Peano type theorem for abstract parabolic equations [☆]

Oleg Zubelevich [1]

*Department of Differential Equations and Mathematical Physics, Peoples Friendship University of Russia,
Ordzhonikidze st., 3, 117198, Moscow, Russia*

Abstract

We consider parabolic problems with non-Lipschitz nonlinearity in the different scales of Banach spaces and prove local-in-time existence theorem.
© 2009 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

MSC: 35K90; 35R05; 35R10

Keywords: Peano theorem; Abstract Cauchy problem; Nonlocal problems; Functional-differential equations; Integro-differential equations;
Quasilinear parabolic equations

1. Introduction

This paper is devoted to quasi-linear parabolic equations with a non-Lipschitz nonlinearity. In the classical setup a quasi-linear initial value parabolic problem has the form

$$u_t = f(t, u, \nabla^k u) + Au, \quad u|_{t=0} = \hat{u}. \tag{1.1}$$

Here A is a linear elliptic operator of order n and the term $\nabla^k u$ symbolizes the derivatives of u up to order k. Besides this, Eq. (1.1) must be provided with the boundary conditions.

If the function $\hat{u}$ belongs to a suitable space, the mapping f is Lipschitz in a certain sense and $k < n$ then problem (1.1) has a unique local-in-time solution. This simple observation easily follows from the contracting mapping principle.

We consider the case when the function f is non-Lipschitz. It is well known that in general situation, in infinitely dimensional Banach space, an initial value problem for differential equation with non-Lipschitz right-hand side does not have solutions [8,18,14]. Nevertheless, as a rule, the initial value problem lives not in a single Banach space but in a scale of Banach spaces and in addition this scale is completely continuous embedded. Such scales for example are the scale of Sobolev spaces, the scale of analytic functions. This observation prompts that to find a solution one should study the problem in the whole scale.

The main mathematical tool we use is a locally convex space version of the Schauder fixed point theorem and theory of scales of Banach spaces. Another approaches to the abstract parabolic problems in the Lipschitz setup contain in [5]. Non-Lipschitz setup has been considered in [7]. There is some extra hypothesis of partial order in the Banach space employed in this article.

2. Main theorem

Consider two scales of Banach spaces $\{E_s, \|\cdot\|_s^E\}_{s>0}$ and $\{G_s, \|\cdot\|_s^G\}_{s>0}$ such that $E_s \subseteq G_s$ for all $s > 0$. All the embeddings $E_{s+\delta} \subseteq E_s$, $\delta > 0$ are completely continuous and

$$\|\cdot\|_s^E \leqslant \|\cdot\|_{s+\delta}^E. \tag{2.1}$$

The parameter s may not necessarily be ran through all the positive real numbers. We do not use the spaces E_s, G_s with big s and one can assume for example that $s \in (0, 1)$. It is just for simplicity's sake that we consider $s > 0$.

Introduce constants $C, T, R > 0, \phi, \alpha \geqslant 0$.

Let $S^t : G_s \to E_s$, $t > 0$ be a strongly continuous linear semigroup in the following sense. For any $u \in E_s$ one has

$$\|S^t u - u\|_s^E \to 0 \quad \text{as } t \searrow 0 \quad \text{and} \quad \|S^t u\|_s^E \leqslant C\|u\|_s^E.$$

Definition 1. The semigroup S^t is said to be parabolic if there exists a constant $\gamma > 1$ such that for any $\delta, t > 0$, $\delta^\gamma < t < T$ we have

$$\|S^t u\|_{s+\delta}^E \leqslant \frac{C}{t^\phi}\|u\|_s^G. \tag{2.2}$$

Let $B_s(r)$ be an open ball of the space E_s with radius r and center at the origin. Suppose a function $f : (0, T] \times \overline{B}_{s+\delta}(R) \to G_s$ to be continuous and such that if $(s + \delta)^\gamma < t \leqslant T$ and $u \in \overline{B}_{s+\delta}(R)$ then the following inequality holds

$$\|f(t, u)\|_s^G \leqslant \frac{C}{\delta^\alpha}. \tag{2.3}$$

Remark 1. A case when

$$\|f(t, u)\|_s^G \leqslant \frac{C}{t^\beta \delta^\alpha}, \quad \beta > 0,$$

is rather usual but since $\delta^\gamma < t$ this case reduces to (2.3): $C/(t^\beta \delta^\alpha) \leqslant C/\delta^{\beta\gamma+\alpha}$.

We proceed with two setups of our problem. The first one is a classical setup and we find classical solutions and the second one is a generalized setup to obtain generalized solutions.

In the generalized setup we are looking for solutions to the following integral equation

$$u(t) = \int_0^t S^{t-\xi} f(\xi, u(\xi)) \, d\xi. \tag{2.4}$$

In the classical setup we make several additional assumptions. Namely, suppose that $G_s = E_s$. Introduce a linear operator $A : E_{s+\delta} \to E_s$ and assume that the semigroup S^t is generated by this operator: $S^t = e^{At}$ such that for any $u \in E_{s+\delta}$ we have

$$\lim_{h \to 0+} \left\|\frac{1}{h}(e^{Ah} - \text{id}_{E_{s+\delta}})u - Au\right\|_s^E = 0. \tag{2.5}$$

In the classical setup our problem has the form

$$u_t = f(t, u) + Au, \tag{2.6}$$

$$u|_{t=0} = 0. \tag{2.7}$$

The sense of initial condition (2.7) will be clear in the sequel.

Now we give a definition.

Definition 2. We shall say that problem (2.6) or (2.4) is parabolic if the semigroup S^t is parabolic and

$$\chi = \phi + \frac{\alpha}{\gamma} < 1.$$

In case of Remark 1 $\chi = \phi + \beta + \alpha/\gamma$.

Let a space $E^1(T)$, $T > 0$, be given by the formula

$$E^1(T) = \bigcap_{0 < s^\gamma < \tau < T} C^1\big((\tau, T), E_s\big). \tag{2.8}$$

This space consists of all functions u that map any number $t \in (0, T)$ to the element $u(t) \in \bigcap_{0 < s^\gamma < t} E_s$ and the restriction $u|_{(\tau, T)}$ belongs to the space $C^1((\tau, T), E_s)$ for all $s \in (0, \tau^{1/\gamma})$.

Theorem 1.

1. *Classical setup. Suppose that problem* (2.6) *is parabolic. Then there exists a constant $T_* > 0$ such that this problem has a solution $u(t) \in E^1(T_*)$, and for any constant $c \in (0, 1)$ one has*

$$\big\| u(t) \big\|_{ct^{1/\gamma}}^E \to 0 \quad as \ t \searrow 0. \tag{2.9}$$

   The function $u(t)$ also solves Eq. (2.4).
2. *Generalized setup. Suppose that problem* (2.4) *is parabolic. Then there exists a constant $T_* > 0$ such that this problem has a solution*

$$u(t) \in E(T_*) = \bigcap_{0 < s^\gamma < \tau < T_*} C\big((\tau, T_*), E_s\big).$$

   In both cases the constants T_* depends only on C, α, γ, ϕ.

The proof of Theorem 1 contains in Sections 3, 4.

Then we apply our theorem to some functional differential equation. To compare our result with the known one we also consider the Navier–Stokes equation.

If A is the classical Laplace operator and the parabolic equation is considered in a suitable domain then $\gamma = 2$ and the inequality from formula (2.8) takes the form $0 < s^2 < \tau$.

The parameter s symbolizes a spatial variable, so that this inequality specifies the parabolic domain in the plane (τ, s). This endows the term "parabolic equation" with the new sense.

Let us remark that if $G_s = E_s = \mathbb{R}^m$, $\| \cdot \|_s^E = | \cdot |$, $s > 0$ and $A = 0$ then Theorem 1 generalizes classical Peano's theorem to the case when the right side of the equation satisfies (2.3) with $s = \delta = (t/3)^{1/\gamma}$, $\alpha = 0$.

3. Preliminaries on functional analysis

In this section we collect several facts from functional analysis. These facts will be useful in Section 4 when we prove Theorem 1.

Consider the spaces

$$C\big([\tau, T], E_{\mu\tau^{1/\gamma}}\big), \quad 0 < \mu < 1, \ 0 < \tau < T,$$

with standard norms. Now we construct the projective limit of these spaces. Define a space $E(T)$ as follows

$$E(T) = \bigcap_{0 < \mu < 1} \bigcap_{0 < \tau < T} C\big([\tau, T], E_{\mu\tau^{1/\gamma}}\big).$$

There is another equivalent definition of the space $E(T)$:

$$E(T) = \bigcap_{0 < s^\gamma < \tau < T} C\big([\tau, T], E_s\big).$$

Being endowed with a collection of seminorms

$$\|u\|_{\tau,\mu} = \max_{\tau \leqslant \xi \leqslant T} \|u(\xi)\|_{\mu\tau^{1/\gamma}}^{E}, \quad u \in E(T), \tag{3.1}$$

the space $E(T)$ becomes a locally convex topological space.

These seminorms obviously satisfy the following inequalities

$$\|u\|_{\tau,\mu} \leqslant \|u\|_{\tau,\mu+\delta}, \quad \delta > 0, \tag{3.2}$$

$$\|u\|_{\tau,r\mu} \leqslant \|u\|_{r^{\gamma}\tau,\mu}, \quad 0 < r \leqslant 1. \tag{3.3}$$

Indeed, formula (3.2) follows from (2.1) directly. Formula (3.3) is a result of the estimate

$$\|u\|_{\tau,r\mu} = \max_{\tau \leqslant \xi \leqslant T} \|u(\xi)\|_{\mu(r^{\gamma}\tau)^{1/\gamma}}^{E} \leqslant \max_{r^{\gamma}\tau \leqslant \xi \leqslant T} \|u(\xi)\|_{\mu(r^{\gamma}\tau)^{1/\gamma}}^{E} = \|u\|_{r^{\gamma}\tau,\mu}.$$

Formulas (3.2), (3.3) imply that the space $E(T)$ is first countable: the topology of this space can be defined by the seminorms (3.1) only with $\mu, \tau \in \mathbb{Q}$.

Recall the Arzela–Ascoli theorem [17]:

Theorem 2. *Let $H \subset C([0, T], X)$ be a set in the space of continuous functions with values in a Banach space X. Assume that the set H is closed, bounded, uniformly continuous and for every $t \in [0, T]$ the set $\{u(t) \in X\}$ is a compact set in the space X. Then the set H is a compact set in the space $C([0, T], X)$.*

Now we shall establish an analogue of this result.

Proposition 1. *Suppose that a set $K \subset E(T)$ is closed. Then K is a compact set if the following two conditions are fulfilled.*

The set K is bounded.

For any $\varepsilon > 0$ and for any $\tau \in (0, T)$, $\mu \in (0, 1)$ there exists a constant $\delta > 0$ such that if $t', t'' \in [\tau, T]$, $|t' - t''| < \delta$ then

$$\sup_{u \in K} \|u(t') - u(t'')\|_{\mu\tau^{1/\gamma}}^{E} < \varepsilon.$$

(This means that K is a uniformly continuous set.)

First prove a lemma.

Lemma 1. *Let $\{v_j\} \subseteq K$ be a sequence. Then for any $\tau \in (0, T)$ the sequence $\{v_j\}$ contains a subsequence that is convergent in all the norms $\|\cdot\|_{\tau,\mu}$, $\mu \in (0, 1)$.*

Proof. Indeed, take an increasing sequence $\mu_k \to 1$, $\mu_1 > 0$ and fix any value of $\tau \in (0, T)$. Since the sequence $\{v_j\}$ is bounded and uniformly continuous in $C([\tau, T], E_{\mu_2\tau^{1/\gamma}})$ then by Theorem 2 it contains a subsequence $\{v_j^1\}$ that is convergent in $C([\tau, T], E_{\mu_1\tau^{1/\gamma}})$.

Further since the sequence $\{v_j^1\}$ is bounded and uniformly continuous in $C([\tau, T], E_{\mu_3\tau^{1/\gamma}})$ one can pick a subsequence $\{v_j^2\} \subseteq \{v_j^1\}$ such that the sequence $\{v_j^2\}$ is convergent in $C([\tau, T], E_{\mu_2\tau^{1/\gamma}})$, etc.

By inequality (3.2) the diagonal sequence $\{v_j^j\}$ converges in all the norms $\|\cdot\|_{\tau,\mu}$, $\mu \in (0, 1)$ with this fixed τ. $\quad\square$

Proof of Proposition 1. A set $P = \mathbb{Q} \cap (0, T)$ is countable. So we can number its elements as follows $P = \{\tau_i\}_{i \in \mathbb{N}}$.

We must show that any sequence $\{u_j\} \subseteq K$ contains a convergent subsequence $\{u_{j_k}\}$.

By Lemma 1 there is a subsequence $\{u_j^1\} \subseteq \{u_j\}$ that is convergent in all the norms $\|\cdot\|_{\tau_1,\mu}$ $\mu \in (0, 1)$. By the same argument there is a subsequence $\{u_j^2\} \subseteq \{u_j^1\}$ that is convergent in all the norms $\|\cdot\|_{\tau_2,\mu}$ $\mu \in (0, 1)$, etc.

The diagonal sequence $\{u_j^j\}$ is convergent in all the norms $\|\cdot\|_{\tau_k,\mu}$, $k \in \mathbb{N}$, $\mu \in (0, 1)$.

By inequality (3.3) the sequence $\{u_j^j\}$ is convergent in all the norms $\|\cdot\|_{\tau,\mu}$, $\tau \in (0, T)$, $\mu \in (0, 1)$.

Proposition 1 is proved. $\quad\square$

Lemma 2. *Let X, Y be Banach spaces. Suppose that $A_a : X \to Y$, $a' > a > 0$ is a collection of bounded linear operators such that for each $x \in X$ we have*

$$\sup_{a' > a > 0} \|A_a x\|_Y < \infty, \qquad \|A_a x\|_Y \to 0 \quad as \ a \to 0.$$

Then for any compact set $B \subset X$ it follows that

$$\sup_{x \in B} \|A_a x\|_Y \to 0 \quad as \ a \to 0.$$

This result is a direct consequence of the Banach–Steinhaus theorem [17,9].

Let us recall a generalized version of the Schauder fixed point theorem.

Theorem 3. *(See [6].) Let W be a closed convex subset of the locally convex space E. Then a compact continuous mapping $f : W \to W$ has a fixed point $\hat{u}$ i.e. $f(\hat{u}) = \hat{u}$.*

4. Proof of Theorem 1

By definition put

$$W(T_*) = \left\{ u \in E(T_*) \mid \|u\|_{\tau,\nu} \leqslant R, \ 0 < \tau < T_*, \ 0 < \nu < 1 \right\}.$$

The constant $T_* > 0$ will be defined.

First we find a fixed point of a mapping

$$F(u) = \int_0^t S^{t-\xi} f\big(\xi, u(\xi)\big) \, d\xi.$$

This fixed point is the generalized solution announced in the second part of the theorem. Then by using formula (2.5) we show that this fixed point is the desired solution to problem (2.6).

Lemma 3. *If the constant T_* is small enough then the mapping F takes the set $W(T_*)$ to itself.*

Proof. Let constants t, s be taken as follows $0 < s < t^{1/\gamma}$, $t \leqslant T_*$. Suppose $u \in W(T_*)$ then estimate a function $v(t) = F(u)$:

$$\big\| v(t) \big\|_s^E \leqslant \int_0^t \big\| S^{t-\xi} f\big(\xi, u(\xi)\big) \big\|_s^E \, d\xi = X + Y, \tag{4.1}$$

here we use the notation

$$X = \int_0^{t-s^\gamma} \big\| S^{t-\xi} f\big(\xi, u(\xi)\big) \big\|_s^E \, d\xi, \qquad Y = \int_{t-s^\gamma}^t \big\| S^{t-\xi} f\big(\xi, u(\xi)\big) \big\|_s^E \, d\xi.$$

To estimate X take constants ε and μ such that

$$0 < \varepsilon < \frac{s}{t^{1/\gamma}} < \mu < 1. \tag{4.2}$$

The constant ε is assumed to be small and the constant μ is assumed to be close to 1.

Let the variables δ and δ' be given by the formulas

$$\delta = s - \varepsilon \xi^{1/\gamma}, \qquad \delta' = \xi^{1/\gamma}(\mu - \varepsilon).$$

Taking into account that $\xi \in (0, t - s^\gamma]$ we see that the variables δ, δ' are positive and

$$s - \delta > 0, \quad s - \delta + \delta' < \xi^{1/\gamma}, \quad \delta < (t - \xi)^{1/\gamma}. \tag{4.3}$$

The inequality in the middle implies that

$$u(\xi) \in \overline{B}_{s-\delta+\delta'}(R) \tag{4.4}$$

and thus the term X is estimated as follows

$$X \leqslant C \int_0^{t-s^\gamma} (t-\xi)^{-\phi} \|f(\xi, u(\xi))\|_{s-\delta}^G d\xi \leqslant C^2 \int_0^{t-s^\gamma} \frac{1}{\delta'^\alpha (t-\xi)^\phi} d\xi$$

$$\leqslant \frac{C^2}{(\mu-\varepsilon)^\alpha} \int_0^{t-s^\gamma} \frac{d\xi}{(t-\xi)^\phi \xi^{\alpha/\gamma}} \bigg|_{\xi=yt} = \frac{C^2 t^{1-\chi}}{(\mu-\varepsilon)^\alpha} \int_0^{1-s^\gamma/t} \frac{dy}{(1-y)^\phi y^{\alpha/\gamma}}$$

$$\leqslant \frac{C^2 J t^{1-\chi}}{(\mu-\varepsilon)^\alpha}, \qquad J = \int_0^1 \frac{dy}{(1-y)^\phi y^{\alpha/\gamma}}. \tag{4.5}$$

We shall estimate the term Y.

Introduce a function ψ by the formula

$$\psi(y) = y^{1/\gamma} + (1-y)^{1/\gamma} - 1.$$

The function ψ is positive on the interval $(0, 1)$. Define a constant I as follows

$$I = \int_0^1 \frac{dy}{(1-y)^\phi (\psi(y))^\alpha}.$$

Let the constant μ be as above. We redefine the variables δ, δ' by the formulas

$$\delta = \mu(t-\xi)^{1/\gamma}, \qquad \delta' = \mu\xi^{1/\gamma} + \delta - s.$$

Now the variable ξ belongs to the interval $[t-s^\gamma, t]$ and thus the variables δ, δ' are positive and satisfy inequalities (4.3).

It is only not trivial to show that the variable δ' is positive. Let us prove this. Indeed,

$$\delta' = \mu\xi^{1/\gamma} + \mu(t-\xi)^{1/\gamma} - s = t^{1/\gamma}\left(\mu y^{1/\gamma} + \mu(1-y)^{1/\gamma} - \frac{s}{t^{1/\gamma}}\right), \tag{4.6}$$

recall that $y = \xi/t$. From (4.6) it follows that

$$\delta' > t^{1/\gamma} \mu \psi(y). \tag{4.7}$$

By the same argument as above, inclusion (4.4) is fulfilled with the new δ and δ'.

We are ready to estimate the term Y. By (4.7) it follows that

$$Y \leqslant C \int_{t-s^\gamma}^t (t-\xi)^{-\phi} \|f(\xi, u(\xi))\|_{s-\delta}^G d\xi \leqslant C^2 \int_{t-s^\gamma}^t \frac{d\xi}{(t-\xi)^\phi \delta'^\alpha}$$

$$\leqslant \frac{C^2 t^{1-\chi}}{\mu^\alpha} \int_{1-s^\gamma/t}^1 \frac{dy}{(1-y)^\phi (\psi(y))^\alpha} \leqslant \frac{C^2 I}{\mu^\alpha} t^{1-\chi}. \tag{4.8}$$

Now the assertion of the lemma follows from formulas (4.1), (4.5) and (4.8). $\quad\square$

Corollary 1. *Formulas* (4.5), (4.8) *imply that if* $0 < s^\gamma < t \leqslant T_*$ *and* $v(t) = F(u)$, $u \in W(T_*)$ *then*

$$\|v(t)\|_s^E \leqslant c_2 t^{1-\chi},$$

here c_2 *is a positive constant independent on* u, t, s.

Lemma 4. *The set $F(W(T_*))$ is precompact in $E(T_*)$.*

Proof. By Proposition 1 it is sufficient to prove that the set $F(W(T_*))$ is uniformly continuous.

Take a function $u \in W(T_*)$ and let $v(t) = F(u)$. We must show that if $t', t'' \geqslant \tau$, $\tau \in (0, T_*)$ then for any $\mu \in (0, 1)$ one has

$$\sup_{u \in W(T_*)} \| v(t') - v(t'') \|_{\mu \tau^{1/\gamma}}^E \to 0, \quad \text{as } |t' - t''| \to 0.$$

Indeed, for definiteness assume that $t'' > t'$ then

$$v(t'') - v(t') = \int_{t'}^{t''} S^{t''-\xi} f(\xi, u) \, d\xi + \left(S^{t''-t'} - \mathrm{id}_{E_s} \right) \int_0^{t'} S^{t'-\xi} f(\xi, u) \, d\xi, \quad s^\gamma < \tau. \tag{4.9}$$

Choose a positive constant δ such that $(s + \delta)^\gamma < \tau$ and using the parabolicity of the semigroup S^t estimate the first term from the right side of this formula

$$\left\| \int_{t'}^{t''} S^{t''-\xi} f(\xi, u) \, d\xi \right\|_s^E \leqslant C \int_{t'}^{t''} (t'' - \xi)^{-\phi} \| f(\xi, u) \|_{s-\delta}^G \, d\xi$$

$$\leqslant C^2 \int_{t'}^{t''} \frac{d\xi}{\delta^\alpha (t'' - \xi)^\phi} = \frac{C^2}{\delta^\alpha (1 - \phi)} (t'' - t')^{1-\phi}.$$

So that the first term in the right side of (4.9) is vanished uniformly.

Consider a set

$$U = \bigcup_{\tau \leqslant t' \leqslant T_*} \left\{ \int_0^{t'} S^{t'-\xi} f(\xi, u) \, d\xi \, \Big| \, u \in W(T_*) \right\}.$$

By Lemma 3 the set U is bounded in any space $E_{\mu' \tau^{1/\gamma}}$ with $1 > \mu' > \mu$ thus it is compact in $E_{\mu \tau^{1/\gamma}}$. By Lemma 2 we get

$$\sup_{w \in U} \| S^{t''-t'} w - w \|_{\mu \tau^{1/\gamma}}^E \to 0, \quad \text{as } t'' - t' \to 0.$$

This shows that the second term in the right side of formula (4.9) is vanished uniformly. $\square$

Corollary 2. *The set $F(W(T_*))$ is uniformly continuous with respect to the variable t.*

Lemma 5. *The mapping $F : W(T_*) \to W(T_*)$ is continuous with respect to the topology of the space $E(T_*)$.*

Proof. Suppose a sequence $\{v_l\} \subset W(T_*)$ to be convergent to the element $v \in W(T_*)$ as $l \to \infty$. We need to show that for any $s^\gamma < \tau < T_*$ the sequence

$$\sup_{\tau \leqslant t \leqslant T_*} \left\| \int_0^t S^{t-\xi} f\big(\xi, v_l(\xi)\big) \, d\xi - \int_0^t S^{t-\xi} f\big(\xi, v(\xi)\big) \, d\xi \right\|_s^E$$

vanishes as $l \to \infty$.

By Corollary 2 the sequence

$$\left\{ \int_0^t S^{t-\xi} f\big(\xi, v_l(\xi)\big) \, d\xi \right\} \tag{4.10}$$

is uniformly continuous on the interval $[\tau, T_*]$. The uniform convergence of such a sequence is equivalent to its pointwise convergence [17]. Thus it is sufficient to prove that sequence (4.10) is convergent in E_s for each $t \in [\tau, T_*]$.

Fix $t \in [\tau, T_*]$ and let constants ε, μ satisfy inequality (4.2). Then using the argument of Lemma 3 write

$$\left\| \int_0^t S^{t-\xi}\big(f\big(\xi, v_l(\xi)\big) - f\big(\xi, v(\xi)\big)\big)\, d\xi \right\|_s^E \leqslant C \int_0^{t-s^\gamma} (t-\xi)^{-\phi} \big\| f\big(\xi, v_l(\xi)\big) - f\big(\xi, v(\xi)\big)\big\|_{\varepsilon\xi^{1/\gamma}}^G \, d\xi$$

$$+ C \int_{t-s^\gamma}^t (t-\xi)^{-\phi} \big\| f\big(\xi, v_l(\xi)\big) - f\big(\xi, v(\xi)\big)\big\|_{s-\mu(t-\xi)^{1/\gamma}}^G \, d\xi. \tag{4.11}$$

Since the function f is continuous, for a fixed ξ we have:

$$(t-\xi)^{-\phi} \big\| f\big(\xi, v_l(\xi)\big) - f\big(\xi, v(\xi)\big)\big\|_{\varepsilon\xi^{1/\gamma}}^G \to 0, \quad \xi \in \big[0, t-s^\gamma\big],$$

$$(t-\xi)^{-\phi} \big\| f\big(\xi, v_l(\xi)\big) - f\big(\xi, v(\xi)\big)\big\|_{s-\mu(t-\xi)^{1/\gamma}}^G \to 0, \quad \xi \in \big[t-s^\gamma, t\big),$$

as $l \to \infty$.

Moreover by formulas (4.5), (4.8) both of these expressions are majorized with the L^1-integrable function:

$$(t-\xi)^{-\phi} \big\| f\big(\xi, v_l(\xi)\big) - f\big(\xi, v(\xi)\big)\big\|_{\varepsilon\xi^{1/\gamma}}^G \leqslant (t-\xi)^{-\phi}\big(\big\| f\big(\xi, v_l(\xi)\big)\big\|_{\varepsilon\xi^{1/\gamma}}^G + \big\| f\big(\xi, v(\xi)\big)\big\|_{\varepsilon\xi^{1/\gamma}}^G\big)$$

$$\leqslant \frac{2C^2}{(\mu-\varepsilon)^\alpha \xi^{\alpha/\gamma}(t-\xi)^\phi},$$

and

$$(t-\xi)^{-\phi} \big\| f\big(\xi, v_l(\xi)\big) - f\big(\xi, v(\xi)\big)\big\|_{s-\mu(t-\xi)^{1/\gamma}}^G \leqslant \frac{2C^2}{t^{\alpha/\gamma}\mu^\alpha(\psi(\xi/t))^\alpha(t-\xi)^\phi}.$$

Therefore by the dominated convergence theorem the integrals in the right side of (4.11) are vanished as $l \to \infty$. □

So by Theorem 3 and Lemmas 3–5 we obtain a fixed point of the mapping F, say u:

$$F(u) = u \in W(T_*).$$

This proves the second part of Theorem 1.

To prove the first one let us show that this fixed point is the solution to problem (2.6). Suppose that $t, t + h > s^\gamma$. First consider the case $h > 0$. Differentiate the function $u(t)$ explicitly:

$$u_t(t) = \lim_{h\to 0} h^{-1}\left(\int_0^{t+h} e^{A(t+h-\xi)} f\big(\xi, u(\xi)\big)\, d\xi - \int_0^t e^{A(t-\xi)} f\big(\xi, u(\xi)\big)\, d\xi \right)$$

$$= \lim_{h\to 0} h^{-1} \int_t^{t+h} e^{A(t+h-\xi)} f\big(\xi, u(\xi)\big)\, d\xi + \lim_{h\to 0} h^{-1}\big(e^{Ah} - \mathrm{id}_{E_s}\big) \int_0^t e^{A(t-\xi)} f\big(\xi, u(\xi)\big)\, d\xi. \tag{4.12}$$

Lemma 3 implies that $\int_0^t e^{A(t-\xi)} f\big(\xi, u(\xi)\big)\, d\xi \in E_{s'}$ with $s^\gamma < s'^\gamma < t, t + h$ hence formula (2.5) gives

$$h^{-1}\big(e^{Ah} - \mathrm{id}_{E_s}\big) \int_0^t e^{A(t-\xi)} f\big(\xi, u(\xi)\big)\, d\xi \to A \int_0^t e^{A(t-\xi)} f\big(\xi, u(\xi)\big)\, d\xi \tag{4.13}$$

in E_s as $h \to 0$.

Let us prove that

$$h^{-1} \int_t^{t+h} e^{A(t+h-\xi)} f\big(\xi, u(\xi)\big)\, d\xi \to f\big(t, u(t)\big) \tag{4.14}$$

in E_s as $h \to 0$.

Indeed, observe that

$$h^{-1} \int_t^{t+h} e^{A(t+h-\xi)} f\big(\xi, u(\xi)\big) \, d\xi - f\big(t, u(t)\big)$$

$$= h^{-1} \left(\int_t^{t+h} e^{A(t+h-\xi)} \big(f\big(\xi, u(\xi)\big) - f\big(t, u(t)\big)\big) \, d\xi + \int_t^{t+h} \big(e^{A(t+h-\xi)} - \mathrm{id}_{E_s}\big) f\big(t, u(t)\big) \, d\xi \right).$$

The first integral in the right side of this formula is estimated as follows:

$$\left\| \int_t^{t+h} e^{A(t+h-\xi)} \big(f\big(\xi, u(\xi)\big) - f\big(t, u(t)\big)\big) \, d\xi \right\|_s^E \leqslant Ch \max_{t \leqslant \xi \leqslant t+h} \big\| f\big(\xi, u(\xi)\big) - f\big(t, u(t)\big) \big\|_s^E = \mathrm{o}(h).$$

Since the semigroup e^{At} is strongly continuous for the second integral we get

$$\left\| \int_t^{t+h} \big(e^{A(t+h-\xi)} - \mathrm{id}_{E_s}\big) f\big(t, u(t)\big) \, d\xi \right\|_s^E \leqslant h \max_{t \leqslant \xi \leqslant t+h} \big\| \big(e^{A(t+h-\xi)} - \mathrm{id}_{E_s}\big) f\big(t, u(t)\big) \big\|_s^E = \mathrm{o}(h).$$

If $h < 0$ then instead of formula (4.12) one must use the following expression

$$u_t(t) = \lim_{h \to 0} h^{-1} \left(\big(\mathrm{id}_{E_s} - e^{-Ah}\big) \int_0^{t+h} e^{A(t+h-\xi)} f\big(\xi, u(\xi)\big) \, d\xi - \int_{t+h}^t e^{A(t-\xi)} f\big(\xi, u(\xi)\big) \, d\xi \right).$$

In this case only the proof of the formula

$$\lim_{h \to 0} h^{-1} \big(\mathrm{id}_{E_s} - e^{-Ah}\big) \int_0^{t+h} e^{A(t+h-\xi)} f\big(\xi, u(\xi)\big) \, d\xi = A \int_0^t e^{(t-\xi)} f\big(\xi, u(\xi)\big) \, d\xi$$

differs from the previous argument.

Let us prove this formula. Obviously we have

$$\big(\mathrm{id}_{E_s} - e^{-Ah}\big) \int_0^{t+h} e^{A(t+h-\xi)} f\big(\xi, u(\xi)\big) \, d\xi = \big(\mathrm{id}_{E_s} - e^{-Ah}\big) u(t) + \big(\mathrm{id}_{E_s} - e^{-Ah}\big) \big(u(t+h) - u(t)\big). \tag{4.15}$$

The set

$$V = \left\{ \frac{u(t+h) - u(t)}{\|u(t+h) - u(t)\|_{s'}^E} \,\Big|\, h \in (h', 0) \right\}$$

with $h' < 0$ close to zero is bounded in $E_{s'}$, $s^\gamma < s'^\gamma < t + h'$. Consequently V is a compact set in E_s. By Lemma 2 the set

$$(A_{-h} - A)V, \qquad A_{-h} = \frac{1}{h}\big(\mathrm{id}_{E_s} - e^{-Ah}\big)$$

is bounded in E_s and thus the set $A_{-h}V$ is also bounded.

Thus taking into account that the function $u(t)$ is continuous we yield

$$\left\| \frac{1}{h}\big(\mathrm{id}_{E_s} - e^{-Ah}\big)\big(u(t+h) - u(t)\big) \right\|_s^E = \big\| u(t+h) - u(t) \big\|_{s'}^E \cdot \left\| A_{-h} \frac{u(t+h) - u(t)}{\|u(t+h) - u(t)\|_{s'}^E} \right\|_s^E = \mathrm{o}(1).$$

For the second term of the right side of (4.15) this implies

$$\big\| \big(\mathrm{id}_{E_s} - e^{-Ah}\big)\big(u(t+h) - u(t)\big) \big\|_s^E = \mathrm{o}(h).$$

The first term of the right side of formula (4.15) is estimated as follows

$$\left\| \left(\mathrm{id}_{E_s} - e^{-Ah} \right) u(t) - h A u(t) \right\|_s^E = \mathrm{o}(h).$$

Substituting formulas (4.13) and (4.14) to (4.12) we see that the function u is a solution to Eq. (2.6). Formula (2.9) follows from Corollary 1.

Theorem 1 is proved.

5. Applications

In the sequel we denote all the inessential positive constants by the same letter c.

5.1. Parabolic equation with gradient nonlinearity

In this section we consider a model example.

Let $M \subset \mathbb{R}^m$ be a bounded domain with smooth boundary ∂M.

Consider the following equation

$$u_t = f(\nabla u) + \Delta u, \qquad u|_{t=0} = \hat{u} \in H_0^{1,q}(M), \qquad u(t, \partial M) = 0, \quad t > 0, \tag{5.1}$$

here $q > 1$.

The function f is continuous in $\mathbb{R}^m$ and for all $z \in \mathbb{R}^m$ we have $|f(z)| \leqslant c(|z|^p + 1)$, $q \geqslant p \geqslant 1$. Note that the function f may not necessarily be a Lipschitz function.

Let us show that if

$$m(p - 1) < q \tag{5.2}$$

then problem (5.1) has a generalized solution from $C([0, T], H_0^{1,q}(M))$, the constant $T > 0$ depends on $\hat{u}$.

If the function f is a Lipschitz function then inequality (5.2) is well known: it corresponds to the subcritical case in the sense of Fujita.

After the change of the unknown function $u = e^{\Delta t} \hat{u} + v$ our problem takes the form

$$v_t = g(t, x, \nabla v) + \Delta v, \quad v|_{t=0} = 0, \quad g(t, x, \nabla v) = f\left(\nabla \left(e^{\Delta t} \hat{u} + v \right) \right). \tag{5.3}$$

Introduce the following fractional order spaces $X^{s,p} = (-\Delta)^{-s/2}(L^p(M))$. Here by $(-\Delta)^{-1}h$ we denote the solution to the problem

$$\Delta u = -h, \quad u(\partial M) = 0.$$

The properties of the spaces $X^{s,p}$ are described in [2–4,15]. See also [5] and references therein. Particularly the spaces $X^{s,p}$ satisfy almost all embedding theorems which the Sobolev spaces $H^{s,p}(M)$ do; $X^{k,p} = H^{k,p}(M) \cap H_0^{1,p}(M)$, $k \in \mathbb{N}$; and the following semigroup lemma is fulfilled.

Lemma 6. *Let $e^{\Delta t} u$ stands for the solution to initial value problem*

$$v_t = \Delta v, \quad v|_{t=0} = u.$$

Regarding boundary conditions we assume either $M \subset \mathbb{R}^m$ is a bounded domain with smooth boundary and then $v(t, \partial M) = 0$, or M is an m-dimensional smooth compact manifold without boundary.

Upon these assumptions one has an estimate

$$\|e^{\Delta t} u\|_{X^{l,a}(M)} \leqslant \frac{c}{t^\mu} \|u\|_{X^{h,b}(M)}, \quad \mu = \frac{m(1/b - 1/a) + l - h}{2}. \tag{5.4}$$

The real numbers a, b, h, l are such that $l \geqslant h$, $1 < b \leqslant a$. Inequality (5.4) holds for all t belonging to some interval $(0, \tilde{t}]$, constant c does not depend on u and t.

Consider problem (5.3) in the scales

$$E_s = X^{1+s_0+s,q}(M), \quad \|\cdot\|_s = \|\cdot\|_{X^{1+s_0+s,q}(M)}, \quad s \in (0, S),$$

and

$$G_s = X^{-\lambda,q}(M), \quad \|\cdot\|^G = \|\cdot\|_{X^{-\lambda,q}(M)},$$

this means that all the spaces G_s coincide with each other, the constants $S > 0$, $s_0 \geqslant 0$ and $0 \leqslant \lambda < m(1 - 1/q)$ to be defined.

Introduce a constant

$$r = \frac{qm}{m + \lambda q} \in (1, q].$$

It is easy to see that

$$L^r(M) \subset X^{-\lambda,q}(M). \tag{5.5}$$

Indeed, like in the case of Sobolev spaces [1], for the fractional order spaces we have

$$X_0^{\lambda,q'}(M) \subset L^{\frac{mq'}{m-\lambda q'}}(M), \quad \frac{1}{q'} + \frac{1}{q} = 1.$$

For conjugated spaces this gives

$$L^r(M) \subset X^{-\lambda,q}(M), \quad \frac{1}{r} + \frac{m - \lambda q'}{mq'} = 1.$$

Then using formula (5.5) estimate the function g:

$$\begin{aligned}
\|g(t, x, \nabla v)\|^G &\leqslant c \|g(t, x, \nabla v)\|_{L^r(M)} \leqslant c\big(\|\nabla(e^{\Delta t}\hat{u} + v)\|_{L^{pr}(M)}^p + 1\big) \\
&\leqslant c\big(\|e^{\Delta t}\hat{u}\|_{X^{1,pr}(M)}^p + \|v\|_{X^{1,pr}(M)}^p + 1\big).
\end{aligned} \tag{5.6}$$

Choose a constant s_0 as follows

$$s_0 = m\left(\frac{1}{q} - \frac{1}{rp}\right).$$

Then the condition $X^{1+s_0,q}(M) \subseteq X^{1,pr}(M)$ is satisfied.

Here we assume that the constant λ is such that we have $q < rp$. By Lemma 6 one has

$$\|e^{\Delta t}\hat{u}\|_{X^{1,pr}(M)}^p \leqslant ct^{-\beta}\|\hat{u}\|_{X^{1,q}(M)}^p, \quad \beta = \frac{m}{2}\left(\frac{p}{q} - \frac{1}{r}\right).$$

If $v \in B_s = \{h \in E_s \mid \|h\|_s^E \leqslant 1\}$ then by all these argument formula (5.6) implies

$$\|g(t, x, \nabla v)\|^G \leqslant \frac{c}{t^\beta}.$$

Another inequality we need is

$$\|e^{\Delta t}w\|_s \leqslant ct^{-\phi}\|w\|^G, \quad \phi = \frac{1 + s_0 + s + \lambda}{2},$$

this formula also follows from Lemma 6.

Proposition 2. *The mapping* $(t, v) \mapsto g(t, x, \nabla v)$ *is a continuous mapping of* $(0, T) \times B_s$ *to* G_s.

Proof. Assume the converse: there exists a sequence (t_k, v_k) such that $t_k \to t \in (0, T)$, $v_k \to v$ in E_s as $k \to \infty$, $v, v_k \in B_s$ and

$$\|g_k(x) - g(x)\|^G \geqslant c > 0, \tag{5.7}$$

here we put $g_k(x) = g(t_k, x, \nabla v_k)$, $g(x) = g(t, x, \nabla v)$.

By the argument above formula (5.7) imply

$$\left\| g_k(x) - g(x) \right\|_{L^r(M)} \geqslant c > 0.$$

Since $\nabla v_k \to \nabla v$ in $L^{pr}(M)$ then there exists a subsequence $\{v_{k'}\} \subseteq \{v_k\}$ such that $\nabla v_{k'} \to \nabla v$ almost every where in M. Thus $|g_{k'}(x) - g(x)|^r \to 0$ almost everywhere in M. Consequently $|g_{k'}(x) - g(x)|^r \to 0$ in measure.

It remains to show that the sequence $|g_{k'}(x) - g(x)|^r$ is uniformly integrable. If we do this then by the Vitali convergence theorem [11,10] it follows that $\|g_{k'}(x) - g(x)\|_{L^r(M)} \to 0$ and this contradiction proves the proposition.

Note that since $v_k, v \in E_s$, $s > 0$, we actually have $\nabla v_k \to \nabla v$, in $L^{pr+\sigma}(M)$ with small $\sigma > 0$. Thus the functions $g_{k'}(x) - g(x)$ belong not only to $L^r(M)$ but also to $L^{r+\varepsilon}(M)$ with small $\varepsilon > 0$ and the sequence $\|g_{k'}(x) - g(x)\|_{L^{r+\varepsilon}(M)}$ is bounded (these observations follow from the same argument as above). The last observation can be rewritten as follows:

$$\sup_{k'} \int_M \left| g_{k'}(x) - g(x) \right|^r æ\left(\left| g_{k'}(x) - g(x) \right| \right) dx = \sup_{k'} \left\| g_{k'}(x) - g(x) \right\|_{L^{r+\varepsilon}(M)}^{r+\varepsilon} < \infty$$

with $æ(y) = y^\varepsilon$. Since the function $æ$ is monotone and unbounded in $\mathbb{R}_+$, this proves the uniform integrability of the sequence $|g_{k'}(x) - g(x)|^r$. $\square$

Now we see that $\alpha = 0$ and to apply Theorem 1 we need $\chi = \phi + \beta < 1$. It is easy to show that the last inequality follows from (5.2) if only the constant S is sufficiently small and the constant λ is chosen to make the expression pr to be sufficiently close to q.

Indeed, let us rewrite inequality (5.2) as $\lambda_0 = m(p-1)/q < 1$. If $p = 1$ then we set $\lambda = 1$. Otherwise, if $q \geqslant p > 1$, we take $\lambda \in (0, \lambda_0)$ such that $0 < \lambda_0 - \lambda \ll 1$. Then

$$\frac{1}{r} = \frac{1}{q} + \frac{\lambda}{m} < \frac{1}{q} + \frac{\lambda_0}{m} = \frac{p}{q}, \quad 0 < \frac{p}{q} - \frac{1}{r} \ll 1.$$

This implies $0 < pr - q \ll 1$, $0 < s_0 \ll 1$, $0 < \beta \ll 1$, and

$$\chi = \phi + \beta < \frac{1 + s_0 + S + \lambda_0}{2} + \beta < 1$$

if $S \ll 1$.

5.2. The scale of analytic functions

Let $\mathbb{T}^m = \mathbb{R}^m/(2\pi\mathbb{Z})^m$ be the m-dimensional torus. All the technique developed below can be transferred almost literally to the case of the problem with zero boundary conditions on the m-dimensional cube.

By $x = (x_1, \ldots, x_m)$ denote an element of $\mathbb{R}^m$.

Let

$$\mathbb{T}_s^m = \left\{ z = x + iy \in \mathbb{C}^m \,\Big|\, x \in \mathbb{T}^m, \ |y| = \left(\sum_{j=1}^m |y_j|^2 \right)^{\frac{1}{2}} < s \right\}$$

be the complex neighborhood of the torus $\mathbb{T}^m$.

Define a set E_s, $s > 0$, as follows $E_s = C(\overline{\mathbb{T}}_s^m) \cap \mathcal{O}(\mathbb{T}_s^m)$. Here $\mathcal{O}(\mathbb{T}_s^m)$ stands for the set of analytic functions in $\mathbb{T}_s^m$.

The set E_s is a Banach space with respect to the norm $\|u\|_s = \max_{z \in \overline{\mathbb{T}}_s^m} |u(z)|$. By the Montel theorem the embeddings $E_{s+\delta} \subset E_s$, $\delta > 0$ are completely continuous. By definition put $E_0 = C(\mathbb{T}^m)$ and $\|\cdot\|_0 = \|\cdot\|_{C(\mathbb{T}^m)}$.

By $H^{k,p}(\mathbb{T}_s^m)$ we denote a subspace of $\mathcal{O}(\mathbb{T}_s^m)$, this subspace consists of functions $u(x, y) = u_1(x, y) + iu_2(x, y)$ with finite Sobolev norm.

For example, the space $H^1(\mathbb{T}_s^m)$ consists of functions u with finite norm

$$\|u\|_{H^1(\mathbb{T}_s^m)}^2 = \int_{\mathbb{T}_s^m} |u(x, y)|^2 \, dx \, dy + \int_{\mathbb{T}_s^m} \sum_{j=1}^m \left| \frac{\partial u(x, y)}{\partial x_j} \right|^2 dx \, dy \tag{5.8}$$

here $dx\,dy = dx_1 \cdots dx_m\,dy_1 \cdots dy_m$. In formula (5.8) we use only derivatives in variable x, but due to analyticity of the function u this norm is equivalent to the standard Sobolev norm on the $2m$-dimensional real manifold:

$$\|u\|^2_{H^1(\mathbb{T}^m_s)} = \int_{\mathbb{T}^m_s} |u(x,y)|^2\,dx\,dy + \int_{\mathbb{T}^m_s} \sum_{j=1}^m \left|\frac{\partial u(x,y)}{\partial x_j}\right|^2 + \left|\frac{\partial u(x,y)}{\partial y_j}\right|^2\,dx\,dy.$$

The main facts on Sobolev spaces remain valid for $H^{k,p}(\mathbb{T}^m_s)$. For example, let Δ stands for the Laplace operator

$$\Delta = \sum_{j=1}^m \partial_j^2, \quad \partial_j = \frac{\partial}{\partial x_j}.$$

Consider the equation

$$\Delta u = f \in H^{k,p}(\mathbb{T}^m),$$

here $H^{k,p}(\mathbb{T}^m)$ is the standard Sobolev space on manifold $\mathbb{T}^m$. This Laplace–Poisson equation has a solution $u \in H^{k+2,p}(\mathbb{T}^m)$, provided $\int_{\mathbb{T}^m} f(x)\,dx = 0$. We denote this solution by $\Delta^{-1}f$.

If one considers the same equation with $f \in H^{k,p}(\mathbb{T}^m_s)$ then $\Delta^{-1}f \in H^{k+2,p}(\mathbb{T}^m_s)$. To verify this observation it is sufficient to apply the previous argument to the equation

$$\Delta\big(u_1(x,y) + iu_2(x,y)\big) = f_1(x,y) + if_2(x,y)$$

for all $|y| < s$ and take into account norm (5.8).

Lemma 7. *There exists a positive constant c such that for any $u \in E_s$, $s \geqslant 0$, the following inequality holds*

$$\big\|e^{t\Delta}u\big\|_{s+\delta} \leqslant c\exp\left(\frac{\delta^2}{4t}\right)\|u\|_s, \quad t,\delta > 0.$$

The constant c depends only on m.

Proof. The assertion of the lemma easily follows from the well-known formula:

$$\big(e^{t\Delta}u\big)(x) = \frac{1}{(4\pi t)^{m/2}} \int_{\mathbb{R}} e^{-(\xi_1-x_1)^2/(4t)}\,d\xi_1 \cdots \int_{\mathbb{R}} e^{-(\xi_m-x_m)^2/(4t)}\,d\xi_m u(\xi).$$

In all these integrals one must shift the contour of integration to the complex plane and then the desired inequality follows from the standard estimates. □

By Lemma 7 the semigroup $e^{t\Delta}$ is parabolic with $\gamma = 2$.

Lemma 8. *Take a constant $\rho \in (0, 1/2]$. For any $\varepsilon \in (0, 2\rho)$ there is a positive constant $c = c(\varepsilon)$ such that if $u \in E_{s+\delta}$ then*

$$\big\|(-\Delta)^{-\rho}\partial_j u\big\|_s \leqslant \frac{c}{\delta^{1-2\rho+\varepsilon}}\|u\|_{s+\delta}, \quad s \geqslant 0,\ \delta > 0, \tag{5.9}$$

$$\big\|(-\Delta)^\rho u\big\|_s \leqslant \frac{c}{\delta^{2\rho+\varepsilon}}\|u\|_{s+\delta}, \tag{5.10}$$

$$\big\|\Delta^{-1}\partial_k\partial_l u\big\|_s \leqslant \frac{c}{\delta^\varepsilon}\|u\|_{s+\delta}. \tag{5.11}$$

Proof. Let us prove formula (5.9). Using the standard facts on Sobolev's spaces we have

$$\big\|(-\Delta)^{-\rho}\partial_j u\big\|_s \leqslant c\big\|(-\Delta)^{-\rho}\partial_j u\big\|_{H^{\varepsilon,p}(\mathbb{T}^m_s)} \leqslant c\|u\|_{H^{\varepsilon+1-2\rho,p}(\mathbb{T}^m_s)}, \quad \varepsilon p > 2m.$$

Then the desired result follows from the interpolation formula and the Cauchy inequality:

$$\|u\|_{H^{\varepsilon+1-2\rho,p}(\mathbb{T}^m_s)} \leqslant c\|u\|^{\varepsilon+1-2\rho}_{H^{1,p}(\mathbb{T}^m_s)}\|u\|^{2\rho-\varepsilon}_{L^p(\mathbb{T}^m_s)}, \quad \|u\|_{H^{1,p}(\mathbb{T}^m_s)} \leqslant \frac{c}{\delta}\|u\|_{s+\delta}.$$

Formulas (5.10), (5.11) is derived in the same way. □

To compare our result with the known one consider the Navier–Stokes equation.

5.3. 3-D Navier–Stokes equation

In this section we use the scale $G_s = E_s = C(\overline{\mathbb{T}}^m_s) \cap \mathcal{O}(\mathbb{T}^m_s)$.

Consider the Navier–Stokes equation in the divergence free setup. After Leray's projection the Navier–Stokes equation takes the well-known form [12]

$$
\begin{aligned}
(u^k)_t &= A^k_l \partial_j (u^j u^l) + \Delta u^k, \quad A^k_l = (\Delta^{-1} \partial_k \partial_l - \delta_{kl}), \\
u^k|_{t=0} &= \hat{u}^k \in H^r(\mathbb{T}^3),
\end{aligned}
\tag{5.12}
$$

where $\delta_{kl} = 1$ for $k = l$ and 0 otherwise; $k, l, j = 1, 2, 3$ we also use the Einstein summation convention.

One can also see that the right-hand side of Eq. (5.12) takes the set of zero mean value functions

$$
\int_{\mathbb{T}^m} u^j(x)\, dx = 0
$$

to itself. Due to the Galilean invariance of this problem (Eq. (5.12) does not change after the substitution $x \mapsto x + ct$, $u^k \mapsto u^k + c$, where c is a constant) without loss of generality we assume that the initial vector field $\hat{u}^k$ is of zero mean value. This gives us possibility to consider our problem on the set of zero mean value functions.

From [16,13] it follows that if $r = 1/2$ then problem (5.12) has a solution $u^i(t, x)$ which is regular in the spatial variables for all $t \in (0, T_*)$. Here T_* is a small positive constant.

Let us show that by Theorem 1 the analytic solution exists for all $r > 1/2$. This indicates that in terms of paper [5] Theorem 1 allows us to carry out only the subcritical case. This is no surprise since Theorem 1 is very general.

In accordance to formula (5.11) the operator A^k_l is estimated as follows:

$$
\| A^k_l u \|_s \leqslant \frac{c}{\delta^\varepsilon} \| u \|_{s+\delta}.
$$

Assume a parameter $\rho \in (0, 1/2)$ to be close $1/2$ and let us change the variable in (5.12):

$$
u^k = u^k_1 + u^k_2, \quad u^k_1 = e^{t\Delta} \hat{u}^k, \quad u^k_2 = (-\Delta)^\rho v^k.
$$

Then

$$
f^k(t, v) = A^k_l \partial_j (-\Delta)^{-\rho} (u^j u^l) = \sum_{\sigma, \tau = 1}^{2} A^k_l \partial_j (-\Delta)^{-\rho} (u^j_\sigma u^l_\tau).
$$

Using Lemma 8 we have

$$
\| f^k \|_s \leqslant \frac{c}{\delta^{\varepsilon+1-2\rho}} \sum_{j,l,\sigma,\tau} \| u^j_\sigma u^l_\tau \|_{s+\delta/2}.
$$

Here $\| u^j_\sigma u^l_\tau \|_{s+\delta/2} \leqslant \| u^j_\sigma \|_{s+\delta/2} \| u^l_\tau \|_{s+\delta/2}$. Therefore, using the elementary inequality $ab \leqslant (a^2 + b^2)/2$, we get

$$
\| f^k \|_s \leqslant \frac{c}{\delta^{\varepsilon+1-2\rho}} \sum_{j,\sigma} \| u^j_\sigma \|^2_{s+\delta/2}.
\tag{5.13}
$$

Estimate these terms separately. Using Lemmas 6, 7 and taking into account that $(s + \delta)^2 < t$ we obtain

$$
\| u^k_1 \|^2_{s+\delta/2} = \| e^{t\Delta} \hat{u}^k \|^2_{s+\delta/2} \leqslant c \| e^{\Delta t/2} \hat{u}^k \|^2_0 \leqslant c \| e^{\Delta t/2} \hat{u}^k \|^2_{H^a(\mathbb{T}^m)} \leqslant \frac{c}{t^{a-r}} \| \hat{u}^k \|^2_{H^r(\mathbb{T}^m)}.
\tag{5.14}
$$

We choose the constants a, r to be close from above to $3/2$ and $1/2$ respectively.

Another term is estimated as follows

$$
\| (-\Delta)^\rho v^k \|^2_{s+\delta/2} \leqslant \frac{c}{\delta^{2(2\rho+\varepsilon)}} \| v^k \|^2_{s+\delta}.
\tag{5.15}
$$

To apply the main theorem, from formulas (5.13) and (5.14) we need to have

$$
\frac{\varepsilon + 1 - 2\rho}{2} + a - r < 1,
$$

and from formulas (5.13) and (5.15) we also need to have

$$2\rho + \varepsilon + \frac{\varepsilon + 1 - 2\rho}{2} < 1.$$

The both inequalities holds if we take

$$r = 1/2 + \varepsilon_1, \quad \rho = 1/2 - \varepsilon_2, \quad a = 3/2 + \varepsilon_3,$$

here $\varepsilon, \varepsilon_j, j = 1, 2, 3$, are suitably small positive constants.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] H. Amann, On abstract parabolic fundamental solutions, J. Math. Soc. Japan 39 (1987) 93–116.
[3] H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in: H. Schmeisser, H. Triebel (Eds.), Function Spaces, Differential Operators and Nonlinear Analysis, in: Teubner Texte zur Mathematik, vol. 133, Teubner, 1993, pp. 9–126. MR 94m:35153.
[4] H. Amann, Linear and Quasilinear Parabolic Problems. Abstract Linear Theory, Birkhäuser Verlag, 1995.
[5] J.M. Arrieta, A.N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier–Stokes and Heat equations, Trans. Amer. Math. Soc. 352 (1) (2000) 285–310.
[6] F.E. Browder, A new generalization of the Schauder fixed point theorem, Math. Ann. 174 (1967) 285–290.
[7] A.N. Carvalho, Abstract parabolic problems in ordered Banach spaces, Cadernos de Mathematica 02, 141–146, March 2001, Artigo Numero SMA No. 104.
[8] J. Diedonné, Deux exeples singuliers d'équations différentielles, Acta Sci. Math. (Szeged) 12 (1950) B 38–40.
[9] N. Dunford, J.T. Schwartz, Linear Operators. Part 1: General Theory, Interscience Publishers, New York, London, 1958.
[10] N. Dunford, J.T. Schwartz, Linear Operators. Part 3: Spectral Operators, Wiley-Interscience, New York, London, 1971.
[11] G. Folland, Real Analysis: Modern Techniques and Their Applications, second ed., Wiley-Interscience, 1999.
[12] U. Frisch, Turbulence. The Legacy of A.N. Kolmogorov, Cambridge University Press, 1995.
[13] H. Fujita, T. Kato, On the Navier–Stokes initial value problem I., Arch. Ration. Mech. Anal. 16 (1964) 269–315. MR 29:3774.
[14] A. Godunov, Peano's theorem in Banach spaces, Funct. Anal. Appl. 9 (1975) 53–55.
[15] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981. MR 83j:35084.
[16] T. Kato, H. Fujita, On the nonstationary Navier–Stokes system, Rend. Sem. Math. Univ. Padova 32 (1962) 243–260. MR 26:495.
[17] L. Schwartz, Analyse Mathématique, Hermann, 1967.
[18] J. Yorke, A continuous differential equation in Hilbert space without existence, Funkcial. Ekvac. 13 (1970) 19–21.