilon > 0$

$$G_\varepsilon(u, A) = \frac{1}{2}\int_K \left(|\nabla u - iAu|^2 + \frac{1}{2}(\operatorname{curl} A - h_{\mathrm{ex}})^2 + \frac{1}{2\varepsilon^2}\left(1 - |u|^2\right)^2\right).$$

Then the problem of minimizing $G_\varepsilon(u, A)$ over $H^1_{\mathrm{per}}$ is well posed.

**Proposition 2.1.** *The minimum of $G_\varepsilon(u, A)$ over $H^1_{\mathrm{per}}$ is achieved.*

We refer to [20] for the proof.

A useful fact is

**Proposition 2.2.** *Given any* $(u, A) \in H^1_{\text{per}}$, *then*

$$\frac{1}{2\pi}\int_K \operatorname{curl} A$$

*is an integer. Moreover, if* $(u_1, A_1)$ *minimizes the Ginzburg–Landau energy with parameters* $\varepsilon > 0$ *and* $h_{\text{ex}} = h_1$, *and if* $(u_2, A_2)$ *minimizes the Ginzburg–Landau energy with parameters* $\varepsilon > 0$ *and* $h_{\text{ex}} = h_2 > h_1$, *then* $n_2 \geqslant n_1$, *where we have set for* $i = 1, 2$

$$n_i = \frac{1}{2\pi}\int_K \operatorname{curl} A_i.$$

**Proof.** The fact that $\int_K \operatorname{curl} A \in 2\pi\mathbb{Z}$ was already mentioned.

For the second statement, denote $G_1$ (resp. $G_2$) the Ginzburg–Landau functional with parameters $(\varepsilon, h_1)$ (resp. $(\varepsilon, h_2)$), then $G_1(u_1, A_1) \leqslant G_1(u_2, A_2)$ and $G_2(u_1, A_1) \geqslant G_2(u_2, A_2)$, hence

$$(G_1 - G_2)(u_1, A_1) \leqslant (G_1 - G_2)(u_2, A_2),$$

which translates as

$$(h_2 - h_1)\int_K \operatorname{curl} A_1 + \frac{1}{2}\int_K \left(h_2^2 - h_1^2\right) \leqslant (h_2 - h_1)\int_K \operatorname{curl} A_2 + \frac{1}{2}\int_K \left(h_2^2 - h_1^2\right),$$

hence the result. □

**Remark 2.1.** As a consequence of the above proposition, for each $\varepsilon > 0$ there is a well-defined value $H_{c_1}(\varepsilon)$ which we call the first critical field, and such that the minimizers of the Ginzburg–Landau functional with parameters $(\varepsilon, h_{\text{ex}})$ satisfy $n = 0$ if $h_{\text{ex}} < H_{c_1}$, and $n \neq 0$ if $h_{\text{ex}} > H_{c_1}$. Note that in the former case, the minimizers are necessarily gauge-equivalent to the constant superconducting solution $u = 1$, $A = 0$, see below.

**Theorem 1.** *Denote by* $(u_\varepsilon, A_\varepsilon)$ *any minimizer of* $G_\varepsilon$ *and let* $h_\varepsilon = \operatorname{curl} A_\varepsilon$. *We define the integer* $n_\varepsilon$ *by*

$$2\pi n_\varepsilon = \int_K h_\varepsilon.$$

*Then the following behaviour of* $h_\varepsilon$, $n_\varepsilon$ *holds, according to the regime considered for the applied field* $h_{\text{ex}}$.

(1) *If* $1 \ll \Delta_{\text{ex}} \ll 1/\varepsilon^2$, *then, as* $\varepsilon \to 0$,

$$\frac{h_\varepsilon}{2\pi n_\varepsilon} \to 1 \quad \text{in } W^{1,p} \text{ for any } p < 2, \text{ and } n_\varepsilon \approx \frac{\Delta_{\text{ex}}}{2\pi}. \tag{2.1}$$

(2) *If* $\Delta_{\text{ex}}$ *is bounded independently of* $\varepsilon$ *then so is* $\|h_\varepsilon\|_{W^{1,p}}$, *for any* $p < 2$. *In particular* $n_\varepsilon$ *is bounded independently of* $\varepsilon$. *If* $\{\varepsilon\}$ *is a subsequence such that* $\{h_\varepsilon\}_\varepsilon$ *converges to* $h_*$ *and* $\Delta_{\text{ex}}$ *converges to a value* $\Delta^*_{\text{ex}}$, *then* $n_\varepsilon \to n_* \in \mathbb{N}$ *along the same subsequence, thus in particular* $n_\varepsilon = n_*$ *for small enough* $\varepsilon$, *and there are* $n_*$ *distinct points* $\{a_i\}_i$ *in* $K$ *such that*

$$-\Delta h_* + h_* = 2\pi\sum_{i=1}^{n_*}\delta_{a_i}. \tag{2.2}$$

*Moreover, denote by* $\mathcal{P}$ *the finite families of points in* $K$ *and for* $\mathbf{p} = (p_1, \ldots, p_k) \in \mathcal{P}$ *let*

$$W(\mathbf{p}) = \lim_{\rho\to 0}\left(\pi n \log\rho + \frac{1}{2}\int_{K\setminus\bigcup_i B(p_i,\rho)} |\nabla h_{\mathbf{p}}|^2 + h_{\mathbf{p}}^2\right) + n\left(\gamma - 2\pi\Delta^*_{\text{ex}}\right), \tag{2.3}$$

*where* $h_{\mathbf{p}}$ *is the unique* $K$*-periodic solution of* $-\Delta h_{\mathbf{p}} + h_{\mathbf{p}} = 2\pi \sum_{i=1}^k \delta_{p_i}$. *Then* $(a_1, \ldots, a_{n_*})$ *minimizes* $W$ *over* $\mathcal{P}$.

*The number* $\gamma$ *in* (2.3) *was introduced in* [8], *we define it as in* [22], *Proposition* 3.11, *as*

$$\gamma = \lim_{R\to+\infty} -\pi \log R + \frac{1}{2} \int\limits_{B(0,R)} |\nabla u_0|^2 + \frac{(1-|u_0|^2)^2}{2}, \tag{2.4}$$

*where* $u_0$ *is the unique solution of* $-\Delta u_0 = u_0(1-|u_0|^2)$ *in* $\mathbb{R}^2$ *of the form* $u_0(r,\theta) = f(r)e^{i\theta}$, *with* $f : \mathbb{R}_+ \to \mathbb{R}_+$.

(3) *There exists a possibly negative* $\Delta_1 \in \mathbb{R}$ *such that if* $\Delta_{\mathrm{ex}} < \Delta_1$ *and* $\varepsilon$ *is small enough, then* $n_* = 0$. *In this case* $(u_\varepsilon, A_\varepsilon)$ *is gauge-equivalent to the Meissner solution* $(1,0)$.

**Remark 2.2.** The first item in the above theorem was proved initially in [7].

The main difference between the periodic case and the case of a bounded domain is that in the former the distribution of vortices is always uniform, and hence may be described by a unique number, namely the number of vortices, at least if it tends to $+\infty$ as $\varepsilon \to 0$. This is a big simplification over the case of a bounded domain, together with the fact that here the number of vortices is given by the integral of $h$.

**Remark 2.3.** The fact that the minimum of $W$ on $\mathcal{P}$ is achieved is a consequence of the above theorem, it could also be derived directly from (2.3). From Proposition 2.2, if $\Delta_2 > \Delta_1 \in \mathbb{R}$ and if $\mathbf{p}_1$ (resp. $\mathbf{p}_2$) minimizes $W$ for $\Delta_{\mathrm{ex}} = \Delta_1$ (resp. $\Delta_{\mathrm{ex}} = \Delta_2$), then the number of points in $\mathbf{p}_2$ is larger than the number of points in $\mathbf{p}_1$. This shows that there are critical values of $\Delta_{\mathrm{ex}}$ for which the number of points for a minimizer experiences a jump, and it would be interesting to show that it jumps by one unit only. This would show the existence of an increasing sequence of critical values $(\Delta_k)_{k\in\mathbb{N}^*}$ for which the number of vortices jumps from $k-1$ to $k$.

Note that at a jump there exists minimizers with different numbers of vortices, hence the minimizer of $W$ need not be unique. Even if $\Delta_{\mathrm{ex}}$ is not a critical value, i.e. if minimizers of $W$ all have the same number of points, if this number is 2 for instance then the symmetry of the periodicity is broken and there are several minimizers as well.

**Remark 2.4.** Regarding the finer structure of vortices, they are expected in general to arrange themselves in periodic lattices, and the hexagonal lattice is supposedly optimal. Several rigorous mathematical results in this direction have been proved (see for instance [3,2,23]). In the periodic setting, the strongest version of this conjecture should be true: If the lattice generated by $(\vec{u}, \vec{v})$ is the hexagonal lattice and if $n_\varepsilon = k^2$ for some integer $k$, then the minimizing configuration should be periodic w.r.t. the vectors $(\vec{u}/k, \vec{v}/k)$. A limiting form of this conjecture would be that – still in the case of a hexagonal lattice – in the case of a bounded number of vortices, and assuming $n_* = k^2$, then $h_*$ is periodic w.r.t. the vectors $(\vec{u}/k, \vec{v}/k)$.

**Remark 2.5.** Note that for a minimizer $(u_\varepsilon, A_\varepsilon)$ of $G_\varepsilon$ and arbitrary positive values of the parameters $\varepsilon$, $h_{\mathrm{ex}}$, we have

$$\frac{1}{2}\int\limits_K (h_\varepsilon - h_{\mathrm{ex}})^2 \leqslant G_\varepsilon(u_\varepsilon, A_\varepsilon) \leqslant G_\varepsilon(1,0) = \frac{1}{2}h_{\mathrm{ex}}^2,$$

where $h_\varepsilon = \operatorname{curl} A_\varepsilon$. Therefore the following holds

$$0 \leqslant n_\varepsilon, \qquad G_\varepsilon(u_\varepsilon, A_\varepsilon) \leqslant \frac{1}{2}h_{\mathrm{ex}}^2. \tag{2.5}$$

Moreover, if $n_\varepsilon = 0$, then $h_\varepsilon = 0$ as well. Then $(u_\varepsilon, A_\varepsilon)$ is gauge equivalent to a configuration $(u'_\varepsilon, 0)$ and $G_\varepsilon(u'_\varepsilon, 0) \leqslant G_\varepsilon(1,0)$. But, since $(1,0)$ is clearly the unique minimizer of $G_\varepsilon$ among configurations $(u, A)$ such that $A = 0$, we deduce that $u'_\varepsilon = 1$ and that $(u_\varepsilon, A_\varepsilon)$ is gauge-equivalent to the Meissner solution.

The paper is organized as follows: In Section 3 we construct a test configuration which will be useful in the proof of case (1) of the theorem. In Section 4, matching the upper bound of the previous section with appropriate lower bounds, we prove case (1) of the theorem, with an $L^2$ instead of $W^{1,p}$ convergence, we also prove an $L^2$ bound for $h_\varepsilon$ in case (2), and we prove case (3) completely. In Section 5 we discuss the improvement from $L^2$ to $W^{1,p}$ convergence without going into the details, since similar arguments appear in [22] in the context of natural, instead of periodic, boundary conditions. Finally in the last section we finish the proof of case (2), i.e. identify the limit $h_*$ as a minimizer of $W$. This involves arguments from [8,9] adapted to the periodic setting, that are in part sketched.

## 3. Upper bound

The theorem is proved by energy comparison with an appropriate test configuration $(v_\varepsilon, B_\varepsilon)$, which will be periodic. It is defined below, rather quickly since this construction appears elsewhere (see for instance [21,7] or [4]). In the case where $\Delta_{\text{ex}} \leqslant C$, which corresponds to cases (2) and (3) of the theorem, the upper bound in (2.5) will suffice, hence we assume from now on that

$$1 \ll \Delta_{\text{ex}} \ll 1/\varepsilon^2.$$

Then, we define $m_\varepsilon$ as the integer such that $\sqrt{m_\varepsilon}$ is the integer part of

$$\sqrt{\frac{\Delta_{\text{ex}}}{2\pi}}.$$

Since $\Delta_{\text{ex}}$ tends to $+\infty$ as $\varepsilon \to 0$ we have

$$m_\varepsilon \approx \frac{\Delta_{\text{ex}}}{2\pi}.$$

We then divide $K$ into $m_\varepsilon$ identical parallelograms generated by the vectors $\vec{u}_\varepsilon = \vec{u}/\sqrt{m_\varepsilon}$ and $\vec{v}_\varepsilon = \vec{v}/\sqrt{m_\varepsilon}$. Denote by $K_\varepsilon$ one of these parallelograms, by $a_\varepsilon$ its center, and by $f_\varepsilon$ the solution in $K_\varepsilon$, with periodic boundary conditions, of the equation

$$-\Delta f_\varepsilon = 2\pi \delta_{a_\varepsilon} - \frac{2\pi}{|K_\varepsilon|}.$$

Note that such a solution exists since the integral of the right-hand side over $K_\varepsilon$ is 0. It is then naturally extended by periodicity to a function $f_\varepsilon$ defined in all of $\mathbb{R}^2$ and satisfying

$$-\Delta f_\varepsilon = 2\pi \sum_{k,\ell\in\mathbb{Z}} \delta_{a_\varepsilon + k\vec{u}_\varepsilon + \ell\vec{v}_\varepsilon} - \frac{2\pi}{|K_\varepsilon|}.$$

Then, we let $B_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ be such that

$$\operatorname{curl} B_\varepsilon = \frac{2\pi}{|K_\varepsilon|}$$

and write $v_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$, where $\rho_\varepsilon$ is periodic with respect to the vectors $(\vec{u}_\varepsilon, \vec{v}_\varepsilon)$ and is defined in $K_\varepsilon$ by $\rho_\varepsilon(x) = \min(|x - a_\varepsilon|/\varepsilon, 1)$, and where $\varphi_\varepsilon$ is such that $-\nabla^\perp f_\varepsilon = \nabla\varphi_\varepsilon - B_\varepsilon$. This latter equation can be solved in the sense that the curl of $(B_\varepsilon - \nabla^\perp f_\varepsilon)$ is

$$2\pi \sum_{k,\ell\in\mathbb{Z}} \delta_{a_\varepsilon + k\vec{u}_\varepsilon + \ell\vec{v}_\varepsilon},$$

hence there exists a function $\varphi_\varepsilon$ defined modulo $2\pi$ except at the points $a_\varepsilon + k\vec{u}_\varepsilon + \ell\vec{v}_\varepsilon$ which satisfies the identity. Then letting $v_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ is legitimate, since precisely from the definition of $\rho_\varepsilon$ we have $\rho_\varepsilon(a_\varepsilon + k\vec{u}_\varepsilon + \ell\vec{v}_\varepsilon) = 0$ for any integers $k, \ell$.

To estimate the energy, note that the integrand of $G_\varepsilon(v_\varepsilon, B_\varepsilon)$ is precisely

$$\frac{1}{2}\left(\rho_\varepsilon^2|\nabla f_\varepsilon|^2 + |\nabla\rho_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\left(1 - \rho_\varepsilon^2\right)^2 + \left|\frac{2\pi}{|K_\varepsilon|} - h_{\text{ex}}\right|^2\right). \tag{3.1}$$

Its integral over $K_\varepsilon$ is easily estimated (see [21,7] or [4]). Putting aside the last term, the contribution from the region where $\rho_\varepsilon \neq 1$, which is $B(a_\varepsilon, \varepsilon)$ is bounded by a constant independent of $\varepsilon$ from scaling arguments. The integrand outside $B(a_\varepsilon, \varepsilon)$ reduces to $\frac{1}{2}|\nabla f_\varepsilon|^2$ which, with the price of an error bounded independently of $\varepsilon$, may be replaced by $\frac{1}{2}|\nabla \log|x - a_\varepsilon||^2$, whose integral over $K_\varepsilon$ is $\pi \log \frac{1}{\sqrt{m_\varepsilon}\varepsilon} + \mathrm{O}(1)$ as $\varepsilon \to 0$. Then, integrating (3.1) over $K_\varepsilon$ yields

$$\pi \log \frac{1}{\sqrt{m_\varepsilon}\varepsilon} + \frac{|K_\varepsilon|}{2}\left|\frac{2\pi}{|K_\varepsilon|} - h_{\text{ex}}\right|^2 + \mathrm{O}(1)$$

as $\varepsilon \to 0$.

Since there are $m_\varepsilon = 1/|K_\varepsilon|$ squares in $K$, we deduce

$$G_\varepsilon(v_\varepsilon, B_\varepsilon) = \pi m_\varepsilon \log \frac{1}{\sqrt{m_\varepsilon}\varepsilon} + \frac{1}{2}|2\pi m_\varepsilon - h_{\rm ex}|^2 + \mathrm{O}(m_\varepsilon).$$

After expanding and replacing $h_{\rm ex} = \Delta_{\rm ex} + |\log\varepsilon|/2$ we find

$$G_\varepsilon(v_\varepsilon, B_\varepsilon) = 2\pi^2 m_\varepsilon^2 - 2\pi m_\varepsilon \Delta_{\rm ex} + \frac{1}{2}h_{\rm ex}^2 + \mathrm{o}\big(m_\varepsilon^2\big).$$

Finally, using the fact that $m_\varepsilon \approx \Delta_{\rm ex}/2\pi$ and tends to $+\infty$ as $\varepsilon \to 0$ we get

$$G_\varepsilon(v_\varepsilon, B_\varepsilon) = \frac{1}{2}\big(h_{\rm ex}^2 - \Delta_{\rm ex}^2\big) + \mathrm{o}\big(\Delta_{\rm ex}^2\big). \tag{3.2}$$

## 4. Lower bound, identification of the limits

**High fields.** The case where $\Delta_{\rm ex} \approx h_{\rm ex}$, or equivalently $|\log\varepsilon| \ll h_{\rm ex} \ll \varepsilon^{-2}$ is particularly simple. In fact in this case one could even get more detailed information (see [21,22]). Indeed the existence of a test configuration satisfying (3.2) implies that $G_\varepsilon(u_\varepsilon, A_\varepsilon)$ is smaller than the right-hand side of (3.2), which is itself $\mathrm{o}(h_{\rm ex}^2)$. it follows in particular that $\|h_\varepsilon - h_{\rm ex}\|^2_{L^2(K)}$ is $\mathrm{o}(h_{\rm ex}^2)$ and therefore, dividing by $h_{\rm ex}^2$, that $h_\varepsilon/h_{\rm ex} \to 1$ in $L^2(K)$, hence also in $L^1(K)$. In particular

$$n_\varepsilon \approx \frac{h_{\rm ex}}{2\pi} \approx \frac{\Delta_{\rm ex}}{2\pi},$$

and of course $h_\varepsilon/(2\pi n_\varepsilon) \to 1$ in $L^2(K)$. In fact in this case we even have strong convergence of $h_\varepsilon/(2\pi n_\varepsilon)$ in $H^1$. Indeed, any critical point of $G_\varepsilon$ satisfies the so-called second Ginzburg–Landau equation

$$-\nabla^\perp h_\varepsilon = j_\varepsilon, \quad \text{where } j_\varepsilon = (iu_\varepsilon, \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon) = \rho_\varepsilon^2(\nabla\varphi_\varepsilon - A_\varepsilon), \tag{4.1}$$

where the last equality is an alternative expression for $j_\varepsilon$ when $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ is not zero. A consequence of (4.1) is that pointwise $|\nabla h_\varepsilon| \leqslant |\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|$. Therefore

$$\frac{1}{2}\int_K \big(|\nabla h_\varepsilon|^2 + |h - h_{\rm ex}|^2\big) \leqslant G_\varepsilon(u_\varepsilon, A_\varepsilon),$$

and we are able to deduce as above, since the right-hand side is $\mathrm{o}(h_{\rm ex}^2)$, that $h_\varepsilon/h_{\rm ex} \to 1$ in $H^1$.

**Low fields.** We now deal with the rest of the cases in the theorem, namely case (1) with $h_{\rm ex} = \mathrm{O}(|\log\varepsilon|)$ and cases (2) and (3). We thus assume the estimate $h_{\rm ex} \leqslant C|\log\varepsilon|$.

First, we recall the following construction which can either be adapted from [15] or [22], Theorem 4.1 to this periodic setting. We first define for any $r \in (0, 1)$ the *free energy* of $(u, A) \in H^1_{\rm per}$ as

$$F_{r,\varepsilon}(u, A) = \frac{1}{2}\int_K \left(|\nabla u - iAu|^2 + r^2h^2 + \frac{1}{2\varepsilon^2}\big(1 - |u|^2\big)^2\right), \tag{4.2}$$

with the notation $h = \operatorname{curl} A$. In particular we have the following relation between $F_{r,\varepsilon}$ and $G_\varepsilon$

$$G_\varepsilon(u, A) = F_{r,\varepsilon}(u, A) + \frac{1 - r^2}{2}\int_K h^2 - h_{\rm ex}\int_K h + \frac{1}{2}h_{\rm ex}^2. \tag{4.3}$$

**Proposition 4.1.** *Assume $G_\varepsilon(u_\varepsilon, A_\varepsilon) \leqslant C|\log\varepsilon|^2$ for any $\varepsilon > 0$, where $(u_\varepsilon, A_\varepsilon) \in H^1_{\rm per}$. Then there exists $\varepsilon_0 > 0$ and $r_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$ and any $1 > r \geqslant \varepsilon^{1/3}$, the following holds.*

*There exists a collection of disjoint closed balls $\mathcal{B} = \{B_i\}_{i\in I}$ which is periodic with respect to $K$ and finite in any compact subset of $\mathbb{R}^2$. Moreover, denoting by $\mathcal{B}_0$ a minimal subset of $\mathcal{B}$ such that $\mathcal{B} = \bigcup_{\tau\in\mathcal{L}} \tau\mathcal{B}_0$,*

(1) *Denoting by $r(\mathcal{B})$ the sum of the radii of balls belonging to a collection $\mathcal{B}$, we have $r(\mathcal{B}_0) = r$.*

(2) *Abusing notation,* $\{||u_\varepsilon| - 1| \geqslant 1/2\} \subset \mathcal{B}$.
(3) *Writing* $d_B = \deg(u, \partial B)$,

$$F_{r,\varepsilon}(u_\varepsilon, A_\varepsilon, K \cap \mathcal{B}) \geqslant \pi D\left(\log \frac{r}{D\varepsilon} - C\right), \tag{4.4}$$

*where* $D = \sum_{B \in \mathcal{B}_0} |d_B|$ *is assumed to be nonzero and* $C$ *is a universal constant.*
(4) *Finally,*

$$D \leqslant C \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon, K)}{|\log \varepsilon|}, \tag{4.5}$$

*where* $C$ *is a universal constant.*

*Finally, if* $r_1 > r_2$ *and* $\mathcal{B}_1$, $\mathcal{B}_2$ *are the corresponding families of balls, then every ball in* $\mathcal{B}_2$ *is included in one of the balls of* $\mathcal{B}_1$.

The above result is adapted from [22], Theorem 4.1, applied with $\alpha = 2/3$. The proof of the equivalent of Theorem 4.1 in the periodic setting poses no difficulty, the way to do it is to consider $u_\varepsilon$ as a section of a complex line bundle over the torus $T = \mathbb{R}^2/\mathcal{L}$, where $\mathcal{L}$ is the group of translations generated by $(\vec{u}, \vec{v})$, and $A_\varepsilon$ as a connection over this bundle. This is described somewhat in [4], Proposition 4.7. Note that we must impose a bound $r < r_0$ which is here to ensure that any ball of radius $r < r_0$ on $T$ is indeed a topological ball, it suffices to take $r_0$ smaller than the injectivity radius of $T$.

We also note the following

**Lemma 4.1.** *For any* $(u, A) \in H^1_{\text{per}}$, *if* $\{B_i\}_{i \in I}$ *is a finite collection of disjoint closed balls in* $\mathbb{R}^2$ *such that* $|u| > 0$ *in* $K \setminus \bigcup_i B_i$ *and such that* $\bigcup_i B_i$ *does not intersect* $\partial K$, *then*

$$\sum_i \deg\bigl(u/|u|, \partial B_i\bigr) = \int_K \operatorname{curl} A.$$

**Proof.** This is standard. By periodicity, there exists two functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ such that

$$u(x + \vec{u}) = u(x) e^{if(x)}, \qquad A(x + \vec{u}) = A(x) + \nabla f(x),$$

and the same relations hold, replacing $\vec{u}$ by $\vec{v}$ and $f$ by $g$.

The sum $D$ of the degrees is the degree of $u/|u|$ restricted to $\partial K$. Letting $\varphi$ be the phase of $u$, we thus have, denoting by $\tau$ the positively oriented unit vector tangent to $\partial K$,

$$D = \int_{\partial K} \partial_\tau \varphi = \int_0^1 \frac{1}{\|\vec{u}\|} \frac{d}{ds}\bigl(\varphi(s\vec{u}) - \varphi(s\vec{u} + \vec{v})\bigr) + \int_0^1 \frac{1}{\|\vec{v}\|} \frac{d}{ds}\bigl(\varphi(\vec{u} + s\vec{v}) - \varphi(s\vec{v})\bigr)\, ds.$$

Then since $\varphi(x + \vec{u}) - \varphi(x) = f(x)$ and $\varphi(x + \vec{v}) - \varphi(x) = g(x)$ it follows that

$$D = \int_0^1 \frac{1}{\|\vec{u}\|} \frac{d}{ds} g(s\vec{u}) + \frac{1}{\|\vec{v}\|} \frac{d}{ds} f(s\vec{v}),$$

and then, since $A(x + \vec{u}) - A(x) = \nabla f(x)$ and $A(x + \vec{v}) - A(x) = \nabla g(x)$,

$$D = \int_{\partial K} A \cdot \tau = \int_K \operatorname{curl} A. \qquad \square$$

Let us now consider for any $\varepsilon > 0$ the minimizer $(u_\varepsilon, A_\varepsilon)$ of $G_\varepsilon$, and let $h_\varepsilon = \operatorname{curl} A_\varepsilon$. For any $r \in (0, 1)$ we may construct using Proposition 4.1 a collection of balls $\mathcal{B}$ of total radius $r$. Moreover if $r$ is small enough, then a translation $\tau$ can be chosen so that $\partial(\tau K)$ does not intersect $\mathcal{B}$. We may then choose as the minimal subset $\mathcal{B}_0$ of

Proposition 4.1 the collection $\mathcal{B}_r$ consisting of the balls in $\mathcal{B}$ which are included in $\tau K$. The previous lemma combined with Proposition 4.1 then implies that

$$F_{r,\varepsilon}(u_\varepsilon, A_\varepsilon, \mathcal{B}_r) \geqslant \pi d_\varepsilon \left(\log \frac{r}{d_\varepsilon \varepsilon} - C\right), \qquad 2\pi d_\varepsilon \geqslant \int_{\tau K} h_\varepsilon = \int_K h_\varepsilon = n_\varepsilon,$$

where $d_\varepsilon = \sum_{B \in \mathcal{B}_r} |d_B|$, and the equality between the integrals over $\tau K$ and $K$ follows from the periodicity of the configurations. Note that since $F_{r,\varepsilon}(u_\varepsilon, A_\varepsilon) \leqslant C h_{\text{ex}}^2$ and $h_{\text{ex}} \leqslant C|\log \varepsilon|$, the a priori bound (4.5) reads

$$d_\varepsilon \leqslant C|\log \varepsilon|.$$

we deduce from the above a lower bound for $G_\varepsilon$:

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \geqslant \pi|\log \varepsilon| n_\varepsilon - 2\pi h_{\text{ex}} n_\varepsilon + \frac{1}{2} h_{\text{ex}}^2 + \frac{1 - r^2}{2} \int_K h_\varepsilon^2 + R_\varepsilon, \tag{4.6}$$

where

$$R_\varepsilon = \pi|\log \varepsilon|(d_\varepsilon - n_\varepsilon) + \pi d_\varepsilon \left(\log \frac{r}{d_\varepsilon} - C\right). \tag{4.7}$$

We now choose a fixed radius $r < 1/4$. Since $d_\varepsilon \leqslant C|\log \varepsilon|$, it is easy to check, that if $d_\varepsilon > 2n_\varepsilon$ and $\varepsilon$ is small enough, then $R_\varepsilon \geqslant 0$ while if $n_\varepsilon \leqslant d_\varepsilon \leqslant 2n_\varepsilon$, then clearly

$$R_\varepsilon \geqslant -C n_\varepsilon (\log n_\varepsilon - C_r), \tag{4.8}$$

where $C_r$ depends on $r$. Returning to the lower bound for $G_\varepsilon$, we write $h_{\text{ex}} = \Delta_{\text{ex}} + |\log \varepsilon|/2$ and obtain

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) \geqslant -2\pi \Delta_{\text{ex}} n_\varepsilon + \frac{1}{2} h_{\text{ex}}^2 + \frac{1 - r^2}{2} \int_K h_\varepsilon^2 + R_\varepsilon. \tag{4.9}$$

Now we distinguish two cases.

$\Delta_{\text{ex}} \leqslant \mathbf{C}$. This corresponds to cases (2) and (3) of the theorem. Combining (4.9) with the upper bound (2.5) and the estimate (4.8) gives

$$\frac{1 - r^2}{2} \int_K h_\varepsilon^2 - C n_\varepsilon (\log n_\varepsilon - C_r) \leqslant 2\pi \Delta_{\text{ex}} n_\varepsilon \leqslant C n_\varepsilon \tag{4.10}$$

but the integral of $h_\varepsilon^2$ over $K$ is bounded below by $4\pi^2 n_\varepsilon^2$ using Cauchy–Schwarz, and thus the above inequality implies that for some $c, C > 0$ independent of $\varepsilon$,

$$c n_\varepsilon^2 - C n_\varepsilon (\log n_\varepsilon + 1) \leqslant 2\pi \Delta_{\text{ex}} n_\varepsilon \leqslant C n_\varepsilon.$$

Thus $n_\varepsilon$ is bounded independently of $\varepsilon$, and going back to (4.10) $h_\varepsilon$ is bounded in $L^2(K)$ independently of $\varepsilon$.

Moreover, since $n_\varepsilon \geqslant 0$, the above inequality implies that if $\Delta_{\text{ex}}$ is smaller than a certain, possibly negative value $\Delta_1 \in \mathbb{R}$ which could be expressed in terms of the constants $c, C$ appearing in the inequality, then $n_\varepsilon = 0$ if $\varepsilon$ is small enough. Then the inequality

$$\frac{1}{2} \int_K (h_\varepsilon - h_{\text{ex}})^2 \leqslant G_\varepsilon(u_\varepsilon, A_\varepsilon) \leqslant G_\varepsilon(1, 0) = \frac{1}{2} h_{\text{ex}}^2$$

implies that $h_\varepsilon = 0$ and $|u_\varepsilon| = 1$, thus $(u_\varepsilon, A_\varepsilon)$ is the Meissner solution, proving case (3) of the theorem.

$\mathbf{1} \ll \Delta_{\text{ex}} \leqslant \mathbf{C}|\log \varepsilon|$. In this case as in the previous one we first combine (4.9) with the upper bound (2.5) to obtain

$$\frac{1 - r^2}{2} \int_K h_\varepsilon^2 - C n_\varepsilon (\log n_\varepsilon - C_r) \leqslant 2\pi \Delta_{\text{ex}} n_\varepsilon,$$

which allows to conclude that $n_\varepsilon/\Delta_{\mathrm{ex}}$ remains bounded as $\varepsilon \to 0$, and then that $h_\varepsilon/\Delta_{\mathrm{ex}}$ is bounded in $L^2(K)$ independently of $\varepsilon$. It remains to identify the limit of both $n_\varepsilon/\Delta_{\mathrm{ex}}$ and $h_\varepsilon/\Delta_{\mathrm{ex}}$. More precisely, from any sequence $\{\varepsilon_n\}_n$ tending to zero we may extract a subsequence such that the limits exist, let us call them $n_*$ and $h_*$. If we compute $h_*$ and $n_*$ independently of the particular subsequence, then we will have proved the convergence, and identified the limits. Since we have the relation

$$2\pi n_* = \int_K h_*, \tag{4.11}$$

it suffices, to finish the proof of the theorem, to prove that $h_* = 1$.

First we may combine the more precise upper bound (3.2) with (4.9) and (4.8) to obtain

$$\frac{1-r^2}{2}\int_K h_\varepsilon^2 - \Delta_{\mathrm{ex}}\int_K h_\varepsilon - Cn_\varepsilon(\log n_\varepsilon - C_r) \leqslant -\frac{1}{2}\Delta_{\mathrm{ex}}^2 + \mathrm{o}\big(\Delta_{\mathrm{ex}}^2\big).$$

Then dividing the above by $\Delta_{\mathrm{ex}}^2$ and letting $\varepsilon \to 0$ we find

$$\frac{1-r^2}{2}\int_K h_*^2 - \int_K h_* \leqslant -\frac{1}{2}.$$

The left-hand side is bounded below by the minimum of the function $-x + x^2/2$, i.e. by $-1/2$. It follows that $h_* = 1$.

## 5. Improved convergence

To improve the convergence of $h_\varepsilon/\Delta_{\mathrm{ex}}$ (resp. $h_\varepsilon$) in case (1) (resp. case (2)) of the theorem, we invoke the classical Jacobian estimate of [16], together with a small-large ball type argument. Note that in the case $h_{\mathrm{ex}} \gg |\log\varepsilon|$, we have already proved the $H^1$ convergence, hence we assume below that $h_{\mathrm{ex}} < C|\log\varepsilon|$.

We recall the London equation satisfied by critical points of the functional $G_\varepsilon$:

$$-\Delta h_\varepsilon + h_\varepsilon = \mu_\varepsilon, \quad \text{where } \mu_\varepsilon = \operatorname{curl} j_\varepsilon + h_\varepsilon, \tag{5.1}$$

where the superconducting current $j_\varepsilon$ is defined in (4.1).

The proof for the improved convergence is then the following. We construct for any $\varepsilon > 0$ using Proposition 4.1 a collection of vortex balls with total radius $r_\varepsilon = \varepsilon^{\frac{1}{3}}$, which we denote $\mathcal{B}'$, and we let

$$\nu_\varepsilon = 2\pi \sum_{B\in\mathcal{B}'} d_B\delta_{a_B}, \tag{5.2}$$

where $a_B$ denotes the center of $B$.

It then follows from the Jacobian estimate (see for instance [22], Theorem 6.2) that for any $\beta \in (0,1)$, we have

$$\lim_{\varepsilon\to 0}(\mu_\varepsilon - \nu_\varepsilon) = 0, \quad \text{in the dual of } C^{0,\beta}.$$

Now we let $d'_\varepsilon = \sum_{B\in\mathcal{B}'}|d_B|$, and we claim that

$$d'_\varepsilon = \mathrm{O}(n_\varepsilon), \quad \text{as } \varepsilon \text{ tends to } 0. \tag{5.3}$$

Assume a moment that this is true (since $n_\varepsilon = \sum_{B\in\mathcal{B}'} d_B$, it amounts to proving that the balls have mostly positive degrees), then writing

$$\mu_\varepsilon = \nu_\varepsilon + (\mu_\varepsilon - \nu_\varepsilon),$$

we find that in the case $1 \ll \Delta_{\mathrm{ex}} \leqslant C|\log\varepsilon|$ and since $n_\varepsilon = \mathrm{O}(\Delta_{\mathrm{ex}})$, the sequence $\{\mu_\varepsilon/\Delta_{\mathrm{ex}}\}_\varepsilon$ is bounded in the dual of $C^{0,\beta}$ for any $\beta \in (0,1)$. By compact embedding of $W^{1,q}$ into $C^{0,\beta}$ for any $q > 2$ and well chosen $\beta$, we deduce that for any $p < 2$ we may extract a convergent subsequence in $W^{-1,p}$. Using (5.1), this gives the convergence of $\{h_\varepsilon/\Delta_{\mathrm{ex}}\}_\varepsilon$ in $W^{1,p}$. The same argument holds, without normalizing by $\Delta_{\mathrm{ex}}$ in case $\Delta_{\mathrm{ex}}$ is bounded independently of $\varepsilon$.

It remains to prove (5.3). This is done by inspecting carefully (4.6) and (4.7), letting $r = \varepsilon^{\frac{1}{3}}$ there, and comparing with the upper bound (2.5). These yield after some simplifications

$$-2\pi n_\varepsilon \Delta_{\mathrm{ex}} + \pi\left[\left(d'_\varepsilon - n_\varepsilon\right)|\log\varepsilon| - \frac{d'_\varepsilon}{3}|\log\varepsilon| - d'_\varepsilon \log d'_\varepsilon - Cd'_\varepsilon\right] \leqslant 0.$$

Since we have assumed $\Delta_{\mathrm{ex}} \leqslant C|\log\varepsilon|$, the above may be written

$$\pi\frac{2}{3}d'_\varepsilon|\log\varepsilon| - Cn_\varepsilon|\log\varepsilon| + \mathrm{o}\left(d'_\varepsilon|\log\varepsilon|\right) \leqslant 0,$$

which implies that $d_\varepsilon = \mathrm{O}(n_\varepsilon)$ as $\varepsilon \to 0$, and concludes the proof of the $W^{1,p}$ convergence.

## 6. The case of a finite number of vortices

We now finish the proof of case (2) of the theorem, i.e. identify $h_*$ as a minimizer of some renormalized energy, in the terminology of [8].

The proof of this fact relies first on the construction of test configurations with prescribed vortices, a rather straightforward task. The second part is to compute a precise expansion of the energy of a minimizer $(u_\varepsilon, A_\varepsilon)$ in terms of the vortex locations. This expansion can be found in [8] for the model without magnetic field and Dirichlet boundary condition, or in the case with magnetic field in [9], the definition of $\gamma$ there is different from (2.4), but a result of Mironescu [19] shows the two are equivalent. The vortex locations in these references is to be understood as the limiting locations as $\varepsilon \to 0$. A similar expansion may be found in [24] for configurations which are not necessarily minimizers of the energy. Another option is to give an expansion of the energy in terms of the actual vortex locations for fixed $\varepsilon$. This is the approach used in [11] in the case without magnetic field or in [22] for the case with magnetic field. The latter approach gives information for each $\varepsilon > 0$, but is less elementary than the former, that we adopt.

The proof we sketch for the convenience of the reader is borrowed from [22,9] and [8].

**Upper bound.** Given $\mathbf{p} = (a_1, \ldots, a_n) \in \mathcal{P}$, the test configuration is constructed as follows. First we let $h$ be the solution of $-\Delta h + h = 2\pi\sum_i \delta_{a_i}$ in $K$ with periodic boundary conditions. We still denote by $h$ the extension of $h$ to $\mathbb{R}^2$ by periodicity (note that $h$ does not depend on $\varepsilon$). Letting

$$\mathcal{A} = \{a_i + k\vec{u} + \ell\vec{v} \mid k,\ \ell \in \mathbb{Z},\ 1 \leqslant i \leqslant n\},$$

we have $-\Delta h + h = 2\pi\sum_{a\in\mathcal{A}} \delta_a$ in $\mathbb{R}^2$.

Then $A$ is chosen such that $\operatorname{curl} A = h$ and $\varphi$ is defined modulo $2\pi$ in $\mathbb{R}^2 \setminus \mathcal{A}$ and such that $\nabla\varphi - A = \nabla^\perp h$. Such a $\varphi$ exists precisely because $\operatorname{curl}(A + \nabla^\perp h) = -\Delta h + h$ is equal to $2\pi\sum_{a\in\mathcal{A}} \delta_a$. Finally we define, for a fixed, arbitrarily chosen $R > 0$,

$$\rho_\varepsilon(x) = \begin{cases} 1 & \text{if } x \notin \bigcup_{a\in\mathcal{A}} B(a, R\varepsilon), \\ \frac{f(|x-a|/\varepsilon)}{f(R)} & \text{if } x \in B(a, R\varepsilon), \end{cases}$$

where $f(r) = |u_0|(r)$, and $u_0$ is the radial vortex which appears in (2.4), the definition of $\gamma$.

Then we let $v_\varepsilon = \rho_\varepsilon e^{i\varphi}$. It is clear that $(v_\varepsilon, A)$ is $K$-periodic since the gauge-invariant quantities $h = \operatorname{curl} A$, $\rho_\varepsilon = |v_\varepsilon|$ are and since $\nabla\varphi - A = \nabla^\perp h$. It remains to evaluate the energy of $(v_\varepsilon, A)$ over $K$. First we note that, since the integral of $h$ over $K$ is $2\pi n$,

$$G_\varepsilon(v_\varepsilon, A) = \frac{h_{\mathrm{ex}}^2}{2} - 2\pi n_\varepsilon h_{\mathrm{ex}} + G'_\varepsilon(v_\varepsilon, A), \tag{6.1}$$

where

$$G'_\varepsilon(v_\varepsilon, A) = \frac{1}{2}\int\limits_K \left(|\nabla v_\varepsilon - iAv_\varepsilon|^2 + h^2 + \frac{1}{2\varepsilon^2}\left(1 - |v_\varepsilon|^2\right)^2\right). \tag{6.2}$$

We evaluate $G'_\varepsilon(v_\varepsilon, A)$. Let $B_i = B(a_i, R\varepsilon)$ and $K_R = K \setminus \bigcup_i B_i$. On $K_R$ we have $|v_\varepsilon| = 1$ and $\nabla\varphi - A = \nabla^\perp h$ from which it follows that the energy density there reduces to $(|\nabla h|^2 + |h|^2)/2$. Therefore, from the definition (2.3), we may write

$$\lim_{\varepsilon\to 0}\left(G'_\varepsilon(v_\varepsilon, A, K_R) + \pi n \log(R\varepsilon)\right) = W(\mathbf{p}) - n\left(\gamma - 2\pi\Delta^*_{\mathrm{ex}}\right).$$

On the other hand, from [22], Chapter 10, we have

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\left(G'_\varepsilon\left(v_\varepsilon, A, \bigcup_i B_i\right) - \pi n\log R - n\gamma\right) = 0.$$

It follows that

$$\lim_{\varepsilon\to 0}\big(G'_\varepsilon(v_\varepsilon, A) + \pi n\log\varepsilon\big) = W(\mathbf{p}) + 2\pi n\Delta^*_{\mathrm{ex}}$$

and therefore

$$\lim_{\varepsilon\to 0}\left(G_\varepsilon(v_\varepsilon, A) - \frac{h^2_{\mathrm{ex}}}{2}\right) = W(\mathbf{p}, n). \tag{6.3}$$

**Lower bound.** Now we assume that $(u_\varepsilon, A_\varepsilon)$ is a minimizer of $G_\varepsilon$ and we try to compute a matching lower bound for (6.3).

First, from (5.1), we know that $h_\varepsilon$ converges in $W^{1,p}$ for any $p < 2$ to the solution of

$$-\Delta h_* + h_* = \mu_*,$$

where $\mu_*$ is the common limit of $\{\mu_\varepsilon\}_\varepsilon$ and $\{\nu_\varepsilon\}_\varepsilon$ as $\varepsilon\to 0$. Then (5.2), (5.3) and the fact that $n_\varepsilon \to n_*$ imply that $\mu_*$ is of the form $2\pi\sum_{i=1}^k d_i\delta_{a_i}$, where $d_i\in\mathbb{Z}$ and $\sum_i d_i = n_*$. Note that the points $a_i$ need not (yet) be distinct.

From the lower and upper bounds (4.6) and (2.5), we draw some further consequences, in view of (4.7). Indeed, in the lower bound (4.6), we have not used all the terms in the integrand of $G_\varepsilon$. More precisely, we have only taken into account outside the balls $\mathcal{B}_r$ the term $(h_\varepsilon - h_{\mathrm{ex}})^2/2$. It follows that a by-product of the upper bound (2.5) is that, as $\varepsilon\to 0$,

$$\frac{1}{2}\int_{K\setminus\mathcal{B}_r} |\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\big(1-|u_\varepsilon|^2\big)^2 \leqslant C.$$

Since this is true for any $r$ (the constant $C$ then depending on $r$), we obtain in particular, letting $\rho_\varepsilon = |u_\varepsilon|$ and since $|\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon| \geqslant |\nabla\rho_\varepsilon|$, that

$$|\nabla\rho_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\big(1-\rho_\varepsilon^2\big)^2$$

is bounded in $L^1_{\mathrm{loc}}(K\setminus\{a_1,\dots,a_n\})$.

Let $r_0$ be half the minimal distance between two points $a_i, a_j$, and fix $r\in(0,r_0)$. It follows from the above and from the strong convergence of $h_\varepsilon$ to $h_*$ in $W^{1,p}$, using a mean-value argument that for any $\varepsilon>0$, there exists $r/2 < r_\varepsilon < r$ such that for every $1\leqslant i\leqslant n$ and letting $\gamma_{i,\varepsilon} = \partial B(a_i,\varepsilon)$,

$$\int_{\gamma_{i,\varepsilon}} |\nabla\rho_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\big(1-\rho_\varepsilon^2\big)^2 = \mathrm{O}(1), \qquad \|h_\varepsilon - h_*\|_{W^{1,p}(\gamma_{i,\varepsilon})} = \mathrm{o}(1), \tag{6.4}$$

as $\varepsilon\to 0$.

We bound from below $G'_\varepsilon(u_\varepsilon, A_\varepsilon, B_{i,\varepsilon})$, where $B_{i,\varepsilon} = B(a_i, r_\varepsilon)$. To this aim we assume that we are in the Coulomb gauge. Then (see [9]), $\{A_\varepsilon\}_\varepsilon$ is bounded in $W^{2,p}(K)$ for any $p<2$, hence in $L^\infty$. Moreover, since

$$|\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|^2 \geqslant |\nabla u_\varepsilon|^2 - 2\big(A_\varepsilon\cdot j_\varepsilon + |A_\varepsilon|^2|u_\varepsilon|^2\big),$$

we deduce that $G'_\varepsilon(u_\varepsilon, A_\varepsilon, B_{i,\varepsilon})$ is bounded below by

$$\frac{1}{2}\int_{B_{i,\varepsilon}}\left(|\nabla u_\varepsilon|^2 + \frac{(1-|u_\varepsilon|^2)^2}{2\varepsilon^2}\right) - C\big(\|A_\varepsilon\|^2_{L^2(B_{i,\varepsilon})} + \|A_\varepsilon\|_{L^4(B_{i,\varepsilon})}\|j_\varepsilon\|_{L^{4'}(B_{i,\varepsilon})}\big),$$

where $j_\varepsilon = (iu_\varepsilon, \nabla u_\varepsilon - iA_\varepsilon u_\varepsilon)$. But from (4.1) we have $j_\varepsilon \to -\nabla^\perp h_*$ in $L^p$ for any $p<2$. It follows easily (using also the $W^{2,p}$ bound for $A_\varepsilon$), that

$$G'_\varepsilon(u_\varepsilon, A_\varepsilon, B_{i,\varepsilon}) \geqslant \frac{1}{2}\int_{B_{i,\varepsilon}}\left(|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\big(1-|u_\varepsilon|^2\big)^2\right) - \delta_{r,\varepsilon},$$

where here and below $\delta_{r,\varepsilon}$ will denote a quantity such that

$$\lim_{r\to 0}\Bigl(\limsup_{\varepsilon\to 0}|\delta_{r,\varepsilon}|\Bigr)=0.$$

Next we blow-up $B_{i,\varepsilon}$ into the unit ball $B_1$, $u_\varepsilon$ becomes $v_\varepsilon$ and

$$\frac{1}{2}\int\limits_{B_{i,\varepsilon}}\left(|\nabla u_\varepsilon|^2+\frac{1}{2\varepsilon^2}\bigl(1-|u_\varepsilon|^2\bigr)^2\right)=\frac{1}{2}\int\limits_{B_1}\left(|\nabla v_\varepsilon|^2+\frac{1}{2\varepsilon'^2}\bigl(1-|v_\varepsilon|^2\bigr)^2\right),$$

where $\varepsilon'=\varepsilon/r_\varepsilon$. From (6.4) we deduce easily that $|v_\varepsilon|\to 1$ uniformly on $\partial B_1$. Moreover, as we have already noted, $(iu_\varepsilon,\nabla u_\varepsilon)=j_\varepsilon+|u_\varepsilon|^2A_\varepsilon$. Using (6.4) and (4.1), it follows that $|(iu_\varepsilon,\nabla u_\varepsilon)-\nabla^\perp h_*|\to 0$ in $L^p(\gamma_{i,\varepsilon})$ as $\varepsilon\to 0$. After blow-up, and using the fact that near $a_i$, $\nabla h_*$ behaves like the gradient of $-d_i\log|x-a_i|$, we deduce, letting $\tau$ denote the unit tangent vector to $\partial B_1$, that

$$\int\limits_{\partial B_1}\bigl|(iv_\varepsilon,\tau\cdot\nabla v_\varepsilon)-d_i\bigr|=\delta_{r,\varepsilon}.$$

But, letting $v_\varepsilon=\rho_\varepsilon e^{i\varphi_\varepsilon}$, we have $(iv_\varepsilon,\tau\cdot\nabla v_\varepsilon)={\rho_\varepsilon}^2\tau\cdot\nabla\varphi_\varepsilon$ and we already noted that $\rho_\varepsilon\to 1$ uniformly on $\partial B_1$. It is then straightforward to deduce that, extracting a subsequence if necessary, there exists $\theta_0\in\mathbb{R}$ such that

$$\bigl\|v_\varepsilon-e^{i(\theta_0+d_i\theta)}\bigr\|_{L^\infty(\partial B_i)}=\delta_{r,\varepsilon}.$$

Extracting again, we may assume that $r_\varepsilon\to r_*\in[r/2,r]$ and we deduce from the above that

$$G'_\varepsilon(u_\varepsilon,A_\varepsilon,B_{i,\varepsilon})\geqslant I(\varepsilon/r_*,d_i)+\delta_{r,\varepsilon},\tag{6.5}$$

where we have set

$$I(\varepsilon,d_i)=\min\left\{\frac{1}{2}\int\limits_{B_1}\left(|\nabla u|^2+\frac{1}{2\varepsilon^2}\bigl(1-|u|^2\bigr)^2\right)\Bigm| u=e^{id_i\theta}\text{ on }\partial B_1\right\}.$$

Note that from the analysis of [8] we have

$$I(\varepsilon,d)=\pi|d||\log\varepsilon|+C_d+o(1)\tag{6.6}$$

as $\varepsilon\to 0$, and in the case $d=\pm 1$ we have from [19]

$$I(\varepsilon,\pm 1)=\pi|\log\varepsilon|+\gamma+o(1).$$

We now compute a lower bound for $G'_\varepsilon(u_\varepsilon,A_\varepsilon,K_{r,\varepsilon})$, where we have set $K_{r,\varepsilon}=K\setminus\bigcup_i B_{i,\varepsilon}$. There, we use the fact that $|\nabla u_\varepsilon-iA_\varepsilon u_\varepsilon|\geqslant|j_\varepsilon|=|\nabla h_\varepsilon|$ (see for instance [22], Lemma 3.3), and the fact that $r_\varepsilon\to r_*$ to deduce

$$G'_\varepsilon(u_\varepsilon,A_\varepsilon,K_{r,\varepsilon})\geqslant\frac{1}{2}\int\limits_{K_{r,\varepsilon}}|\nabla h_\varepsilon|^2+|h_\varepsilon|^2,$$

and thus

$$\liminf_{\varepsilon\to 0}G'_\varepsilon(u_\varepsilon,A_\varepsilon,K_{r,\varepsilon})\geqslant\frac{1}{2}\int\limits_{K\setminus\bigcup_i B(a_i,r_*)}|\nabla h_*|^2+h_*^2.$$

Adding the above to (6.5), for $1\leqslant i\leqslant n$, we find in view of (6.6) that

$$G'_\varepsilon(u_\varepsilon,A_\varepsilon)\geqslant\frac{1}{2}\int\limits_{K\setminus\bigcup_i B(a_i,r_*)}\bigl(|\nabla h_*|^2+h_*^2\bigr)+\sum_{i=1}^n\left(\pi|d_i|\log\frac{r_*}{\varepsilon}+C_{d_i}\right)+\delta_{r,\varepsilon}.$$

Then, in view of (6.1) we find for $\varepsilon$ small enough and letting $D=\sum_i|d_i|$,

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) - \frac{1}{2}h_{\text{ex}}^2 \geqslant \frac{1}{2}\int_{K\setminus\bigcup_i B(a_i,r_*)} \left(|\nabla h_*|^2 + h_*^2\right) + \sum_{i=1}^{n}\big(\pi|d_i|\log r_* + C_{d_i}\big) + \pi(D-n_*)|\log\varepsilon| - 2\pi n_* \Delta_{\text{ex}} + \delta_{r,\varepsilon}. \tag{6.7}$$

And from the upper bound given by (6.3) the left-hand side is bounded independently of $\varepsilon$. A first consequence, obtained by taking the limit $\varepsilon \to 0$ is that $D = n_*$ hence every $d_i$ is positive. A second consequence is that

$$\frac{1}{2}\int_{K\setminus\bigcup_i B(a_i,r_*)} \left(|\nabla h_*|^2 + h_*^2\right) + \pi n_* \log r_*$$

remains bounded as $r_* \to 0$, which implies (see [8] or [9]) that $d_i = +1$ for every $i$. Then we obtain (2.2) and since $C_{d_i} = \gamma$ when $d_i = \pm 1$, taking successively the limits $\varepsilon \to 0$ and then $r \to 0$ in (6.7) we find, letting $\mathbf{p} = (a_1, \ldots, a_n)$,

$$\liminf_{\varepsilon\to 0}\left(G_\varepsilon(u_\varepsilon, A_\varepsilon) - \frac{1}{2}h_{\text{ex}}^2\right) \geqslant W(\mathbf{p}).$$

Comparing with (6.3) we conclude that $\mathbf{p}$ minimizes $W$.

**Note added in proof**

Recently, M. Kurzke and D. Spirn have studied the minimization of the Ginzburg–Landau functional in the high-kappa limit in a domain of size tending either to 0 or infinity as kappa tends to infinity (paper to appear in SIAM J. Math. Anal.). In the case of large domains, they obtain in particular estimates for the first critical field consistent with ours.

## Acknowledgements

The authors would like to express their thanks to the referees for their particularly useful suggestions, in particular for mentioning many of the references in the introduction.

## References

[1] A. Abrikosov, On the magnetic properties of superconductors of the second type, Soviet Phys. JETP 5 (1957) 1174–1182.
[2] A. Aftalion, X. Blanc, F. Nier, Lowest Landau level functional and Bargmann spaces for Bose–Einstein condensates, J. Funct. Anal. 241 (2) (2006) 661–702.
[3] A. Aftalion, S. Serfaty, Lowest Landau level approach in superconductivity for the Abrikosov lattice close to $H_{c_2}$, Selecta Math. (N.S.) 13 (2) (2007) 183–202.
[4] S. Alama, L. Bronsard, E. Sandier, On the shape of interlayer vortices in the Lawrence–Doniach model, Trans. Amer. Math. Soc. 360 (2008) 1–34.
[5] Y. Almog, On the bifurcation and stability of periodic solutions of the Ginzburg–Landau equations in the plane, SIAM J. Appl. Math. 61 (1) (2000) 149–171.
[6] Y. Almog, Abrikosov lattices in finite domains, Commun. Math. Phys. 262 (3) (2006) 677–702.
[7] H. Aydi, Doctoral Dissertation, Université Paris-XII, 2004.
[8] F. Bethuel, H. Brezis, F. Hélein, Ginzburg–Landau Vortices, Progress in Nonlinear Partial Differential Equations and Their Applications, vol. 13, Birkhäuser Boston, Boston, 1994.
[9] F. Bethuel, T. Rivière, Vortices for a variational problem related to superconductivity, Ann. Inst. H. Poincaré Anal. Non Linéaire 12 (3) (1995) 243–303.
[10] E.H. Brandt, The flux-line lattice in superconductors, Rep. Prog. Phys. 58 (1995) 1465–1594.
[11] M. Comte, P. Mironescu, The behavior of a Ginzburg–Landau minimizer near its zeroes, Calc. Var. Partial Differential Equations 4 (4) (1996) 323–340.
[12] M.M. Doria, J.E. Gubernatis, D. Rainer, Solving the Ginzburg–Landau equations by simulated annealing, Phys. Rev. B 41 (1990) 6335–6340.
[13] Q. Du, M.D. Gunzburger, J.S. Peterson, Modeling and analysis of a periodic Ginzburg–Landau model for type-II superconductors, SIAM J. Appl. Math. 53 (3) (1993) 689–717.
[14] M. Dutour, Bifurcation vers l'état d'Abrikosov et diagramme de phase, Thèse Orsay. Available online at: http://xxx.lanl.gov/abs/math-ph/9912011, 1999.
[15] R.L. Jerrard, Lower bounds for generalized Ginzburg–Landau functionals, SIAM J. Math. Anal. 30 (4) (1999) 721–746.
[16] R.L. Jerrard, H.M. Soner, The Jacobian and the Ginzburg–Landau energy, Calc. Var. Partial Differential Equations 14 (2) (2002) 151–191.
[17] W.H. Kleiner, L.M. Roth, S.H. Autler, Bulk solution of Ginzburg–Landau equations for type II superconductors: Upper critical field region, Phys. Rev. 133 (1964) A1226–A1227.

[18] G. Lasher, Series solution of the Ginzburg–Landau equations for the Abrikosov mixed state, Phys. Rev. 2 140 (1965) A523–A528.
[19] P. Mironescu, Les minimiseurs locaux pour l'équation de Ginzburg–Landau sont à symétrie radiale, C. R. Acad. Sci. Paris, Ser. I 323 (6) (1996) 593–598.
[20] F. Odeh, Existence and bifurcation theorems for the Ginzburg–Landau equations, J. Math. Phys. 8 (1967) 2351–2356.
[21] E. Sandier, S. Serfaty, On the energy of type-II superconductors in the mixed phase, Rev. Math. Phys. 12 (9) (2000) 1219–1257.
[22] E. Sandier, S. Serfaty, Vortices in the Magnetic Ginzburg–Landau Model, Progress in Nonlinear Differential Equations and Their Applications, vol. 70, Birkhäuser Boston Inc., Boston, MA, 2007.
[23] E. Sandier, S. Serfaty, From the Ginzburg–Landau model to vortex lattice problems, in preparation.
[24] S. Serfaty, Local minimizers for the Ginzburg–Landau energy near critical magnetic field, part I, Comm. Contemp. Math. 1 (2) (1999) 213–254, part II, 295–333.
[25] M. Tinkham, Introduction to Superconductivity, second ed., McGraw–Hill, New York, 1996.