# Decay of geometry for Fibonacci critical covering maps of the circle [☆]

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**Abstract**

We study the growth of $Df^n(f(c))$ when $f$ is a Fibonacci critical covering map of the circle with negative Schwarzian derivative, degree $d \geqslant 2$ and critical point $c$ of order $\ell > 1$. As an application we prove that $f$ exhibits exponential decay of geometry if and only if $\ell \leqslant 2$, and in this case it has an absolutely continuous invariant probability measure, although not satisfying the so-called Collet–Eckmann condition.
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**Résumé**

Nous étudions la croissance de $Df^n(f(c))$ lorsque $f$ est un revêtement critique de Fibonacci du cercle avec dérivée Schwarzienne négative, degré $d \geqslant 2$ et point critique $c$ d'ordre $\ell > 1$. Comme application nous démontrons que $f$ exhibe une décroissance exponentielle de géométrie si et seulement si $\ell \leqslant 2$, et dans ce cas $f$ a une mesure de probabilité invariante absolument continue, sans satisfaire la condition de Collet–Eckmann.
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## 1. Introduction

A *critical covering map* of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is a $C^r$ ($r \geqslant 1$) covering map $f : \mathbb{S}^1 \to \mathbb{S}^1$ with just one critical point, say $c$, which must be of inflection type. In the present work, we will consider critical covering maps of degree $d \geqslant 2$. This kind of map has been considered before, for example in [20–22,30]. Under the point of view of dynamical systems such critical coverings with neither wandering intervals nor periodic attractors are all topologically conju-

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gate to the map of the circle induced by the map $x \mapsto dx$ of the real line $\mathbb{R}$. This implies in particular that the set $\mathcal{B}(f) := \{x \in \mathbb{S}^1 \colon \omega(x) = \mathbb{S}^1\}$ is a countable intersection of open and dense subsets of $\mathbb{S}^1$. But metric properties of the underlying dynamics as the Lebesgue measure of $\mathcal{B}(f)$ or the growth of $Df^n(f(c))$, as $n \to \infty$, depend on the order and combinatorial behavior of the forward critical orbit. Here we will consider the case of critical points of order $\ell > 1$ with the *Fibonacci combinatorics*, which will be defined below.

The Fibonacci combinatorics appeared before in the context of unimodal maps related to a question posed by J. Milnor [28] about the classification of the measure-theoretical attractors in dynamical systems on compact spaces. Among the quadratic real polynomials $Q_\alpha(x) = \alpha x(1 - x)$ there is one whose turning point has this combinatorics and, as it implies a strong recurrence of the critical point, it was considered in [14] as a candidate to exhibit a *wild attractor*. A wild attractor for $Q_\alpha$ (also called *absorbing Cantor attractor* in [13]) is a compact invariant Cantor set whose basin of attraction is a meager subset of $[0, 1]$ with full Lebesgue measure [28]. It was proved in [24] that a quadratic real polynomial with the Fibonacci combinatorics has no wild attractor. This was generalized later for any combinatorics [23] (see also [7,10–13,16,26]). On the other hand, in [1] it was proved that if a unimodal real polynomial of degree big enough has the Fibonacci combinatorics then it has a wild attractor.

Under mild hypotheses a critical covering map $f$ as above is ergodic with respect to the Lebesgue measure [35], that is, every totally invariant set has Lebesgue measure zero or one. Then, if it has an absolutely continuous invariant probability measure it has no wild attractor and the topological conjugacy between $f$ and a map of the circle induced by $x \mapsto dx$ is absolutely continuous.

In order to prove that there is an absolutely continuous invariant probability measure one examines the derivative on the orbit of the critical value and tries to show that it undergoes a relative expansion, for example by showing that the map satisfies the so-called *Collet–Eckmann condition*, stating that the derivative on the critical orbit grows exponentially, or weaker conditions as the summability condition in [2,31] and also [3]. This may be done either by directly examining the derivative for some specially chosen moments of the orbit (at closest returns, for example) or by studying the geometric aspects of sequences of inductively defined return maps around the critical point. In the latter case, if some metric relations between successive stages decay exponentially fast then one says that $f$ exhibits *exponential decay of geometry* (see below for a precise definition), the opposite situation being *bounded geometry*, where scales do not decay at all. Exponential decay of geometry and topological additional hypotheses (as finiteness of central returns) are related to the expansion of the derivative on the critical orbit and have also been a way of proving existence of absolutely continuous invariant probability measures (see [27] for the unimodal case). We also mention that in the proof of denseness of hyperbolicity in the logistic family $Q_\alpha$ (see [8,9,25]), a central problem in one-dimensional dynamics, the exponential decay of geometry played an important role.

Several questions about dynamics on the real line were treated with tools from complex analysis. But these tools can be applied just to a somewhat narrow set of cases which do not include situations where the order of the critical point is a general real number greater than one. It is also a natural aim that real analysis is enough to solve the questions from real dynamics. In this line of thought the non-existence of wild attractors for negative Schwarzian derivative Fibonacci unimodal maps with critical point of order $1 < \ell \leqslant 2$ was proved by G. Keller and T. Nowicki in [17] with just real analysis (in fact the result extends a little above $\ell = 2$). Moreover, the exponential decay of geometry was proved by W. Shen in [33] for non-renormalizable unimodal maps with any combinatorics with critical point of order $1 < \ell \leqslant 2$. Shen strongly uses the natural symmetry around the critical point that arises in the unimodal case, contrary to what happens with the inflection critical point of critical covering maps.

Here the decay of geometry for a map with an inflection critical point is considered for the first time. We prove that Fibonacci critical covering maps of the circle of degree $d \geqslant 2$ having critical point of order $1 < \ell \leqslant 2$ display decay of geometry. Moreover, our result is sharp since in [22] it was proved that Fibonacci critical coverings have bounded geometry for every $\ell > 2$. This is similar to the unimodal case where bounded geometry for the Fibonacci combinatorics follows for every $\ell > 2$ from a result in [17].

We also mention that the study of the growth of $|Df^n(f(c))|$ connected to the question of the existence of an absolutely continuous invariant probability measure and the regularity of topological conjugacies appeared at least three decades ago, see for example [5,15,18,19,29,32,34] and others references in the books [4] and [6].

## 2. Basic concepts and main results

We look at the circle $\mathbb{S}^1$ as the quotient space $\mathbb{R}/\mathbb{Z}$ with the orientation and metric induced from the real line $\mathbb{R}$. Given a $C^1$ critical covering map $f : \mathbb{S}^1 \to \mathbb{S}^1$ of degree $d \geqslant 2$ we choose one of its fixed points, say $p$, and set $I_1 := \mathbb{S}^1 \setminus \{p\}$. The distance between 2 points $x$ and $y$ in $I_1$ will be denoted by $|x - y|$ and the length of an interval $I \subset I_1$ will be denoted by $|I|$.

Note that $f$ has $d$ branches mapping onto $I_1$ (below we will identify $I_1$ with $(0, 1) \subset \mathbb{R}$ in order to consider homeomorphic extensions of these branches and prove a technical but essential lemma). From now on we will assume that the critical point $c$ of $f$ is recurrent and define the sequence $I_1 \supset I_2 \supset I_3 \supset \cdots \{c\}$ such that, for $n \geqslant 1$, the interval $I_{n+1}$ is a component of the domain of the first return map $\phi_n$ to $I_n$. If $\phi_n(c) \in I_{n+1}$, for some $n \geqslant 1$, we say that $n$ is a *central return moment*. The *critical return time* $s_n$ is defined by $\phi_n(c) = f^{s_n}(c)$. If the sequence $s_1, s_2, s_3, s_4, \ldots$ coincides with the Fibonacci sequence $1, 2, 3, 5, \ldots$ the critical covering map $f$ is called a *Fibonacci critical covering map*. It is not difficult to show that Fibonacci critical covering maps have no central returns.

One of the main geometric aspects of one-dimensional dynamics with critical points is the evolution of the *scaling factor* $\mu_n = |I_{n+1}|/|I_n|$, particularly in the subsequence $(n_i)_i$ of non-central return moments (which is the sequence itself in the case of Fibonacci critical covering maps). If $\mu_{n_i} \to 0$ exponentially fast as $i \to \infty$ we say that $f$ has *exponential decay of geometry*.

Let $\mathcal{C}_d$ denote the set of $C^1$ critical coverings $f$ of degree $d \geqslant 2$ which also have the following properties:

- The critical point $c$ is recurrent and not periodic.
- There exist a $C^1$ map $\psi : I_1 \to I_1$ satisfying $\lim_{x \to c} \psi(x) = 0$ and real constants $\vartheta > 0$ and $\ell > 1$ such that

$$f(x) = f(c) + \vartheta \, \mathrm{sgn}(x - c)|x - c|^{\ell}\big(1 + \psi(x)\big) \tag{1}$$

  for every $x$ in a neighborhood of $c$.
- Restricted to $\mathbb{S}^1 \setminus \{c\}$, the map $f$ is $C^3$ and has negative Schwarzian derivative.

Remark that a map $f \in \mathcal{C}_d$ cannot have neither a wandering interval nor a periodic attractor, see [6] and also [35]. Our main result claims the exponential growth of $Df^{s_n}(f(c))$ as $n \to \infty$ for Fibonacci critical coverings.

**Theorem 1.** *There are constants $\rho > 1$ and $C > 0$ such that if $f \in \mathcal{C}_d$ has the Fibonacci combinatorics and order $\ell \in (1, 2]$ then $Df^{s_n}(f(c)) \geqslant C\rho^n$, for all $n \geqslant 1$.*

Three consequences are the following.

**Theorem 2.** *If $f \in \mathcal{C}_d$ has the Fibonacci combinatorics and order $\ell \in (1, 2]$ then $f$ has an absolutely continuous invariant probability measure.*

**Theorem 3.** *If $f \in \mathcal{C}_d$ has the Fibonacci combinatorics and order $\ell \in (1, 2]$ then $f$ exhibits exponential decay of geometry.*

**Theorem 4.** *If $f \in \mathcal{C}_d$ has the Fibonacci combinatorics and order $\ell \in (1, 2]$ then the $\omega$-limit set $\omega(c)$ of its critical point is a minimal invariant Cantor set with zero Hausdorff dimension.*

In order to prove Theorem 1 we find a recursive difference equation (12) involving $Df^{s_n}(f(c))$ and the parameter $\lambda_n = |f^{s_n}(c) - c|/|f^{s_{n+2}}(c) - c|$. Then we proceed as in [17] to get the exponential growth of $Df^{s_n}(f(c))$ with $n$. Using a classical argument, we show that the sequence of derivatives $Df^k(f(c))$ goes to infinity as $k \to \infty$ and this is exactly the criterion in [3] assuring the existence of an absolutely continuous invariant probability measure, as stated in Theorem 2. Finally, Theorem 1 together with distortion control imply the third and the fourth theorems.

A technical difference with respect to the work of Keller and Nowicki [17] is that we do not get an *a priori* numeric lower bound for $\lambda_n$, and in fact we do not need it for the proof. It suffices to use that $\lambda_n$ is uniformly bounded away from one, which is a consequence of the real bounds (see Theorem 9 below). Another difference is a cross-ratio argument where an asymmetric huge extendibility allowed by topological properties plays a central role. For this

argument, the assumption of negative Schwarzian derivative is essential, since the range of extendibility goes much beyond the neighborhood of the critical point.

We also derive some more precise statements about the geometry and the derivative growth in the Fibonacci combinatorics, as a consequence of these theorems and a technical result proved in [22] for any $\ell > 1$. This result states, in particular, that the two components of $I_n \setminus I_{n+1}$ are comparable with $I_n$. Hence $I_{n+1}$ is not only exponentially small with respect to $I_n$ but also with respect to these two adjacent components.

The derivative growth presents three distinct pictures, depending on the value of $\ell$, as the following theorems tell.

**Theorem 5.** *Let $f \in \mathcal{C}_d$ with the Fibonacci combinatorics and $\ell \in (1, 2)$. Then*

$$\lim_{n \to \infty} \frac{1}{n} \log \log Df^{s_n}\big(f(c)\big) = \mu,$$

*where*

$$\mu = \frac{1 + \sqrt{1 + 4\ell}}{2\ell}.$$

*In particular, the geometry decays super exponentially.*

This theorem says that $Df^{s_n}(f(c))$ grows super exponentially with $n$, for $\ell < 2$. But as $s_n$ grows with $\omega^n$, where $\omega = \frac{1+\sqrt{5}}{2}$ is the golden number, and $\mu = \mu(\ell)$ decreases from $\omega$, when $\ell = 1$, to 1, when $\ell = 2$, then we can conclude that $Df^{s_n}(f(c))$ does not grow exponentially with $s_n$. Therefore, as a consequence, $f$ is not Collet–Eckmann. But we can extend this conclusion to all iterates, by the following theorem.

**Theorem 6.** *Let $f \in \mathcal{C}_d$ with the Fibonacci combinatorics and $\ell \in (1, 2)$. Then*

$$\lim_{k \to \infty} \frac{\log \log Df^k(f(c))}{\log k} = \frac{\log \mu}{\log \omega} < 1,$$

*where $\mu = \mu(\ell)$ is given as in Theorem 5 and $\omega$ is the golden number. In particular, $f$ does not satisfy the Collet–Eckmann condition.*

For $\ell = 2$ the exponential growth of $Df^{s_n}(f(c))$ with $n$ is the best that we can hope.

**Theorem 7.** *Let $f \in \mathcal{C}_d$ with the Fibonacci combinatorics and $\ell = 2$. Then the growth of $Df^{s_n}(f(c))$ is not more than exponential. In particular, the geometry decays with a ratio which is not more than exponential.*

Finally, for $\ell > 2$ we conclude that $Df^{s_n}(f(c))$ and the geometry are bounded. This last assertion, in particular, is also proved in [22].

**Theorem 8.** *Let $f \in \mathcal{C}_d$ with the Fibonacci combinatorics and $\ell > 2$. Then $Df^{s_n}(f(c))$ is bounded, and in particular the geometry is bounded.*

This paper is organized as follows. In Section 3 we review the technical tools related to the uniform distortion control of powers of $f$ which are widely used in one-dimensional dynamics. The so-called real bounds play a central role: they state an *a priori* property on the geometry of the induced sequence of first return maps. In Section 4 we discuss in more details the Fibonacci combinatorics, which is explicitly used in the proofs. The facts stated in this section are very similar to the Fibonacci combinatorics of unimodal maps. In Section 5 we state the extendibility properties of a first entry map from the critical value to a neighborhood of the critical point which will be essential to get lower and upper bounds for its derivative, through a cross-ratio argument. These bounds are obtained in Section 6 and used to show Theorem 1, in Section 7. In this section we also prove the existence of an absolutely continuous invariant probability measure, the exponential decay of geometry and the zero Hausdorff dimension of $\omega(c)$. Finally, in Section 8 we derive, using a result in [22], the more precise consequences of these findings, which are stated in Theorems 5, 6, 7 and 8.

## 3. Cross-ratio, distortion control

Koebe Principles are the most important tools to controlling distortion and finding lower bounds for derivatives that will ultimately prove Theorem 1. We list some important definitions and results that may be found, for example, in [6].

Let $J, T \subset I_1 \subset \mathbb{S}^1$ be a pair of intervals such that $J \subset T$ and $T \setminus J$ has 2 non-empty connected components $L$ and $R$. We say that $J$ is $\alpha$-*well inside* $T$ if $\min\{|L|, |R|\} \geqslant \alpha|J|$. The *cross-ratio* of $J$ and $T$ is the ratio

$$C(T, J) = \frac{|T|}{|J|} \frac{|L|}{|R|}.$$

Let $h : T \to \mathbb{S}^1$ be a diffeomorphism onto its image and define

$$B(h, T, J) = \frac{C(h(T), h(J))}{C(T, J)}.$$

If $h$ is $C^3$ and has negative Schwarzian derivative then $B(h, T, J) > 1$. As a consequence, if $J \subset T$ is degenerated to a point $x$ we have

$$\left| Dh(x) \right| \geqslant \frac{|h(L)||h(R)|}{|h(T)|} \frac{|T|}{|L||R|}. \tag{2}$$

On the other hand, if we degenerate $R$ to a point $x$ then

$$\left| Dh(x) \right| \leqslant \frac{|h(T)||h(J)|}{|h(L)|} \frac{|L|}{|T||J|}. \tag{3}$$

There is also the Macroscopic Koebe Principle saying that if $h(J)$ is $\alpha$-well inside $h(T)$ then $J$ is also $\alpha$-well inside $T$. Under the same condition the distortion is controlled by

$$\frac{Dh(x)}{Dh(y)} \leqslant \left( \frac{1+\alpha}{\alpha} \right)^2, \tag{4}$$

for all $x, y \in J$.

In [20] and [35] one may find the following important result, that will be used everywhere throughout this work.

**Theorem 9** (Real bounds). *There is $\alpha = \alpha(\ell) > 0$ (which can be uniform for $\ell \leqslant \ell_0$, for every $\ell_0 > 1$), such that if $f \in \mathcal{C}_d$ has order $\ell$ then $I_{n+1}$ is $\alpha$-well inside $I_n$ for every non-central return moment $n$ after two non-central return moments. In particular, if $f$ has the Fibonacci combinatorics, this is true for every $n \geqslant 3$.*

Again we consider $\phi_n$, the first return map to the interval $I_n$ associated to the critical covering $f \in \mathcal{C}_d$. All branches of $\phi_n$ map their domains diffeomorphically onto $I_n$ except by the *critical branch* $\phi_n|_{I_{n+1}}$, which is a $C^1$ homeomorphism onto $I_n$. In the case of the critical branch, even if the distortion cannot be bounded because of the presence of the critical point, the corollary of the next lemma guarantees that it can be written as the composition of $f$ with a map (a power of $f$) having uniformly bounded distortion.

**Lemma 10.** *If $f \in \mathcal{C}_d$ and $x \in I_1 \setminus I_{n+1}$ is a point such that there exists the smallest $k \geqslant 1$ such that $f^k(x) \in I_{n+1}$ then there is an interval $T$ containing $x$ which is mapped by $f^k$ diffeomorphically onto $I_n$.*

**Proof.** The statement follows from the fact that $I_n$ is a nice interval in the sense of Martens [26].  □

**Lemma 11.** *There is $K = K(\ell) > 0$ such that if $f \in \mathcal{C}_d$ has the Fibonacci combinatorics and order $\ell > 1$ then for every $x \in I_1 \setminus I_{n+1}$, $n \geqslant 3$, and $k$ the first positive integer such that $f^k(x) \in I_{n+1}$ there is an interval $J$ to which $x$ belongs such that $f^k(J) = I_{n+1}$ and*

$$\frac{Df^j(y)}{Df^j(z)} \leqslant K,$$

*for all $y, z \in J$ and $j = 0, 1, 2, \ldots, k$. The constant $K$ is given by*

$$K = \left( \frac{1 + \alpha}{\alpha} \right)^2,$$

*where $\alpha = \alpha(\ell) > 0$ is the constant given in Theorem 9.*

**Proof.** By Lemma 10 there is an interval $T$ such that $x \in T$ and $f^k|_T$ is a diffeomorphism onto $I_n$. Let $J$ be the pre-image of $I_{n+1}$ under $f^k|_T$. By Theorem 9, $I_{n+1}$ is $\alpha$-well inside $I_n$. Now we apply the Macroscopic Koebe Principle to $h = f^{k-j}|_{f^j(T)}$ and conclude that $f^j(J)$ is $\alpha$-well inside $f^j(T)$, for every $j = 0, 1, \ldots, k$. Finally the distortion control of (4) applied to $h = f^j|_T$ implies the lemma.    $\square$

## 4. The Fibonacci combinatorics

The domain of $\phi_n$ has a connected component $M_{n+1}$, called *post critical domain*, which contains $\phi_n(c)$. Following the terminology in [9], a branch of $\phi_n$ which is a restriction of a branch of $\phi_{n-1}$ is called an *immediate branch*.

**Lemma 12.** *If $f \in \mathcal{C}_d$ has the Fibonacci combinatorics then the following properties hold*:

(1) *The post critical domain of $\phi_n$ is an immediate branch. In particular $\phi_{n+1}(c) = \phi_{n-1} \circ \phi_n(c)$, for all $n \geqslant 2$.*
(2) *Two consecutive post critical domains $M_n$ and $M_{n+1}$ (and in particular $\phi_{n-1}(c)$ and $\phi_n(c)$), $n \geqslant 1$, are on opposite sides of $c$ inside $I_1$.*

**Proof.** Due to the fact that the critical branch of $\phi_n$ is a homeomorphism from $I_{n+1}$ onto $I_n$, for all $n \geqslant 1$, $\phi_n$ has exactly one immediate branch, for every $n \geqslant 2$. As $d \geqslant 2$ the first return map $\phi_1$ has exactly $d$ branches and all these branches are restrictions of $f$. Now $\phi_2$ has its critical branch with return time equal to 2 (since by the hypothesis $s_2 = 2$), its immediate branch with return time equal to 1 and any of its branches which is not an immediate branch with return time at least equal to 2. Then, as $s_3 = 3$, $\phi_3(c) = \phi_1 \circ \phi_2(c)$. By induction, assume, for $n \geqslant 2$, that the immediate branch of $\phi_n$ has return time $s_{n-1}$ and that the return time of its others branches (including the critical branch) are at least $s_n$. So, as $s_{n+1} = s_{n-1} + s_n$, the post critical branch of $\phi_n$ must be its immediate branch. Also, except for its immediate and critical branches, all the other branches of $\phi_{n+1}$ will have return time greater or equal than $2s_n$, which is greater than $s_{n+1}$, and the induction follows.

The second statement follows from the fact that the branches of $\phi_n$ preserve orientation and the fact that its post critical branch is an immediate branch.    $\square$

The following lemma illustrates the strong recurrence property of Fibonacci critical coverings.

**Proposition 13.** *If $g \in \mathcal{C}_d$ has a sequence of return times $s_1, s_2, s_3, s_4, \ldots$ smaller (with respect to the lexicographic order) than the Fibonacci sequence $1, 2, 3, 5, \ldots$ then $g$ has central returns.*

**Proof.** We argue that if $g \in \mathcal{C}_d$ has a sequence of return times $s_1, s_2, s_3, s_4, \ldots$ smaller (with respect to the lexicographic order) than the Fibonacci sequence $1, 2, 3, 5, \ldots$ then $g$ has central returns. Among the branches of the first return map of a critical covering the immediate branch has the smallest possible return time. So, if a critical covering has no central returns its sequence of critical returns is at least as big as the Fibonacci sequence $1, 2, 3, 5, \ldots$.    $\square$

The following lemma is one of the applications of Lemma 11 in the Fibonacci case.

**Lemma 14.** *Let $K = K(\ell) > 0$ be the constant given in Lemma 11. If $f \in \mathcal{C}_d$ has the Fibonacci combinatorics and order $\ell > 1$ then*

$$K^{-1} < \frac{Df^j(f^{s_{n+1}}(f(c)))}{Df^j(f(c))} < K,$$

*for all $n \geqslant 4$ and for all $j = 0, 1, \ldots, s_n - 1$.*

**Proof.** As $\phi_n|_{I_{n+1}}$ has $c$ as its only critical point and $c$ is not periodic, then the interval $J = f(I_{n+1})$ is diffeomorphically mapped onto $I_n$ by $f^{s_n-1}$, it is disjoint from $I_n$, it is the first entry map into $I_n$ and it contains the points $f(c)$ and $f(f^{s_{n+1}}(c))$, since $c$ and $f^{s_{n+1}}(c)$ are in $I_{n+1}$. Therefore the lemma follows directly from Lemma 11 (applied to $I_n$).   $\square$

## 5. Extendibility around the critical value

From now on it will be always assumed in the statements that we are considering a function $f \in \mathcal{C}_d$ with the Fibonacci combinatorics.

We define $f^m(c) = c_m$, for $m \geqslant 1$, the points of the critical orbit. Among them there are the points $c_{s_n}$, that belong to $M_{n+1} \subset I_n$ and are in alternate sides of $c$, accordingly to the second statement of Lemma 12. We also define, for each $n \geqslant 1$, the point $z_n = (\phi_n|_{I_{n+1}})^{-1}(c)$. The points $z_n$ will help us to find the right cross-ratios in order to evaluate $Df^{s_n-1}(f(c))$.

**Lemma 15** (*Definition and placement of $z_n$*). *Let $z_n = (\phi_n|_{I_{n+1}})^{-1}(c)$, for every $n \geqslant 1$. Then $z_n$ is between $c_{s_{n-1}}$ and $c_{s_{n+1}}$.*

**Proof.** As $\phi_n(c) = c_{s_n} \in M_{n+1}$ it is easy to see that $z_n$ and $c_{s_n}$ are in opposite sides of $c$. That $z_n$ is between $c$ and $c_{s_{n-1}}$ follows from the fact that $c_{s_{n-1}} \in M_n$ and $z_n \in I_n$, and that they are both in the same side of $c$. That $c_{s_{n+1}}$ is between $z_n$ and $c$ follows from the monotonicity of $\phi_n|_{I_{n+1}}$, that send the points $z_n$, $c_{s_{n+1}}$, $c$ to $c$, $c_{s_{n+2}}$, $c_{s_n}$, in this order, and $c_{s_{n+2}}$ is between $c$ and $c_{s_n}$.   $\square$

We now identify $I_1$ with the interval $(0, 1)$ in the real line, in order to consider extendibility properties of the return map $\phi_n|_{I_{n+1}}$ not restricted by the circle topology. Regarding as an interval map, $f$ has $d$ branches in $(0, 1)$, two of them being $\phi_1|_{M_2}$ and $\phi_1|_{I_2}$.

This identification allows that *all results will be valid also in the context of interval maps with $d$ branches*. These maps are in correspondence with circle maps with (in general) $d$ points of discontinuity of the derivative. But these discontinuity points will make no difference in the arguments.

Let $g_0$ and $g_1$ be the lifts of $f$ that extend $\phi_1|_{M_2}$ and $\phi_1|_{I_2}$, respectively, to the whole line. Their difference is constant and equal to some integer $\tau$ with absolute value smaller than $d$. More precisely, defining the unitary vector $e_n = \frac{c_{s_n}-c}{|c_{s_n}-c|}$ then

$$g_1 = g_0 + \tau e_1, \tag{5}$$

for some $\tau \in \{1, \ldots, d-1\}$.

Therefore if we define by induction $g_n = g_{n-2} \circ g_{n-1}$, for $n \geqslant 2$, then $g_n$ is the homeomorphic extension of $\phi_n|_{I_{n+1}}$ to the whole line. We claim that

$$g_n \circ g_{n-1} = g_{n-1} \circ g_n + (d-1)\tau e_{n+1}. \tag{6}$$

For $n = 1$ the assertion follows after explicitly developing the equality

$$g_1 \circ g_0 = (g_0 + \tau e_1) \circ (g_1 - \tau e_1).$$

If it is true for $n$ then

$$g_{n+1} \circ g_n = g_{n-1} \circ g_n \circ g_n = (g_n \circ g_{n-1} - (d-1)\tau e_{n+1}) \circ g_n = g_n \circ g_{n+1} + (d-1)\tau e_{n+2}.$$

Another auxiliary point is $x_n$, a pre-image of one of the boundary points of $(0, 1)$.

**Lemma 16** (*Definition and placement of $x_n$*). *For $n \geqslant 1$ let $\partial_n$ be the boundary point of $(0, 1)$ which is opposite to $c_{s_n}$ with respect to $c$ and let $x_n = g_n^{-1}(\partial_n)$. Then*

(1)  $x_n$ *is between $c_{s_{n-1}}$ and $z_n$, for every $n \geqslant 2$;*
(2)  $x_n = g_{n-1}^{-1}(x_{n-2})$, *for every $n \geqslant 3$.*

**Proof.** (1) It is enough to prove that $g_n(x_n)$ is between $g_n(c_{s_{n-1}})$ and $g_n(z_n) = c$. But by (6)

$$g_n(c_{s_{n-1}}) = c_{s_{n+1}} - (d-1)\tau e_n,$$

that is, $g_n(c_{s_{n-1}})$ is outside $(0,1)$ and in the same side of $\partial_n$ with respect to $c$. The first claim follows.

(2) By definition $g_n(x_n) = \partial_n = \partial_{n-2}$, hence $g_{n-2}(g_{n-1}(x_n)) = \partial_{n-2}$. This implies that $g_{n-1}(x_n) = x_{n-2}$, proving the second claim.  □

**Lemma 17.** *For $n \geqslant 3$, the restriction of $g_n$ to $(x_n, z_{n-1})$ is a $C^1$ homeomorphism onto $(\partial_n, c_{s_{n-2}})$ having $c$ as its unique critical point.*

**Proof.** Although the statement is for $n \geqslant 3$, we first look at $g_1$ and $g_2$. Let $U_1 = I_2$ and $U_2 = g_1^{-1}(M_2)$. Then $g_1|_{U_1}$ is a $C^1$ homeomorphism onto $(0,1)$ having $c$ as its unique critical point. The same is true for $g_2|_{U_2}$, since $g_2 = g_0 \circ g_1$ and $g_0|_{M_2}$ is a diffeomorphism onto $(0,1)$. Notice that $x_1 \in \partial U_1$, $x_2 \in \partial U_2$, and $U_2 \subset U_1$.

Now we let $U_n = (x_n, z_{n-1})$, for every $n \geqslant 3$. We claim that $z_n$ and $x_{n+1}$ belong to $\overline{U}_n$, for all $n \geqslant 1$, hence $U_{n+1} \subset U_n$ for every $n \geqslant 1$. This assertion immediately follows in the case $n \geqslant 3$ from the localization Lemmas 15 and 16. Moreover, $z_1 \in U_1 = I_2$ since $z_1 = (\phi_1|_{I_2})^{-1}(c)$ and $z_2 \in U_2 = g_1^{-1}(M_2)$, since $I_3 \subset g_1^{-1}(M_2)$ and $z_2 = (\phi_2|_{I_3})^{-1}(c)$. That $x_2 \in \overline{U}_1$ follows from $x_2 \in \partial U_2$ and $U_2 \subset U_1$. That $x_3 \in \overline{U}_2$ follows from $x_3 = g_2^{-1}(x_1)$ (by the second assertion of Lemma 16), $g_2(U_2) = (0,1)$ and $x_1 \in (0,1)$.

We claim that $g_n|_{U_n}$ is a $C^1$ homeomorphism with unique critical point $c$ for all $n \geqslant 1$. As for $n = 1, 2$ it is already proved, assume by induction that $n \geqslant 3$ and that the claim is valid for $n-1$ and $n-2$. First $U_n \subset U_{n-1}$ implies that $g_{n-1}|_{U_n}$ is a $C^1$ homeomorphism with unique critical point $c$. Moreover, as $z_{n-1} \in \partial U_n$ and $g_{n-1}(z_{n-1}) = c$ then $g_{n-1}(U_n)$ intersects just one component of $U_{n-2} \setminus \{c\}$. But in fact $g_{n-1}(U_n)$ is exactly this component, since $g_{n-1}(x_n) = x_{n-2} \in \partial U_{n-2}$, by the second assertion of Lemma 16. Then $g_{n-2}$ restricted to $g_{n-1}(U_n)$ is a diffeomorphism onto its image and the composition $g_n = g_{n-2} \circ g_{n-1}$, restricted to $U_n$, is a $C^1$ homeomorphism onto its image having $c$ as its unique critical point.

As $g_n(x_n) = \partial_n$ and $g_n(z_{n-1}) = g_{n-2}(c) = c_{s_{n-2}}$ the image of $g_n|_{U_n}$ is the interval given in the assertion.  □

It is the following corollary of Lemma 17 that will be explicitly used when estimating the derivative growth using cross-ratio expansion. The notation introduced in the lemma closely follows previous works on this subject, for example [1].

**Lemma 18.** *For $n \geqslant 2$ let $x_n^f = f(x_n)$, $z_n^f = f(z_n)$, $c_{s_n}^f = f(c_{s_n})$, $c^f = f(c)$ (in this case, $x_n$, $z_n$, $c_{s_n}$ and $c$ are all in the closure of $I_2$, and $f$ means the continuous extension of the branch $\phi_1|_{I_2}$ to $\overline{I}_2$). Then, for every $n \geqslant 3$, $f^{s_n-1}$ diffeomorphically sends $(x_n^f, z_{n-1}^f)$ onto $(\partial_n, c_{s_{n-2}})$. Moreover, the points $c^f$ and $c_{s_{n+1}}^f$ are sent to $c_{s_n}$ and $c_{s_{n+2}}$.*

**Proof.** By Lemma 17, $f^{s_n}$ sends $(x_n, z_{n-1})$ homeomorphically onto $(\partial_n, c_{s_{n-2}})$ with $c$ as its unique critical point. As $c$ is a critical point for the first iterate, if $f^{s_n-1}$ restricted to $(x_n^f, z_{n-1}^f)$ had a critical point then there would exist $j < s_n$ such that $f^j(c) = c$. This would be a contradiction, since $c$ is not periodic.

Moreover, $f^{s_n-1}(c^f) = f^{s_n}(c) = c_{s_n}$ and

$$f^{s_n-1}(c_{s_{n+1}}^f) = f^{s_n}(c_{s_{n+1}}) = f^{s_{n+2}}(c) = c_{s_{n+2}}.  □$$

## 6. Derivative on the critical orbit

We will control the growth of $Df^{s_n}(f(c))$ stated in Theorem 1 using the expansion of suitably chosen (degenerated) cross-ratios, that will allow us to construct difference equations, similarly as was done in [17].

We use the notation introduced in Lemma 18 (following [1]), where $w^f = f(w)$, for every point $w \in I_2$, and define also the distances $d_n := |c_{s_n} - c|$, $d_n^f := |c_{s_n}^f - c^f|$, but just once we use $x_n^f$ to denote $|x_n^f - c|$. We also introduce the parameter

$$\lambda_n = \frac{d_n}{d_{n+2}},$$

that will play an important role in the proof. Since $c_{s_n}$ and $c_{s_{n+2}}$ are in the same side of $\{c\}$, $\lambda_n$ is always greater than one. But the real bounds of Theorem 9 give more.

**Lemma 19.** *Let $\alpha > 0$ be as in Theorem* 9, $f \in \mathcal{C}_d$ *with the Fibonacci combinatorics and $\lambda_n$ defined as above. Then*

$$\lambda_n \geqslant 1 + \alpha$$

*for every $n \geqslant 2$.*

**Proof.** There is a component $R$ of $I_{n+1} \setminus I_{n+2}$ between $c_{s_{n+2}} \in I_{n+2}$ and $c_{s_n} \in I_n \setminus I_{n+1}$. Theorem 9 implies that $|R| \geqslant \alpha |I_{n+2}|$. Then

$$d_n \geqslant R + d_{n+2} \geqslant \alpha |I_{n+2}| + d_{n+2} \geqslant (\alpha + 1) d_{n+2}. \qquad \square$$

The local approximation given in (1) gives useful estimates involving the $d_n^f$'s and also the derivative of $f$ at the $c_{s_n}$'s. One of the estimates also uses Lemma 19.

**Lemma 20.** *For sufficiently high n, there are numbers $\theta_n^{(1)}$, $\theta_n^{(2)}$ and $\theta_n^{(3)}$ going to 1 as $n \to \infty$, such that*

$$d_n^f = \theta_n^{(1)} \vartheta d_n^\ell, \tag{7}$$

$$Df(c_{s_n}) = \theta_n^{(2)} \vartheta \ell d_n^{\ell-1}, \tag{8}$$

$$\frac{d_n^f}{d_n^f - d_{n+2}^f} = \theta_n^{(3)} \frac{d_n^\ell}{d_n^\ell - d_{n+2}^\ell} = \theta_n^{(3)} \frac{1}{1 - \lambda_n^{-\ell}}, \tag{9}$$

*where $\vartheta$ is the constant given in* (1).

**Proof.** Eq. (7) follows directly from (1). From the same equation we get that

$$Df(x) = \vartheta \ell |x - c|^{\ell-1} \left(1 + \psi_1(x)\right) \tag{10}$$

for $x$ in a neighborhood of the critical point, where $\psi_1$ is a continuous function such that $\lim_{x \to c} \psi_1(x) = 0$, and this proves (8).

Let $\xi(a, x) = \frac{1-x}{1-ax}$, for $x \in [0, (1+\alpha)^{-1}]$ and $a < 1 + \alpha$, where $\alpha > 0$ is the constant given in Lemma 19. We have

$$\frac{d_n^f}{d_n^f - d_{n+2}^f} = \frac{1}{1 - \frac{d_{n+2}^f}{d_n^f}} = \frac{1}{1 - \frac{\theta_{n+2}^{(1)}}{\theta_n^{(1)}} \lambda_n^{-\ell}} = \theta_n^{(3)} \frac{1}{1 - \lambda^{-\ell}},$$

where $\theta_n^{(3)} = \xi(\frac{\theta_{n+2}^{(1)}}{\theta_n^{(1)}}, \lambda^{-\ell})$, which is defined for $n$ sufficiently high, since $\theta_n^{(1)} \to 1$. As $\xi(a, \cdot)$ uniformly tends to 1 as $a \to 1$ then $\theta_n^{(3)} \to 1$. $\square$

Now taking into account Lemma 18 we apply (2) to the diffeomorphism $f^{s_n-1}$ in the interval $(x_n^f, c^f)$ and obtain a fundamental lower bound for its derivative.

**Lemma 21.** *For sufficiently high n, there is $K_n^{(1)}$ going to 1 as $n \to \infty$ such that*

$$Df^{s_n-1}\left(c_{s_{n+1}}^f\right) \geqslant K_n^{(1)} \frac{d_n - d_{n+2}}{d_{n+1}^f} \frac{d_{n-1}^f}{d_{n-1}^f - d_{n+1}^f}.$$

**Proof.** Let $T_n = (x_n^f, c^f)$ with subintervals $L_n = (x_n^f, c_{s_{n+1}}^f)$, $R_n = (c_{s_{n+1}}^f, c^f)$ and $J_n = \{c_{s_{n+1}}^f\}$ (degenerated), which are sent, under $f^{s_n-1}$, to the intervals $(\partial_n, c_{s_n})$, $(\partial_n, c_{s_{n+2}})$, $(c_{s_{n+2}}, c_{s_n})$ and $\{c_{s_{n+2}}\}$, respectively. Eq. (2) yields

$$Df^{s_n-1}\left(c_{s_{n+1}}^f\right) \geqslant \frac{|f^{s_n-1}(L_n)|}{|f^{s_n-1}(T_n)|} \frac{d_n - d_{n+2}}{d_{n+1}^f} \frac{x_n^f}{x_n^f - d_{n+1}^f}.$$

As $|\partial_n - c|$ alternates between two positive values and $c_{s_n} \to 0$ (since $c$ is recurrent), then it follows that $K_n^{(1)} \equiv |f^{s_n-1}(L_n)|/|f^{s_n-1}(T_n)| \to 1$ as $n \to \infty$. Moreover, as $x_n^f < d_{n-1}^f$, by Lemma 16,

$$\frac{x_n^f}{x_n^f - d_{n+1}^f} = \frac{1}{1 - \frac{d_{n+1}^f}{x_n^f}} > \frac{1}{1 - \frac{d_{n+1}^f}{d_{n-1}^f}} = \frac{d_{n-1}^f}{d_{n-1}^f - d_{n+1}^f},$$

and the assertion follows. $\quad\square$

At this point but only for later use we also give, in the following lemma, an upper bound for $Df^{s_n-1}(c^f)$, using (3).

**Lemma 22.** *For sufficiently high $n \geqslant 1$,*

$$Df^{s_n-1}\big(c^f\big) \leqslant \big(K_n^{(1)}\big)^{-1} \frac{d_n - d_{n+2}}{d_{n+1}^f} \frac{d_{n-1}^f - d_{n+1}^f}{d_{n-1}^f},$$

*where $K_n^{(1)}$ is the same as in Lemma* 21.

**Proof.** Let $T_n = (x_n^f, c^f)$ with subintervals $L_n = (x_n^f, c_{s_{n+1}}^f)$, $J_n = (c_{s_{n+1}}^f, c^f)$ and $R_n = \{c^f\}$ (degenerated), which are sent, under $f^{s_n-1}$, to the intervals $(\partial_n, c_{s_n})$, $(\partial_n, c_{s_{n+2}})$, $(c_{s_{n+2}}, c_{s_n})$ and $\{c_{s_n}\}$, respectively. Eq. (3) yields

$$Df^{s_n-1}\big(c^f\big) \leqslant \frac{|f^{s_n-1}(T_n)|}{|f^{s_n-1}(L_n)|} \frac{d_n - d_{n+2}}{d_{n+1}^f} \frac{x_n^f - d_{n+1}^f}{x_n^f}$$

and the result follows similarly as in the proof of Lemma 21. $\quad\square$

We define the auxiliary parameter

$$q_n \equiv \frac{Df^{s_n-1}(c_{s_{n+1}}^f)}{Df^{s_n-1}(c^f)},$$

which is bounded away from zero and from infinity, by Lemma 14. The next lemma will be the starting point to obtain the exponential growth of $Df^{s_n-1}(c^f)$ with $n$ for $1 < \ell \leqslant 2$.

**Lemma 23.** *For sufficiently high $n$, there is $K_n^{(2)} \to 1$ as $n \to \infty$ such that*

$$\frac{Df^{s_{n+1}}(c^f)^2}{Df^{s_n}(c^f)Df^{s_{n-1}}(c^f)} \geqslant K_n^{(2)} \ell^2 \lambda_{n-1}^{2-\ell} \frac{q_{n-1}}{q_n} \frac{1 - \lambda_n^{-1}}{1 - \lambda_{n-2}^{-\ell}} \frac{1 - \lambda_{n-1}^{-1}}{1 - \lambda_{n-1}^{-\ell}}. \tag{11}$$

**Proof.** A straightforward calculation using the chain rule and $s_{n+1} = s_n + s_{n-1}$ shows that the left-hand side of (11) is equal to

$$\frac{q_{n-1}}{q_n} Df^{s_{n-1}-1}\big(c_{s_n}^f\big) Df^{s_n-1}\big(c_{s_{n+1}}^f\big) \frac{Df(c_{s_{n+1}})^2 Df(c_{s_n})}{Df(c_{s_{n-1}})}.$$

Eq. (8) implies that

$$\frac{Df(c_{s_{n+1}})^2 Df(c_{s_n})}{Df(c_{s_{n-1}})} = \theta_n^{(4)} \vartheta^2 \ell^2 \frac{d_{n+1}^{2\ell-2} d_n^{\ell-1}}{d_{n-1}^{\ell-1}},$$

where $\theta_n^{(4)} \to 1$ as $n \to \infty$. Using Lemma 21 for the two remaining factors of this product and also Lemma 20 the Inequality follows, with

$$K_n^{(2)} = \frac{\theta_n^{(4)} \theta_{n-1}^{(3)} \theta_{n-2}^{(3)}}{\theta_{n+1}^{(1)} \theta_n^{(1)}} K_n^{(1)} K_{n-1}^{(1)}. \quad\square$$

Finally, the next lemma will be useful at obtaining finer results on the growth of the derivative along the orbit of the critical value.

**Lemma 24.** *There is $C > 0$ such that, for sufficiently high $n$,*

$$C^{-1}\lambda_{n-1}^{2-\ell} \leqslant \frac{Df^{s_{n+1}}(c^f)^2}{Df^{s_n}(c^f)Df^{s_{n-1}}(c^f)} \leqslant C\lambda_{n-1}^{2-\ell}.$$

**Proof.** The first inequality is a consequence of Lemmas 14 and 19 applied to (11) of Lemma 23. The second inequality can be obtained in two stages: first by following the same arguments as in the proof of Lemma 23, using Lemma 22. And then using Lemmas 14 and 19 again.   □

## 7. Exponential growth of derivative

### 7.1. Exponential growth

We aim at proving the exponential growth of $Df^{s_n}(c^f)$ with $n$, which is the same as finding a positive $\liminf$ for $\frac{1}{n}\log Df^{s_n}(c^f)$. From Lemma 23, there is $n_0$ such that $n \geqslant n_0$ implies

$$\left[\frac{Df^{s_{n+1}}(c^f)}{1-\lambda_n^{-1}}\right]^2 \left[\frac{Df^{s_n}(c^f)}{1-\lambda_{n-1}^{-1}}\right]^{-1} \left[\frac{Df^{s_{n-1}}(c^f)}{1-\lambda_{n-2}^{-1}}\right]^{-1} \geqslant \frac{Q_{n-1}}{Q_n}\sigma_n, \tag{12}$$

where

$$Q_n = q_n\left(1-\lambda_n^{-1}\right)$$

and

$$\sigma_n = K_n^{(2)}\ell^2\lambda_{n-1}^{2-\ell}\frac{1-\lambda_{n-1}^{-1}}{1-\lambda_{n-1}^{-\ell}}\frac{1-\lambda_{n-2}^{-1}}{1-\lambda_{n-2}^{-\ell}} \tag{13}$$

(for the sake of precision, as $K_n^{(2)}$ is defined only for high $n$ in Lemma 23, let $\sigma_n$ be the minimum (positive) value such that (12) is true for the finite set of values of $n$ where $K_n^{(2)}$ is not defined). Taking logarithms on both sides and defining $\delta_n = \frac{1}{2}\log Q_n$, $\epsilon_n = \frac{1}{2}\log\sigma_n$ and

$$X_n = \log\left[\frac{Df^{s_n}(c^f)}{1-\lambda_{n-1}^{-1}}\right],$$

we obtain

$$X_{n+1} - \frac{1}{2}X_n - \frac{1}{2}X_{n-1} \geqslant \delta_{n-1} - \delta_n + \epsilon_n. \tag{14}$$

The following lemma gives an explicit "solution" of this inequation and gives an inferior limit for $\frac{X_n}{n}$ when the $\delta_n$'s are bounded.

**Lemma 25.** *Let $(X_n)_{n\geqslant 0}$ be a sequence satisfying* (14), *for arbitrary $X_0$ and $X_1$. Then*

$$X_n \geqslant \alpha_n X_0 + (1-\alpha_n)X_1 + \sum_{j=1}^{n-1}\alpha_{n-j}(\delta_{j-1}-\delta_j) + \sum_{j=1}^{n-1}\alpha_{n-j}\epsilon_j, \tag{15}$$

*for $n \geqslant 2$, where $\alpha_0 = 0$, $\alpha_1 = 1$ and, by induction, $\alpha_n = \frac{1}{2}(\alpha_{n-1}+\alpha_{n-2})$, for every $n \geqslant 2$. If moreover $|\delta_n| \leqslant c$ for every $n \geqslant 0$ and $\liminf_n \epsilon_n > -\infty$ then*

$$\liminf_n \frac{X_n}{n} \geqslant \frac{2}{3}\liminf_n \epsilon_n.$$

**Proof.** The proof of (15) can be done by straightforward induction. Remark that $\alpha_n \to \frac{2}{3}$ when $n \to \infty$ and in fact

$$\alpha_n - \frac{2}{3} = (-1)^{n+1}\frac{2}{3}2^{-n}.$$

In particular the sequence $\alpha_n$ is bounded, implying that the first two terms of the right-hand side of this inequation go to zero when divided by $n$. The third term can be written as

$$\sum_{j=1}^{n-1}\left(\alpha_{n-j} - \frac{2}{3}\right)(\delta_{j-1} - \delta_j) + \frac{2}{3}\sum_{j=1}^{n-1}\delta_{j-1} - \delta_j.$$

The first sum is bounded by a sum of a finite geometric progression (with ratio $\frac{1}{2}$) and the second is equal to $\frac{2}{3}(\delta_0 - \delta_{n-1})$, hence it is also bounded. Therefore the third term also goes to zero when divided by $n$. The last term is treated in the same way: it is equal to

$$\sum_{j=1}^{n-1}\left(\alpha_{n-j} - \frac{2}{3}\right)\epsilon_j + \frac{2}{3}\sum_{j=1}^{n-1}\epsilon_j.$$

The $\liminf$ of the first sum divided by $n$ is greater or equal than 0 since the sum is uniformly bounded away from $-\infty$, and the $\liminf$ of the second sum divided by $n$ is greater or equal $\liminf_n \epsilon_n$. $\quad\square$

Now we use this lemma to obtain a positive lower bound for $\liminf_n \frac{X_n}{n}$. We know that the $\delta_n$'s are bounded, since $\delta_n = \frac{1}{2}\log q_n(1 - \lambda_n^{-1})$, $q_n$ is uniformly bounded away from 0 and from $\infty$, by Lemma 14, and $1 - \lambda_n^{-1}$ is smaller than one and also bounded away from 0, by Lemma 19. The next step is to prove that $\liminf_n \epsilon_n > 0$ whenever $\ell \in (1, 2]$.

**Lemma 26.** *Let $\ell \in (1, 2]$ and take $\epsilon_n = \frac{1}{2}\log\sigma_n$, where $\sigma_n$ is defined as in (13). Then there is $\epsilon > 0$ such that*

$$\liminf_n \epsilon_n \geqslant \epsilon.$$

**Proof.** Let $\psi_\ell : [1, +\infty) \to \mathbb{R}$ be defined by $\psi_\ell(1) = 1$ and

$$\psi_\ell(t) = \ell\frac{1 - t^{-1}}{1 - t^{-\ell}},$$

for $t > 1$. It is a Calculus' exercise to show that $\psi_\ell$ is continuous, strictly increasing with $\lim_{t\to\infty}\psi_\ell(t) = \ell$, and also that $\ell \mapsto \psi_\ell(t_0)$ is strictly increasing for every $t_0 > 1$. With this definition, $\sigma_n = K_n^{(2)}\lambda_{n-1}^{2-\ell}\psi_\ell(\lambda_{n-1})\psi_\ell(\lambda_{n-2})$ and, as $K_n^{(2)} \to 1$ it suffices to show that $\lambda_{n-1}^{2-\ell}\psi_\ell(\lambda_{n-1})\psi_\ell(\lambda_{n-2})$ is uniformly greater than one. We divide the proof into two cases: (i) $1 < \ell < \frac{3}{2}$ and (ii) $\frac{3}{2} \leqslant \ell \leqslant 2$. In the first case, we use $\psi_\ell(x) \geqslant 1$, so that the product is greater or equal than $\lambda_{n-1}^{2-\ell}$, which in turn is greater or equal than $\sqrt{1 + \alpha}$, by Lemma 19. In the second case, the first factor is just assumed to be greater or equal than one (it is equal to one if $\ell = 2$), but $\psi_{\frac{3}{2}}(1 + \alpha)^2$ is a lower bound for the rest of the product, using the properties of $\psi_\ell$ stated above and also Lemma 19. This proves the lemma with $\epsilon = \min\{\sqrt{1 + \alpha}, \psi_{\frac{3}{2}}(1 + \alpha)^2\}$. $\quad\square$

Therefore we conclude that

$$\liminf_n \frac{X_n}{n} \geqslant \frac{2\epsilon}{3},$$

for $\ell \in (1, 2]$, where $\epsilon > 0$ is given by Lemma 26. But once again we use Lemma 19: as $1 - \lambda_{n-1}^{-1}$ is greater or equal than $\frac{\alpha}{1+\alpha}$ this implies

$$\liminf_n \frac{1}{n}\log Df^{s_n}(c^f) \geqslant \frac{2\epsilon}{3}$$

and the proof of Theorem 1 is complete.

### 7.2. The invariant measure

Lemma 27 below shows a classic summability condition on the critical orbit which has been used as a criterion to prove the existence of an absolutely continuous invariant probability measure. We refer to the work [3] and references therein, since it includes the case of maps with inflection critical points. There, in fact, the authors prove the existence with the weaker condition that $Df^j(c^f)$ goes to infinity as $j \to \infty$.

**Lemma 27.** *If $f \in \mathcal{C}_d$ is a Fibonacci critical covering with order $\ell \in (1, 2]$ then $\sum_{j=0}^{\infty} Df^j(f(c))^{-\frac{1}{\ell}}$ converges.*

**Proof.** Define the monotone sequence $H_n = \sum_{j=0}^{s_n - 1} Df^j(c^f)^{-\frac{1}{\ell}}$, which is a subsequence of the partial sums of the series in the statement, and note that

$$H_{n+1} = H_n + Df^{s_n}(c^f)^{-\frac{1}{\ell}} \sum_{j=0}^{s_{n-1}-1} Df^j(c_{s_n}^f)^{-\frac{1}{\ell}}.$$

By Lemma 14,

$$H_{n+1} \leqslant H_n + K Df^{s_n}(c^f)^{-\frac{1}{\ell}} \sum_{j=0}^{s_{n-1}-1} Df^j(c^f)^{-\frac{1}{\ell}} = H_n + K Df^{s_n}(c^f)^{-\frac{1}{\ell}} H_{n-1},$$

and, as $H_{n-1} \leqslant H_n$, we conclude that

$$H_{n+1} \leqslant \prod_{k=1}^{n} \left(1 + K Df^{s_k}(c^f)^{-\frac{1}{\ell}}\right).$$

The exponential growth of Theorem 1 shows that $H_n$ is uniformly bounded, hence the sum of the statement converges. $\quad\square$

### 7.3. Exponential decay of geometry

We decompose the proof into three parts, since the first two will be used in Section 8.

**Lemma 28.** *There is $n_0$ such that*

$$2^{-\ell-1}\vartheta|I_n|^\ell \leqslant |f(I_n)| \leqslant 4\vartheta|I_n|^\ell$$

*for all $n \geqslant n_0$, where $\vartheta > 0$ is the constant of (1). Moreover, for every interval $J \subset I_n \setminus \{c\}$,*

$$|f(J)| \geqslant \frac{\vartheta}{2}|J|^\ell.$$

**Proof.** Write $I_n = (c - a, c + b)$ for positive $a$ and $b$. As in the proof of (7),

$$\frac{\vartheta}{2}(a^\ell + b^\ell) \leqslant |f(I_n)| \leqslant 2\vartheta(a^\ell + b^\ell)$$

for large $n$. But $a$ or $b$ is greater or equal than $\frac{|I_n|}{2}$, and from this it follows the first inequality of the first expression. On the other hand, both $a$ and $b$ are smaller than $|I_n|$, and this implies the second inequality.

Now write $J = (c + a, c + b)$, with $b > a > 0$ (without loss of generality). Then, using (10) and $n$ big,

$$|f(J)| \geqslant \frac{\vartheta \ell}{2} \int_a^b |x|^{\ell-1}\,dx \geqslant \frac{\vartheta \ell}{2} \int_0^{b-a} |u|^{\ell-1}\,du = \frac{\vartheta}{2}|J|^\ell. \quad\square$$

**Lemma 29.** *Let $K > 0$ be the constant of Lemma* 11, $\vartheta$ *the constant of* (1). *There is $n_0$ such that*

$$\frac{K^{-1}\vartheta^{-1}}{4}\frac{|I_n|}{|I_{n+1}|^\ell} \leqslant Df^{s_n-1}\left(c^f\right) \leqslant 2^{\ell+1}K\vartheta^{-1}\frac{|I_n|}{|I_{n+1}|^\ell}$$

*for all $n \geqslant n_0$.*

**Proof.** As $f^{s_n-1}$ sends the interval $f(I_{n+1})$ diffeomorphically onto $I_n$ with bounded distortion, by Lemmas 10 and 11, it suffices to combine this information with Lemma 28.　□

**Proof of Theorem 3.** Let $n_0$ be given by Lemma 29 and such that $Df(c_{s_n}) \leqslant 2\vartheta\ell d_n^{\ell-1}$, given by (8). By the Chain Rule

$$\left|Df^{s_n}\left(c^f\right)\right| \leqslant 2^{\ell+2}K\ell\frac{|I_n|}{|I_{n+1}|^\ell}d_n^{\ell-1},$$

where $K > 0$ is the constant of Lemma 11. As $f^{s_n}(c) = c_{s_n} \in I_n$, then $d_n^{\ell-1}|I_n| \leqslant |I_n|^\ell$. Then we get

$$Df^{s_n}\left(c^f\right) \leqslant 2^{\ell+2}K\ell\left(\frac{|I_n|}{|I_{n+1}|}\right)^\ell. \tag{16}$$

As the left-hand side grows exponentially, by Theorem 1, then the exponential decay of geometry of Theorem 3 follows.　□

We can also prove the exponential decay of the ratio $\frac{|M_{n+1}|}{|I_n|}$, which follows from the following lemma combined with Theorem 3.

**Lemma 30.** *If n is big enough then*

$$\frac{|I_n|}{|I_{n+1}|} \leqslant 8K\left(\frac{|I_{n+1}|}{|M_{n+2}|}\right)^\ell, \tag{17}$$

*where $K > 0$ is the constant of Lemma* 11.

**Proof.** Observe that the intervals $f(M_{n+2}) \subset f(I_{n+1})$ are mapped onto the intervals $I_{n+1} \subset I_n$ by $f^{s_n-1}$, with bounded distortion, see the proof of Theorem 3. This shows that

$$\frac{|I_n|}{|I_{n+1}|} \leqslant K\frac{|f(I_{n+1})|}{|f(M_{n+2})|},$$

$K$ is the constant of Lemma 11. By Lemma 28, if $n$ is big then

$$\left|f(I_{n+1})\right| \leqslant 4\vartheta|I_{n+1}|^\ell,$$

where $\vartheta$ is the constant of (1). On the other hand $|f(M_{n+2})| \geqslant \frac{\vartheta}{2}|M_{n+2}|^\ell$, by Lemma 28, and the result follows.　□

### 7.4. Hausdorff dimension

The critical $\omega$-limit set $\omega(c)$ is a subset of

$$\Lambda = \bigcap_{n=1}^\infty \Lambda_n,$$

where

$$\Lambda_n = \left(\bigcup_{i=0}^{s_n-1} f^i(I_{n+1})\right) \cup \left(\bigcup_{j=0}^{s_{n-1}-1} f^j(M_{n+1})\right).$$

This, together with the fact that a Fibonacci critical covering $f \in \mathcal{C}_d$ has no wandering intervals imply that $\Lambda$ is a minimal invariant Cantor set and $\omega(c) = \Lambda$.

The components of $\Lambda_n$ are a covering of $\omega(c)$, for each $n \geqslant 1$. On the other hand, the intervals $f^i(I_{n+1})$ with $1 \leqslant i < s_n$ and $f^j(M_{n+1})$ with $1 \leqslant j < s_{n-1}$ are connected components of the domain of the first entry map $\mathcal{E}_n$ to $I_n$. Moreover, $f^i(I_{n+1})$ contains $f^i(I_{n+2})$ and $f^i(M_{n+2})$, while $f^j(M_{n+1})$ contains $f^{j+s_n}(I_{n+2})$. The intervals $f^i(M_{n+2})$ and $f^{j+s_n}(I_{n+2})$ are mapped by $\mathcal{E}_n$ onto $I_{n+1}$ while the interval $f^i(I_{n+2})$ is mapped by $\mathcal{E}_n$ into $M_{n+1}$.

Since the branches of $\mathcal{E}_n$ have uniformly bounded distortion, by Lemmas 10 and 11, and using Theorem 3, we get that the size of the intervals in the union $\Lambda_n$ decay at least as $a^{n^2}$, for all $n \geqslant 1$ and some $a < 1$. The number of these intervals grows at most as $2^n$ and then we conclude that the Hausdorff dimension of $\omega(c)$ is zero.

## 8. Further results

In [22] it was proved a stronger result on *a priori* bounds involved in the Fibonacci geometry which will be the basis to finer estimates for the derivative growth and the decay of geometry. Let $G_{n+1}^M$, $G_{n+1}^0$ and $G_{n+1}^I$ be the three connected components of $I_n \setminus (I_{n+1} \cup M_{n+1})$, where $G_{n+1}^0$ is the component between $I_{n+1}$ and $M_{n+1}$ and $G_{n+1}^M$ is the lateral component adjacent to $M_{n+1}$. Let

$$r_n = \max \left\{ \frac{|I_n|}{|G_{n+1}^M|}, \frac{|I_n|}{|G_{n+1}^0|}, \frac{|I_n|}{|G_{n+1}^I|} \right\}.$$

**Lemma 31** *(Levin–Świątek). For every $\ell > 1$ there is $r = r(\ell) > 0$ such that*

$$\limsup_n r_n \leqslant r,$$

*for every $f \in \mathcal{C}_d$ with the Fibonacci combinatorics and order $\ell$. In particular, for every such $f$, there is $R = R(f)$ such that $r_n \leqslant R$ for all $n \geqslant 1$.*

In particular, Theorem 3 and Lemma 30 imply that $I_{n+1}$ and $M_{n+1}$ exponentially decrease not only with respect to $I_n$ but also with respect to these components.

A basic consequence is that the distance of $c_{s_n}$ to $c$, that we called $d_n$, is comparable to $|I_n|$. As $c_{s_n} \in I_n$ we already know that $d_n \leqslant |I_n|$. The following lemma states the opposite inequality, up to a constant.

**Lemma 32.** *Let $f \in \mathcal{C}_d$ have the Fibonacci combinatorics and order $\ell > 1$. Then there is $n_0$ such that*

$$d_n \geqslant \frac{|I_n|}{2r},$$

*for every $n \geqslant n_0$, where $r = r(\ell)$ is given by Lemma 31.*

**Proof.** As $c_{s_n} \in M_{n+1}$ and $c \in |I_{n+1}|$ then $d_n > |G_{n+1}^0|$. By Lemma 31, there is $n_0$ such that $r_n \leqslant 2r$ for every $n \geqslant n_0$, in particular $|I_n| \leqslant 2r|G_{n+1}^0|$.   $\square$

This lemma, in turn, immediately allows one to relate ratios between $d_n$'s and ratios between $|I_n|$'s. For example,

$$(2r)^{-1} \frac{d_n}{d_{n+1}} \leqslant \frac{|I_n|}{|I_{n+1}|} \leqslant 2r \frac{d_n}{d_{n+1}} \tag{18}$$

for large $n$ (or with another $f$-dependent constant for any $n \geqslant 1$).

Moreover, Lemma 32 gives, for large $n$,

$$Df(c_{s_n}) \geqslant \frac{\vartheta}{2} \ell \left( \frac{|I_n|}{2r} \right)^{\ell-1},$$

implying, by Lemma 29, that there is a constant $C_1 > 0$ such that

$$Df^{s_n}(c^f) \geqslant C_1^{-1} \left( \frac{|I_n|}{|I_{n+1}|} \right)^{\ell}, \tag{19}$$

which is the counterpart of (16). This inequation, together with (16) and (18) gives a constant $C_2 > 0$ such that for large $n$

$$C_2^{-1}\left(\frac{d_n}{d_{n+1}}\right)^\ell \leqslant Df^{s_n}(c^f) \leqslant C_2\left(\frac{d_n}{d_{n+1}}\right)^\ell. \tag{20}$$

This, in turn, implies the existence of a constant $C_3 > 0$ such that

$$-\frac{C_3}{\ell} \leqslant \log \lambda_n - \frac{1}{\ell} \log Df^{s_n}(c^f) - \frac{1}{\ell} \log Df^{s_{n+1}}(c^f) \leqslant \frac{C_3}{\ell}. \tag{21}$$

If we define $X_n = \log Df^{s_n}(c^f)$ this, together with Lemma 24, gives a constant $C_4 > 0$ such that

$$-C_4 \leqslant X_{n+1} - \frac{1}{\ell}(X_{n-1} + X_n) \leqslant C_4, \tag{22}$$

for large $n$. With this inequation we will derive precise estimates on the growth of the derivative on the critical orbit. We start by the stating the following lemma, which can be easily proved by induction.

**Lemma 33.** *Let $(X_n)_n$ be a sequence satisfying*

$$X_{n+1} \leqslant \frac{1}{\ell}(X_{n-1} + X_n) + \Delta, \tag{23}$$

*for $\Delta \in \mathbb{R}$, and let $(Y_n)_n$ be another sequence satisfying $Y_0 = X_0$, $Y_1 = X_1$ and $Y_{n+1} = \frac{1}{\ell}(Y_n + Y_{n-1}) + \Delta$. Then*

$$X_n \leqslant Y_n,$$

*for all $n \geqslant 0$. Similarly, if the inequality in (23) is inverted then $X_n \geqslant Y_n$.*

Hence, estimating the growth of $Y_n$ gives estimates about the growth of $X_n$. The sequence $Y_n$, in turn, can be studied by the two-dimensional affine iteration $(u_n, v_n) \mapsto (v_n, \frac{1}{\ell}(u_n + v_n) + \Delta)$, by making $u_n = Y_{n-1}$ and $v_n = Y_n$. If $\ell \neq 2$ this iteration has the unique fixed point $(Y_*, Y_*)$, where

$$Y_* = \frac{-\Delta}{\frac{2}{\ell} - 1},$$

and eigenvalues

$$\mu_\pm = \mu_\pm(\ell) = \frac{1 \pm \sqrt{1 + 4\ell}}{2\ell},$$

where the eigenvectors associated to $\mu_\pm$ are multiples of $(1, \mu_\pm)$. The eigenvalue $\mu_-$ is negative with absolute value smaller than one, for every $\ell > 1$, giving a contractive direction. On the other hand, $\mu_+$ decreases with $\ell$ from the golden number $\omega = \frac{1+\sqrt{5}}{2}$, when $\ell = 1$, to zero, as $\ell \to \infty$, being greater than 1 for $\ell < 2$ and smaller than one for $\ell > 2$. This gives an expanding or contracting direction, for $\ell \in (1, 2)$ and $\ell > 2$, respectively. From this it is easy to conclude that:

(1) if $\ell \in (1, 2)$ and if $Y_0$ and $Y_1$ are both greater than $Y_*$ then $Y_n > 0$ for large $n$ and $\lim_n \frac{1}{n} \log Y_n = \mu_+$;
(2) if $\ell > 2$ then $|Y_n|$ is bounded.

In the case $\ell = 2$ the recurrence $Y_{n+1} = \frac{1}{2}(Y_n + Y_{n-1}) + \Delta$ has the explicit solution $Y_n = \frac{2\Delta}{3}(n-1) +$ (*a converging term*), arguing as in the proof of Lemma 25. From these considerations, we derive the conclusions in the three cases: $1 < \ell < 2$, $\ell = 2$, and $\ell > 2$.

**Proof of Theorem 5.** If $X_n = \log Df^{s_n}(c^f)$ then the sequence $(X_n)_n$ satisfies (22), for large $n$ and some (universal) $C_4 > 0$. Let $Y_* = \frac{C_4}{\frac{2}{\ell} - 1}$, which is positive, by the hypothesis. By Theorem 1, $X_n$ grows at least linearly, then there is some (perhaps) larger $n_0$ such that $X_{n_0}$ and $X_{n_0+1}$ are both greater than $Y_*$. Now let $(Y_n)_n$ be a sequence satisfying

$Y_0 = X_{n_0}$, $Y_1 = X_{n_0+1}$ and $Y_{n+1} = \frac{1}{\ell}(Y_{n-1} + Y_n) - C_4$. Then, by Lemma 33, $X_n \geqslant Y_{n-n_0}$ for every $n \geqslant n_0$. By the considerations preceding this proof, $\liminf_n \frac{1}{n} \log X_n \geqslant \mu_+(\ell) = \mu$.

On the other hand, if we redefine $(Y_n)_n$ in such a way that $Y_{n+1} = \frac{1}{\ell}(Y_{n-1} + Y_n) + C_4$ then we conclude that $\limsup_n \frac{1}{n} \log X_n \leqslant \mu_+(\ell) = \mu$.

That the geometry decays super exponentially immediately follows from (16).  $\square$

**Proof of Theorem 6.** Let $k \geqslant 1$ and $n_0$ be such that $s_{n_0} \leqslant k < s_{n_0+1}$. If $k = s_{n_0}$ then $Df^k(c^f) = Df^{s_{n_0}}(c^f)$, otherwise

$$Df^k(c^f) = Df^{s_{n_0}}(c^f) Df^{k-s_{n_0}}(c_{s_{n_0}}^f).$$

Let $k_1 = k - s_{n_0}$. As $k_1 < s_{n_0+1} - s_{n_0} = s_{n_0-1}$ then we can apply Lemma 11 to obtain

$$\left(1 + \frac{1}{\alpha_{n_0}}\right)^{-2} \leqslant \frac{Df^{k_1}(c_{s_{n_0}}^f)}{Df^{k_1}(c^f)} \leqslant \left(1 + \frac{1}{\alpha_{n_0}}\right)^2,$$

where $\alpha_{n_0}$ is such that $I_{n_0-1}$ is $\alpha_{n_0}$-well inside $I_{n_0-2}$. Now let $n_1$ such that $s_{n_1} \leqslant k_1 < s_{n_1+1}$ (certainly $n_1 \leqslant n_0 - 2$, since $k_1 < s_{n_0-1}$, see above). Then either $k_1 = s_{n_1}$ and then we stop or

$$Df^{k_1}(c^f) = Df^{s_{n_1}}(c^f) Df^{k_2}(c_{s_{n_1}}^f)$$

and

$$\left(1 + \frac{1}{\alpha_{n_0}}\right)^{-2} \leqslant \frac{Df^{k_1}(c_{s_{n_0}}^f)}{Df^{k_1}(c^f)} \leqslant \left(1 + \frac{1}{\alpha_{n_0}}\right)^2,$$

where $\alpha_{n_1}$ is such that $I_{n_1-1}$ is $\alpha_{n_1}$-well inside $I_{n_1-2}$. Following this procedure, by induction, there will be some $j \geqslant 0$ such that $k_j = s_{n_j}$. Consequently, $Df^k(c^f)$ and

$$Df^{s_{n_0}}(c^f) Df^{s_{n_1}}(c^f) \cdots Df^{s_{n_j}}(c^f)$$

will differ by a multiplicative constant

$$\prod_{i=0}^{j} \left(1 + \frac{1}{\alpha_{n_i}}\right)^2.$$

This constant is uniformly bounded (independently of $k$), since the $\alpha_{n_i}$'s exponentially grow with $i$, by Theorem 3 and Lemma 31. If $M > 0$ is such a constant, then

$$-\log M \leqslant \log Df^k(c^f) - \log Df^{s_{n_0}}(c^f) \sum_{i=0}^{j} \frac{\log Df^{s_{n_i}}(c^f)}{\log Df^{s_{n_0}}(c^f)} \leqslant \log M.$$

By Theorem 5, $\log Df^{s_n}(c^f)$ is exponential with $n$, then for large $k$ the sum in the last expression is positive and uniformly bounded from zero and infinity.

One of the inequations then gives

$$\limsup_k \frac{\log \log Df^k(c^f)}{\log k} \leqslant \limsup_k \frac{\log \log Df^{s_{n_0}}(c^f)}{\log k}.$$

But

$$1 \leqslant \frac{k}{s_{n_0}} < \frac{s_{n_0+1}}{s_{n_0}}$$

and this last quotient is bounded (it approaches the golden number $\omega$), then $k$ and $s_{n_0}$ are comparable up to a multiplicative constant. Moreover, $s_{n_0}$ is comparable to $\omega^{n_0}$. These considerations imply that

$$\limsup_k \frac{\log \log Df^k(c^f)}{\log k} \leqslant \limsup_{n_0} \frac{\log \log Df^{s_{n_0}}(c^f)}{n_0 \log \omega} = \frac{\log \mu}{\log \omega}.$$

By a similar reasoning, we obtain $\liminf_k \frac{\log\log Df^k(c^f)}{\log k} \geqslant \frac{\log\mu}{\log\omega}$ and the theorem is proved. $\quad\square$

**Proof of Theorem 8.** We know that $X_n = \log Df^{s_n}(c^f)$ satisfies (22), and by Lemma 33 that $X_n$ is bounded from above by $Y_n$, where the sequence $(Y_n)_n$ satisfies the equality $Y_{n+1} = \frac{1}{\ell}(Y_n + Y_{n-1}) + C_4$. As we concluded that $Y_n$ is bounded then $X_n$ is bounded, and the conclusion follows.

    That the geometry is bounded follows from (19). $\quad\square$

**Proof of Theorem 7.** If $X_n = \log Df^{s_n}(c^f)$ then $X_{n+1} \leqslant \frac{1}{2}(X_n + X_{n-1}) + C_4$, accordingly to (22). Once again we conclude that $X_n$ is bounded by $Y_n$ satisfying the corresponding equality and, as remarked above, $Y_n$ grows linearly with $n$. Then $X_n$ cannot grow faster than linearly and the theorem follows. $\quad\square$

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