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On polyharmonic maps into spheres in the critical dimension ☆

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Abstract

We prove that every polyharmonic map $u \in W^{m,2}(\mathbb{B}^n, \mathbb{S}^{N-1})$ is smooth in the critical dimension $n = 2m$. Moreover, in every dimension n, a weak limit $u \in W^{m,2}(\mathbb{B}^n, \mathbb{S}^{N-1})$ of a sequence of polyharmonic maps $u_j \in W^{m,2}(\mathbb{B}^n, \mathbb{S}^{N-1})$ is also polyharmonic.

The proofs are based on the equivalence of the polyharmonic map equations with a system of lower order conservation laws in divergence-like form. The proof of regularity in dimension $2m$ uses estimates by Riesz potentials and Sobolev inequalities; it can be generalized to a wide class of nonlinear elliptic systems of order $2m$.

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1. Introduction

In this paper we study extrinsic m-polyharmonic maps from $\mathbb{B}^n$ into an $(N-1)$-dimensional round sphere $\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N\colon\ |x|^2 = 1\}$ — that is, roughly speaking, critical points of the functional

$$E_m(u) = \frac{1}{2}\int_{\mathbb{B}^n} \left|D^m u\right|^2 dx, \quad u:\mathbb{B}^n \to \mathbb{R}^N,\ u\left(\mathbb{B}^n\right) \subset \mathbb{S}^{N-1}, \tag{1.1}$$

with respect to variations in the range. Since $m \geqslant 2$ is fixed in the sequel, and we do not investigate *intrinsic* polyharmonic maps, we drop the adjective and m, and adopt the following definition.

Definition 1.1. We say that a map $u \in W^{m,2}(\mathbb{B}^n, \mathbb{S}^{N-1})$ is *polyharmonic* iff

$$\frac{d}{dt} E_m\left(\pi_{\mathbb{S}^{N-1}}(u + t\psi)\right)\Big|_{t=0} = 0 \quad \text{for all } \psi \in C_0^\infty\left(\mathbb{B}^n, \mathbb{R}^N\right), \tag{1.2}$$

where $\pi_{\mathbb{S}^{N-1}}(y) = y/|y|$ denotes the nearest point projection of $\mathbb{R}^N$ onto $\mathbb{S}^{N-1}$.

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Our paper is devoted to regularity of such maps in the critical dimension $n = 2m$ and to their weak convergence; this research has been prompted in 2007 by Andreas Gastel's lecture on the results of his research on polyharmonic map flow [8]. Before stating the results, let us briefly sketch the perspective.

Polyharmonic mappings are (one of possible) natural generalizations of harmonic mappings. The main difficulty in the study of regularity of minimizers and critical points of E_m, and other analytical issues associated with this functional, is that the nonlinearity of the Euler–Lagrange equations (1.2) is just integrable. In dimensions $n > 2m$ there is no hope to obtain even partial regularity in general: there exist examples, see [18], of harmonic maps $u \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ which are everywhere discontinuous. In the dimension $n = 2m$ the nonlinearity is critical. Examples, see e.g. Frehse [7], show that general elliptic systems of that type may have discontinuous solutions even if the solutions belong to $L^\infty \cap VMO$.

However, F. Hélein proved that harmonic mappings from a two-dimensional disk into a compact Riemannian manifold $\mathcal{N}$ are smooth.[1] Such mappings satisfy the system

$$-\Delta u = A(u)(\nabla u, \nabla u), \tag{1.3}$$

where $A(u)$ stands for the second fundamental form of the target manifold. In particular, if $\mathcal{N}$ is a round sphere or a homogeneous space, then (1.3) is equivalent to a system of conservation laws in divergence form. This implies that $A(u)(\nabla u, \nabla u)$ is not only integrable (this follows by Schwarz inequality from *a priori* assumptions), but in fact lies in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. Also for a general targets, the key ingredient of his proof is the use of Coulomb moving frames in order to expose Jacobian-like structure of the critical terms. This, by a combination of results of [5], the duality of $\mathcal{H}^1(\mathbb{R}^n)$ and *BMO*, and the embedding of appropriate Morrey spaces into *VMO*, allows one to absorb (locally) these critical terms.

Hélein's result has been extended by Evans [6] and Bethuel [3] to *stationary* harmonic maps in higher dimensions. Their proofs also rely on symmetries of the nonlinearity that, via the duality of $\mathcal{H}^1(\mathbb{R}^n)$ and *BMO*, lead to cancellation phenomena and hence allow one to deal with the critical nonlinear terms. Recently, new proofs of these results — based on conservation laws and avoiding direct use of Hardy space and *BMO* duality — have been discovered by Rivière [19] and Rivière and Struwe [20].

For $m = 2$, the critical points of (1.1) are known as (extrinsic) *biharmonic maps*.[2] Chang, Wang and Yang [4] proved that such mappings from a 4-dimensional disc into a sphere are smooth, and that stationary mappings from higher-dimensional disks are Hölder continuous outside a closed singular set of Hausdorff codimension 4. These results have been generalized to arbitrary Riemannian target manifolds by Wang [27]; the codimension estimate has been recently improved from 4 to 5 by Scheven [22]. Another proof for maps from 4-dimensional domains into spheres has been given by the second author of this paper in [25]. A new proof of regularity of biharmonic maps in dimension 4 has been discovered by Lamm and Rivière [15].

For $m > 2$ there are very few results. First, the paper of Gastel [8] extends earlier results on harmonic map flow to the polyharmonic case and establishes the existence of the unique eternal solution, regular except at finitely many time instants. Next, there is a preprint of Angelsberg and Pumberger [2] who prove smoothness of polyharmonic maps which are small in an appropriate Morrey norm, under a rather strong extra assumption that $D^m u$ is integrable with some power larger than 2 (in the critical dimension $n = 2m$ this immediately implies continuity of u). Up to our knowledge, no other regularity or existence results have been known up to now.

Then, very recently, Gastel and Scheven [9] have proved the regularity of both extrinsic and intrinsic m-polyharmonic maps from $n = 2m$ dimensional domains into general compact Riemannian manifolds. Their proof is based on Wang's generalization of Hélein's moving frame technique, combined with higher order estimates for moving frames which are obtained via a clever application of Lorentz space estimates for Hodge decomposition and Uhlenbeck's gauge theorem [26]. This allows them to obtain the decay estimates for Lorentz $L^{2,\infty}$ norm of $\sum_j |D^j u|^{m/j}$ on small balls, which is enough to obtain Hölder continuity of u.

The present work was essentially completed when the authors have learned about the results of [9]. Due to the symmetry of $\mathbb{S}^{N-1}$, our proof is somewhat simpler and shorter, and in fact we use only standard tools of harmonic analysis, which can be applied in basically the same way to obtain regularity of a large class of nonlinear elliptic

[1] See his book [14] for an excellent account of this topic.
[2] These are usually defined as critical points of $\int |\Delta u|^2$, but these two definitions are equivalent.

systems of order $2m$ in dimension $n = 2m$. As a byproduct, we are also able to prove that a weak limit of polyharmonic mappings in $W^{m,2}$ is polyharmonic. Here are the results.

Theorem 1.2. *If a sequence of polyharmonic maps $u_k \in W^{m,2}(\mathbb{B}^n, \mathbb{S}^{N-1})$ converges weakly in $W^{m,2}$ to $u \in W^{m,2}(\mathbb{B}^n, \mathbb{S}^{N-1})$, then u is polyharmonic, as well.*

Theorem 1.3. *Let $n = 2m$. Then every polyharmonic map $u \in W^{m,2}(\mathbb{B}^n, \mathbb{S}^{N-1})$ is smooth.*

In fact, our proof of the latter result can be generalized to obtain the following.

Theorem 1.4. *Let $n = 2m \geqslant 4$. Assume that $u \in L^\infty \cap W^{m,2}(\mathbb{B}^n, \mathbb{R}^N)$ solves the elliptic system*

$$Lu = \sum_{|\alpha|=m} E_\alpha \cdot D^\alpha u + F\big(u, Du, \ldots, D^m u\big), \tag{1.4}$$

where L is an elliptic operator with smooth coefficients, $E_\alpha \in L^2(\mathbb{B}^n, \mathbb{R}^N)$ satisfy a higher order cancellation condition

$$\sum_{|\alpha|=m} D^\alpha E_\alpha = 0 \quad \text{in } \mathbb{B}^n, \tag{1.5}$$

and $F(\cdot)$ is smooth and satisfies the growth condition

$$\big|F\big(u, Du, \ldots, D^m u\big)\big| \lesssim \sum_{\boldsymbol{j}=(j_1,\ldots,j_m)\in J} \prod_{k=1}^{m} \big|D^k u\big|^{j_k}, \tag{1.6}$$

where J is a fixed finite set of m-tuples $\boldsymbol{j} = (j_1, \ldots, j_m)$ of nonnegative reals j_k such that

$$\sum_{k=1}^{m} k j_k = n \quad \text{and} \quad 0 \leqslant j_m < 2 \quad \text{for each } \boldsymbol{j} \in J. \tag{1.7}$$

Then u is smooth in $\mathbb{B}^n$.

Remark. Conditions (1.6) and (1.7) combined with standard Gagliardo–Nirenberg interpolation inequalities imply that $F(u, Du, \ldots, D^m u)$ is of class L^1 whenever $u \in L^\infty \cap W^{m,2}$. To see this, one applies Young's inequality with exponents $n(kj_k)^{-1}$ to each term in the sum in (1.6). The point is that the right-hand side of (1.6) does not contain the *squares* of $D^m u$, and therefore, by Young's inequality again, for each $\varepsilon > 0$ we have

$$\big|F\big(u, Du, \ldots, D^m u\big)\big| \leqslant \varepsilon \big|D^m u\big|^2 + C(\varepsilon) \sum_{k=1}^{m-1} \big|D^k u\big|^{n/k} \tag{1.8}$$

for some constant $C(\varepsilon)$. It is much easier to deal with integrals of critical powers of low order derivatives; see e.g. [16,17] and [21] for other examples of that phenomenon.

The main idea in the proof of Theorem 1.3 has its origin in the paper of Hajłasz and the second author [13] on subelliptic p-harmonic maps into spheres. The same idea was later reworked and applied in other contexts, for biharmonic maps and for higher order differential operators, in [24,25]. We rewrite the Euler–Lagrange equations of (1.1) in a particular divergence-like form (see Section 3), and use a test function quadratic in u. Careful inspection of all the emerging terms reveals a general structure which is, in fact, similar to (1.4) and leads to local reverse Hölder inequalities for derivatives of u. This point is, in fact, rather delicate and requires the use of generalized Riesz potentials to cope with the terms of the form $\sum E_\alpha D^\alpha u$; here, cancellation properties are crucial. Gehring's lemma gives higher integrability of $D^m u$; that, in turn, implies higher integrability of all lower order derivatives of u. Next, we prove the existence of derivatives of order $m+1, \ldots, 2m-1$ in various L^p spaces. Standard bootstrap and Schauder theory arguments conclude the proof of smoothness of u.

The rest of the paper is organized as follows. In Section 2 we explain some notation and recall the necessary tools. Section 3 contains various equivalent forms of the Euler–Lagrange equations of polyharmonic maps and the proof of

Theorem 1.2. In Section 4 we prove reverse Hölder's inequalities for $V = \sum_j |D^j u|^{n/j}$. In Section 5 we explain in some detail how to pass from higher integrability of $D^m u$ to smoothness, since this part of the proof is not entirely trivial.

2. Notation and tools

Barred integrals denote averages, i.e.

$$\fint_{B(x,r)} u\,dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} u\,dy,$$

where $B(x,r)$ stands for an open ball with a center $x \in \mathbb{R}^n$ and a radius r. From time to time we write B_r instead of $B(x,r)$ and $(u)_{B_r}$ instead of $\fint_{B_r} u\,dy$.

If A and B are two positive expressions, we write $A \lesssim B$ if $A \leqslant C \cdot B$ for some constant C which depends only on n, N and possibly on the exponents of integrability which enter into the definitions of A and B. (In the computations in Section 4, all constants denoted by C depend in fact *only* on n and N.)

Notation for derivatives. Greek letters α, β and γ denote multiindices in $\mathbb{R}^n$. We employ the commonly used abbreviations: $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ is the length of a multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where all α_i are nonnegative integers; we write $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $x \in \mathbb{R}^n$. For $v \in W^{k,1}_{\mathrm{loc}}$, $k = 1, 2, \ldots,$ we write

$$T_z^k v(y) = \sum_{|\beta| \leqslant k} D^\beta v(z) \frac{(y-z)^\beta}{\beta!}$$

to denote the Taylor polynomial of v; moreover,

$$T_A^k v(y) := \fint_A T_z^k v(y)\,dz$$

denotes the averaged Taylor polynomial of v.

The letter D with *latin* superscripts is used to denote the whole collection of partial derivatives of given order. Thus, for $v : \mathbb{R}^n \supset \Omega \to \mathbb{R}$, $D^k v := (D^\alpha v)_{|\alpha|=k}$ stands for a vector valued function whose range is $\mathbb{R}^{M_k}$, where $M_k := \sum_{|\alpha|=k} 1$ is the number of all multiindices of length k.

Sobolev's inequalities in the critical dimension. We record two simple consequences of Sobolev's embedding. (Related interpolation inequalities are used for the polyharmonic map flow, see [8, Section 3].)

Lemma 2.1. *Let $n = 2m$, $u \in W^{m,2}(\Omega, \mathbb{R}^N)$, $\Omega \subset \mathbb{R}^n$. Fix an arbitrary $\delta > 0$. There exists a number $r_0 = r_0(\varepsilon, u) > 0$ and a constant C (which depends only on n) such that for each $k = 1, 2, \ldots, m-1$ and each ball $B_r = B(a,r) \subset \Omega$ with $r \in (0, r_0)$ we have*

$$\int_{B_r} |D^k u|^{n/k}\,dx \leqslant \delta \int_{B_r} |D^m u|^2\,dx + C r^n \sum_{j=k}^{m-1} \biggl(\fint_{B_r} |D^j u|\,dx \biggr)^{n/j}. \tag{2.1}$$

Proof. Since n/k is the Sobolev conjugate exponent of $n/(k+1)$ for each $k = 1, 2, \ldots, m-1$, where $m = n/2$, Sobolev's inequality yields

$$\begin{aligned}
\int_{B_r} |D^k u|^{n/k}\,dx &\lesssim \int_{B_r} \bigl|D^k u - (D^k u)_{B_r}\bigr|^{n/k}\,dx + r^n \bigl|(D^k u)_{B_r}\bigr|^{n/k} \\
&\lesssim \biggl(\int_{B_r} |D^{k+1} u|^{n/(k+1)}\,dx \biggr)^{\frac{k+1}{k}} + r^n \bigl|(D^k u)_{B_r}\bigr|^{n/k} \\
&\lesssim \phi(r,u) \int_{B_r} |D^{k+1} u|^{n/(k+1)}\,dx + r^n \bigl|(D^k u)_{B_r}\bigr|^{n/k},
\end{aligned}$$

where

$$\phi(r,u) := \sup_{B(y,r)\subset\Omega}\left[\max_{k=1,2,\ldots,m-1}\left(\int\limits_{B(y,r)} \left|D^{k+1}u\right|^{n/(k+1)}\,dx\right)^{\frac{1}{k}}\right]. \tag{2.2}$$

By the absolute continuity of integral, $\phi(r,u)\to 0$ as $r\to 0$. Thus, the lemma follows easily by induction. □

We recall also the Gagliardo–Nirenberg interpolation inequalities in an endpoint case.

Theorem 2.2. *Assume that $u\in W^{m,p}(\mathbb{R}^n)$ for some $p\geqslant 1$ and $1\leqslant k\leqslant m$, $k,m\in\mathbb{N}$. If $u\in L^\infty(\mathbb{R}^n)$, then $D^k u\in L^q(\mathbb{R}^n)$ for $q=\frac{m}{k}p$ and*

$$\left\|D^k u\right\|_{L^q}^2 \leqslant C\|u\|_{L^\infty}^{1-\theta}\left\|D^m u\right\|_{L^p}^{\theta} \quad \text{where } \theta = k/m, \tag{2.3}$$

for some constant $C=C(k,m,p,n)$.

Riesz potentials and fractional integration. We will extensively use the theory of Riesz operators. For the reader's convenience we state the basic facts.

Definition 2.3. Let $a\in(0,n)$. The Riesz potential operator of order a is an integral operator I_a defined as

$$I_a f(x) = \frac{1}{\gamma(a)}\int\limits_{\mathbb{R}^n} |x-y|^{a-n} f(y)\,dy \tag{2.4}$$

where

$$\gamma(a) = 2^a\pi^{n/2}\frac{\Gamma(a/2)}{\Gamma(\frac{n}{2}-\frac{a}{2})}.$$

Theorem 2.4 *(Fractional Integration Theorem). Let $a\in(0,n)$, $1<p<q<\infty$. Then the Riesz potential operator*

$$I_a : L^p \to L^q \quad \text{where } \frac{1}{q} = \frac{1}{p} - \frac{a}{n} \tag{2.5}$$

is bounded.

In the suite, we shall use a more refined version of Riesz potentials, discussed by Hajłasz and Koskela [12] and applied in the manner we need e.g. in [24]. The definition and properties given below are essentially rewritten from the latter paper.

Definition 2.5. For $g\in L^p(B_{20r})$ the generalized Riesz potential $I_{\nu,p}g(y)$, where $\nu>0$, is given by

$$I_{\nu,p}g(y) = \sum_{l=-\infty}^{[\log_2 9r]} 2^{l\nu}\left(\;\fint\limits_{B(y,2^l)} \left|g(x)\right|^p dx\right)^{1/p}.$$

The above integral is well defined for any $y\in B_{2r}$

We have (see e.g. [12])

Lemma 2.6. *Assume that $g\in L^q(B_{20r})$ and that $0<p<q<n/\nu$. Then $I_{\nu,p}g\in L^{q^*}(B_{2r})$, where $q^*=nq/(n-\nu q)$ and*

$$\|I_{\nu,p}g\|_{L^{q^*}(B_{2r})} \lesssim \|g\|_{L^q(B_{20r})}.$$

Finally, we shall use the well-known Gehring–Giaquinta–Modica Lemma on self improving property of reverse Hölder's inequalities and Campanato characterization of Hölder continuous functions. Both facts can be found e.g. in the book of Giaquinta [10].

3. Euler–Lagrange equations and weak convergence

To write down the Euler–Lagrange equation which follows from the definition (1.2), note that, as $|u| = 1$ a.e.,

$$\frac{d}{dt}\left(\frac{u+t\psi}{|u+t\psi|}\right)\bigg|_{t=0} = \psi - \langle u, \psi\rangle u.$$

Thus, differentiating under the integral sign, we obtain

$$\sum_{|\alpha|=m} \sum_{k=1}^{N} \int_{\mathbb{B}^n} D^\alpha u^k\, D^\alpha \psi^k\, dx = \sum_{|\alpha|=m} \sum_{k=1}^{N} \sum_{i=1}^{N} \int_{\mathbb{B}^n} D^\alpha u^k D^\alpha\bigl(u^k u^i \psi^i\bigr)\, dx \tag{3.1}$$

for every $\psi \in C_0^\infty(\mathbb{B}^n, \mathbb{R}^N)$. Equivalently,

$$\sum_{|\alpha|=m} \int_{\mathbb{B}^n} D^\alpha u^j\, D^\alpha \phi\, dx = \sum_{|\alpha|=m} \sum_{k=1}^{N} \int_{\mathbb{B}^n} D^\alpha u^k D^\alpha\bigl(\phi u^j u^k\bigr)\, dx \tag{3.2}$$

for every $j = 1, \dots, N$ and every $\phi \in C_0^\infty(\mathbb{B}^n)$.

Lemma 3.1. *Let $u \in W^{m,2}(\mathbb{B}^n, \mathbb{R}^N)$. The following conditions are equivalent*:

1. *u is polyharmonic, i.e.* (3.2) *holds*;
2. *the identity*

$$\sum_{|\alpha|=m} \int_{\mathbb{B}^n} \bigl(D^\alpha u^j D^\alpha\bigl(\theta u^l\bigr) - D^\alpha\bigl(\theta u^j\bigr) D^\alpha u^l\bigr)\, dx = 0 \tag{3.3}$$

holds for all $j, l = 1, \dots, N$ and $\theta \in C_0^\infty(\mathbb{B}^n)$.

Proof. 1. ⇒ 2. By density (3.2) holds for every $\phi \in W^{m,2}(\mathbb{B}^n)$ which is bounded. We set $\phi = \theta u^l$, where $\theta \in C_0^\infty(\mathbb{B}^n)$. Then Eq. (3.2) takes form

$$\sum_{|\alpha|=m} \int_{\mathbb{B}^n} D^\alpha u^j\, D^\alpha\bigl(\theta u^l\bigr)\, dx = \sum_{|\alpha|=m} \sum_{k=1}^{N} \int_{\mathbb{B}^n} D^\alpha u^k\, D^\alpha\bigl(\theta u^l u^j u^k\bigr)\, dx.$$

The right-hand side is symmetric with respect to l and j. Obviously, the left-hand side must have the same property. Thus we obtain

$$\sum_{|\alpha|=m} \int_{\mathbb{B}^n} \bigl(D^\alpha u^j D^\alpha\bigl(\theta u^l\bigr) - D^\alpha\bigl(\theta u^j\bigr) D^\alpha u^l\bigr)\, dx = 0.$$

2. ⇒ 1. Again, by a density argument, we may use $\theta \in L^\infty \cap W^{m,2}(\mathbb{B}^n)$. Taking $\theta = \phi u^l$ with $\phi \in C_0^\infty(\mathbb{B}^n)$ we obtain

$$\sum_{|\alpha|=m} \int_{\mathbb{B}^n} \bigl(D^\alpha u^j D^\alpha\bigl(\phi\bigl(u^l\bigr)^2\bigr) - D^\alpha\bigl(\phi u^j u^l\bigr) D^\alpha u^l\bigr)\, dx = 0$$

for every $j, l = 1 \dots N$. Summing over $l = 1 \dots N$ and using the constraints $\sum_{l=1}^N (u^l)^2 = 1$ leads to (3.3). □

We may rewrite Eq. (3.3) in yet another convenient equivalent form, namely

$$\sum_{\substack{|\alpha|=m \\ \alpha \geqslant \beta > 0}} \int_{\mathbb{B}^n} D^\beta \vartheta\, F^{jl}_{\alpha\beta}\, dx = 0 \quad \text{for all } j, l = 1, \dots, N \text{ and all } \vartheta \in C_0^\infty(\Omega), \tag{3.4}$$

where

$$F^{jl}_{\alpha\beta} \equiv F^{jl}_{\alpha\beta}(u) := \binom{\alpha}{\beta}\big(D^\alpha u^j D^{\alpha-\beta}u^l - D^\alpha u^l D^{\alpha-\beta}u^j\big). \tag{3.5}$$

Proof of Theorem 1.2. If $u_k \in W^{m,2}(\mathbb{B}^n, S^{N-1})$ converges weakly to $u \in W^{m,2}(\mathbb{B}^n, S^{N-1})$, then, by the Rellich–Kondrashov compactness theorem, $D^{\alpha-\beta}u_k \to D^{\alpha-\beta}u$ strongly in L^2 for every α with $|\alpha| = m$ and every $\beta > 0$. Thus, $F^{jl}_{\alpha\beta}(u_k) \to F^{jl}_{\alpha\beta}(u)$ in the sense of distributions, and therefore one can pass to the limit in the polyharmonic map equation (3.4). This completes the proof. □

4. Reverse Hölder's inequalities and energy decay

The key result needed to prove continuity of a polyharmonic map u and higher integrability of its derivatives $D^k u$, $k = 1, 2, \dots, m$, is the following reverse Hölder's inequality.

Lemma 4.1. *Assume that $u \in W^{m,2}(\mathbb{B}^n, \mathbb{S}^{N-1})$, $n = 2m$, is a polyharmonic map. There exists a constant C_0, depending only on n and N, such that for every $\varepsilon > 0$ there exists a number $r_0 = r_0(\varepsilon, u) > 0$ with the following property*:

$$\int_{B_r} V^2\,dx \leqslant \frac{C_0}{\varepsilon}\bigg(\int_{B_{2r}} V^p\,dx\bigg)^{2/p} + \varepsilon \int_{B_{20r}} V^2\,dx \tag{4.1}$$

for all radii $r \in (0, r_0)$, where

$$V := \sum_{j=1}^m \big|D^j u\big|^{m/j}$$

and $1 < p := 2n/(n+1) < 2$.

Proof. We shall use the third equivalent form of the Euler–Lagrange equation, i.e. (3.4),

$$\sum_{\substack{|\alpha|=m\\ \alpha\geqslant\beta>0}} \int_{\mathbb{B}^n} D^\beta \vartheta F^{jl}_{\alpha\beta}(u)\,dx = 0 \quad \text{for all } j, l = 1, \dots, N \text{ and all } \vartheta \in C_0^\infty(\Omega), \tag{4.2}$$

where the $F^{jl}_{\alpha\beta}(u)$ are defined by (3.5). Throughout the whole proof, C and C_i denote various constants which depend *only* on n and N.

Step 1. The test function and separation of different terms. For fixed j, l, we use

$$\vartheta := \zeta u^l \tilde{u}^j, \quad \text{where } \tilde{u}^j := u^j - T^{m-1}_{B_{2r}} u^j, \tag{4.3}$$

as the test function in (4.2). Here, $\zeta \in C_0^\infty(B_{2r})$ is a standard nonnegative cut-off function with $\zeta \equiv 1$ on B_r and $|D^k(\zeta)| \lesssim r^{-k}$ for $k = 1, 2, \dots, m$; by $T^{m-1}_{B_{2r}} u^j$ we denote, as usual, the mean value of the Taylor polynomial of u^j of order $m - 1$ over the ball B_{2r}. Using Leibniz' formula, we split

$$D^\beta \vartheta = \Phi_1^\beta + \Phi_2^\beta + \Phi_3^\beta, \tag{4.4}$$

where

$$\Phi_1^\beta := u^l D^\beta\big(\zeta \tilde{u}^j\big), \qquad \Phi_3^\beta := \zeta \tilde{u}^j D^\beta u^l, \tag{4.5}$$

and

$$\Phi_2^\beta := \sum_{\substack{\beta_1,\beta_2>0\\ \beta_1+\beta_2=\beta}} \binom{\beta}{\beta_1} D^{\beta_1} u^l D^{\beta_2}\big(\zeta \tilde{u}^j\big). \tag{4.6}$$

Inserting these expressions into (4.2) and summing with respect to j and l, we obtain an identity of the form

$$W_1 + W_2 + W_3 = 0, \tag{4.7}$$

where

$$W_i := \sum_{j,l} \sum_{\alpha,\beta}{}' \int_{\mathbb{B}^n} F^{jl}_{\alpha\beta}(u)\, \Phi^\beta_i \, dx \quad \text{for } i = 1, 2, 3, \tag{4.8}$$

and the summation in $\sum'_{\alpha,\beta}$ is performed over all α, β such that $\alpha \geqslant \beta > 0$, $|\alpha| = m$.

Before proceeding further, let us now give an informal explanation of the structure of the whole proof. The splitting (4.4) is arranged in such a way that W_3 corresponds to the crucial part of the critical nonlinearity in the polyharmonic map equation; to cope with this term, one really has to use the structure of this equation and a subtle estimate in terms of Riesz potentials. This part of estimates is based on *cancellation*. On the other hand, the leading term W_1 gives the integral $\int \zeta |D^m u|^2 \, dx$ up to a perturbation term which can be controlled by more or less standard applications of Hölder's, Poincaré's and Sobolev's inequalities. W_2 is just a perturbation term, which can be controlled in an analogous way. The estimates of W_1 and W_2 employ the constraints $|u|^2 = 1$ but otherwise are based only on *growth* properties.

To see all that, we further decompose these terms, starting with W_1 and then W_2.

Step 2. The leading term. In W_1, we separate the terms with $\alpha = \beta$ from the remaining ones to obtain

$$W_1 = W_{1,1} + W_{1,2}$$

where

$$W_{1,1} := \sum_{\substack{1 \leqslant j,l \leqslant N \\ |\alpha| = m}} \int_{\mathbb{B}^n} \Phi^{jl}_{\alpha\alpha}(u) u^l D^\alpha \big(\zeta \tilde{u}^j\big) \, dx, \tag{4.9}$$

$$W_{1,2} := \sum_{j,l} \sum_{\substack{|\alpha| = m \\ \alpha > \beta > 0}} \int_{\mathbb{B}^n} \Phi^{jl}_{\alpha\beta}(u) u^l D^\beta \big(\zeta \tilde{u}^j\big) \, dx. \tag{4.10}$$

We insert the definition (3.5) of $F^{jl}_{\alpha\beta}$ into (4.9) and deal with the sum $W_{1,1}$, using the constraints $|u|^2 = 1$ a.e., their consequence

$$\sum_l u^l D^\alpha u^l = -\frac{1}{2} \sum_{0 < \gamma < \alpha} \sum_l \binom{\alpha}{\gamma} D^\gamma u^l D^{\alpha - \gamma} u^l, \tag{4.11}$$

and Leibniz' formula in the following way:

$$\begin{aligned} W_{1,1} &\equiv \sum_{\substack{1 \leqslant j,l \leqslant N \\ |\alpha| = m}} \int_{\mathbb{B}^n} \big(u^l D^\alpha u^j - u^j D^\alpha u^l\big) u^l D^\alpha \big(\zeta \tilde{u}^j\big) \, dx \\ &= \sum_{\substack{1 \leqslant j,l \leqslant N \\ |\alpha| = m}} \int_{\mathbb{B}^n} \big(u^l\big)^2 D^\alpha u^j D^\alpha \big(\zeta \tilde{u}^j\big) \, dx + \frac{1}{2} \sum_{\substack{1 \leqslant j,l \leqslant N \\ |\alpha| = m \\ 0 < \gamma < \alpha}} \int_{\mathbb{B}^n} \binom{\alpha}{\gamma} u^j D^\gamma u^l D^{\alpha - \gamma} u^l D^\alpha \big(\zeta \tilde{u}^j\big) \, dx \\ &\geqslant \int_{B_{2r}} \zeta \big|D^m u\big|^2 \, dx - C\big(S_1 + S_2(\zeta \tilde{u}, u, u)\big), \end{aligned} \tag{4.12}$$

where the two sums S_1 and $S_2(\cdot)$ — which do not contain the *squares* of m-th derivatives of u — are given by

$$S_1 := r^{-k} \sum_{k=1}^{m} \int_{B_{2r}} \big|D^m u\big| \big|D^{m-k} \tilde{u}\big| \, dx, \tag{4.13}$$

$$S_2(f, g, h) := \sum_{k=1}^{m-1} \int_{B_{2r}} \big|D^m f\big| \big|D^{m-k} g\big| \big|D^k h\big| \, dx \quad \text{for } f, g, h \in L^\infty \cap W^{m,2}. \tag{4.14}$$

Now, we estimate S_1 in a standard way, applying Hölder's and Sobolev's inequalities to obtain

$$S_1 \lesssim \sum_{k=1}^{m} r^{n-k} \Big(\fint_{B_{2r}} \big|D^{m-k}\tilde{u}_k\big|^{p_k'}\,dx \Big)^{\frac{1}{p_k'}} \Big(\fint_{B_{2r}} \big|D^m u\big|^{p_k}\,dx \Big)^{\frac{1}{p_k}}$$
$$\lesssim \sum_{k=1}^{m} r^{n} \Big(\fint_{B_{2r}} \big|D^{m}u\big|^{p_k}\,dx \Big)^{\frac{1}{p_k}} \Big(\fint_{B_{2r}} \big|D^m u\big|^{p_k}\,dx \Big)^{\frac{1}{p_k}}, \tag{4.15}$$

which holds provided we choose each p_k such as to have $p_k' = p_k^{*\cdots*} = np/(n-kp)$. This condition yields $p_k = 2n/(n+k)$; in particular, for each $0 < k \leqslant m$ we have $p_k \leqslant p_1 \equiv p := 2n/(n+1) < 2$. Thus, by Hölder's inequality, we can estimate

$$S_1 \leqslant Cr^n \Big(\fint_{B_{2r}} \big|D^m u\big|^{\frac{2n}{n+1}} \Big)^{\frac{n+1}{n}}. \tag{4.16}$$

Step 3. Lower order terms. The estimate of $S_2(\zeta\tilde{u}, u, u)$ is very similar to the estimates of $W_{1,2}$ and W_2. It is easy to check that, by triangle inequality,

$$|W_{1,2}| + |W_2| \leqslant C\big(S_2(u,u,\zeta\tilde{u}) + S_3(u,u,u,\zeta\tilde{u})\big), \tag{4.17}$$

where $S_3(\cdot)$ is defined by

$$S_3(f,g,h,\phi) := \sum_{\substack{k+l+t=m\\ k,l,t\geqslant 1}} \int_{B_{2r}} \big|D^m f\big|\big|D^k g\big|\big|D^l h\big|\big|D^t \phi\big|\,dx \tag{4.18}$$

whenever $f,g,h,\phi \in L^\infty \cap W^{m,2}$. Our general aim now is to estimate $S_2(\cdot)$ and $S_3(\cdot)$ by

$$\text{(a small constant)} \cdot \int_{B_{\sigma r}} \big|D^m u\big|^2 + \text{lower order, harmless terms,} \tag{4.19}$$

with $\sigma = 2$. This is done in a fairly routine way, using Young's inequality and Lemma 2.1. Here are the details.

Fix a small number $\eta > 0$, $\eta << \varepsilon$. The value of η shall be specified later on.

Applying Young's inequality with exponents 2, n/k and $n/(m-k)$, we obtain

$$S_2(\zeta\tilde{u}, u, u) \leqslant \eta \int_{B_{2r}} \big|D^m(\zeta\tilde{u})\big|^2\,dx + \frac{C}{\eta} S_4(u),$$

where

$$S_4(f) := \sum_{k=1}^{m-1} \int_{B_{2r}} \big|D^k f\big|^{n/k}\,dx \quad \text{for } f \in L^\infty \cap W^{m,2}. \tag{4.20}$$

Leibniz' formula and Poincaré's inequality yield

$$\int_{B_{2r}} \big|D^m(\zeta\tilde{u})\big|^2\,dx \leqslant C \int_{B_{2r}} \big|D^m u\big|^2\,dx \tag{4.21}$$

(we use the bounds $|D^k\zeta| \lesssim r^{-k}$ here), so that

$$S_2(\zeta\tilde{u}, u, u) \leqslant C\eta \int_{B_{2r}} \big|D^m u\big|^2\,dx + \frac{C}{\eta} S_4(u), \tag{4.22}$$

where C depends only on n and N.

Next, applying Young's inequality in a similar way, we obtain

$$S_2(u,u,\zeta\tilde u) \leqslant C\eta \int\limits_{B_{2r}} \left|D^m u\right|^2 dx + \frac{C}{\eta} S_4(u) + \frac{C}{\eta} S_4(\zeta\tilde u), \tag{4.23}$$

$$S_3(u,u,u,\zeta\tilde u) \leqslant C\eta \int\limits_{B_{2r}} \left|D^m u\right|^2 dx + \frac{C}{\eta} S_4(u) + \frac{C}{\eta} S_4(\zeta\tilde u). \tag{4.24}$$

It remains now to obtain appropriate estimates of $S_4(u)$ and $S_4(\zeta\tilde u)$. Applying Lemma 2.1 for $\delta := \eta^2/(C_1 m)$, we obtain

$$\frac{C_1}{\eta} S_4(u) \leqslant \eta \int\limits_{B_{2r}} \left|D^m u\right|^2 + \frac{C_2}{\eta} r^n \sum_{k=1}^{m-1} \biggl(\fint\limits_{B_{2r}} \left|D^k u\right| dx \biggr)^{n/k} \tag{4.25}$$

for all radii $r \in (0, r_1)$, where $r_1 = r_1(\eta, u) > 0$.

Next, by Sobolev inequality for compactly supported functions,

$$S_4(\zeta\tilde u) = \sum_{k=1}^{m-1} \int\limits_{B_{2r}} \left|D^k(\zeta\tilde u)\right|^{n/k} dx \lesssim \sum_{k=1}^{m-1} \biggl(\int\limits_{B_{2r}} \left|D^m(\zeta\tilde u)\right|^2 dx \biggr)^{m/k} \overset{(4.21)}{\lesssim} \sum_{k=1}^{m-1} \biggl(\int\limits_{B_{2r}} \left|D^m u\right|^2 dx \biggr)^{m/k}.$$

Since all the exponents m/k in the last sum above are greater than 1, we can use absolute continuity of the integral to conclude that

$$\frac{C_1}{\eta} S_4(\zeta\tilde u) \leqslant \eta \int\limits_{B_{2r}} \left|D^m u\right|^2 dx \tag{4.26}$$

for all radii $r \in (0, r_2)$, where $r_2 = r_2(\eta, u) > 0$.

Step 4. A combined estimate of $W_1 + W_2$. We are now in a position to plug the estimates (4.25) and (4.26) of $S_4(\cdot)$ into the right-hand sides of (4.22), (4.23) and (4.24) to obtain

$$S_2(\zeta u, u, u) + |W_{1,2}| + |W_2| \leqslant C\eta \int\limits_{B_{2r}} \left|D^m u\right|^2 + \frac{C}{\eta} r^n \sum_{k=1}^{m-1} \biggl(\fint\limits_{B_{2r}} \left|D^k u\right| dx \biggr)^{n/k}$$

for all $r < \min(r_1, r_2)$. Combining this estimate with (4.12) and (4.16), we obtain the following estimate of the leading term and all the lower order perturbations:

$$\begin{aligned} W_1 + W_2 &\geqslant \int\limits_{B_{2r}} \zeta \left|D^m u\right|^2 dx - C_3\eta \int\limits_{B_{2r}} \left|D^m u\right|^2 dx - C_4 r^n \biggl(\fint\limits_{B_{2r}} \left|D^m u\right|^{\frac{2n}{n+1}} \biggr)^{\frac{n+1}{n}} - \frac{C_4}{\eta} r^n \sum_{k=1}^{m-1} \biggl(\fint\limits_{B_{2r}} \left|D^k u\right| dx \biggr)^{n/k} \\ &\geqslant \int\limits_{B_{2r}} \zeta \left|D^m u\right|^2 dx - C_3\eta \int\limits_{B_{2r}} \left|D^m u\right|^2 dx - \frac{C_5}{\eta} r^n \biggl(\fint\limits_{B_{2r}} |V|^{\frac{2n}{n+1}} dx \biggr)^{\frac{n+1}{n}} \end{aligned} \tag{4.27}$$

where $V = \sum_{k=1}^m |D^k u|^{m/k}$. In the last step, we used Hölder's inequality and the obvious properties of $s \mapsto s^q$ for $s \in [0,\infty)$ and $q > 1$.

Step 5. Employing cancellation, i.e. the estimates of W_3. This is the heart of the proof. We now pass to the most troublesome term

$$W_3 = \sum_{j,l} {\sum_{\alpha,\beta}}' \int\limits_{\Omega} \zeta \tilde u^j D^\beta u^l F^{jl}_{\alpha\beta}(u)\, dx, \tag{4.28}$$

where the summation $\sum'_{\alpha,\beta}$ is, as in (4.7), performed over α, β such that $|\alpha| = m$, $0 < \beta \leqslant \alpha$.

Our general aim is to prove that, for sufficiently small r, each of the N^2 terms $W_{3,jl}$ of W_3,

$$W_{3,jl} := {\sum_{\alpha,\beta}}' \int\limits_{\Omega} \zeta \tilde u^j D^\beta u^l F^{lj}_{\alpha\beta}\, dx, \tag{4.29}$$

can be estimated by a small multiple of $\int_{B_{20r}} |D^m u|^2$.

From now on we fix j and l. Set $\phi := \zeta \tilde{u}^j$. We shall use the representation formula

$$\phi(x) = \int_{B_{2r}} K(x-y) D^m \phi(y)\, dy, \quad \left|D^\gamma K(x-y)\right| \lesssim |x-y|^{-m-|\gamma|}. \tag{4.30}$$

(Such a formula can be obtained for smooth compactly supported functions, using the fundamental solution of $(\Delta)^m$ in $\mathbb{R}^n$ and integration by parts. The constants in estimates of $D^\gamma K$ depend only on n and γ.)

Let $\zeta_1 \in C_0^\infty$ be such that $\zeta_1 \equiv 1$ on B_{2r}, $\zeta_1 \equiv 0$ off B_{3r}, $D^k \zeta_1 \lesssim r^{-k}$. We can safely multiply the integrand of (4.29) by ζ_1, as the support of ζ (and thus of ϕ) is contained in B_{2r}:

$$\begin{aligned} W_{3,jl} &= {\sum_{\alpha,\beta}}' \int_{\mathbb{R}^n} \int_{B_{2r}} \zeta_1(x) K(x-y) D^m \phi(y) D^\beta u^l(x) F_{\alpha\beta}^{jl}(x)\, dx\, dy \\ &= \int_{B_{2r}} D^m \phi(y) {\sum_{\alpha,\beta}}' \int_{\mathbb{R}^n} \zeta_1(x) K(x-y) D^\beta u^l(x)\, F_{\alpha\beta}^{jl}(x)\, dx\, dy. \end{aligned} \tag{4.31}$$

Since $u \in W^{m,2}$, we have $D^m \phi \in L^2$, and $\|D^m \phi\|_{L^2} \lesssim \|D^m u\|_{L^2(B_{2r})}$ by Poincaré's inequality. We thus face the following crucial question: does

$$A(y) := {\sum_{\alpha,\beta}}' \int_{\mathbb{R}^n} \zeta_1(x) K(x-y) D^\beta u^l(x) F_{\alpha\beta}^{jl}(x)\, dx \tag{4.32}$$

belong to $L^2(B_{2r})$, with possibly good estimates of its L^2-norm? In order to provide a positive answer, and to obtain

$$|W_{3,jl}| \lesssim \left\|D^m u\right\|_{L^2(B_{2r})} \|A\|_{L^2(B_{2r})}, \tag{4.33}$$

let us fix $y \in B_{2r}$ and consider a Whitney decomposition

$$\mathbb{R}^n \setminus \{y\} = \bigcup_{i\in I} B_i,$$

where for all $i \in I$ we have $B_i = B(a_i, r_i)$ and $r_i = \frac{1}{1000}|a_i - y|$. In particular, $|x-y| \approx r_i$ for every $x \in B_i$; this yields

$$\left|D^\gamma K(x-y)\right| \lesssim r_i^{-m-|\gamma|} \quad \text{for all } x \in B_i,\ i \in I.$$

By J we denote the set of all these indices $i \in I$ for which $B_i \cap B_{3r} \neq \emptyset$ (recall that $\zeta_1 \in C_0^\infty(B_{3r})$).

Next, we choose a Whitney partition of unity $\theta_i \in C_0^\infty(B_i)$, $\sum \theta_i \equiv 1$ on $\mathbb{R}^n \setminus \{y\}$, $|D^s \theta_i| \lesssim r_i^{-s}$, $s = 1, 2, \ldots,$ $i \in I$. Moreover, we assume that the family $\{10B_i \mid i \in I\}$ has finite overlap property: there exists an $L = L(n)$ such that

$$\sum_{i\in I} \chi_{10B_i}(x) \leqslant L \quad \text{for all } x \in \mathbb{R}^n.$$

Then, using the Euler equation (4.2) for $\vartheta(x) = \zeta_1(x) K(x-y) \theta_i(x)[u(x) - T_{B_i}^{|\beta|-1} u]$ on each ball B_i, we observe that

$$\begin{aligned} \left|A(y)\right| &:= \left| {\sum_{\alpha,\beta}}' \int_{\mathbb{R}^n} \zeta_1(x) K(x-y) D^\beta u^l(x) \tilde{F}_{\alpha\beta}^{jl}(x)\, dx \right| \\ &\leqslant \sum_{\substack{|\alpha|=m \\ \alpha \geqslant \beta \geqslant \gamma > 0}} \sum_{i\in J} \left| \int_{\mathbb{R}^n} \binom{\beta}{\gamma} D^\gamma (\zeta_1 K \theta_i) D^{\beta-\gamma} \left[u - T_{B_i}^{|\beta|-1} u\right] \tilde{F}_{\alpha\beta}^{jl}\, dx \right| \\ &\lesssim \sum_{\substack{|\alpha|=m \\ \alpha \geqslant \beta \geqslant \gamma > 0}} \sum_{i\in J} r_i^{-m-|\gamma|} \int_{B_i} \left|D^{\beta-\gamma}\left[u - T_{B_i}^{|\beta|-1} u\right]\right| \left|F_{\alpha\beta}^{jl}\right| dx \\ &=: \sum_{\substack{|\alpha|=m \\ \alpha \geqslant \beta \geqslant \gamma > 0}} A_{\alpha\beta\gamma}(y). \end{aligned} \tag{4.34}$$

We fix α, β and γ and deal with each term $A_{\alpha\beta\gamma}(y)$ separately. We need to consider several possible cases:

Case 1: $\alpha=\beta=\gamma$. Fix $i\in J$. The i-th term of the sum $A_{\alpha\beta\gamma}(y)$ can be estimated as follows:

$$\begin{aligned}\int_{B_i} r_i^{-2m}\big|u-T_{B_i}^{m-1}u\big|\big|F_{\alpha\alpha}^{jl}\big|\,dx &\lesssim \fint_{B_i}\big|u-T_{B_i}^{m-1}u\big|\big|D^m u\big|\,dx\\ &\lesssim\Big(\fint_{B_i}\big|u-T_{B_i}^{m-1}u\big|^q\,dx\Big)^{\frac1q}\Big(\fint_{B_i}\big|D^m u\big|^p\,dx\Big)^{\frac1p}\\ &\lesssim r_i^m\Big(\fint_{B_i}\big|D^m u\big|^{q_{*\cdots*}}\,dx\Big)^{\frac{1}{q_{*\cdots*}}}\Big(\fint_{B_i}\big|D^m u\big|^p\,dx\Big)^{\frac1p}\end{aligned}\tag{4.35}$$

using first Hölder's, then Sobolev's inequality. The exponents in Hölder's inequality are chosen in such a way as to have $q_{*\cdots*}=p$, that is $q=\frac{np}{n-mp}=\frac{p}{p-1}$. In our case $m=\frac n2$, which gives $p=\frac43$, $q=4$. Altogether we estimate our term with

$$\int_{B_i} r_i^{-2m}\big|u-T_{B_i}^{m-1}u\big|\big|F_{\alpha\alpha}^{jl}\big|\,dx\lesssim r_i^m\Big(\fint_{B_i}\big|D^m u\big|^{\frac43}\,dx\Big)^{\frac32}\lesssim\Big(\int_{B_i}\frac{|D^m u(x)|^{4/3}\,dx}{|x-y|^{n-n/3}}\Big)^{\frac32}.\tag{4.36}$$

Summing over $i\in J$ and applying the inequality $\sum {c_i}^{3/2}\leqslant(\sum c_i)^{3/2}$, we obtain an estimate by a Riesz potential

$$\big|A_{\alpha\beta\gamma}(y)\big|\lesssim\big(I_{\frac n3}\big(\chi_{B_{4r}}\big|D^m u\big|^{\frac43}\big)(y)\big)^{\frac32}=\Big(\int_{B_{4r}}\frac{|D^m u(x)|^{4/3}\,dx}{|x-y|^{n-n/3}}\Big)^{\frac32}.$$

This term, by the fractional integration theorem (see Section 2, Theorem 2.4), lies in $L^2(B_{2r})$, and

$$\|A_{\alpha\beta\gamma}\|_{L^2(B_{2r})}\lesssim\big\|\big(I_{\frac n3}\big(\chi_{B_{4r}}\big|D^m u\big|^{\frac43}\big)\big)^{\frac32}\big\|_{L^2(B_{2r})}\lesssim\big\|D^m u\big\|^2_{L^2(B_{4r})}.\tag{4.37}$$

Case 2: $\alpha=\beta>\gamma>0$. As in the previous case, we first fix $i\in I$. In order to simplify the notation we shall write $|\gamma|=s$, $\beta-\gamma=\delta$. Using the standard properties of Taylor polynomials,

$$D^{\beta-\gamma}\big[u-T_{B_i}^{|\beta|-1}u\big]=D^{\beta-\gamma}u-T_{B_i}^{|\gamma|-1}D^{\beta-\gamma}u,$$

we can estimate the i-th term of (4.34) in the following way:

$$\begin{aligned}r_i^{-m-s}\int_{B_i}\big|D^\delta\big[u-T_{B_i}^{|\beta|-1}u\big]\big|\big|F_{\alpha\beta}^{jl}\big|\,dx &\lesssim r_i^{-m-s}\int_{B_i}\big|D^\delta u-T_{B_i}^{s-1}D^\delta u\big|\big|D^m u\big|\,dx\\ &\lesssim r_i^{m-s}\Big(\fint_{B_i}\big|D^\delta u-T_{B_i}^{s-1}D^\delta u\big|^q\,dx\Big)^{\frac1q}\Big(\fint_{B_i}\big|D^m u\big|^p\,dx\Big)^{\frac1p}\\ &\lesssim r_i^m\Big(\fint_{B_i}\big|D^s\big(D^\delta u\big)\big|^{q_{*\cdots*}}\,dx\Big)^{\frac{1}{q_{*\cdots*}}}\Big(\fint_{B_i}\big|D^m u\big|^p\,dx\Big)^{\frac1p}.\end{aligned}\tag{4.38}$$

We choose p and q similarly as in (4.35), to get $q_{*\cdots*}=p$, that is $q=\frac{np}{n-sp}=\frac{p}{p-1}$. This yields $p=\frac{2n}{n+s}$; in particular, for every choice of γ we have $p\leqslant\frac{2n}{n+1}$. Moreover, $|D^sD^\delta u|\leqslant|D^m u|$, and we continue the estimates like in (4.36):

$$\begin{aligned}r_i^{-m-s}\int_{B_i}\big|D^\delta\big[u-T_{B_i}^{|\beta|-1}u\big]\big|\big|F_{\alpha\beta}^{jl}\big|\,dx &\lesssim r_i^m\Big(\fint_{B_i}\big|D^m u\big|^{\frac{2n}{n+1}}\,dx\Big)^{1+\frac1n}\\ &\lesssim\Big(\int_{B_i}\frac{|D^m u|^{\frac{2n}{n+1}}}{|x-y|^{n-m\frac{n}{n+1}}}\,dx\Big)^{1+\frac1n}.\end{aligned}\tag{4.39}$$

In the same way as in Case 1, summing over $i \in J$ we obtain an estimate by a Riesz potential

$$\left|A_{\alpha\beta\gamma}(y)\right| \lesssim I_a\left(\chi_{B_{4r}}\left|D^m u\right|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{n}} = \left(\int_{B_{4r}} \frac{|D^m u(x)|^{\frac{2n}{n+1}}\,dx}{|x-y|^{n-a}}\right)^{1+\frac{1}{n}}, \tag{4.40}$$

where $a = mn/(n+1)$. Like before, Theorem 2.4 gives $I_a(\chi_{B_{4r}}|D^m u|^{\frac{2n}{n+1}}) \in L^{\frac{2(n+1)}{n}}$, and

$$\|A_{\alpha\beta\gamma}\|_{L^2(B_{2r})} \lesssim \left\|D^m u\right\|^2_{L^2(B_{4r})}, \tag{4.41}$$

as in Case 1.

Case 3: $\alpha > \beta \geqslant \gamma > 0$. In this case we need to estimate

$$\left|A_{\alpha\beta\gamma}(y)\right| := \sum_{i\in J} r_i^{-m-|\gamma|} \int_{B_i} \left|D^{\beta-\gamma}\left(u - T^{|\beta|-1}_{B_i} u\right)\right| |F_{\alpha\beta}|\,dx. \tag{4.42}$$

We shall, as before, use abbreviations: $s = |\gamma|$, $\delta = \beta - \gamma$, $t = |\alpha - \beta|$. By Hölder's and Sobolev's inequalities,

$$\begin{aligned}\left|A_{\alpha\beta\gamma}(y)\right| &\lesssim \sum_{i\in J} r_i^{m-s} \fint_{B_i} \left|D^\delta u - T^{s-1} D^\delta u\right|\left|D^m u\right|\left|D^t u\right| dx \\ &\lesssim \sum_{i\in J} r_i^m \left(\fint_{B_i} \left|D^{m-t} u\right|^{p_{1,s,*}} dx\right)^{\frac{1}{p_{1,s,*}}} \left(\fint_{B_i} \left|D^m u\right|^{p_2} dx\right)^{\frac{1}{p_2}} \left(\fint_{B_i} \left|D^t u\right|^{p_3} dx\right)^{\frac{1}{p_3}}.\end{aligned} \tag{4.43}$$

The exponent $p_{1,s,*}$, obtained by first using Hölder's, and then Sobolev's inequality, is equal to $np_1/(n+sp_1)$. As $s \geqslant 1$, we can use Hölder's inequality once again, in order to lose the dependence on s, replacing $p_{1,s,*}$ by its maximal possible value $p_{1,*} := np_1/(n+p_1)$.

We may choose the exponents in Hölder's inequality in such a way that $p_3 = \frac{mp_2}{t}$, $p_{1,*} = \frac{mp_2}{m-t}$. The condition that p_1, p_2, p_3 are Hölder conjugate implies that $p_2 = \frac{4m}{2m+1} = \frac{2n}{n+1} < 2$.

Next we estimate the sum of products in (4.43) by a product of three sums, splitting r_i^m into them to obtain

$$\begin{aligned}\left|A_{\alpha\beta\gamma}(y)\right| &\lesssim \left(\sum_{i\in J} r_i^{\nu_1}\left(\fint_{B_i}\left|D^{m-t}u\right|^{p_{1,*}}\right)^{\frac{1}{p_{1,*}}}\right) \times \left(\sum_{i\in J} r_i^{\nu_2}\left(\fint_{B_i}\left|D^m u\right|^{p_2}\right)^{\frac{1}{p_2}}\right) \\ &\quad\times \left(\sum_{i\in J} r_i^{m-\nu_1-\nu_2}\left(\fint_{B_i}\left|D^t u\right|^{p_3}\right)^{\frac{1}{p_3}}\right).\end{aligned} \tag{4.44}$$

We estimate each of the sums by the means of generalized Riesz potentials $I_{\nu,p}$ as in [24, Proof of Theorem 2], grouping the balls B_i with radii $r_i \approx 2^{-\ell}$. (For each fixed ℓ, the number of such balls B_i is bounded by a constant depending only on n, and they are all contained in $B(y, \mathrm{const}\cdot 2^{-\ell})$.) This leads to the following inequality

$$\left|A_{\alpha\beta\gamma}(y)\right| \lesssim I_{\nu_1,\frac{mp_2}{m-t}}\left(\left|D^{m-t}u\right|\right) \times I_{\nu_2,p_2}\left(\left|D^m u\right|\right) \times I_{m-\nu_1-\nu_2,\frac{mp_2}{t}}\left(\left|D^t u\right|\right). \tag{4.45}$$

Gagliardo–Nirenberg inequalities give $|D^{m-t}u| \in L^{2m/(m-t)}(B_{20r})$, $|D^t u| \in L^{2m/t}(B_{20r})$. By assumption $|D^m u| \in L^2(B_{20r})$. We choose ν_1 and ν_2 in such a way that the assumptions of Lemma 2.6 are satisfied, i.e. that

$$0 < \nu_1 < m - t, \quad 0 < \nu_2 < m, \quad \nu_1 + \nu_2 > m - t,$$

and the lemma yields

$$\begin{aligned} f_1 &:= I_{\nu_1,\frac{mp_2}{m-t}}\left(\left|D^{m-t}u\right|\right) \in L^{\frac{2m}{m-t-\nu_1}}(B_{2r}) =: L^{q_1}(B_{2r}), \\ f_2 &:= I_{\nu_2,p_2}\left(\left|D^m u\right|\right) \in L^{\frac{2m}{m-\nu_2}}(B_{2r}) =: L^{q_2}(B_{2r}), \\ f_3 &:= I_{m-\nu_1-\nu_2,\frac{mp_2}{t}}\left(\left|D^t u\right|\right) \in L^{\frac{2m}{t-(m-\nu_1-\nu_2)}}(B_{2r}) =: L^{q_3}(B_{2r}),\end{aligned} \tag{4.46}$$

with the following estimates

$$\|f_i\|_{L^{q_i}(B_{2r})} \lesssim \left\|D^{k_i}u\right\|_{L^{n/k_i}(B_{20r})}, \quad i=1,2,3;\ k_1=m-t,\ k_2=m,\ k_3=t. \tag{4.47}$$

One can easily check that $\frac{q_1}{2}, \frac{q_2}{2}, \frac{q_3}{2}$ are Hölder conjugate. Thus, conditions (4.45)–(4.47) combined with Hölder and Young's inequalities imply that $A_{\alpha\beta\gamma}(y) \in L^2$ and

$$\|A_{\alpha\beta\gamma}\|_{L^2(B_{2r})} \lesssim \left\|D^m u\right\|_{L^2(B_{20r})} \sum_{j=1}^{m-1} \left\|D^j u\right\|^2_{L^{n/j}(B_{20r})}. \tag{4.48}$$

Step 6. Conclusion. Gathering the estimates (4.37), (4.41) and (4.48) of $A_{\alpha\beta\gamma}$ obtained in the three cases above, plugging them into (4.33), and using absolute continuity of integral, we conclude that

$$|W_3| < C_6\eta \int_{B_{20r}} \left|D^m u\right|^2 dx \tag{4.49}$$

for all $0 < r < r_3$, where r_3 is chosen so that

$$\sup_{B(a,20r_3)\subset\mathbb{B}^n} \max_{j=1,2,\dots,m} \left\|D^j u\right\|_{L^{n/j}(B(a,20r_3))} < \min(1,\eta).$$

Combining (4.49) with (4.27), and choosing $\eta = \varepsilon/2m(C_3+C_6)$, we obtain

$$\int_{B_r} \left|D^m u\right|^2 dx \leqslant \frac{\varepsilon}{2m} \int_{B_{20r}} \left|D^m u\right|^2 dx + \frac{C}{\varepsilon} r^n \left(\fint_{B_{2r}} |V|^{\frac{2n}{n+1}} dx \right)^{\frac{n+1}{n}}. \tag{4.50}$$

It is now a routine job to apply Lemma 2.1 with sufficiently small $\delta = \delta(\varepsilon, m)$ in order to incorporate lower order derivatives of u into the left-hand side of (4.50), and to complete the whole proof. □

Combining Lemma 4.1 with Gehring–Giaquinta–Modica Lemma on self improving property of reverse Hölder's inequalities and with Sobolev–Morrey embedding theorem, we obtain the following.

Corollary 4.2. *If $u \in W^{m,2}(\mathbb{B}^n, \mathbb{S}^{N-1})$ is a polyharmonic map, then u is locally Hölder continuous. Moreover, there exists an exponent $q > 2$ such that*

$$D^s u \in L^{\frac{mq}{s}}_{\mathrm{loc}}\left(\mathbb{B}^n\right), \quad s = 1,2,\dots,m.$$

5. From continuity to smoothness

Let $u \in W^{m,2}(\mathbb{B}^n, \mathbb{S}^{N-1})$ be a polyharmonic map. By Corollary 4.2, we may assume u is Hölder continuous and

$$D^s u \in L^{\frac{mq}{s}}, \quad s = 1,2,\dots,m, \tag{5.1}$$

for some fixed $q > 2$. To prove that $u \in C^\infty$, we first establish existence of $D^{m+t}u$ in appropriate Lebesgue spaces (see Section 5.1). Next, we apply linear elliptic estimates and classical bootstrap reasoning based on Schauder theory to prove that u is smooth.

5.1. Existence of higher order derivatives

We shall prove, by induction with respect to t, that

$$D^{m+t}u \in L^{\frac{2m}{m+t}}_{\mathrm{loc}}\left(\mathbb{B}^n\right), \quad t = 0,1,\dots,m-1. \tag{5.2}$$

Let

$$L = (-1)^m \sum_{|\alpha|=m} D^{2\alpha}.$$

To achieve (5.2), we shall prove by induction another claim. Namely, it turns out that for every $\zeta \in C_0^\infty(\mathbb{B}^n)$ and every $t = 0, 1, \dots, m-1$ one has

$$\langle Lu^j, \zeta\rangle = \sum_{\Lambda\in\mathcal{A}_t}\sum_{k=1}^N c_{\Lambda kt}\int_{\mathbb{B}^n} D^{\lambda_0}\zeta\, D^{\lambda_1}u^k D^{\lambda_2}u^k D^{\lambda_3}u^j\,dx, \tag{5.3}$$

where the coefficients $c_{\Lambda kt}$ are constant and depend only on n, N, t, and $\mathcal{A}_t$ denotes the set of these quadruples $\Lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ of multiindices λ_i which satisfy the following conditions:

$$\lambda_0 \leqslant m-t-1, \quad |\lambda_i| \leqslant m+t \quad \text{for } i = 1, 2, 3, \tag{5.4}$$

$$|\lambda_0| + |\lambda_1| + |\lambda_2| + |\lambda_3| = 2m, \tag{5.5}$$

$$|\lambda_i| \geqslant m \quad \text{for at least one } i \in \{1, 2, 3\}. \tag{5.6}$$

Remark. It is easy to see that if $\Lambda \in \mathcal{A}_t$, then $|\lambda_i| \in (0, m]$ for some $i \in \{1, 2, 3\}$.

For $t = 0$ (5.2) does hold. To verify (5.3) for $t = 0$, we use the constraints $|u|^2 = 1$ to rewrite the polyharmonic map equation (3.2) as follows:

$$\langle Lu^j, \zeta\rangle := \sum_{|\alpha|=m}\int_{\mathbb{B}^n} D^\alpha u^j\, D^\alpha\zeta\,dx \overset{(3.2)}{=} \sum_{|\alpha|=m}\sum_{k=1}^N\int_{\mathbb{B}^n} D^\alpha u^k D^\alpha\big(\zeta u^j u^k\big)\,dx =: \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \sum_{|\alpha|=m}\sum_{k=1}^N\int_{\mathbb{B}^n} u^j u^k D^\alpha u^k D^\alpha\zeta\,dx$$

and

$$\Sigma_2 = \sum_{|\alpha|=m}\sum_{k=1}^N\sum_{\beta<\alpha}\binom{\alpha}{\beta}\int_{\mathbb{B}^n} D^\alpha u^k D^\beta\zeta\, D^{\alpha-\beta}\big(u^j u^k\big)\,dx.$$

The second sum, Σ_2, already has the required form (5.3) for $t = 0$. Invoking (4.11), we can replace $\sum_k u^k D^\alpha u^k$ in Σ_1 by

$$-\frac{1}{2}\sum_{0<\gamma<\alpha}\sum_k\binom{\alpha}{\gamma}D^\gamma u^k D^{\alpha-\gamma}u^k.$$

Then, after one integration by parts, moving one partial derivative from ζ to the other terms, we rewrite Σ_1 in the required form (i.e., using only the derivatives of ζ of order $m-1$).

Thus, for $t = 0$ both (5.2) and (5.3) are satisfied.

Fix now some $t \in \{0, 1, \dots, m-2\}$ and assume that (5.2) and (5.3) hold for that t. As it is easy to see, (5.3) combined with (5.1) and Hölder's inequality imply that for each $r \in (0, 1)$ the distribution Lu^j can be extended to a continuous linear functional

$$Lu^j \in \big(W_0^{m-t-1,2m/(m-t-1)}(B_r)\big)^*, \quad B_r \equiv B(0, r). \tag{5.7}$$

Note that $W_0^{m-t-1,2m/(m-t-1)} \not\subset L^\infty$ and to guarantee the boundedness of terms

$$\int_{B_r} \zeta D^{\lambda_1}u^k D^{\lambda_2}u^k D^{\lambda_3}u^j\,dx$$

for $|\lambda_1| + |\lambda_2| + |\lambda_3| = 2m$, $\max|\lambda_i| \leqslant m+t$, one really must use (5.1); mere knowledge that $D^s u \in L^{2m/s}$ does not suffice here.

To show that (5.7) implies (5.2) for $t+1$, we shall apply the following useful lemma.

Lemma 5.1. *Let $k = 1, 2, \dots,$ $1 < p < \infty$ and $pq = p + q$. If U is a smooth bounded domain in $\mathbb{R}^n$ and*

$$\Phi \in \big(W_0^{k,p}(U)\big)^*,$$

then there exist functions $(v_\beta)_{|\beta|=k} \in (L^q(U))^{M_k}$, where $M_k := \sum_{|\beta|=k} 1$ is the number of all multiindices of length k, such that

$$\Phi(\zeta) = \sum_{|\beta|=k} \int_U v_\beta D^\beta \zeta \, dx, \quad \zeta \in C_0^\infty(U).$$

Sketch of the proof. It is well known, see e.g. [1, Theorem 3.8], that

$$\Phi(\zeta) = \sum_{|\beta|\leqslant k} \int_U v_\beta D^\beta \zeta \, dx, \quad \zeta \in C_0^\infty(U), \tag{5.8}$$

with $v_\beta \in L^q(U)$ for each β. To replace v_β with $|\beta| \leqslant k-1$ by zeroes — possibly changing the v_β with $|\beta| = k$ in (5.8), of course! — one repeatedly solves linear elliptic equations and applies [11, Theorem 9.15]. Full details are left to the reader; here is a hint: if v_0 in $L^q(U)$, then $-\Delta w_0 = v_0$ has a unique weak solution $w_0 \in W_0^{1,q}(U) \cap W^{2,q}(U)$, and

$$\int_U \zeta v_0 \, dx = \int_U \nabla\zeta \nabla w_0 \, dx$$

for each $\zeta \in C_0^\infty(U)$. Using this identity, one can remove the term corresponding to $|\beta| = 0$ from (5.8); other terms with $|\beta| \leqslant k-1$ can be treated similarly. □

Remark. Note that the assumption $1 < p < \infty$ (which is not necessary to write down the representation formula (5.8)) is crucial in the above proof.

Combining Lemma 5.1 and (5.7), we see that for every fixed $r \in (0, 1)$

$$\langle Lu^j, \zeta\rangle = \sum_{|\beta|=m-t-1} \int_{B_r} v_{\beta,t} D^\beta \zeta \, dx \quad \text{for all } \zeta \in C_0^\infty(B_r), \tag{5.9}$$

where $v_{\beta,t} \in L^{2m/(m+t+1)}(B_r)$ for each $|\beta| = m-t-1$.

Now, fix a multiindex γ with $|\gamma| = m+t+1$. It is an easy (formal) exercise in Fourier analysis to show that

$$\widehat{D^\gamma u^j}(\xi) = \sum_{|\beta|=m-t-1} m_{\beta,t}(\xi)\hat{v}_{\beta,t}, \tag{5.10}$$

where the multipliers $m_{\beta,t}$ are given by

$$m_{\beta,t}(\xi) = \text{const} \cdot \frac{\xi^{\beta+\gamma}}{\sigma_L(\xi)}$$

and $\sigma_L(\xi) := \sum_{|\alpha|=m} \xi^{2\alpha} \approx |\xi|^{2m}$. As $|\gamma| + |\beta| = 2m$, all $m_{\beta,t}$ satisfy the assumptions of Marcinkiewicz–Hörmander multiplier theorem, see e.g. [23, Theorem 3.2], and the operator

$$T : L^p \ni (v_\beta)_{|\beta|=m-t-1} \mapsto D^\gamma u^j \in L^p, \quad p = 2m/(m+t+1),$$

is continuous. Since $r \in (0, 1)$ is arbitrary, this means that the derivative

$$D^\gamma u^j \in L^{2m/(m+t+1)}_{\text{loc}}\big(\mathbb{B}^n\big).$$

(To check that (5.10) in fact implies the existence of $D^\gamma u^j$, note that (5.9) is linear and consider smooth convolution approximations $u^j * \varphi_\varepsilon$, $v_{\beta,t} * \varphi_\varepsilon$ of u^j and all $v_{\beta,t}$. Since T is linear and continuous, $D^\gamma(u^j * \varphi_\varepsilon)$ converge in L^p as $\varepsilon \to 0$, and the limit is equal to $D^\gamma u^j$.)

Hence, we have established (5.2) for $t+1$. To obtain (5.3) for $t+1$, just perform an integration by parts, to take away one derivative from ζ.

Thus, (5.2) does indeed hold for all $t = 0, 1, \dots, m-1$.

5.2. Bootstrap: some details

By (5.2) for $t = m-1$, the polyharmonic map equation can be written as

$$Lu^j = \sum_{\Lambda,k} c_{\Lambda,k} D^{\lambda_1} u^k D^{\lambda_2} u^k D^{\lambda_3} u^j, \tag{5.11}$$

where the summation is performed over $k = 1, \dots, N$ and over the set of all triples $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$ of multiindices λ_i such that

$$|\lambda_1| + |\lambda_2| + |\lambda_3| = 2m, \quad |\lambda_i| \leqslant 2m-1 \quad \text{for all } i,$$
$$|\lambda_j| \geqslant m \quad \text{for at least one } j \in \{1,2,3\}.$$

Integrability conditions (5.1) and (5.2) imply that

$$Lu = F \in L^{r_0}_{\mathrm{loc}}\big(\mathbb{B}^n, \mathbb{R}^N\big) \quad \text{for some } r_0 > 1. \tag{5.12}$$

Assume w.l.o.g. that r_0 is irrational. Since $D^{2m}u = T(Lu) = TF$, where T is a Calderon–Zygmund singular integral operator, we have $D^{2m}u \in L^{r_0}_{\mathrm{loc}}$. By definition, $D^{2m-1}u \in W^{1,r_0}_{\mathrm{loc}}$; Sobolev's embedding gives $D^{2m-1}u \in L^{q_0}_{\mathrm{loc}}$, where $\frac{1}{q_0} = \frac{1}{r_0}(1 - \frac{r_0}{n})$. Since $u \in L^\infty$, we may apply standard Gagliardo–Nirenberg inequalities to obtain

$$D^j u \in L^{(2m-1)q_0/j}_{\mathrm{loc}}, \quad j = 0, 1, 2, \dots, 2m-1.$$

By Hölder inequality, this means that whenever $|\lambda_1| + |\lambda_2| + |\lambda_3| = 2m = n$, we have

$$D^{\lambda_1} u^k D^{\lambda_2} u^k D^{\lambda_3} u^j \in L^{r_1}_{\mathrm{loc}},$$

where

$$\frac{1}{r_1} = \frac{1}{q_0} \frac{|\lambda_1| + |\lambda_2| + |\lambda_3|}{2m-1} = \frac{1}{q_0} \frac{n}{n-1} = \frac{1}{r_0} \frac{n - r_0}{n-1}.$$

Iterating this procedure, we prove that

$$D^{2m-1}u \in L^{q_j}_{\mathrm{loc}}, \qquad Lu = F \in L^{r_{j+1}}_{\mathrm{loc}}, \qquad D^{2m}u \in L^{r_{j+1}}_{\mathrm{loc}},$$

where $r_0 > 1$ is fixed above, the increasing sequences r_j, q_j are defined by

$$r_{j+1} := r_j \frac{n-1}{n - r_j}, \quad q_j := r_j \frac{n}{n - r_j} \quad \text{for } j = 0, 1, 2, \dots, M,$$

and M is chosen so that $r_M < n < r_{M+1}$. Thus, by Morrey–Sobolev embeddings theorem, we obtain $D^{2m-1}u \in C^\alpha_{\mathrm{loc}}$ for $\alpha = 1 - n/r_{M+1} > 0$. All derivatives of u of lower orders are continuous, too.

Thus, $Lu = F \in C^\alpha_{\mathrm{loc}}$, and smoothness of u follows from Schauder theory.

6. A generalization to other elliptic systems

In this section, we briefly sketch the proof of Theorem 1.4, leaving numerous technical details to the interested reader.

The first step is to prove that the conclusion of Lemma 4.1 holds also for solutions of (1.4). We fix $B(a,r) \equiv B_r$, $r < \frac{1}{20}\operatorname{dist}(a, \partial\mathbb{B}^n)$ and test (1.4) with

$$\psi := \zeta\big(u - T^{m-1}_{B_{2r}} u\big)$$

where ζ denotes, as in Section 4, a nonnegative cutoff function of class $C^\infty_0(B_{2r})$, with $\zeta \equiv 1$ on B_r and $|\nabla\zeta| \lesssim r^{-1}$.

Estimates of critical nonlinearity, part I. Since u is bounded by assumption, we have

$$\|\psi\|_{L^\infty} \leqslant 2\|u\|_{L^\infty} + C \sum_{k=1}^{m-1} r^k \fint_{B_{2r}} \big|D^k u\big|\, dx \leqslant 2\|u\|_{L^\infty} + C \sum_{k=1}^{m-1} \bigg(\int_{B_{2r}} \big|D^k u\big|^{n/k}\, dx \bigg)^{k/n} \tag{6.1}$$

by Hölder's inequality. The last expression does not exceed $3\|u\|_{L^\infty}$ when r is sufficiently small. Thus, one can estimate the integral

$$\int |\psi|\big|F(u, Du, \dots, D^m u)\big|\,dx,$$

using (1.8) and mimicking the arguments used in Step 3 of the proof in Section 4.

Estimates of critical nonlinearity, part II. To estimate the term containing $\sum_\alpha E_\alpha D^\alpha$, we apply the following lemma.

Lemma 6.1. *Assume that $n = 2m \geqslant 2$, $\varrho > 0$ and $a \in \mathbb{R}^n$. If $u \in W^{m,2}(B(a, 10\varrho))$, and if $E = (E_\alpha)_{|\alpha|=m} \in L^2(B(a, 10\varrho))$ satisfies the cancellation condition $\nabla^m \cdot E = 0$, i.e.*

$$\sum_{|\alpha|=m} \int E_\alpha D^\alpha \varphi\,dx = 0 \quad \textit{for all } \varphi \in C_0^\infty\big(B(a, 10\varrho)\big),$$

then there exists a constant $C = C(n)$ such that for all functions $\psi \in W_0^{m,2}(B(a, r))$ we have

$$\left| \int_{B(a,\varrho)} \psi \sum_{|\alpha|=k} E_\alpha D^\alpha u\,dx \right| \leqslant C \big\|D^m \psi\big\|_{L^2} \|E\|_{L^2} \big\|D^m u\big\|_{L^2}; \tag{6.2}$$

the norm of $D^m\psi$ is taken on the smaller ball $B(a, \varrho)$, and two other norms, of E and $D^m u$, on the larger ball $B(a, 10\varrho)$.

The proof of this lemma is very similar to the proof of [24, Theorem 2.2]. One can obtain it, mimicking Step 4 of the proof of Lemma 4.1 given in Section 4.

Thus, on sufficiently small balls with radius $0 < \varrho < \varrho_0 = \varrho_0(\varepsilon, E)$, we have

$$\left| \int_{B_\varrho} \psi \sum_{|\alpha|=k} E_\alpha D^\alpha u\,dx \right| \leqslant \frac{\varepsilon}{C} \big\|D^m \psi\big\|_{L^2(B_\varrho)} \big\|D^m u\big\|_{L^2(B_{10\varrho})} \leqslant \varepsilon \int_{B_{10\varrho}} \big|D^m u\big|\,dx \tag{6.3}$$

(the first inequality follows from Lemma 6.1 and the absolute continuity of the integral, the second one from Poincaré's inequality). We use this estimate for $\varrho = 2r$.

These two observations allow one to estimate the right-hand side of (1.4), i.e.,

$$\int |\psi|\big|F\big(u, Du, \dots, D^m u\big)\big|\,dx + \left| \int \psi \sum_{|\alpha|=k} E_\alpha D^\alpha u\,dx \right|, \tag{6.4}$$

by

$$C\varepsilon \int_{B_{20r}} \big|D^m u\big|^2\,dx + \text{lower order terms} \tag{6.5}$$

(when r is sufficiently small).

The rest of the proof is standard: we combine the inequality (6.4) $\leqslant$ (6.5) with routine estimates of the left-hand side $\langle Lu, \psi\rangle$ to obtain a reverse Hölder's inequality, which implies continuity of u and higher integrability of its derivatives. Smoothness of u follows next from a bootstrap argument, similar to the one presented in Section 5. All details are left to the reader.

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