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Pulsating traveling waves in the singular limit of a reaction–diffusion system in solid combustion ☆

R. Monneau [a,*], G.S. Weiss [b]

[a] *Ecole Nationale des Ponts et Chaussées, CERMICS, 6 et 8 avenue Blaise Pascal, Cité Descartes, Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, France*

[b] *Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo, 153-8914 Japan*

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Abstract

We consider a coupled system of parabolic/ODE equations describing solid combustion. For a given rescaling of the reaction term (the high activation energy limit), we show that the limit solution solves a free boundary problem which is to our knowledge new.

In the time-increasing case, the limit coincides with the Stefan problem with spatially inhomogeneous coefficients. In general it is a parabolic equation with a memory term.

In the first part of our paper we give a characterization of the limit problem in one space dimension. In the second part of the paper, we construct a family of pulsating traveling waves for the limit one phase Stefan problem with periodic coefficients. This corresponds to the assumption of periodic initial concentration of reactant.

MSC: 35K55; 35R35

Keywords: Solid combustion; Singular limit; Pulsating traveling wave; High activation energy; Stefan problem; Free boundary; Self-propagating high temperature synthesis

1. Introduction

For $\varepsilon > 0$, we consider the system

$$\begin{aligned} \partial_t u_\varepsilon - \Delta u_\varepsilon &= \frac{1}{\varepsilon} v_\varepsilon g_\varepsilon(u_\varepsilon), \\ \partial_t v_\varepsilon &= -\frac{1}{\varepsilon} v_\varepsilon g_\varepsilon(u_\varepsilon), \end{aligned} \tag{1}$$

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* Corresponding author.
E-mail addresses: monneau@cermics.enpc.fr (R. Monneau), gw@ms.u-tokyo.ac.jp (G.S. Weiss).
URLs: http://cermics.enpc.fr/~monneau/home.html (R. Monneau), http://www.ms.u-tokyo.ac.jp/~gw (G.S. Weiss).

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where u_ε is the normalized temperature, v_ε is the normalized concentration of the reactant and the non-negative nonlinearity g_ε describes the reaction kinetics. This model has been extensively used in solid combustion (to analyze SHS, i.e. Self-propagating High temperature Synthesis), see for instance Logak and Loubeau [7] and the references therein. We study the limit of high activation energy as $\varepsilon \to 0$, where we can take for instance

$$g_\varepsilon(z) := \begin{cases} \exp((1-1/(z+1))/\varepsilon), & z > -1, \\ 0, & z \leqslant -1 \end{cases}$$

but can also consider more general functions g_ε. In this particular case, the non-negative temperature is actually $u_\varepsilon + 1$ (see Section 6.1).

The analysis of the high activation energy limit has been introduced in the pioneering work of Zeldovich and Frank-Kamenetskii [11]. The rigorous asymptotic analysis for a system with finite Lewis model has been done by Berestycki, Nicolaenko and Scheurer in [3]. In the case of infinite Lewis number (which corresponds to system (1) of solid combustion), Logak and Loubeau proved in [7] the existence of a (planar) traveling wave for system (1) in one space dimension and gave a rigorous proof of convergence of this planar traveling wave in the limit $\varepsilon \to 0$.

The present paper consists of two parts. In the first part (Sections 4–6), we study the high activation energy limit $\varepsilon \to 0$ on a bounded domain, and in the second part (Sections 7, 8) we study pulsating traveling waves for the limit equation on the whole space.

More precisely we show in Section 5 (cf. Theorem 5.1) that in one space dimension, each limit u of u_ε solves the Stefan problem

$$\partial_t u - v^0 \partial_t \chi = \Delta u \tag{2}$$

where v^0 is the initial value of v and χ is the memory term $\chi = H(\operatorname{esssup}_{(0,t)} u(\cdot, x))$ where H is the heavyside function. In higher space dimensions (in Section 4), we get less information on the memory term χ, except in the case that $\partial_t u_\varepsilon \geqslant 0$ in which we show that any limit u still satisfies (2) (see Theorem 4.1). In Section 6, we apply our results to two cases considered in the literature.

In Section 7, we show the existence of pulsating traveling waves solutions of (2) for periodic v^0. Such pulsating traveling waves exist for any velocity and any direction of propagation. In Section 8, in the case $v^0 = \text{constant} > 0$, we give a result of non-existence of non-trivial pulsating traveling waves.

We conclude with miscellaneous remarks in Section 9 and present in Appendix A a result on formal stability of the planar wave for the limit one-phase Stefan problem in one space dimension.

Our approach to the problem is first to reduce the system to a single equation with a right-hand side which turns out to be the time derivative of a term which is non-local in time.

In the first part of the paper, using some a priori estimates, we show the compactness in L^1 (in space–time) of any truncation of u_ε and its convergence to a solution of the limit problem. One main difficulty consists in proving that $\chi = 0$ in the region where $\operatorname{esssup}_{(0,t)} u(\cdot, x)$ is negative for the limit problem, with $u_\varepsilon(t_\varepsilon, x_\varepsilon)$ being possibly positive for x_ε close to x and $t_\varepsilon < t$. To do this, we analyze the "burnt zone" $u_\varepsilon > \kappa$ (in space–time) for negative κ. Based on the comparison principle and on some integral estimates, we show in one space dimension that if the measure of the "burnt zone" is small enough in a parabolic cylinder, then in a cylinder of smaller radius it must be the empty set.

Our construction of pulsating traveling waves for the one-phase Stephan problem in the second part of the paper is based on an integration in time which reduces the problem to an obstacle problem. We then approximate this obstacle problem by a reaction–diffusion equation for which existence of pulsating traveling waves has been proved in [2] by Berestycki and Hamel. We conclude the proof passing to the limit in the approximation. The ingredients used involve Harnack inequality and blow-up arguments. Finally, in order to obtain non-existence of pulsating waves for constant v^0 we use a Liouville technique.

2. Notation

Throughout this article $\mathbf{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$. $B_r(x)$ will denote the open n-dimensional ball of center x, radius r and volume $r^n \omega_n$. When the center is not specified, it is assumed to be 0.

When considering a set A, χ_A shall stand for the characteristic function of A, while ν shall typically denote the outward normal to a given boundary. We will use the distance pardist with respect to the parabolic metric $d((t,x),(s,y)) = \sqrt{|t-s| + |x-y|^2}$.

The operator ∂_t will mean the partial derivative of a function in the time direction, Δ the Laplacian in the space variables and $\mathcal{L}^n$ the n-dimensional Lebesgue measure.

Finally $\mathbf{W}^{2,1}_p$ denotes the parabolic Sobolev space as defined in [6].

3. Preliminaries

In what follows, Ω is a bounded C^1-domain in $\mathbf{R}^n$ and

$$u_\varepsilon \in \bigcap_{T\in(0,+\infty)} \mathbf{W}^{2,1}_2\big((0,T)\times\Omega\big)$$

is a strong solution of the equation

$$\partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) = -v^0_\varepsilon(x)\partial_t \exp\left(-\frac{1}{\varepsilon}\int_0^t g_\varepsilon\big(u_\varepsilon(s,x)\big)\,ds\right),$$
$$u_\varepsilon(0,\cdot) = u^0_\varepsilon \quad \text{in } \Omega, \qquad \nabla u_\varepsilon\cdot\nu = 0 \quad \text{on } (0,+\infty)\times\partial\Omega; \tag{3}$$

here g_ε is a non-negative function on $\mathbf{R}$ satisfying:

(0) g_ε is for each $\varepsilon\in(0,1)$ piecewise continuous with only one possible jump at z_0, $g_\varepsilon(z_0-) = g_\varepsilon(z_0) = 0$ in case of a jump, and g_ε satisfies for each $\varepsilon\in(0,1)$ and for every $z\in\mathbf{R}$ the bound $g_\varepsilon(z)\leqslant C_\varepsilon(1+|z|)$.
(1) $g_\varepsilon/\varepsilon \to 0$ as $\varepsilon\to 0$ on each compact subset of $(-\infty,0)$.
(2) For each compact subset K of $(0,+\infty)$ there is $c_K>0$ such that $\min(g_\varepsilon, c_K)\to c_K$ uniformly on K as $\varepsilon\to 0$.

The initial data satisfy $0\leqslant v^0_\varepsilon\leqslant C<+\infty$, v^0_ε converges in $L^1(\Omega)$ to v^0 as $\varepsilon\to 0$, $(u^0_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2(\Omega)$, it is uniformly bounded from below by a constant $u_{\min}$, and it converges in $L^1(\Omega)$ to u^0 as $\varepsilon\to 0$.

Remark 3.1. Assumption (0) guarantees existence of a global strong solution for each $\varepsilon\in(0,1)$.

4. The high activation energy limit

The following theorem has been proved in [9]. Let us repeat the statements and its proof for the sake of completeness.

Theorem 4.1. *The family $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is for each $T\in(0,+\infty)$ precompact in $L^1((0,T)\times\Omega)$, and each limit u of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ as a sequence $\varepsilon_m\to 0$, satisfies in the sense of distributions the initial-boundary value problem*

$$\partial_t u - v^0\partial_t\chi = \Delta u \quad \text{in } (0,+\infty)\times\Omega, \tag{4}$$
$$u(0,\cdot) = u^0 + v^0 H\big(u^0\big) \quad \text{in } \Omega, \qquad \nabla u\cdot\nu = 0 \quad \text{on } (0,+\infty)\times\partial\Omega,$$

where

$$\chi(t,x)\begin{cases} \in[0,1], & \operatorname{esssup}_{(0,t)} u(\cdot,x)\leqslant 0,\\ =1, & \operatorname{esssup}_{(0,t)} u(\cdot,x) > 0,\end{cases}$$

and H is the maximal monotone graph

$$H(z)\begin{cases} =0, & z<0,\\ \in[0,1], & z=0,\\ =1, & z>0.\end{cases}$$

Moreover, χ is increasing in time and u is a supercaloric function.

If $(u_\varepsilon)_{\varepsilon\in(0,1)}$ satisfies $\partial_t u_\varepsilon\geqslant 0$ in $(0,T)\times\Omega$, then u is a solution of the Stefan problem for supercooled water, i.e.

$$\partial_t u - v^0\partial_t H(u) = \Delta u \quad \text{in } (0,+\infty)\times\Omega.$$

Remark 4.2. Note that assumption (1) is only needed to prove the second statement "If ...".

Remark 4.3. It is interesting to observe that even in the time-increasing case our singular limit *selects certain solutions* of the two-phases Stefan problem. For example, $u(t) = (\kappa - 1)\chi_{\{t<1\}} + \kappa\chi_{\{t>1\}}$ is for each $\kappa \in (0,1)$ a perfectly valid solution of the two-phases Stefan problem, but, as easily verified, it cannot be obtained from the ODE

$$\partial_t u_\varepsilon(t) = -\partial_t \exp\left(-\frac{1}{\varepsilon}\int_0^t \exp\left(\left(1 - 1/\left(\left(u_\varepsilon(s)+1\right)^+\right)\right)/\varepsilon\right) ds\right) \quad \text{as } \varepsilon \to 0.$$

Proof. *Step 0 (Uniform bound from below)*: Since u_ε is supercaloric, it is bounded from below by the constant $u_{\min}$.

Step 1 ($L^2((0,T)\times\Omega)$-bound): The time-integrated function $v_\varepsilon(t,x) := \int_0^t u_\varepsilon(s,x)\,ds$, satisfies

$$\partial_t v_\varepsilon(t,x) - \Delta v_\varepsilon(t,x) = w_\varepsilon(t,x) + u_\varepsilon^0(x) \tag{5}$$

where w_ε is a measurable function satisfying $0 \leqslant w_\varepsilon \leqslant C$. Consequently

$$\int_0^T\int_\Omega (\partial_t v_\varepsilon)^2 + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2(T) = \int_0^T\int_\Omega \left(w_\varepsilon + u_\varepsilon^0\right)\partial_t v_\varepsilon \leqslant \frac{1}{2}\int_0^T\int_\Omega (\partial_t v_\varepsilon)^2 + \frac{T}{2}\int_\Omega \left(C + \left|u_\varepsilon^0\right|\right)^2,$$

implying

$$\int_0^T\int_\Omega u_\varepsilon^2 \leqslant T\int_\Omega \left(C + \left|u_\varepsilon^0\right|\right)^2. \tag{6}$$

Step 2 ($L^2((0,T)\times\Omega)$-bound for $\nabla\min(u_\varepsilon, M)$): For

$$G_M(z) := \begin{cases} z^2/2, & z < M, \\ Mz - M^2/2, & z \geqslant M, \end{cases}$$

and any $M \in \mathbf{N}$,

$$\int_\Omega G_M(u_\varepsilon) - G_M\left(u_\varepsilon^0\right) + \int_0^T\int_\Omega \left|\nabla\min(u_\varepsilon, M)\right|^2 = \int_0^T\int_\Omega -v_\varepsilon^0 \min(u_\varepsilon, M)\partial_t \exp\left(-\frac{1}{\varepsilon}\int_0^t g_\varepsilon\left(u_\varepsilon(s,x)\right) ds\right).$$

As $\partial_t \exp(-\frac{1}{\varepsilon}\int_0^t g_\varepsilon(u_\varepsilon(s,x))\,ds) \leqslant 0$, we know that $\partial_t \exp(-\frac{1}{\varepsilon}\int_0^t g_\varepsilon(u_\varepsilon(s,x))\,ds)$ is bounded in $L^\infty(\Omega; L^1((0,T)))$, and

$$\int_0^T\int_\Omega -v_\varepsilon^0 \min(u_\varepsilon, M)\partial_t \exp\left(-\frac{1}{\varepsilon}\int_0^t g_\varepsilon\left(u_\varepsilon(s,x)\right) ds\right) \leqslant C\int_\Omega \sup_{(0,T)} \max\left(\min(u_\varepsilon, M), 0\right) \leqslant CM\mathcal{L}^n(\Omega).$$

Step 3 (Compactness): Let $\chi_M : \mathbf{R} \to \mathbf{R}$ be a smooth non-increasing function satisfying $\chi_{(-\infty,M-1)} \leqslant \chi_M \leqslant \chi_{(-\infty,M)}$ and let Φ_M be the primitive such that $\Phi_M(z) = z$ for $z \leqslant M-1$ and $\Phi_M \leqslant M$. Moreover, let $(\phi_\delta)_{\delta\in(0,1)}$ be a family of mollifiers, i.e. $\phi_\delta \in C_0^{0,1}(\mathbf{R}^n; [0,+\infty))$ such that $\int \phi_\delta = 1$ and $\operatorname{supp}\phi_\delta \subset B_\delta(0)$. Then, if we extend u_ε and v_ε^0 by the value 0 to the whole of $(0,+\infty)\times\mathbf{R}^n$, we obtain by the homogeneous Neumann data of u_ε that

$$\begin{aligned}
&\partial_t\left(\Phi_M(u_\varepsilon) * \phi_\delta\right)(t,x) \\
&\quad = \left(\left(\chi_M(u_\varepsilon)\left(\chi_\Omega \Delta u_\varepsilon - v_\varepsilon^0 \partial_t \exp\left(-\frac{1}{\varepsilon}\int_0^t g_\varepsilon\left(u_\varepsilon(s,x)\right) ds\right)\right)\right) * \phi_\delta\right)(t,x) \\
&\quad = \int_{\mathbf{R}^n} \chi_M(u_\varepsilon)(t,y)\left(\chi_\Omega(y)\Delta u_\varepsilon(t,y) - v_\varepsilon^0(y)\partial_t \exp\left(-\frac{1}{\varepsilon}\int_0^t g_\varepsilon\left(u_\varepsilon(s,y)\right) ds\right)\right)\phi_\delta(x-y)\,dy
\end{aligned}$$

$$= \int_{\mathbf{R}^n} \phi_\delta(x-y)\left(-\chi_M'\big(u_\varepsilon(t,y)\big)\chi_\Omega(y)\big|\nabla u_\varepsilon(t,y)\big|^2 - \chi_M\big(u_\varepsilon(t,y)\big)v_\varepsilon^0(y)\partial_t \exp\left(-\frac{1}{\varepsilon}\int_0^t g_\varepsilon\big(u_\varepsilon(s,y)\big)\,ds\right)\right)$$
$$+ \chi_M\big(u_\varepsilon(t,y)\big)\chi_\Omega(y)\nabla u_\varepsilon(t,y)\cdot\nabla\phi_\delta(x-y)\,dy.$$

Consequently

$$\int_0^T\int_{\mathbf{R}^n}\big|\partial_t\big(\Phi_M(u_\varepsilon)*\phi_\delta\big)\big| \leqslant C_1(\Omega, C, M, \delta, T)$$

and

$$\int_0^T\int_{\mathbf{R}^n}\big|\nabla\big(\Phi_M(u_\varepsilon)*\phi_\delta\big)\big| \leqslant C_2(\Omega, M, \delta, T).$$

It follows that $(\Phi_M(u_\varepsilon)*\phi_\delta)_{\varepsilon\in(0,1)}$ is for each (M,δ,T) precompact in $L^1((0,T)\times\mathbf{R}^n)$.

On the other hand

$$\int_0^T\int_{\mathbf{R}^n}\big|\Phi_M(u_\varepsilon)*\phi_\delta - \Phi_M(u_\varepsilon)\big| \leqslant C_3\left(\delta^2\int_0^T\int_\Omega\big|\nabla\Phi_M(u_\varepsilon)\big|^2\right)^{\frac{1}{2}} + 2(M-u_{\min})T\mathcal{L}^n\big(B_\delta(\partial\Omega)\big)$$
$$\leqslant C_4(C,\Omega,u_{\min},M,T)\delta.$$

Combining this estimate with the precompactness of $(\Phi_M(u_\varepsilon)*\phi_\delta)_{\varepsilon\in(0,1)}$ we obtain that $\Phi_M(u_\varepsilon)$ is for each (M,T) precompact in $L^1((0,T)\times\mathbf{R}^n)$. Thus, by a diagonal sequence argument, we may take a sequence $\varepsilon_m\to 0$ such that $\Phi_M(u_{\varepsilon_m})\to z_M$ a.e. in $(0,+\infty)\times\mathbf{R}^n$ as $m\to\infty$, for every $M\in\mathbf{N}$. At a.e. point of the set $\{z_M < M-1\}$, u_{ε_m} converges to z_M. At each point (t,x) of the remainder $\bigcap_{M\in\mathbf{N}}\{z_M\geqslant M-1\}$, the value $u_{\varepsilon_m}(t,x)$ must for large m (depending on (M,t,x)) be larger than $M-2$. But that means that on the set $\bigcap_{M\in\mathbf{N}}\{z_M\geqslant M-1\}$, the sequence $(u_{\varepsilon_m})_{m\in\mathbf{N}}$ converges a.e. to $+\infty$. It follows that $(u_{\varepsilon_m})_{m\in\mathbf{N}}$ converges a.e. in $(0,+\infty)\times\Omega$ to a function $z:(0,+\infty)\times\Omega\to\mathbf{R}\cup\{+\infty\}$. But then, as $(u_{\varepsilon_m})_{m\in\mathbf{N}}$ is for each $T\in(0,+\infty)$ bounded in $L^2((0,T)\times\Omega)$, $(u_{\varepsilon_m})_{m\in\mathbf{N}}$ converges by Vitali's theorem (stating that a.e. convergence and a non-concentration condition in L^p imply in bounded domains L^p-convergence) for each $p\in[1,2)$ in $L^p((0,T)\times\Omega)$ to the weak L^2-limit u of $(u_{\varepsilon_m})_{m\in\mathbf{N}}$. It follows that $\mathcal{L}^{n+1}(\bigcap_{M\in\mathbf{N}}\{z_M\geqslant M-1\}) = \mathcal{L}^{n+1}(\{u=+\infty\}) = 0$.

Step 4 (Identification of the limit equation in $\operatorname{esssup}_{(0,t)} u > 0$*)*: Let us consider $(t,x)\in(0,+\infty)\times\Omega$ such that $u_{\varepsilon_m}(s,x)\to u(s,x)$ for a.e. $s\in(0,t)$ and $u(\cdot,x)\in L^2((0,t))$. In the case $\operatorname{esssup}_{(0,t)} u(\cdot,x) > 0$, we obtain by Egorov's theorem and assumption (2) that $\exp(-\frac{1}{\varepsilon_m}\int_0^t g_{\varepsilon_m}(u_{\varepsilon_m}(s,x))\,ds)\to 0$ as $m\to\infty$.

Step 5 (The case $\partial_t u_\varepsilon\geqslant 0$*)*: Let (t,x) be such that $u_{\varepsilon_m}(t,x)\to u(t,x)=\lambda<0$: Then by assumption (1),

$$\exp\left(-\frac{1}{\varepsilon_m}\int_0^t g_{\varepsilon_m}\big(u_{\varepsilon_m}(s,x)\big)\,ds\right) \geqslant \exp\left(-t\frac{\max_{[u_{\min},\lambda/2]} g_{\varepsilon_m}}{\varepsilon_m}\right)\to 1 \quad \text{as } m\to\infty. \qquad \square$$

5. Complete characterization of the limit equation in the case of one space dimension

The aim of this main section is the following theorem:

Theorem 5.1. *Suppose in addition to the assumptions at the beginning of Section 4 that the space dimension $n=1$ and that the initial data u_ε^0 converge in C^1 to a function u^0 satisfying $\nabla u^0\neq 0$ on $\{u^0=0\}$. Then the family $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is for each $T\in(0,+\infty)$ precompact in $L^1((0,T)\times\Omega)$, and each limit u of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ as a sequence $\varepsilon_m\to 0$, satisfies in the sense of distributions the initial-boundary value problem*

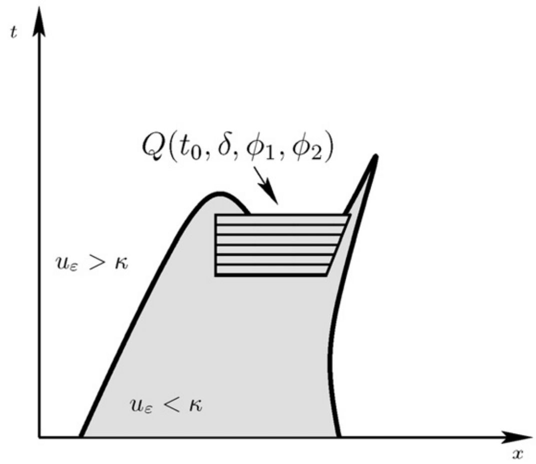

Fig. 1. Clearing out.

$$\partial_t u - v^0 \partial_t \chi = \Delta u \quad \text{in } (0, +\infty) \times \Omega,$$
$$u(0, \cdot) = u^0 + v^0 H\big(u^0\big) \quad \text{in } \Omega, \qquad \nabla u \cdot \nu = 0 \quad \text{on } (0, +\infty) \times \partial\Omega, \tag{7}$$

where H is the maximal monotone graph

$$H(z) \begin{cases} = 0, & z < 0, \\ \in [0, 1], & z = 0, \quad \text{and} \\ = 1, & z > 0 \end{cases}$$

$$\chi(t, x) = H\Big(\operatorname*{esssup}_{(0,t)} u(\cdot, x)\Big) \begin{cases} = 0, & \operatorname{esssup}_{(0,t)} u(\cdot, x) < 0, \\ \in [0, 1], & \operatorname{esssup}_{(0,t)} u(\cdot, x) = 0, \\ = 1, & \operatorname{esssup}_{(0,t)} u(\cdot, x) > 0. \end{cases}$$

Although we assume the space dimension from now on to be 1, we keep the multi-dimensional notation for the sake of convenience. Moreover we extend u_ε by even reflection at the lateral boundary to a space-periodic solution on $[0, +\infty) \times \mathbf{R}$.

We start out with some elementary lemmata:

Lemma 5.2 *(Clearing out). There exists a continuous increasing function $\omega : [0, 1) \to [0, +\infty)$ such that $\omega(0) = 0$ and the following holds: suppose that $\kappa < 0$, that $\varepsilon \leqslant \omega(|\kappa|)$, that $\delta \in (0, 1)$ and that $u_\varepsilon \leqslant (1 + \omega(\delta))\kappa$ on the parabolic boundary of the domain $Q(t_0, \delta, \phi_1, \phi_2) := \{(t, x)\colon\ 0 \leqslant t_0 - 2\delta < t < t_0, \phi_1(t) < x < \phi_2(t)\}$, where $\phi_1 < \phi_2$ are C^1-functions. Then $u_\varepsilon \leqslant \kappa$ in $Q(t_0, \delta, \phi_1, \phi_2)$ (cf. Fig. 1).*

Proof. Comparing u_ε in $Q(t_0, \delta, \phi_1, \phi_2)$ to the solution of the ODE

$$y'(t) = C g_\varepsilon(y)/\varepsilon, \; y(t_0 - 2\delta) = \big(1 + \omega(\delta)\big)\kappa$$

we obtain the statement of the lemma. □

Lemma 5.3. *For almost all $\kappa < 0$ the level set $((0, +\infty) \times \Omega) \cap \{u_\varepsilon = \kappa\}$ is a locally finite union of C^1-curves. For such κ we define the set*

$$\begin{aligned} S_{\kappa,\varepsilon} := \big\{ &(t, x) \in (0, +\infty) \times \Omega)\colon\ u_\varepsilon(t, x) > \kappa \\ &\text{and there is no } (t_0, \delta, \phi_1, \phi_2) \in [0, +\infty) \times (0, 1) \times C^1 \times C^1 \\ &\text{such that } u_\varepsilon \leqslant \kappa \text{ on the parabolic boundary of the domain } Q(t_0, \delta, \phi_1, \phi_2) \big\} \end{aligned}$$

(cf. Fig. 2). Then $\partial S_{\kappa,\varepsilon} = \bigcup_{j=1}^{N_{\kappa,\varepsilon}}$ graph $(g_{j,\kappa,\varepsilon})$ where $g_{j,\kappa,\varepsilon} : [0, T_{j,\kappa,\varepsilon}] \to \mathbf{R}$ are piecewise C^1-functions and $N_{\kappa,\varepsilon}$ is for small ε bounded by a constant depending only on the limit u^0 of the initial data.

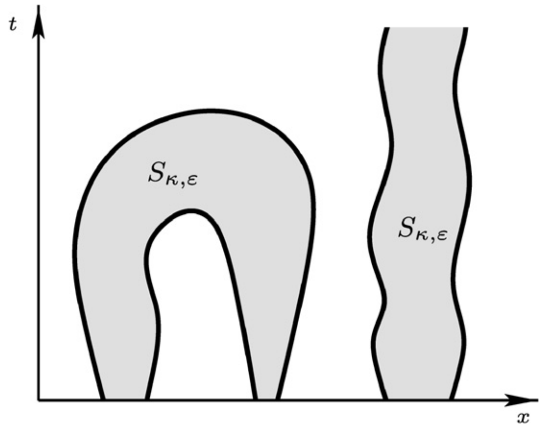

Fig. 2. The set $S_{\kappa,\varepsilon}$.

Remark 5.4. For illustration of the definition of $S_{\kappa,\varepsilon}$, imagine the set $\{u_\varepsilon > \kappa\}$ filled with water in a (t,x)-plane where t represents the height. Our modification of $\{u_\varepsilon > \kappa\}$ means then that the water is now allowed to flow out through the "bottom" $\{t = 0\}$.

Proof of Lemma 5.3. By the definition of $S_{\kappa,\varepsilon}$ and by the fact that u_ε is supercaloric, each connected component of $\partial S_{\kappa,\varepsilon}$ is a piecewise C^1-curve and touches $\{t = 0\}$. Therefore the number of connected components is for small $\varepsilon > 0$ bounded by a constant $\tilde{N}$ depending only on the limit u^0 of the initial data.

Let us consider one connected component γ of $\partial S_{\kappa,\varepsilon}$. By the definition of $S_{\kappa,\varepsilon}$ and by the fact that u_ε is supercaloric, the derivative of the time-component of the piecewise C^1-curve γ can change its sign at most once! Thus we can define for each curve γ one or two piecewise C^1-functions of time such that γ is the union of the graphs of the two functions. The total number of graphs $N_{\kappa,\varepsilon}$ is therefore bounded by $2\tilde{N}$. □

Proof of Theorem 5.1. By Theorem 4.1 we only have to prove that $\chi = 0$ in the set $\{\operatorname{esssup}_{(0,t)} u(\cdot,x) < 0\}$. The main problem is to exclude "peaking" of the solution u_ε, i.e. tiny sets where $u_\varepsilon > \kappa$. Here we show that in the case of one space dimension, "peaking" is not possible. More precisely, if the measure of the set $u_\varepsilon > \kappa$ is small in a parabolic cube, then u_ε is strictly negative in the cube of half the radius, uniformly in ε. The proof is carried out in two steps:

Step 1: Let $(\varepsilon_m)_{m\in\mathbf{N}}$ be the subsequence in the proof of Theorem 4.1. As a.e. point $(t,x) \in ((0,+\infty)\times\mathbf{R}) \cap \{u < 0\}$ is a Lebesgue point of the set $\{u < 0\}$, we may assume that there exists $\kappa < 0$ such that for any $\theta \in (0,1)$, sufficiently small $r_0 > 0$ and every $\varepsilon_m \in (0,\varepsilon_0)$,

$$\mathcal{L}^2\big(\big((t-2r_0,t)\times B_{2r_0}(x)\big)\cap\{u_{\varepsilon_m} < 2\kappa\}\big) \geqslant \theta\mathcal{L}^2\big(\big((t-2r_0,t)\times B_{2r_0}(x)\big)\big).$$

Step 2: Suppose now that $((t-r_0,t)\times B_{r_0}(x)) \cap \{u_{\varepsilon_m} > \kappa\} \neq \emptyset$ (where κ is chosen such that $\{u_{\varepsilon_m} = \kappa\}$ and $\{u_{\varepsilon_m} = 2\kappa\}$ are locally finite unions of C^1-curves): then $((t-r_0,t)\times B_{r_0}(x)) \cap \partial S_{\kappa,\varepsilon_m}$ and $((t-r_0,t)\times B_{r_0}(x)) \cap \partial S_{2\kappa,\varepsilon_m}$ must by Lemma 5.3 be connected to the parabolic boundary of $(t-2r_0,t)\times B_{2r_0}(x)$. The $L^2((0,T)\times\Omega)$-bound for $\nabla\min(u_\varepsilon, M)$, the fact that $\mathcal{L}^2(((t-2r_0,t)\times B_{2r_0}(x))\cap\{u_{\varepsilon_m} < 2\kappa\}) \geqslant \theta\mathcal{L}^2(((t-2r_0,t)\times B_{2r_0}(x)))$ and Lemma 5.2 imply now (see Fig. 3) that there must be an "almost horizontal" component of $\partial S_{\kappa,\varepsilon_m}$ (cf. Fig. 4) with the following properties: for any $\delta \in (0,1)$, there are $t - r_0 < t_1 < t_2 < t_3 < t$ such that (see Fig. 5) $t_3 - t_1 \to 0$ as $\varepsilon_m \to 0$, for some j

$$\left|g_{j,\kappa,\varepsilon_m}(t_2) - g_{j,\kappa,\varepsilon_m}(t_1)\right| \geqslant c_1 > 0,$$

and

$$\mathcal{L}^1\big(\big\{y \in B_{r_0}(x)\colon u_{\varepsilon_m}(t_3,y) > 2\kappa\big\}\big) \leqslant \delta,$$
$$\int_{B_{r_0}(x)\cap\{u_{\varepsilon_m}(t_3,\cdot)>2\kappa\}} \left|u_{\varepsilon_m}(t_3,y)\right| dy \leqslant \delta.$$

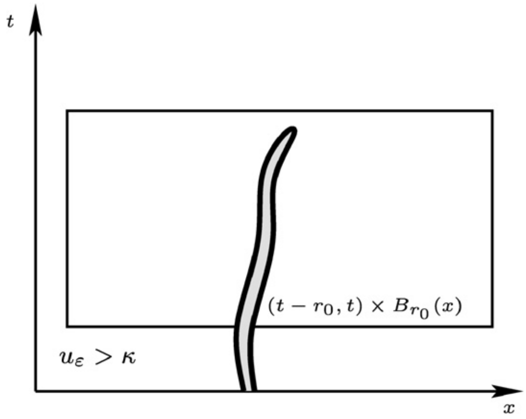

Fig. 3. Situation excluded by the $L^2(W^{1,2})$-estimate.

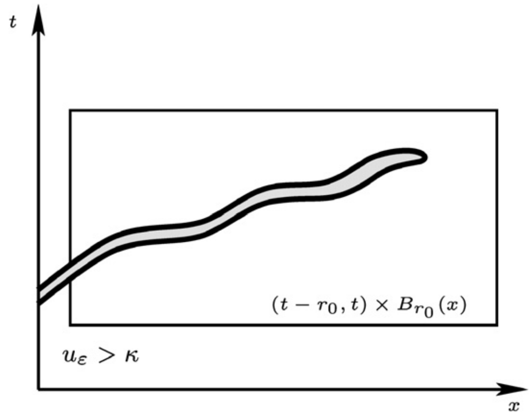

Fig. 4. The main task is to exclude almost horizontal propagation.

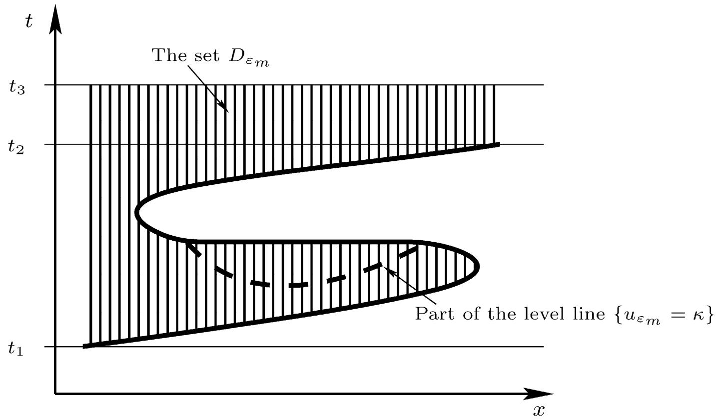

Fig. 5. The set D_{ε_m}.

We may assume that $c_1 < r_0$, that $g_{j,\kappa,\varepsilon_m}(t_2) = \sup_{(t_1,t_2)} g_{j,\kappa,\varepsilon_m}$, that $g_{j,\kappa,\varepsilon_m}(t_2) > g_{j,\kappa,\varepsilon_m}(t_1)$ and that $u_{\varepsilon_m}(s, y) > \kappa$ for some $d > 0$ and $(s, y) \in (t_1, t_2) \times B_{r_0}(x)$ such that $g_{j,\kappa,\varepsilon_m}(s) < y < d + g_{j,\kappa,\varepsilon_m}(s)$. We define the set $D_{\varepsilon_m} := \{(s, y)\colon\ t_1 < s < t_3,\ y < g_{j,\kappa,\varepsilon_m}(s) \text{ for } s \in (t_1, t_2) \text{ and } y < g_{j,\kappa,\varepsilon_m}(t_2) \text{ for } s \in [t_2, t_3)\}$ (cf. Fig. 5) and the cut-off function $\phi(y) := \max(0, \min(y - g_{j,\kappa,\varepsilon_m}(t_1), g_{j,\kappa,\varepsilon_m}(t_2) - y))$. It follows that

$$
\begin{aligned}
c_1^2\kappa/4 + 2\delta + \mathrm{o}(1) &\geqslant \mathrm{o}(1) + \int_{g_{j,\kappa,\varepsilon_m}(t_1)}^{g_{j,\kappa,\varepsilon_m}(t_2)} \phi(y)\big(u_{\varepsilon_m}(t_3, y) - \kappa\big)\, dy \\
&\geqslant \mathrm{o}(1) + \int_{g_{j,\kappa,\varepsilon_m}(t_1)}^{g_{j,\kappa,\varepsilon_m}(t_2)} \phi(y) u_{\varepsilon_m}(t_3, y)\, dy - \kappa \int_{t_1}^{t_2} \phi\big(g_{j,\kappa,\varepsilon_m}(s)\big) g'_{j,\kappa,\varepsilon_m}(s)\, ds \\
&\geqslant \int_{g_{j,\kappa,\varepsilon_m}(t_1)}^{g_{j,\kappa,\varepsilon_m}(t_2)} \phi(y) u_{\varepsilon_m}(t_3, y)\, dy - \int_{t_1}^{t_2} \phi\big(g_{j,\kappa,\varepsilon_m}(s)\big) u_{\varepsilon_m}\big(s, g_{j,\kappa,\varepsilon_m}(s)\big) g'_{j,\kappa,\varepsilon_m}(s)\, ds \\
&\geqslant \int_{D_{\varepsilon_m}} \phi \partial_t u_{\varepsilon_m} \geqslant \int_{D_{\varepsilon_m}} \phi \Delta u_{\varepsilon_m}
\end{aligned}
$$

$$= -\int_{D_{\varepsilon_m}} \nabla\phi \cdot \nabla u_{\varepsilon_m} + \int_{t_1}^{t_2} \phi\big(s, g_{j,\kappa,\varepsilon_m}(s)\big)\partial_x u_{\varepsilon_m}\big(s, g_{j,\kappa,\varepsilon_m}(s)\big)\chi_{\{u_{\varepsilon_m}(s,g_{j,\kappa,\varepsilon_m}(s))=\kappa\}}\,ds$$
$$\geqslant -\int_{D_{\varepsilon_m}} \nabla\phi \cdot \nabla u_{\varepsilon_m} \to 0 \quad \text{as } \varepsilon_m \to 0,$$

a contradiction for small ε_m provided that δ has been chosen small enough; in the third inequality we used Lemma 5.2, and the convergence to 0 is due to the uniform $L^2(W^{1,2})$-bound. □

6. Applications

Here we mention two examples of different systems leading to the same limit.

For the convergence results below we assume that the space dimension is 1.

6.1. The Matkowsky–Sivashinsky scaling

We apply our result to the scaling in [8, Eq. (2)], i.e. the following system of solid combustion

$$\begin{aligned}&\partial_t u_N - \Delta u_N = (1-\sigma_N)N v_N \exp\big(N(1-1/u_N)\big),\\ &\partial_t v_N = -N v_N \exp\big(N(1-1/u_N)\big),\end{aligned} \tag{8}$$

where the normalized temperature u_N and the normalized concentration v_N are non-negative, $(\sigma_N)_{N\in\mathbf{N}} \Subset [0,1)$ (in the case $\sigma_N \uparrow 1$, $N \uparrow \infty$ the limit equation in the scaling as it is would be the heat equation, but we could still apply our result to $u_N/(1-\sigma_N)$) and the activation energy $N \to \infty$.

Setting $u_{\min} := -1$, $\varepsilon := 1/N$, $u_\varepsilon := u_N - 1$ and

$$g_\varepsilon(z) := \begin{cases} \exp((1-1/(z+1))/\varepsilon), & z > -1,\\ 0, & z \leqslant -1\end{cases}$$

and integrating the equation for v_N in time, we see that the assumptions of Theorem 5.1 are satisfied and we obtain that each limit u_∞, σ_∞ of u_N, σ_N satisfies

$$\partial_t u_\infty - (1-\sigma_\infty)v^0\partial_t H\Big(\operatorname*{esssup}_{(0,t)} u_\infty\Big) = \Delta u_\infty \quad \text{in } (0,+\infty)\times\Omega,$$
$$u_\infty(0,\cdot) = u^0 + v^0 H\big(u^0\big) \quad \text{in } \Omega, \qquad \nabla u_\infty \cdot \nu = 0 \quad \text{on } (0,+\infty)\times\partial\Omega, \tag{9}$$

where v^0 are the initial data of v_∞. Moreover, χ is increasing in time and u_∞ is a supercaloric function.

6.2. Another scaling with temperature threshold

Here we consider (cf. [1, pp. 109–110]), i.e. the following system of solid combustion

$$\begin{aligned}&\partial_t\theta_N - \Delta\theta_N = (1-\sigma_N)NY_N \exp\big(\big(N(1-\sigma_N)(\theta_N-1)\big)/\big(\sigma_N + (1-\sigma_N)\theta_N\big)\big)\chi_{\{\theta_N>\bar\theta\}},\\ &\partial_t Y_N = -(1-\sigma_N)NY_N \exp\big(\big(N(1-\sigma_N)(\theta_N-1)\big)/\big(\sigma_N + (1-\sigma_N)\theta_N\big)\big)\chi_{\{\theta_N>\bar\theta\}}\end{aligned} \tag{10}$$

where $N(1-\sigma_N) \gg 1$, $\sigma_N \in (0,1)$ and the constant $\bar\theta \in (0,1)$ is a threshold parameter at which the reaction sets in.

Setting $u_{\min} = -1$, $\varepsilon := 1/(N(1-\sigma_N))$, $\kappa(\varepsilon) := 1-\sigma_N$, $u_\varepsilon := \theta_N - 1$,

$$g_\varepsilon(z) := \begin{cases} \exp((z/(\kappa(\varepsilon)z+1))/\varepsilon), & z > \bar\theta - 1,\\ 0, & z \leqslant \bar\theta - 1\end{cases}$$

and integrating the equation for Y_N in time, we see that the assumptions of Theorem 5.1 are satisfied and we obtain that each limit u_∞ of u_N satisfies

$$\partial_t u_\infty - v^0 \partial_t H\Big(\operatorname*{esssup}_{(0,t)} u_\infty\Big) = \Delta u_\infty \quad \text{in } (0,+\infty)\times\Omega,$$

$$u_\infty(0,\cdot) = u^0 + v^0 H\big(u^0\big) \quad \text{in } \Omega, \qquad \nabla u_\infty\cdot\nu = 0 \quad \text{on } (0,+\infty)\times\partial\Omega, \tag{11}$$

where v^0 are the initial data of v_∞. Moreover, χ is increasing in time and u_∞ is a supercaloric function.

7. Existence of pulsating waves

The aim of this section is to construct pulsating waves for the limit problem. For the sake of clarity we have chosen not to present the most general result in the following theorem. Moreover we confine ourselves to the one-phase case.

Theorem 7.1 *(Existence of pulsating waves). Let us consider a Hölder continuous function v^0 defined on $\mathbf{R}^n$ that satisfies*

$$v^0(x) \geqslant 1 \quad \textit{and} \quad v^0(x+k) = v^0(x) \quad \textit{for every } k\in\mathbf{Z}^n,\ x\in\mathbf{R}^n.$$

Given a unit vector $e\in\mathbf{R}^n$ and a velocity $c>0$, there exists a solution $u(t,x)$ of the one-phase problem

$$\begin{cases} \partial_t u - v^0\partial_t\chi_{\{u\geqslant 0\}} = \Delta u \quad \textit{on } \mathbf{R}\times\mathbf{R}^n,\\ \partial_t u\geqslant 0 \quad \textit{and} \quad -\mu_0 := -\displaystyle\int_{[0,1)^n} v^0 \leqslant u\leqslant 0, \end{cases} \tag{12}$$

which satisfies

$$\begin{cases} u(t,x+k) = u\Big(t-\dfrac{e\cdot k}{c}, x\Big) \quad \textit{for every } k\in\mathbf{Z}^n, \quad (t,x)\in\mathbf{R}\times\mathbf{R}^n,\\ u(t,x) = 0 \quad \textit{for } x\cdot e - ct\leqslant 0 \quad \textit{and} \quad \displaystyle\limsup_{x\cdot e-ct\to+\infty} u(t,x) = -\mu_0, \end{cases} \tag{13}$$

where the last limit is uniform as $x\cdot e - ct$ tends to $+\infty$.

Let us transform the problem by the so-called Duvaut transform (see [10]), setting $w(t,x) = -\int_t^{+\infty} u(s,x)\,ds$. In this section we will prove the existence of a pulsating wave w. More precisely, Theorem 7.1 is a corollary of the following result which will be proved later.

Theorem 7.2 *(Pulsating waves for the obstacle problem). Under the assumptions of Theorem 7.1, there exists a function $w(t,x)$ solving the obstacle problem*

$$\begin{cases} \partial_t w = \Delta w - v^0\chi_{\{w>0\}} \quad \textit{on } \mathbf{R}\times\mathbf{R}^n,\\ w\geqslant 0, \quad -\mu_0\leqslant\partial_t w\leqslant 0, \quad \partial_{tt} w\geqslant 0, \end{cases} \tag{14}$$

with the conditions

$$\begin{cases} w(t,x+k) = w\Big(t-\dfrac{e\cdot k}{c}, x\Big) \quad \textit{for every } k\in\mathbf{Z}^n,\ (t,x)\in\mathbf{R}\times\mathbf{R}^n,\\ w(t,x) = 0 \quad \textit{for } x\cdot e-ct\leqslant 0 \quad \textit{and} \quad \partial_t w(t,x)\to-\mu_0 \quad \textit{as } x\cdot e-ct\to+\infty. \end{cases} \tag{15}$$

The convergence is uniform as $x\cdot e - ct$ tends to $+\infty$.

Proof of Theorem 7.1. Simply set $u(t,x) := \partial_t w(t,x)$ with w given by Theorem 7.2, and use the fact that $\chi_{\{u<0\}} = \chi_{\{w>0\}}$. To check this last property, it is sufficient to exclude the case where $w(t_0,x_0)>0$ and $\partial_t w(t_0,x_0)=0$ at some point (t_0,x_0): using the fact that $\partial_t w$ is caloric in $\{w>0\}$ as well as the strong maximum principle, we deduce that $\partial_t w(t,x) = 0$ for $t\in(-\infty,t_0]$ and x in a neighborhood of x_0. This contradicts the last line of (15). □

Proof of Theorem 7.2. We will prove the existence of an unbounded solution w in six steps, approximating w by bounded solutions of a truncated equation, for which we can apply the existence of pulsating fronts due to Berestycki and Hamel [2].

Step 1: Approximation by bounded solutions and estimates of the velocity. For any $0 < A < M$, let us start by approximating the function $\chi_{(0,+\infty)}$ by the characteristic function $g = \chi_{(0,A]}$. In that case we can compute explicitly the *traveling wave* (ϕ, c_0) (unique up to translations of ϕ) of

$$c_0\phi' = \phi'' - g(\phi), \quad \phi' \leqslant 0 \quad \text{on } \mathbf{R}, \qquad \phi(-\infty) = M \quad \text{and} \quad \phi(+\infty) = 0. \tag{16}$$

Let us define for $c_0 > 0$, $M > 0$, $s_0 \in (-\infty, 0)$ and $s_1 \in (s_0, 0)$

$$\phi(s) = \begin{cases} M(1 - e^{c_0(s-s_0)}) & \text{for } s \in (-\infty, s_1],\\ \frac{1}{c_0^2}(e^{c_0 s} - 1 - c_0 s) & \text{for } s \in [s_1, 0],\\ 0 & \text{for } s \in [0, +\infty). \end{cases}$$

For any $A \in (0, M)$ and for suitable s_0, s_1, we see that ϕ is continuous and satisfies $\phi(s_1) = A$, which fixes the parameter s_1 as a function of A. Moreover we see that ϕ is of class C^1 if and only if $s_1 = -c_0 M$ and

$$M - A = \frac{1}{c_0^2}\left(1 - e^{-c_0^2 M}\right). \tag{17}$$

Thus A is determined in terms of the velocity c_0 and M. The above calculations show in particular that $\phi(c_0 t - e\cdot x)$ is a good bounded approximation of the solution of (14), (15) in the case $v^0 = 1$, i.e. the traveling wave case.

In all that follows let A be given by (17).

Now, when the function g in (16) is replaced by a Lipschitz continuous function whose support is a compact interval, there are known results on the existence of pulsating waves. For such g, it is possible to apply Theorem 1.13 of Berestycki and Hamel [2], which states the existence (and uniqueness up to translation in time) of bounded pulsating solutions traveling at a unique velocity. Bearing that in mind, we define g_M as a Lipschitz regularization of the characteristic function g such that supp $g_M = [0, A]$ and – for later use –

$$\begin{aligned} &g_M = 1 \quad \text{on } [1/M, A/2], \quad 0 \leqslant g_M \leqslant 1 \quad \text{on } \mathbf{R}, \quad \text{and}\\ &g_M' \geqslant 0, \qquad g_M'' \leqslant 0 \quad \text{on } (0, A/2). \end{aligned} \tag{18}$$

Let us call c_M^0 the unique velocity of the traveling wave equation (16) with g replaced by g_M. As (ϕ, c_0) can be shown to be unique up to translations of ϕ), $c_M^0 \to c_0$ as $g_M \to g$.

Then there exists by Theorem 1.13 of [2] a bounded pulsating wave w_M traveling at velocity c_M such that

$$\begin{cases} \partial_t w_M = \Delta w_M - v^0 g_M(w_M) \quad \text{on } \mathbf{R}\times\mathbf{R}^n,\\ \partial_t w_M \leqslant 0 \quad \text{and} \quad \limsup\limits_{x\cdot e - c_M t \to -\infty} w_M(t,x) = 0 \leqslant w_M \leqslant M = \liminf\limits_{x\cdot e - c_M t \to +\infty} w_M(t,x) \quad \text{and}\\ w_M(t, x+k) = w_M\left(t - \dfrac{e\cdot k}{c_M}, x\right) \quad \text{for every } k \in \mathbf{Z}^n,\ (t,x) \in \mathbf{R}\times\mathbf{R}^n. \end{cases}$$

From the assumption $1 \leqslant v^0$ and the comparison principle (see Lemma 3.2 and 3.4 of [2]) we infer that the velocities satisfy the ordering $c_M^0 \leqslant c_M$. Similarly, defining $\lambda = \|v^0\|_{L^\infty}$ and comparing to $w_M(\lambda t, \sqrt{\lambda} x)$, we get $c_M \leqslant c_M^0\sqrt{\lambda}$.

Furthermore the above comparison principles tell us that the velocity c_M (resp. c_M^0) is continuous and non-decreasing in M.

For all that follows let $c > 0$ be an arbitrary but fixed velocity for which we want to construct the pulsating wave. Then, for any $M > 0$ we can adjust $A \in (0, M)$ such that for c_0 defined by (17),

$$c = c_M \in \left[c_0/2, 2c_0\sqrt{\left\|v^0\right\|_{L^\infty}}\,\right].$$

In order to pass to the limit as $M \to +\infty$, we need to get some bounds on the solution first. To this end, rotating space–time proves to be very convenient:

Step 2: Space–time transformation and first estimates on the time derivatives. Let us introduce the function $\tilde{w}_M$ defined by $\tilde{w}_M(s,x) = w_M(\frac{s+e\cdot x}{c}, x)$ which is periodic in x and satisfies

$$L\tilde{w}_M = v^0(x) g_M(\tilde{w}_M) \quad \text{with } L\tilde{w}_M = \Delta\tilde{w}_M + \partial_{ss}\tilde{w}_M - 2\partial_{e,s}\tilde{w}_M - c\partial_s\tilde{w}_M$$

and $\lim_{s\to-\infty}\tilde w_M(s,x)=M$, $\lim_{s\to+\infty}\tilde w_M(s,x)=0$ uniformly with respect to x. Using (18), we obtain $L\partial_s\tilde w_M - c_1\partial_s\tilde w_M=0$ and $L\partial_{ss}\tilde w_M - c_1\partial_{ss}\tilde w_M\leqslant 0$ on $\{\tilde w_M<A/2\}$ for $c_1(s,x)=v^0(x)g_M'(\tilde w_M(s,x))\geqslant 0$ on this set. We deduce from the maximum principle (see Lemmas 3.2 and 3.4 of [2]) that for any $s_0\in\mathbf{R}$ such that $\sup_{[s_0,+\infty)\times\mathbf{R}^n}\tilde w_M\leqslant A/2$,

$$\inf_{[s_0,+\infty)\times\mathbf{R}^n}\partial_s\tilde w_M=\inf_{\mathbf{R}^n}\partial_s\tilde w_M(s_0,\cdot)\quad\text{and}\quad\inf_{[s_0,+\infty)\times\mathbf{R}^n}\partial_{ss}\tilde w_M=\min\Bigl(0,\inf_{\mathbf{R}^n}\partial_{ss}\tilde w_M(s_0,\cdot)\Bigr).\tag{19}$$

Step 3: Bound of the solution from above. From the fact that g_M is bounded by 1, and from the Harnack inequality, we deduce that there exists a constant $C_H\in(1,+\infty)$ such that for any $r>0$ and for any point (t_0,x_0)

$$\sup_{B_r(x_0)} w_M\bigl(t_0-r^2,\cdot\bigr)\leqslant C_H\Bigl(\inf_{B_r(x_0)} w_M(t_0,\cdot)+r^2\lambda\Bigr)\tag{20}$$

where $\lambda=\|v^0\|_{L^\infty}$. For $\tilde w_M$ that means that – setting $s_0=ct_0-e\cdot x_0$ –

$$\sup_{y\in B_{\sqrt{n}/2}(0)}\tilde w_M\bigl(s_0-cr^2-e\cdot y,x_0+y\bigr)\leqslant C_H\Bigl(\inf_{y\in B_{\sqrt{n}/2}(0)}\tilde w_M(s_0-e\cdot y,x_0+y)+r^2\lambda\Bigr)$$

for $r\geqslant\sqrt{n}/2$. We will now use the fact that the unit cell $(-1/2,1/2)^n$ is contained in the ball $B_{\sqrt{n}/2}(0)$. Using first the monotonicity of $\tilde w_M$ in the variable s, and second the periodicity of $\tilde w_M(\tau,y)$ in y, we get for $\tau_0:=s_0-\sqrt{n}/2$

$$\sup_{\mathbf{R}^n}\tilde w_M\bigl(\tau_0-cr^2+\sqrt{n},\cdot\bigr)\leqslant C_H\Bigl(\inf_{\mathbf{R}^n}\tilde w_M(\tau_0,\cdot)+r^2\lambda\Bigr).\tag{21}$$

By a translation in time we may assume that

$$0=\inf\bigl\{\tau:\ \tilde w_M(s,x)\leqslant 1/M\ \text{for}\ s\geqslant\tau,\ x\in\mathbf{R}^n\bigr\}\tag{22}$$

and get the bound

$$\tilde w_M(s,x)\leqslant\max(1/M,\alpha-\beta s)\tag{23}$$

for some constants $\alpha,\beta\in(0,+\infty)$ and every large positive M.

Step 4: Passing to the limit. By estimate (23), we can pass to the limit as $M\to+\infty$ and obtain $M-A\to 1/c_0^2$. Moreover, passing to a subsequence if necessary, $\tilde w_M$ converges locally in $\mathbf{W}^{2,1}_p$ to $\tilde w$ satisfying

$$\begin{cases}\partial_t w\leqslant 0\quad\text{and}\quad\displaystyle\limsup_{x\cdot e-ct\to-\infty} w(t,x)=0\leqslant w\quad\text{and}\\[2mm] w(t,x+k)=w\Bigl(t-\dfrac{e\cdot k}{c},x\Bigr)\quad\text{for every}\ k\in\mathbf{Z}^n,\ (t,x)\in\mathbf{R}\times\mathbf{R}^n.\end{cases}$$

Furthermore, we obtain for w related to $\tilde w$ by $\tilde w(s,x)=w(\frac{s+e\cdot x}{c},x)$ that

$$w_t=\Delta w-v^0\chi_{\{w>0\}};$$

here we used the fact that w, being locally a $\mathbf{W}^{2,1}_p$-function, satisfies $\partial_t w=0=\Delta w$ a.e. on the set $\{w=0\}$.

In order to conclude $\partial_{tt}w\geqslant 0$ the following non-degeneracy property will prove to be necessary:

Step 5: Non-degeneracy property and bound from below. Let us assume that $w_M(t_0,x_0)\in(1/M,A/2)$. Using the fact that $v^0(x)g_M(z)\geqslant 1$ for $z\in[1/M,A/2]$, we can use the usual parabolic maximum principle, comparing $\max(w_M,1/M)$ to the function

$$h(t,x)=w_M(t_0,x_0)+\frac{1}{4n}|x-x_0|^2+\frac{1}{4n}(t_0-t)$$

on the set

$$\{1/M<w_M<A/2\}\cap Q_r^-(t_0,x_0),$$

where $Q_r^-(t_0,x_0)=\{(t,x):\ t_0-r^2\leqslant t\leqslant t_0,\ |x-x_0|\leqslant r\}$. We get for every $r>0$ the following non-degeneracy property:

$$\sup_{Q_r^-(t_0,x_0)} w_M\geqslant\min\Bigl(w_M(t_0,x_0)+\frac{1}{4n}r^2,A/2\Bigr).\tag{24}$$

Combined with the Harnack-type inequality (21) for some radius $r' < r$, we obtain the following bound from below:

$$\tilde{w}_M(s,x) \geqslant \alpha' > 0 \quad \text{for } s \leqslant s_1 < 0 \tag{25}$$

for some constants α' and s_1 and every large M.

Step 6: Further estimates on the time derivative of the limit solution. By the bound from above in Step 3, we obtain that

$$\left|w(t,x)\right| \leqslant C_1 + C_2\bigl(|t| + |x|\bigr), \tag{26}$$

where C_1 and C_2 are finite positive constants. Let now $(t_k, x_k) \in \{w > 0\}$ be a sequence such that

$$ct_k - x_k \cdot e \to -\infty.$$

Then by the result in Step 5,

$$d_k := \operatorname{pardist}\bigl((t_k, x_k), \partial\{w > 0\}\bigr) \geqslant c_3\sqrt{|t_k| + |x_k|^2} \tag{27}$$

for some constant $c_3 > 0$. So w is a solution of $\partial_t w - \Delta w = -v^0$ in $Q_{d_k}(t_k, x_k)$. Defining

$$z_k(t,x) := \frac{w(t_k + d_k^2 t, x_k + d_k x)}{d_k^2},$$

(26) and (27) imply that z_k is a solution of $\partial_t z_k - \Delta z_k = -v^0(x_k + d_k x)$ in $Q_1(0)$ satisfying

$$\sup_{Q_1(0)} |z_k| \leqslant C_4,$$

where C_4 is a constant not depending on k. Consequently $\partial_t w(t_k, x_k) = \partial_t z_k(0)$ is bounded, implying that $\limsup_{(t,x)\in\{w>0\},\, ct - x\cdot e \to -\infty} |\partial_t w(t,x)| < +\infty$. Passing if necessary to a subsequence, we obtain by the periodicity of v^0 a limit z satisfying $\partial_t z - \Delta z = -\mu_0$ in $Q_1(0)$. Moreover we infer from the fact that $\tilde{w}$ is periodic in the space variables that z is constant in the space variables. Thus $\partial_t z \equiv -\mu_0$ in $Q_1(0)$.

From regularity theory of caloric functions it follows that

$$\lim_{(t,x)\in\{w>0\},\, ct - x\cdot e \to -\infty} \partial_{tt} w = 0.$$

But then a combination of the comparison principle (19) and of (25) yield

$$-\mu_0 \leqslant \partial_t w \leqslant 0 \quad \text{on } \mathbf{R} \times \mathbf{R}^n$$

and

$$\partial_{tt} w \geqslant 0 \quad \text{on } \mathbf{R} \times \mathbf{R}^n.$$

This ends the proof of Theorem 7.2. □

8. Non-existence of pulsating waves in the case of constant initial concentration

In the time increasing case, we consider solutions u of the one-phase limit problem with constant initial concentration in *any finite dimension*, i.e. (in the case $v^0 = 1$)

$$\partial_t u - \partial_t \chi_{\{u \geqslant 0\}} = \Delta u \quad \text{in } \mathbf{R} \times \mathbf{R}^n, \tag{28}$$

and prove that u cannot be a non-trivial pulsating wave in the sense of (12), (13). More precisely:

Theorem 8.1 *(Non-existence of pulsating waves for constant initial concentration). Let u be a solution of* (12), (13) *in dimension $n \geqslant 1$ with $v^0 = \text{constant} > 0$. Then $u(t,x) = u(t - e \cdot x/c, 0)$, i.e. u is a planar wave.*

Proof. We set $w(t,x) = -\int_t^{+\infty} u(s,x)\,ds \geqslant 0$. From the proof of Theorem 7.1 we know that $w > 0$ if and only if $u < 0$. As $\partial_t u \geqslant 0$ we obtain

$$\partial_t w = \Delta w - v^0 \chi_{\{w>0\}},$$

and w satisfies (14), (15). For any $\xi \in \mathbf{R}^n$, we define the “tangential difference”

$$z^\xi(t,x) = w\left(t - \frac{e\cdot\xi}{c}, x-\xi\right) - w(t,x)$$

which satisfies

$$(\partial_t - \Delta)z^\xi = -az^\xi, \quad \text{where } 0 \leqslant a = \begin{cases} 0 & \text{if } z^\xi(t,x) = 0, \\ v^0\left(\dfrac{\chi_{\{w(t-\frac{e\cdot\xi}{c},x-\xi)>0\}} - \chi_{\{w(t,x)>0\}}}{w(t-\frac{e\cdot\xi}{c},x-\xi) - w(t,x)}\right) & \text{if } z^\xi(t,x) \neq 0. \end{cases} \tag{29}$$

From (14), (15) and the definition of z^ξ we infer that

$$\left|\partial_t z^\xi\right| \leqslant 2\mu_0 = 2v^0 \quad \text{in } \mathbf{R}^{n+1},$$
$$\partial_t z^\xi(t,x) \to 0 \quad \text{uniformly in } t,x,\xi \text{ as } ct - e\cdot x \to -\infty.$$

Moreover (15) and (25) as well as the definition of z^ξ tell us that for some $s_0 \in (0,+\infty)$ not depending on ξ,

$$(\partial_t - \Delta)z^\xi = 0 \quad \text{in } \left\{|ct - e\cdot x| > s_0\right\}.$$

Furthermore we obtain from the comparison principle (see Lemmata 3.2 and 3.4 in [2]) that

$$\left|\partial_t z^\xi(t,x)\right| \leqslant 2v^0 e^{ct-e\cdot x+s_0} \tag{30}$$

and – integrating this estimate for $t \in (-\infty, \frac{s_0+e\cdot x}{c})$ and using that $z^\xi = 0$ in $t \geqslant \frac{s_0+e\cdot x}{c}$ – we obtain that z^ξ is bounded on $\mathbf{R}^{n+1}$ by a constant not depending on ξ.

Liouville's theorem for the heat equation implies therefore that for each sequence (t_m, x_m) such that $ct_m - e\cdot x_m \to -\infty$, $z^\xi(t_m + \cdot, x_m + \cdot)$ converges locally uniformly in $\mathbf{R}^{n+1}$ (and uniformly with respect to ξ) to a constant K depending on the choice of ξ and the sequence (t_m, x_m). As we know that $\int_{x+[0,1)^n} z^\xi(t,y)\,dy = 0$ for every $(t,x) \in \mathbf{R}^{n+1}$ (see (15)), it follows that $K = 0$ and that

$$z^\xi(t+\cdot, x+\cdot) \to 0 \quad \text{locally uniformly in } \mathbf{R}^{n+1} \text{ as } ct - e\cdot x \to -\infty;$$

the convergence is also uniform with respect to ξ.

Finally we define

$$\eta(t,x) := \sup_{\xi\in\mathbf{R}^n} \left|z^\xi(t,x)\right|.$$

The function η is by (29) a bounded subcaloric function. Moreover, by construction,

$$\partial_y \eta\left(t - \frac{e\cdot y}{c}, x - y\right) \equiv 0.$$

But then $\eta(t,x) = f(ct - e\cdot x)$, $cf' - f'' \leqslant 0$ in $\mathbf{R}$, $f \in W^{1,1}_{\mathrm{loc}}(\mathbf{R})$, $\lim_{s\to-\infty} f(s) = \lim_{s\to+\infty} f(s) = 0$ and f is bounded from above, implying that $f = 0$, that $\eta \equiv 0$ and that $w(t - \frac{e\cdot\xi}{c}, x-\xi) - w(t,x) = 0$ for every $t \in \mathbf{R}$ and $x, \xi \in \mathbf{R}^n$. We obtain the corresponding result for u. $\square$

9. Conclusion and open questions

Let us conclude with a comparison to blow-up in semilinear heat equations, as the main problem arising in our convergence proof, i.e. excluding “peaking of the solution” or burnt zones with very small measure, resembles the blow-up phenomena in semilinear heat equations. One could therefore hope to apply methods used to exclude blow-up in low dimensions in order to exclude peaking, say in two dimensions. There are however problems: First, here, we are dealing not with a single solution but with the one-parameter family u_ε concentrating at some “peak” as ε gets smaller. Second, the ε-problem is not a scalar equation but a degenerate system. Third, in contrast to blow-up, peaking would not necessarily imply u_ε going to $+\infty$. Fourth, our limit problem is a two-phase problem while most known results for blow-up in semilinear heat equations assume the solution to be non-negative. Fifth, in our problem it does not make much sense studying the onset of burning, say the first time when $u_\varepsilon \geqslant -\varepsilon$, whereas studying the time of first blow-up can be very reasonable for semilinear heat equations. For the same reason blow-up in our case is always incomplete and there is always the non-trivial blow-up profile of the traveling wave. The last and most important difference is that while semilinear heat equations are parabolic and therefore well-posed in a sense, our limit problem contains a backward component making it *ill-posed* (the memory term with the function χ).

Concerning open questions, the most pressing task is of course to study for space dimension $n \geqslant 2$ the existence or non-existence of "peaking" of the solution in the negative phase. A related question is whether $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in L^∞ in the case of uniformly bounded initial data. Although this seems natural, it is not clear how to prevent concentration close to the interface.

Another challenge is to use the information on the limit problem gained in the present paper to construct pulsating waves for the ε-problem.

Uniqueness for the limit problem (the Stefan problem with memory term) in general seems unlikely. One might however ask whether time-global uniqueness holds in the case that u is strictly increasing in the x_1-direction. By the result in [5] for the ill-posed Hele–Shaw problem, time-local uniqueness is likely to be true here, too.

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Appendix A. Formal stability in the case of one space dimension and constant initial concentration

Recall that for the ε-problem (1) in one space dimension, instability of the planar wave for a special linearization (and high activation energy) is due to [4]. On the contrary, the result of this appendix suggests (formally) that for the limit one-phase problem (i.e. $\varepsilon = 0$) in one space dimension, the planar wave is stable.

More precisely, we consider here solutions u of the following equation

$$\partial_t u - \partial_t \chi_{\{u \geqslant 0\}} = \partial_{xx} u \quad \text{in } \mathbf{R} \times \mathbf{R} \tag{31}$$

that are close to the traveling wave solution

$$\bar{u}(t,x) = -\max\bigl(1 - e^{-c(x-ct)}, 0\bigr) \tag{32}$$

moving with velocity $c > 0$.

Proposition A.1 *(Formal linear stability of one-dimensional traveling waves). The traveling wave $\bar{u}$ given by* (32) *is formally linearly stable with respect to Eq.* (31).

Remark A.2. In higher dimensions this result is no longer true. It is well known that a fingering instability occurs. However the pulsation phenomenon with which we are concerned in the present paper appears already in dimension 1.

Formal proof of Proposition A.1. Let us consider solutions u of (31) satisfying

$$\begin{cases} u(t,x) = 0 & \text{for } x \leqslant s(t), \\ u(t,x) < 0 & \text{for } x > s(t), \\ u(t,x) \to -1 & \text{as } x - s(t) \to +\infty, \\ s'(t) = -\partial_x u(t, s(t) + 0^+) \geqslant 0. \end{cases} \tag{33}$$

Let us remark that a simple analysis shows that we do not have a comparison principle for solutions of (33).

In order to analyze the stability we transform (33) by

$$v(t,y) := u\bigl(t, y + s(t)\bigr);$$

v satisfies $v(t,y) = 0$ for $y \leqslant 0$ and

$$\begin{cases} v(t,0) = 0, \\ v(t,y) < 0 & \text{for } y > 0, \\ v(t,y) \to -1 & \text{as } y \to +\infty, \\ \partial_t v = \partial_{yy} v + s'(t)\partial_y v & \text{on } \mathbf{R} \times (0,+\infty), \\ s'(t) = -\partial_y v(t, 0^+). \end{cases} \tag{34}$$

We now consider for $t > 0$ a perturbation of the traveling wave

$$\bar{v}(y) = -\max\bigl(1 - e^{-cy}, 0\bigr)$$

with velocity $c > 0$. In the formal expansion

$$\begin{cases} s(t) = ct + \varepsilon\gamma(t) + \mathrm{O}(\varepsilon^2), \\ v = \bar{v} + \varepsilon w + \mathrm{O}(\varepsilon^2), \end{cases}$$

the first order terms $w(t, y), \gamma(t)$ formally satisfy

$$\begin{cases} w(t,0) = 0, \\ w(t,y) \to 0 \quad \text{as } y \to +\infty, \\ \partial_t w = \partial_{yy} w + c\partial_y w + \gamma'(t)\partial_y \bar{v} \quad \text{in } (0,+\infty) \times (0,+\infty), \\ \gamma'(t) = -\partial_y w(t, 0^+). \end{cases}$$

Let us look for solutions of the form

$$\begin{cases} w(t,y) = e^{\lambda t} W(y), \\ \gamma'(t) = e^{\lambda t}, \end{cases}$$

where $\mathrm{Re}(\lambda) \geqslant 0$. We obtain

$$W'' + cW' - \lambda W = ce^{-cy},$$

i.e.

$$W(y) = -\frac{c}{\lambda} e^{-cy} + \sum_{\pm} A_\pm e^{\mu_\pm y},$$

where

$$\mu_\pm = -c/2 \pm \sqrt{c^2/4 + \lambda} \quad \text{and} \quad \mathrm{Re}\Bigl(\sqrt{c^2/4 + \lambda}\Bigr) > c/2.$$

The function W can only be bounded if $A_+ = 0$ and, by $W(0) = 0$,

$$W(y) = -\frac{c}{\lambda}\bigl(e^{-cy} - e^{\mu_- y}\bigr).$$

Finally the relation $\gamma'(t) = -\partial_y w(t, 0)$ implies

$$1 = \frac{c}{\lambda}(-c - \mu_-).$$

The unique solution of this equation is $\lambda = 0$. Thus we formally proved stability of traveling waves. □

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