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Good Banach spaces for piecewise hyperbolic maps via interpolation

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Abstract

We introduce a weak transversality condition for piecewise $C^{1+\alpha}$ and piecewise hyperbolic maps which admit a $C^{1+\alpha}$ stable distribution. We show bounds on the essential spectral radius of the associated transfer operators acting on classical anisotropic Sobolev spaces of Triebel–Lizorkin type which are better than previously known estimates (when our assumption on the stable distribution holds). In many cases, we obtain a spectral gap from which we deduce the existence of finitely many physical measures with basin of total measure. The analysis relies on standard techniques (in particular complex interpolation) but gives a new result on bounded multipliers. Our method applies also to piecewise expanding maps and to Anosov diffeomorphisms, giving a unifying picture of several previous results on a simpler scale of Banach spaces.

Keywords: Piecewise hyperbolic maps; Transfer operators; Spectral gap; Physical measures; SRB measures; Complex interpolation; Bounded multipliers

Proving the existence of physical measures and studying their statistical properties is an important task in dynamical systems. In this paper, we shall be concerned with maps with singularities (that is, discontinuities in the map or its derivatives). We shall assume that the map is piecewise smooth relative to a finite partition, and the most challenging case is when this partition does not have a Markov-type property.

For one-dimensional piecewise expanding maps, the space of functions of bounded variation has proved a very powerful tool, since the transfer operator acting on it has a spectral gap. This readily implies the existence of finitely many physical measures whose basins have full measure, as well as numerous other consequences. This functional approach has been extended to higher dimensional piecewise expanding maps, under stronger assumptions (the counter-examples of Tsujii [30] and Buzzi [11] show that *some* additional assumption is necessary), by considering various functional spaces (see the work of Keller, Góra and Boyarsky, Saussol, Buzzi, Tsujii, Cowieson [23,19,25,10,31,17]). On the other hand, a more elementary approach, involving a more detailed study of the dynamics and how sets are cut by the discontinuities, was developed by Young and Chernov [34,14], culminating in the article of Buzzi and Maume-Deschamps [12] where the existence of physical measures (or more generally equilibrium measures) was proved under very weak additional assumptions.

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For piecewise hyperbolic maps, finding good functional spaces on which the transfer operator has a spectral gap is a more complicated task, and the story went in the other direction, with the elementary (but very involved) arguments of Chernov and Young [13,34,14] coming first. Indeed, even for *smooth* hyperbolic dynamics, good spaces of distributions were only introduced a few years ago by Gouëzel and Liverani and Baladi and Tsujii [20,5,21,6], following the pioneering work of Blank, Keller and Liverani [8]. These spaces cannot be used for piecewise hyperbolic systems because they are not invariant under multiplication by the characteristic function of a set with smooth boundary. Only very recently, a good functional space was constructed by Demers and Liverani [18], for two-dimensional piecewise hyperbolic maps. However, the arguments in this last paper are close in spirit to the previous ones [34,14], in the sense that pieces of stable or unstable manifolds are iterated by the dynamics, and the way they are cut by the discontinuities has to be studied in a very careful way. In particular, to ensure sufficiently precise control, an essential assumption in [34,14,18] is transversality between stable or unstable manifolds and discontinuity hypersurfaces.

In this paper, we show that, under mild additional assumptions, the transfer operator of piecewise hyperbolic maps in arbitrary dimensions has a spectral gap on classical functional spaces $\mathcal{H}_p^{t,t_-}$, for suitable indices $t_- < 0 < t$ and $1 < p < \infty$. These spaces are anisotropic Sobolev spaces in the Triebel–Lizorkin class [33,27]. Moreover, we are able to replace the strong transversality assumption from [34,14,18] with a much weaker one, formulated in terms of the geometry of stable manifolds and discontinuity hypersurfaces: for instance, we allow discontinuity sets coinciding with pieces of stable manifolds. Of course, this implicitly assumes the existence of stable manifolds, and this may be the main current restriction of our approach: we require stable manifolds to exist everywhere, and to depend in a piecewise $C^{1+\alpha}$ way on the point for some $\alpha > 0$. (See also Remarks 3 and 11.)

The main novelty in this work is that, as in [25,17], we do not need to study precisely the dynamics. (It suffices that the hyperbolicity dominates the complexity growth, as measured by (3) and (4).) In particular, we do not iterate single stable or unstable manifolds (contrary to [34,14,18]), and we do not need to match nearby stable or unstable manifolds. Indeed, everything comes from the functional analytic framework. This makes it possible to get a short self-contained proof working in any dimension and with very weak transversality assumptions.

Our spaces $\mathcal{H}_p^{t,t_-}$ (or more precisely their $\tilde{\mathcal{H}}_p^{t_+,t}$ version, see Remark 11) are the same the first named author considered in [4] (with the notation $W^{t,t_+,p}$) to study smooth hyperbolic maps. The main new observation that we shall use is (Lemma 23) that these spaces are stable under multiplication by characteristic functions of nice sets, if the smoothness indices in the definition of the space are small enough with respect to the integrability index ($0 < t < 1/p$ and $0 > t_- > -1 + 1/p$). This property is well known (see the thesis [26] of Strichartz, and also [24, §4.6.3]) for classical Sobolev spaces where $t_- = 0$, and we will exploit some ideas in [26] to prove that it extends to our spaces. For this, we use complex interpolation arguments to extend easily to our spaces estimates that are straightforward for the standard Sobolev spaces. Interpolation also makes it possible to generalize the basic estimates in [4] to arbitrary differentiability (see Appendix A). Another helpful technical ingredient is the use of a "zooming" norm (45) (based on a rather standard localization principle, see Lemma 28) which allows us to go further than [4], which only dealt with specific transfer operators.

We do not believe that our upper bounds on the essential spectral radius are also lower bounds in general. However, we note that for a (nonnecessarily Markov) piecewise linear map of the unit square given by a hyperbolic matrix A of maximal eigenvalue $\lambda > 1$ (see Section 2.2), we find for each $\epsilon > 0$ a space on which the essential spectral radius of the ordinary (Perron–Frobenius) transfer operator is $\leqslant \lambda^{-1/2+\epsilon}$. This is sharper than the results in [18] (which give a bound of $\lambda^{-1/3+\epsilon}$), and may well be the optimal bound (in the strong sense of meromorphic extensions of the corresponding zeta function or essential decorrelation rate [15]). We refer also to Subsection 2.2 for examples of conservative and dissipative (sloppy) baker maps to which our results apply. (Note that none of the previous results apply to sloppy baker maps, because the transversality assumption needed in [34,18], etc. is not satisfied.)

Our proof extends the results of [4] to $C^{1+\alpha}$ Anosov diffeomorphisms with $C^{1+\alpha}$ stable and/or unstable distributions, and general C^α weights (see Remark 27). Let us also mention that our results apply to piecewise expanding and piecewise $C^{1+\alpha}$ maps for $0 < \alpha < 1$ (without transversality assumptions, but under the hypothesis that the dynamical complexity does not grow too fast), giving yet another functional space on which the results of Saussol and Cowieson [25,17], e.g., hold. This space is simply the usual Sobolev space $\mathcal{H}_p^t$ for $1 < p < \infty$ and $0 < t < \min(1/p, \alpha)$. Hence, introducing exotic spaces to study piecewise expanding maps is not necessary. This remark seems to be new even for one-dimensional piecewise expanding maps. (For *smooth* expanding maps in arbitrary dimensions, the transfer operator was studied on Sobolev spaces in [3].)

The paper is organized as follows. Section 1 contains the definitions (Definition 1) of the dynamics T considered (in particular, the condition on the stable foliation) and the spaces (Definition 9) $\mathcal{H}_p^{t,t_-}$, as well as our weak transversality condition (Definition 5), and our main result. This main result, Theorem 12, gives a bound on the essential spectral radius of the transfer operator acting on $\mathcal{H}_p^{t,t_-}$. We give in Corollary 13 the consequences of our main result on the existence of finitely many physical measures with total ergodic basin (based on a key result given in Appendix B), as well as variants of this main result under assumptions on the unstable foliation. Section 2 is devoted to a discussion of several examples, illustrating our conditions. In Section 3, we recall various classical results in functional analysis. Section 4 is the heart of the paper: it contains the basic bounds (multiplication by a function, composition by a smooth map preserving the stable foliation, multiplication by the characteristic function of a nice set) which lead to Lasota–Yorke type inequalities. In Section 5, we exploit these bounds, using a new "zooming" trick made possible by the localization property of our spaces, to prove Theorem 12.

1. Statements

Notations. If B is a Banach space, we denote the norm of an element f of B by $\|f\|_B$. In this paper, a function defined on a closed subset of a manifold is said to be C^k or C^∞ if it admits an extension to a neighborhood of this closed subset, which is C^k or C^∞ in the usual sense.

1.1. The setting

Let X be a Riemannian manifold of dimension d, and let X_0 be a compact subset of X. Let also $0 \leqslant d_s \leqslant d$ and $\alpha > 0$. We call C^1 hypersurface with boundary a codimension one C^1 submanifold of X with boundary (i.e., every point of this set has a neighborhood diffeomorphic either to $\mathbb{R}^{d-1}$ or $\mathbb{R}^{d-2} \times [0,\infty)$). For a closed subset K of X_0 we shall consider *integrable $C^{1+\alpha}$ distributions* of d_s-dimensional subspaces E^s on K. By definition, this means that for each x in a neighborhood of K, $E^s(x)$ is a d_s-dimensional vector subspace of the tangent space $\mathcal{T}_x X$, the map $x \mapsto E^s(x)$ is $C^{1+\alpha}$ and, for any $x \in K$, there exists a unique submanifold of dimension d_s containing x, defined on a neighborhood of x, and everywhere tangent to E^s. We will denote this local submanifold by $W^s_{\mathrm{loc}}(x)$, and by $W^s_\epsilon(x)$ we will mean the ball of size ϵ around x in this submanifold.

Definition 1 *(Piecewise hyperbolic maps with stable distribution).* For $\alpha > 0$, we say that a map $T : X_0 \to X_0$ is a piecewise $C^{1+\alpha}$ hyperbolic map with smooth stable distribution if

- There exists an integrable $C^{1+\alpha}$ distribution of d_s-dimensional subspaces E^s on a neighborhood of X_0.
- There exists a finite number of disjoint open subsets $O_1, \dots, O_I$ of X_0, covering Lebesgue-almost all X_0, whose boundaries are unions of finitely many compact C^1 hypersurfaces with boundary.
- For $1 \leqslant i \leqslant I$, there exists a $C^{1+\alpha}$ map T_i defined on a neighborhood of $\overline{O_i}$, which is a diffeomorphism onto its image, such that T coincides with T_i on O_i.
- For any $x \in \overline{O_i}$, there exists $\lambda_s(x) < 1$ such that, for any $v \in E^s(x)$, $DT_i(x)v \in E^s(T_i(x))$ and $|DT_i(x)v| \leqslant \lambda_s(x)|v|$.
- There exists a family of cones $C^u(x)$, depending continuously on $x \in X_0$, with $C^u(x) + E^s(x) = \mathcal{T}_x X$, such that, for any $x \in \overline{O_i}$, $DT_i(x)C^u(x) \subset C^u(T_i(x))$, and there exists $\lambda_u(x) > 1$ such that $|DT_i(x)v| \geqslant \lambda_u(x)|v|$ for any $v \in C^u(x)$.

See Remark 11 and Subsection 1.4 regarding the replacement of E^s by E^u and C^u by C^s in the above definition. Note that we do not assume that T is continuous or injective on X_0.

When $d_s = 0$, the map T is piecewise expanding. When $d_u = 0$, it is piecewise contracting (we shall see that our results are not very useful in this case). In the intermediate case, there are at the same time contracted and expanded directions. We will denote by $\lambda_{s,n}(x) < 1$ and $\lambda_{u,n}(x) > 1$ the weakest contraction and expansion constants of T^n at x.

Remark 2. The requirement that E^s is defined everywhere and $C^{1+\alpha}$ is extremely strong. Indeed, for a generic piecewise hyperbolic map, the stable direction depends only measurably on the point: the examples to which our theory applies belong therefore to a very narrow class, including most notably piecewise expanding maps, and some hyperbolic piecewise linear maps (see below for more details). However, when E^s is $C^{1+\alpha}$, its integrability is not a real issue, due to the hyperbolicity of the map. Indeed, if E^s is $C^{1+\alpha}$, it is automatically integrable at points that have a nontrivial stable manifold not cut by discontinuities (by the usual graph transform argument). When this set of points is dense, then the distribution is integrable, since integrability of $C^{1+\alpha}$ distributions is an infinitesimal property.

Remark 3. It is possible to weaken our assumption slightly, by requiring only that E^s is $C^{1+\alpha}$ on each set O_i (this can be really weaker than requiring that E^s is globally $C^{1+\alpha}$ in some specific situations, for instance when T has a Markov-like property, i.e., the preimage of a singularity set is contained in another singularity set). Indeed, our proofs still work under this weaker assumption (one should just slightly modify the definition of the Banach space we use). It is also possible to apply directly our results to this more general setting, by working on a different manifold, as follows. Assume that T is a piecewise hyperbolic map for which E^s is $C^{1+\alpha}$ on each set O_i, but not globally. Start from the disjoint union of the sets $\overline{O_i}$, and glue them together at all the points $x \in \overline{O_i} \cap \overline{O_j}$ such that E^s is $C^{1+\alpha}$ on a neighborhood of x. Then T induces a piecewise hyperbolic map on this new manifold, for which the stable distribution is globally $C^{1+\alpha}$. Indeed, since T is $C^{1+\alpha}$ on each set O_i, the set $T(O_i)$ intersects the boundaries of the sets O_j only at places where E^s is $C^{1+\alpha}$. Hence, the places in the original manifold where $\overline{O_i}$ and $\overline{O_j}$ are cut apart are not an obstruction to extending T to the new manifold. The assumption on the C^u can be similarly weakened.

We shall also need a weak transversality condition on the boundaries of the sets O_i. Some kind of transversality condition appears in every work dealing with piecewise hyperbolic maps ([34,18], etc.), although we do not know any "counter-examples". We shall use the following notion.

Definition 4 *(L-generic vector in E^s).* Let $K \subset X_0$ be a compact hypersurface with boundary and let $L \in \mathbb{Z}_+$. For $x \in K \setminus \partial K$, we say that a vector $a \in E^s(x)$ is L-*generic* with respect to K if, for any C^1 vector field v defined on a neighborhood of x, with $v(x) = a$ and $v(y) \in E^s(y)$ for any y, there exists a smaller neighborhood of x in which the intersection of Lebesgue almost every integral line of v with K has at most L points.

By "Lebesgue almost every integral line of v has some property (P)", we mean the following: let A be the set of points x such that the integral line of v through x does not have property (P), then A has Lebesgue measure 0. Equivalently, this can be formulated by requiring that the intersection of A with a transversal to the vector field v has zero measure in this transversal.

Definition 5 *(Weak transversality condition for E^s).* Let $T : X_0 \to X_0$ be a piecewise hyperbolic map with smooth stable distribution. We say that T satisfies the *weak transversality condition* if there exists $L > 0$ such that, for any $K \subset \bigcup_{i=1}^{I} \partial O_i$ which is a hypersurface with boundary, there exists a larger hypersurface with boundary K' (containing K in its interior) such that, for any $x \in K' \setminus \partial K'$, the set of tangent vectors at x that are L-generic with respect to K' has full Lebesgue measure in $E^s(x)$.[1]

The small enlargement K' of K is simply a technical point in the definition, to avoid problems at the boundary of K.

If the boundary of each O_i is a finite union of smooth hypersurfaces $K_{i1}, \dots, K_{ik_i}$, each of which is transversal to the stable direction (in the sense that $E^s(x)$ is never contained in $\mathcal{T}_x K_{ij}$), then T satisfies the weak transversality condition. However, the converse does not hold. For instance, we have the following result:

Proposition 6. *Assume that $d_s = 1$ (so that the stable manifolds are curves), and that T is a piecewise hyperbolic map with smooth stable distribution. Then T satisfies the weak transversality condition if there exists $\epsilon > 0$ such that*

$$\sup_{1 \leqslant i \leqslant I} \left\| \operatorname{Card}\left(W^s_\epsilon(x) \cap \partial O_i\right) \right\|_{L_\infty(\mathrm{Leb})} < \infty. \tag{1}$$

[1] We could replace "full Lebesgue measure" in this definition by "generic in the sense of Baire" (i.e., contains a countable intersection of dense open sets), all the following results would hold true as well, with the same proofs.

Hence, tangencies to the boundaries of the O_i's are allowed, and even flat tangencies or pieces of the boundary coinciding with W^s. The only problematic situation is when a boundary oscillates around the stable manifold, cutting it into infinitely many small pieces, as in the following pathological case: assume that $d = 2$, that E^s is the vertical direction, and that $O = \{(x, y) \mid x > y^2 \sin(1/y)\}$.

Remark 7. In fact, our proofs work even if the boundaries of the sets O_i are not piecewise smooth hypersurfaces, allowing for instance conical points, and under even weaker transversality assumptions (one should only be able to check the conclusion of our Lemma 31 below). However, it is unclear how to formulate the most general condition which would be of practical interest.

To get a result on the physical measures of finitely differentiable maps T, it is necessary to add *some* assumption on the asymptotic dynamical complexity, already for piecewise expanding maps in dimension two or higher (see [25,12,16,30] and [11]). We shall use the following way to quantify the complexity.

Let $\mathbf{i} = (i_0, \dots, i_{n-1}) \in \{1, \dots, I\}^n$. We define inductively sets $O_{\mathbf{i}}$ by $O_{(i_0)} = O_i$, and

$$O_{(i_0,\dots,i_{n-1})} = \{x \in O_{i_0} \mid T_{i_0}x \in O_{(i_1,\dots,i_{n-1})}\}. \tag{2}$$

Let also $T_{\mathbf{i}} = T_{i_{n-1}} \circ \cdots \circ T_{i_0}$, it is defined on a neighborhood of $O_{\mathbf{i}}$.

We define the complexity at the beginning[2]

$$D_n^b = \max_{x \in X_0} \operatorname{Card}\{\mathbf{i} = (i_0, \dots, i_{n-1}) \mid x \in \overline{O_{\mathbf{i}}}\}, \tag{3}$$

and the complexity at the end

$$D_n^e = \max_{x \in X_0} \operatorname{Card}\{\mathbf{i} = (i_0, \dots, i_{n-1}) \mid x \in \overline{T^n(O_{\mathbf{i}})}\}. \tag{4}$$

(For a globally invertible map T we have $D_n^e(T, \{O_i, i\}) = D_n^b(T^{-1}, \{T(O_i), i\})$. For $T(x) = 2x \bmod 1$ on [0, 1] we have $D_n^e = 2^n$, but fortunately this quantity plays no role when $d_s = 0$.)

In the piecewise expanding case, it is known that, for generic maps, the complexities D_n^b increase slowly, and therefore do not play an important role in the spectral formula (6) below (see [17]). Such a result should also hold for piecewise hyperbolic map, although it is not proved yet.

1.2. The main spectral result

We shall use spaces $\mathcal{H}_p^{t,t_-}$ which were first introduced in a dynamical setting in [4] (the local version of these spaces belongs to the Triebel–Lizorkin class, see [33,2,27] for earlier mentions of these spaces in functional analysis). Section 4 is devoted to a precise study of these spaces, and the statements in the following definition are justified there.

Let $\mathcal{F}$ denote the Fourier transform in $\mathbb{R}^d$. We will write a point $z \in \mathbb{R}^d$ as $z = (x, y)$ where $x = (z_1, \dots, z_{d_u})$ and $y = (z_{d_u+1}, \dots, z_d)$. In the same way, an element ζ of the dual space of $\mathbb{R}^d$ will be written as $\zeta = (\xi, \eta)$. The subspaces $\{x\} \times \mathbb{R}^{d_s}$ of $\mathbb{R}^d$ will sometimes be referred to as the "stable leaves" in $\mathbb{R}^d$. We say that a diffeomorphism sends stable leaves to stable leaves if its derivative has this property.

Definition 8 *(Local spaces H_p^{t,t_-}).* For $1 < p < \infty$, $t, t_- \in \mathbb{R}$, we define a space H_p^{t,t_-} of distributions in $\mathbb{R}^d$ as the (tempered) distributions u such that

$$\mathcal{F}^{-1}\left(\left(1 + |\xi|^2 + |\eta|^2\right)^{t/2}\left(1 + |\eta|^2\right)^{t_-/2}\mathcal{F}u\right) \in L_p,$$

with its canonical norm, i.e., the L_p norm of the expression above.

We will simply write H_p^t instead of $H_p^{t,0}$. This is the classical Sobolev space, see, e.g., [29] for many properties of this very classical space.

[2] In the language of [9], D_n^b is the multiplicity of the collection of sets $\{\overline{O_{\mathbf{i}}} \mid \mathbf{i} \in \{1, \dots, N\}^n\}$.

If $t \geqslant 0$, $t + t_- \leqslant 0$ and $t + |t_-| < \alpha \leqslant 1$, we shall see that H_p^{t,t_-} is invariant under $C^{1+\alpha}$ diffeomorphisms sending stable leaves to stable leaves (Remark 26). Hence, we can glue such spaces locally together in appropriate coordinate patches, to define a space $\mathcal{H}_p^{t,t_-}$ of distributions on the manifold:

Definition 9 *(Spaces $\mathcal{H}_p^{t,t_-}$ of distributions on X).* Let $t \geqslant 0$, $t + t_- \leqslant 0$ and $t + |t_-| < \alpha < 1$. Fix a finite number of $C^{1+\alpha}$ charts $\kappa_1, \dots, \kappa_J$ whose derivatives send E^s to $\{0\} \times \mathbb{R}^{d_s}$, and whose domains of definition cover a compact neighborhood of X_0, and a partition of unity $\rho_1, \dots, \rho_J$, such that the support of ρ_j is compactly contained in the domain of definition of κ_j, and $\sum \rho_j = 1$ on X_0. The space $\mathcal{H}_p^{t,t_-}$ is then the space of distributions[3] u supported on X_0 such that $(\rho_j u) \circ \kappa_j^{-1}$ belongs to H_p^{t,t_-} for all j, endowed with the norm

$$\|u\|_{\mathcal{H}_p^{t,t_-}} = \sum \big\| (\rho_j u) \circ \kappa_j^{-1} \big\|_{H_p^{t,t_-}}. \tag{5}$$

Changing the charts and the partition of unity gives an equivalent norm on the same space of distributions by Lemma 22 and Remark 26. To fix ideas, we shall view the charts and partition of unity as fixed.

Remark 10. The intuition behind this definition is that an element of H_p^{t,t_-} is C^{t+t_-} in the y direction (and the y direction is coded in the definition of the space), and C^t in the direction transverse to y (but there is no preferred transverse direction). Hence, an element of $\mathcal{H}_p^{t,t_-}$ is roughly C^{t+t_-} in the stable direction, and C^t in the transverse direction (which corresponds intuitively to the unstable direction, even though it is not always properly defined). If $t + t_- < 0$ and $t > 0$, we may therefore hope that the transfer operator acting on $\mathcal{H}_p^{t,t_-}$ has good spectral properties.

Remark 11. Note that [4] considers a slightly different space, where the stable and unstable direction and the signs of t and $t + t_-$ are exchanged. This choice is completely innocent, we also get the same results for the space of [4] (for maps with smooth unstable distribution) in Theorem 15. Intuitively, this space $\tilde{\mathcal{H}}^{t_+,t}$ is composed of functions which are C^{t+t_+} in the unstable direction (when this direction is well defined), and C^t in the transverse direction. Hence, this space behaves well when $t + t_+ > 0$ and $t < 0$.

Finally, if the stable and unstable directions exist and are smooth, we also define below a space $\tilde{\tilde{\mathcal{H}}}^{t_+,t_-}$ of functions which are essentially C^{t_+} in the unstable direction and C^{t_-} in the stable direction. This space behaves well when $t_+ > 0$ and $t_- < 0$, and yields more precise estimates than the previous spaces (but under the stronger assumption of the existence of both stable and unstable directions), see Theorem 16.

Our main result follows (recall the notation (3)–(4)):

Theorem 12 *(Spectral theorem for smooth stable distributions). Let $\alpha \in (0, 1]$. Let T be a piecewise $C^{1+\alpha}$ hyperbolic map with smooth stable distribution, satisfying the weak transversality condition. Let $1 < p < \infty$ and let t, t_- be so that $1/p - 1 < t_- < 0 < t < 1/p$, $t + t_- < 0$ and $t + |t_-| < \alpha$.*

Let $g : X_0 \to \mathbb{C}$ be a function such that the restriction of g to any O_i admits a C^α extension to $\overline{O_i}$. Define an operator $\mathcal{L}_g$ acting on bounded functions by $(\mathcal{L}_g u)(x) = \sum_{Ty=x} g(y)u(y)$. Then $\mathcal{L}_g$ acts continuously on $\mathcal{H}_p^{t,t_-}$. Moreover, its essential spectral radius is at most

$$\lim_{n\to\infty} \big(D_n^b\big)^{1/(pn)} \big(D_n^e\big)^{(1/n)(1-1/p)} \big\| g^{(n)} \,\big|\det DT^n\big|^{1/p} \max\big(\lambda_{u,n}^{-t}, \lambda_{s,n}^{-(t+t_-)}\big) \big\|_{L_\infty}^{1/n}, \tag{6}$$

where $g^{(n)} = \prod_{j=0}^{n-1} g \circ T^j$.

When we say that $\mathcal{L}_g$ acts continuously on $\mathcal{H}_p^{t,t_-}$, we should be more precise. We mean that, for any $u \in \mathcal{H}_p^{t,t_-} \cap L_\infty(\mathrm{Leb})$, then $\mathcal{L}_g u$, which is defined as a bounded function, still belongs to $\mathcal{H}_p^{t,t_-}$ and satisfies $\|\mathcal{L}_g u\|_{\mathcal{H}_p^{t,t_-}} \leqslant$

[3] On a manifold, the space of *generalized functions* supported in X_0, i.e., elements in the dual of the space of measures with smooth densities with respect to Lebesgue measure, and the space of *generalized densities* supported in X_0, i.e., elements in the dual of the space of smooth functions, are isomorphic if X_0 is compact: taking Leb any smooth Riemannian measure then $f \mapsto f\,\mathrm{dLeb}$ gives an isomorphism. "Distributions supported in X_0" (not to be confused with the integrable distributions of subspaces in Definition 1) refers in this paper to generalized functions (this avoids some Jacobians when changing variables).

$C\|u\|_{\mathcal{H}_p^{t,t_-}}$. Since the set of bounded functions is dense in $\mathcal{H}_p^{t,t_-}$ (by Lemma 18), the operator $\mathcal{L}_g$ can therefore be extended to a continuous operator on $\mathcal{H}_p^{t,t_-}$.

Note that the limit in (6) exists by submultiplicativity. Of course, we can bound $\lambda_{s,n}$ and $\lambda_{u,n}^{-1}$ by λ^n, where $\lambda < 1$ is the weakest rate of contraction/expansion of T. In some cases, it will be important to use the more precise expression given above (see, e.g., Example 3 below).

The restriction $1/p - 1 < t_- < 0 < t < 1/p$ is exactly designed so that the space $\mathcal{H}_p^{t,t_-}$ is stable under multiplication by characteristic functions of nice sets, see Lemma 23. While this feature will be used in an essential way in the proof, it also implies (see Remark 35 in Appendix B) that Dirac measures (or more generally measures supported on nice hypersurfaces) do not belong to the space $\mathcal{H}_p^{t,t_-}$. We cannot exclude that measures supported on nasty hypersurfaces belong to $\mathcal{H}_p^{t,t_-}$, but this will not be a problem.

1.3. Physical measures

The physical measures of T are by definition the probability measures μ such that there exists a set A of positive Lebesgue measure such that, for all $x \in A$, $1/n \sum_{k=0}^{n-1} \delta_{T^k x}$ converges weakly to μ.

The physical measures of T are often studied through the transfer operator $\mathcal{L}_{1/|\det DT|}$. (Note that the dual of $\mathcal{L}_{1/|\det DT|}$ preserves Lebesgue measure.) Theorem 12 becomes in this setting:

Corollary 13. *Under the assumptions of Theorem 12, assume that*

$$\lim_{n\to\infty} \left(D_n^b\right)^{1/(np)} \left(D_n^e\right)^{(1/n)(1-1/p)} \left\| \max\left(\lambda_{u,n}^{-t}, \lambda_{s,n}^{-(t+t_-)}\right) \left|\det DT^n\right|^{1/p-1} \right\|_{L_\infty}^{1/n} < 1. \tag{7}$$

Then the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ acting on $\mathcal{H}_p^{t,t_-}$ is < 1.

Together with classical arguments, this implies the following:

Theorem 14. *Under the assumptions of Theorem 12, if (7) holds, then T has a finite number of physical measures, which are invariant and ergodic, whose basins cover Lebesgue almost all X_0. Moreover, if μ is one of these measures, there exist an integer k and a decomposition $\mu = \mu_1 + \cdots + \mu_k$ such that T sends μ_j to μ_{j+1} for $j \in \mathbb{Z}/k\mathbb{Z}$, and the probability measures $k\mu_j$ are exponentially mixing for T^k and Hölder test functions.*

The deduction of this theorem from Corollary 13 is essentially folklore, but the proofs of similar results in the literature (e.g., in [8,18]) rely on additional properties of the system (existence of stable manifolds almost everywhere, and estimates of the measure of neighborhoods of singularities) that we have not established (although they certainly hold true). Instead, we prove in Appendix B a general theorem (Theorem 33) that guarantees the existence of finitely many physical measures whenever the transfer operator has a spectral gap on a space of distributions, and show (Lemma 34) that this general theorem holds in our setting. The interest of this argument is that it also applies to nonhyperbolic situations, such as (perturbations of the operators in) [32].

The results in this subsection answer the question in [4, Remark 1.1], in a much more general framework.

1.4. Hyperbolic maps with a smooth unstable distribution

Just like in Definition 1, we can define piecewise $C^{1+\alpha}$ hyperbolic maps with smooth unstable distribution. More precisely, we require the existence of an invariant integrable $C^{1+\alpha}$ distribution of d_u-dimensional subspaces $E^u(x)$ along which the dynamics is uniformly expanding, and a transverse cone $C^s(x)$ along which the dynamics is contracting. (In particular, if T is noninvertible, we assume that the unstable manifolds are independent of a choice of sequences of inverse branches.) We say that such a map satisfies the weak transversality condition for E^u if Definition 5 holds with E^u instead of E^s.

Our results also apply to such maps (by the same techniques used to prove Theorem 12), but on the space of distributions $\tilde{\mathcal{H}}^{t_+,t}$ whose norm is given in charts by $\|\mathcal{F}^{-1}((1+|\xi|^2)^{t_+/2}(1+|\xi|^2+|\eta|^2)^{t/2}\mathcal{F}u)\|_{L_p}$. More precisely:

Theorem 15 *(Spectral theorem for smooth unstable distributions). Let $\alpha \in (0,1]$. Let T be a piecewise $C^{1+\alpha}$ hyperbolic map with smooth unstable distribution, satisfying the weak transversality condition with E^s replaced by E^u. Let $1 < p < \infty$ and let t_+, t be so that $1/p - 1 < t < 0 < t_+ < 1/p$, $t + t_+ > 0$ and $|t| + t_+ < \alpha$.*

Let $g : X_0 \to \mathbb{C}$ be a function such that the restriction of g to any O_i admits a C^α extension to $\overline{O_i}$. Define an operator $\mathcal{L}_g$ acting on bounded functions by $(\mathcal{L}_g u)(x) = \sum_{Ty=x} g(y)u(y)$. Then $\mathcal{L}_g$ acts continuously on $\tilde{\mathcal{H}}_p^{t_+,t}$. Moreover, its essential spectral radius is at most

$$\lim_{n\to\infty} \left(D_n^b\right)^{1/(pn)} \left(D_n^e\right)^{(1/n)(1-1/p)} \left\| g^{(n)} \left|\det DT^n\right|^{1/p} \max\left(\lambda_{u,n}^{-(t+t_+)}, \lambda_{s,n}^{-t}\right)\right\|_{L_\infty}^{1/n}.$$

In particular, if

$$\lim_{n\to\infty} \left(D_n^b\right)^{1/(np)} \left(D_n^e\right)^{(1/n)(1-1/p)} \left\| \max\left(\lambda_{u,n}^{-(t+t_+)}, \lambda_{s,n}^{-t}\right) \left|\det DT^n\right|^{1/p-1}\right\|_{L_\infty}^{1/n} < 1,$$

then the spectral radius of $\mathcal{L}_{1/|\det DT|}$ acting on $\tilde{\mathcal{H}}_p^{t_+,t}$ is < 1. This implies that T has a finite number of ergodic physical measures whose basins cover Lebesgue almost all X_0. Moreover, if μ is one of these measures, there exist an integer k and a decomposition $\mu = \mu_1 + \cdots + \mu_k$ such that T sends μ_j to μ_{j+1} for $j \in \mathbb{Z}/k\mathbb{Z}$, and the probability measures $k\mu_j$ are exponentially mixing for T^k and Hölder test functions.

This theorem is proved just like Theorem 12. More precisely, the only nontrivial modification to be made is in the first step of the proof of Lemma 25, where the computation for the linear contribution is slightly different.

Finally, similar results hold for maps that have at the same time smooth stable and unstable distributions (and satisfy the weak transversality condition in both directions), as follows. Under this stronger assumption, let $\tilde{\tilde{\mathcal{H}}}_p^{t_+,t_-}$ be the space of distributions whose norm is given in charts by $\|\mathcal{F}^{-1}((1+|\xi|^2)^{t_+/2}(1+|\eta|^2)^{t_-/2}\mathcal{F}u)\|_{L_p}$. When $\tilde{\tilde{\mathcal{H}}}_p^{t_+,t_-}$ is well-defined, we have $\mathcal{H}_p^{t_+,t_-} \subset \tilde{\tilde{\mathcal{H}}}_p^{t_+,t_-}$ and $\tilde{\mathcal{H}}_p^{t_+,t_-} \subset \tilde{\tilde{\mathcal{H}}}_p^{t_+,t_-}$. (For the first inclusion, note that $(1+|\xi|^2)^{t_+} \leqslant (1+|\xi|^2+|\eta|^2)^{t_+}$ if $t_+ \geqslant 0$ and proceed as in the first step of the proof of Lemma 25.)

Theorem 16 *(Spectral theorem when both distributions are smooth). Let T be a piecewise $C^{1+\alpha}$ hyperbolic map with smooth stable and unstable distributions, satisfying the weak transversality conditions for E^s and E^u for $\alpha \in (0,1]$. Let $1 < p < \infty$ and let t_+, t_- be so that $1/p - 1 < t_- < 0 < t_+ < 1/p$, and $|t_-| + t_+ < \alpha$.*

Let $g : X_0 \to \mathbb{C}$ be a function such that the restriction of g to any O_i admits a C^α extension to $\overline{O_i}$. Define an operator $\mathcal{L}_g$ acting on bounded functions by $(\mathcal{L}_g u)(x) = \sum_{Ty=x} g(y)u(y)$. Then $\mathcal{L}_g$ acts continuously on $\tilde{\tilde{\mathcal{H}}}_p^{t_+,t_-}$. Moreover, its essential spectral radius is at most

$$\lim_{n\to\infty} \left(D_n^b\right)^{1/(pn)} \left(D_n^e\right)^{(1/n)(1-1/p)} \left\| g^{(n)} \left|\det DT^n\right|^{1/p} \max\left(\lambda_{u,n}^{-t_+}, \lambda_{s,n}^{-t_-}\right)\right\|_{L_\infty}^{1/n}. \tag{8}$$

The results on physical measures follow analogously. It should be noted that the results of Theorem 16 are stronger than Theorems 12 and 15, since the exponents t_+ and t_- appear independently in the estimate (8).

Once again, this theorem follows from the techniques we will use to prove Theorem 12.

2. Examples

Let us look at some applications of our results to $\mathcal{L}_{1/|\det DT|}$.

2.1. General examples

Example 1. On $[-1,1] \times \{0,1\}$, let $T(x,j) = (x/2, j)$ if $x \neq 0$, and $T(0,j) = (0, 1-j)$. This fits in our framework. Since the complexities D_n^b and D_n^e are always equal to 2, Theorem 12 gives the following bound for the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ on the classical Sobolev space $\mathcal{H}_p^{t_-}$:

$$\lim_{n\to\infty} \left\| \lambda_{s,n}^{-t_-} \left|\det DT^n\right|^{1/p-1}\right\|_{L_\infty}^{1/n} = 2^{t_- + 1 - 1/p}. \tag{9}$$

Since $t_- < 0$ is restricted by $t_- > 1/p - 1$, this bound is > 1, hence useless. This is not surprising since the physical measure, the Dirac masses at $(0,0)$ and $(0,1)$, do not belong to $\mathcal{H}_p^{t_-}$ if $1/p - 1 < t_- < 0$ (see Remark 35).

This was to be expected since the conclusion of Theorem 14 is false: the map T has two physical measures, the Dirac masses at $(0,0)$ and $(0,1)$, but these measures are not invariant!

It is nevertheless interesting to see where precisely our arguments fail. Let $\tilde{T}(x,j)=(x/2,j)$, then the transfer operators associated to T and $\tilde{T}$ acting on distributions coincide on C^∞ functions (since the difference at 0 is not seen by the integration against smooth functions). Since $\tilde{T}$ is continuous, there is no truncation term in its transfer operator, hence the results of Theorem 12 hold for the full range $t_-<0$, without the restriction $t_->1/p-1$ (with the same proof). In particular, for $t_-=-1$ and $p=2$, we get a bound $1/\sqrt{2}$ for the essential spectral radius of $\mathcal{L}_{1/\det DT}(T)=\mathcal{L}_{1/\det D\tilde{T}}(\tilde{T})$ acting on $\mathcal{H}_2^{-1}$, and Corollary 13 holds. The problem comes up in the deduction of the properties of physical measures from this bound on the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$: we need to check that the physical measures do not give weight to the discontinuities of the map, to apply Theorem 33. This is ensured by Lemma 34 when $t_->1/p-1$, but does not hold for $t_-=-1$ and $p=2$.

Example 2. Assume that $d_s=0$, i.e., T is piecewise expanding. In this case, we can take $\lambda_s=0$, and the value of t_- is irrelevant (in fact, the space $\mathcal{H}_p^{t,t_-}$ does not depend on t_-, and is the classical Sobolev space $\mathcal{H}_p^t$). The following proposition is deduced from Corollary 13 by choosing carefully the parameters t and p.

Proposition. *If T is piecewise C^2, if $d_s=0$ and $\lim\|\lambda_{u,n}^{-1}\|_{L_\infty}^{1/n}\cdot\lim(D_n^b)^{1/n}<1$, then there exist $0<t<1/p<1$ such that the spectral radius of $\mathcal{L}_{1/|\det DT|}$ acting on $\mathcal{H}_p^t$ is <1. In particular, Theorem 14 applies.*

Proof. When ϵ tends to 0, the bound on the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ acting on $\mathcal{H}_{(1-\epsilon)^{-1}}^{1-2\epsilon}$, given by Corollary 13, converges at most to $\lim_{n\to\infty}\|\lambda_{u,n}^{-1}\|_{L_\infty}^{1/n}\cdot\lim_{n\to\infty}(D_n^b)^{1/n}$. Hence, it is <1 for small enough ϵ. □

In the proof of the above proposition, we use parameters t and p very close to 1, but we are "morally" working with $\mathcal{H}_1^1$. This is not surprising since this space is essentially a space of functions with one derivative in L_1, i.e., a space of functions of bounded variation. It is well known that functions of bounded variation are useful to study piecewise expanding maps, see [16]. This proposition is analogous to results proved in [25,16] for different Banach spaces.

Example 3. When $\det DT=1$ and D_n^e, D_n^b grow subexponentially fast, then it is clear from Corollary 13 that the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ is <1 on any space $\mathcal{H}_p^{t,t_-}$ (as soon as $t>0$ and $t+t_-<0$). In some situations, it is possible to weaken (or even remove) the assumption that $\det DT=1$. For example if the unstable direction is smooth then Theorem 15 implies the following result.

Proposition. *Let T be a piecewise C^2 hyperbolic map with smooth unstable distribution satisfying the weak transversality condition, and such that D_n^e and D_n^b grow subexponentially. Assume that there exist $N>0$ and $\gamma<1$ such that $\lambda_{s,N}\leqslant\gamma|\det DT^N|$. Then there exist $p\in(1,\infty)$ and $1/p-1<t<0<t_+<1/p$ such that the essential spectral radius of $\mathcal{L}_{1/|\det DT|}$ acting on $\tilde{\mathcal{H}}_p^{t_+,t}$ is <1. In particular, T has finitely many physical measures whose basins contain Lebesgue almost every point.*

The assumption $\lambda_{s,N}\leqslant\gamma|\det DT^N|$ is satisfied whenever $d_s=1$ and $d_u>0$, or whenever $\det DT=1$.

Proof. We will take p very close to 1, $t=1/p-1+\epsilon$ and $t_+=1/p-\epsilon$ for $\epsilon>0$ very small.

We have

$$\left|\det DT^N\right|^{1/p-1}\lambda_{s,N}^{-t}\leqslant\left(\gamma^{-1}\lambda_{s,N}\right)^{1/p-1}\lambda_{s,N}^{-(1/p-1)-\epsilon}=\gamma^{1-1/p}\lambda_{s,N}^{-\epsilon}. \tag{10}$$

Since $\gamma<1$, this quantity is <1 if ϵ is small enough (in terms of p).

Moreover,

$$\left|\det DT^N\right|^{1/p-1}\lambda_{u,N}^{-(t_++t)}=\left|\det DT^N\right|^{1/p-1}\lambda_{u,N}^{1-2/p}. \tag{11}$$

When $p\to 1$, this quantity converges to $\lambda_{u,N}^{-1}<1$.

Hence, it is possible to choose p and ϵ such that

$$\big\| |\det DT^N|^{1/p-1} \max(\lambda_{s,N}^{-t}, \lambda_{u,N}^{-(t+t_+)}) \big\|_{L_\infty} < 1. \tag{12}$$

This concludes the proof. □

2.2. Piecewise linear maps

In this paragraph, we describe an explicit class of maps for which the assumptions of the previous theorems are satisfied. Let A be a $d \times d$ matrix with no eigenvalue of modulus 1. It acts on $\mathbb{R}^d$ in a hyperbolic way, with best expansion/contraction constants $\lambda_u > 1$ and $\lambda_s < 1$. Let X_0 be a polyhedral region of $\mathbb{R}^d$, and define a map T on X_0 by cutting it into finitely many polyhedral subregions $O_1, \dots, O_N$, applying A to each of them, and then mapping $AO_1, \dots, AO_N$ back into X_0 by translations.

Let $J(n)$ be the covering multiplicity of T^n, i.e., the maximal number of preimages of a point under T^n. It is submultiplicative, hence the limit $J = \lim_{n\to\infty} J(n)^{1/n}$ exists.

Proposition 17. *The map T is a piecewise hyperbolic map with smooth stable and unstable distributions* (*given by the eigenspaces of A corresponding to eigenvalues of modulus* < 1, *resp.* > 1). *It satisfies the weak transversality conditions for both stable and unstable distributions. Moreover, if $J\lambda_s < |\det A|$, there exist $1 < p < \infty$, and t_+, t_- so that $1/p - 1 < t_- < 0 < t_+ < 1/p$ and such that the essential spectral radius of $\mathcal{L}_{1/\det|DT|}$ acting on $\tilde{\tilde{\mathcal{H}}}_p^{t_+,t_-}$ is < 1. Therefore, T satisfies the conclusions of Theorem* 14.

As an example of such a map, one can take $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Cutting the torus $\mathbb{T}^2$ into finitely many squares, applying A to each of these squares, and then permuting the images of the squares, one obtains a bijection of the torus (for which $J = 1$). Hence, Proposition 17 applies. The novelty with respect to previous works such as [34,14,18] is that the sides of the squares can be taken parallel to the stable or unstable directions.

Proof of Proposition 17. The weak transversality conditions are direct consequences of the definitions.

Let K be the total number of the sides of the polyhedra O_i. Around any point x, the boundaries of the sets $O_{(i_0,\dots,i_{n-1})}$ are preimages of theses sides by one of the maps $A, \dots, A^{n-1}$, which gives at most nK possible directions. Hence, the claim p. 105 in [9] gives $D_n^b \leqslant 2(nK)^d$. This quantity grows subexponentially. In the same way, $D_n^e \leqslant 2J(n)(nK)^d$.

By Theorem 16, the essential spectral radius of $\mathcal{L}_{1/\det A}$ acting on $\tilde{\tilde{\mathcal{H}}}_p^{t_+,t_-}$ (for suitable values of p, t_+, t_-) is bounded by $J^{1-1/p}|\det A|^{1/p-1} \max(\lambda_u^{-t_+}, \lambda_s^{-t_-})$. Let us take $t_+ = 1/p - \epsilon$, $t_- = 1/p - 1 + \epsilon$ and p close to 1. Then $1/p - 1 < t_- < 0 < t_+ < 1/p$, hence Theorem 16 applies and yields the following bound for the essential spectral radius:

$$|\det A|^{1/p-1} J^{1-1/p} \max(\lambda_u^{-1/p+\epsilon}, \lambda_s^{1-1/p-\epsilon}). \tag{13}$$

If p is close to 1 and ϵ is small enough, this quantity is < 1 under the assumptions of the proposition. (Note that if $\det A = J = 1$, choosing $p = 2$ and $t_+ = 1/2 - \epsilon$, $t_- = -1/2 + \epsilon$ gives better bounds.) □

The standard conservative (piecewise affine) baker's map on the unit square is given by $T(x, y) = (2x, y/2)$ for $0 \leqslant x < 1/2$ and $T(x, y) = (2x - 1, (y + 1)/2)$ for $1/2 \leqslant x \leqslant 1$. It fits in the model of this subsection, for a diagonal matrix A with eigenvalues 2 and 1/2. The baker has an obvious Markov partition with two pieces, and can thus be analyzed by a (Lipschitz) symbolic model, which gives an essential decorrelation rate of $2^{-1/2}$ for Lipschitz observables. (The physical measure is just Lebesgue measure.) The proof of the previous proposition gives a bound $2^{-1/2+\epsilon}$ for the essential spectral radius of $\mathcal{L}_{1/\det A}$ on $\tilde{\tilde{\mathcal{H}}}_2^{1/2-\epsilon,-1/2+\epsilon}$ for arbitrarily small $\epsilon > 0$ (here $J = 1$, $\det A = 1$, $\lambda_u = 2$ and $\lambda_s = 1/2$). For a dissipative baker $T(x, y) = (2x, y/3)$ for $0 \leqslant x < 1/2$ and $T(x, y) = (2x - 1, (y + 2)/3)$ for $1/2 \leqslant x \leqslant 1$ ($\lambda_u = 2$ and $\lambda_s = 1/3$, $\det A = 2/3$ and $J = 1$), the proof of the above proposition gives a bound $2^{-1+\epsilon+(\log 3/\log 6)}$ for the essential spectral radius on $\tilde{\tilde{\mathcal{H}}}_p^{1/p-\epsilon,1/p-1+\epsilon}$ for $p = \log 6/\log 3$. (Note that the dimension of the attractor is strictly between 1 and 2 in this case.) The above two examples are piecewise affine hyperbolic maps with a finite Markov partition. But the following variant, that we shall call a "sloppy baker", does not have a finite Markov partition: let (a,b) be a point in the interior of the unit square and put $T(x,y)=(2x+a, y/2+b) \bmod 1$ for $0 \leqslant x < 1/2$ and $T(x,y)=(2x-1+a, (y+1)/2+b) \bmod 1$ for $0 \leqslant x < 1$. For almost all (a,b), the sloppy baker does not have a finite Markov partition. However, our estimate gives the same bound $2^{-1/2+\epsilon}$ for the essential spectral radius on $\tilde{\mathcal{H}}_2^{1/2-\epsilon,-1/2+\epsilon}$. Similarly, one may consider a dissipative sloppy baker, and we recover the same estimates.

3. Tools of functional analysis

In this section, we recall some classical notions of functional analysis (interpolation theory and properties of Triebel spaces), that will be useful in the next sections to study the space $H_p^{t,t-}$ and to prove our main result.

3.1. Complex interpolation

We first recall some notations and definitions from the classical complex interpolation theory of Lions, Calderón and Krejn (see, e.g., [28]). A pair $(\mathcal{B}_0,\mathcal{B}_1)$ of Banach spaces is called an interpolation couple if they are both continuously embedded in a linear Hausdorff space $\mathcal{B}$. For any interpolation couple $(\mathcal{B}_0,\mathcal{B}_1)$, we let $L(\mathcal{B}_0,\mathcal{B}_1)$ be the space of all linear operators $\mathcal{L}$ mapping $\mathcal{B}_0+\mathcal{B}_1$ to itself so that $\mathcal{L}|_{\mathcal{B}_j}$ is continuous from $\mathcal{B}_j$ to itself for $j=0,1$. For an interpolation couple $(\mathcal{B}_0,\mathcal{B}_1)$ and $0<\theta<1$, we denote by $[\mathcal{B}_0,\mathcal{B}_1]_\theta$ the complex interpolation space of parameter θ. We recall the definition: set $S=\{z\in\mathbb{C} \mid 0<\Re z<1\}$, and introduce the normed vector space

$$F(\mathcal{B}_0,\mathcal{B}_1)=\Big\{f:S\to\mathcal{B}_0+\mathcal{B}_1,\ \text{analytic, extending continuously to } \bar{S},\ \text{with } \sup_{z\in\bar{S}}\big\|f(z)\big\|_{\mathcal{B}_0+\mathcal{B}_1}<\infty,\ \text{and}$$
$$t\mapsto f(j+it) \text{ is continuous from } (-\infty,\infty) \text{ to } \mathcal{B}_j,\ j=0,1,$$
$$\text{and } \|f\|_{F(\mathcal{B}_0,\mathcal{B}_1)}:=\max_{j=0,1}\Big(\sup_t\big\|f(j+it)\big\|_{\mathcal{B}_j}\Big)<\infty\Big\}.$$

Then the complex interpolation space is defined for $\theta\in(0,1)$ by

$$[\mathcal{B}_0,\mathcal{B}_1]_\theta:=\big\{u\in\mathcal{B}_0+\mathcal{B}_1 \mid \exists f\in F(\mathcal{B}_0,\mathcal{B}_1) \text{ with } f(\theta)=u\big\}, \tag{14}$$

normed by

$$\|u\|_{[\mathcal{B}_0,\mathcal{B}_1]_\theta}=\inf_{f(\theta)=u}\|f\|_{F(\mathcal{B}_0,\mathcal{B}_1)}. \tag{15}$$

It is well known (see, e.g., [28, §1.9]) that $(\mathcal{B}_0,\mathcal{B}_1)\mapsto[\mathcal{B}_0,\mathcal{B}_1]_\theta$ is an exact interpolation functor of type θ, in the following sense: for any interpolation couple $(\mathcal{B}_0,\mathcal{B}_1)$ and every $\mathcal{L}\in L(\mathcal{B}_0,\mathcal{B}_1)$ we have

$$\|\mathcal{L}\|_{[\mathcal{B}_0,\mathcal{B}_1]_\theta\to[\mathcal{B}_0,\mathcal{B}_1]_\theta}\leqslant\|\mathcal{L}\|_{\mathcal{B}_0\to\mathcal{B}_0}^{1-\theta}\|\mathcal{L}\|_{\mathcal{B}_1\to\mathcal{B}_1}^{\theta}\quad\forall\theta\in(0,1). \tag{16}$$

The above bound will be used several times throughout this work.

3.2. A class of Sobolev-like spaces containing the local spaces $H_p^{t,t-}$

Let S be the Schwartz space of C^∞ rapidly decaying functions. Its dual S' is the space of tempered distributions.

Let M be the set of functions a from $\mathbb{R}^d$ to $\mathbb{R}_+$ such that there exists $C>0$ such that, for all multi-indices $\gamma=(\gamma_1,\dots,\gamma_d)$ with $\gamma_j\in\{0,1\}$, and all $\zeta\in\mathbb{R}^d$,

$$\left|\prod_{j=1}^d\big(1+\zeta_j^2\big)^{\gamma_j/2}D^\gamma a(\zeta)\right|\leqslant Ca(\zeta). \tag{17}$$

For $a\in M$ and $p\in(1,\infty)$, let us define a space H_p^a as the space of all tempered distributions u such that $\mathcal{F}^{-1}(a\mathcal{F}u)$ belongs to L_p, with its canonical norm

$$\|u\|_{H_p^a}=\big\|\mathcal{F}^{-1}(a\mathcal{F}u)\big\|_{L_p(\mathbb{R}^d)}. \tag{18}$$

These spaces were introduced and studied by Triebel in [27], in a slightly more general setting involving another parameter q (under a different form [27, Definition 2.3/4], but Theorem 5.1/2 and Remark 5.1 there shows that it is equivalent to the previous description for $q = 2$).

Among other things, Triebel proved the following results concerning these spaces:

Lemma 18. *For any $a \in M$ and $1 < p < \infty$, the space S is contained in H_p^a, and dense.*

Proof. This is proved in Theorem 3.2/2 and Remark 3.2/2 in [27]. □

For $t, t_- \in \mathbb{R}$, the function

$$a_{t,t_-}(\xi, \eta) = \left(1 + |\xi|^2 + |\eta|^2\right)^{t/2}\left(1 + |\eta|^2\right)^{t_-/2} \tag{19}$$

belongs to M. Then H_p^{t,t_-} from Definition 8 is just $H_p^{a_{t,t_-}}$, and the previous lemma says that S is dense in H_p^{t,t_-}.

Proposition 19 *(Interpolation). For any a_0, $a_1 \in M$, p_0, $p_1 \in (1, \infty)$ and $\theta \in (0, 1)$, the interpolation space $[H_{p_0}^{a_0}, H_{p_1}^{a_1}]_\theta$ is equal to H_p^a for $a = a_0^{1-\theta} a_1^\theta$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$.*

Proof. This is [27, Theorem 4.2/2]. □

We will also use the following straightforward lemma. (Note that if $a \in M$ then $1/a \in M$, see, e.g., [27, Lemma 2.1/1].)

Lemma 20 *(Duality). For any $a \in M$ and $1 < p < \infty$, the dual of the space H_p^a is $H_{p'}^{1/a}$ for $1/p + 1/p' = 1$.*

3.3. Multiplier theorems

In order to understand the spaces H_p^a, an essential tool is provided by Fourier multiplier theorems. The following Marcinkiewicz multiplier theorem (see, e.g., [27, Theorem 2.4/2]) will be sufficient for our purposes.

Theorem 21. *Let $b \in C^d(\mathbb{R}^d)$ satisfy $|\zeta^\gamma D^\gamma b(\zeta)| \leqslant B$ for all multi-indices $\gamma = (\gamma_1, \dots, \gamma_d)$ with $\gamma_j \in \{0, 1\}$, and all $\zeta \in \mathbb{R}^d$. Then, for all $p \in (1, \infty)$, there exists a constant $C(p, d)$ such that, for any $u \in L_p$,*

$$\left\| \mathcal{F}^{-1}(b\mathcal{F}u) \right\|_{L_p} \leqslant CB\|u\|_{L_p}. \tag{20}$$

4. Towards Lasota–Yorke bounds on the local space H_p^{t,t_-}

Aiming at the proof of Theorem 12 on transfer operators, we describe in Subsections 4.1 and 4.2 how the local spaces H_p^{t,t_-}, which are the building blocks of our spaces of distributions, behave under multiplication by a smooth function or by the characteristic function of a nice set, as well as under composition with a smooth map preserving the stable leaves. Then, in Section 4.3, we state and prove a localization principle on H_p^{t,t_-} that we were not able to find in the literature and which plays a key part in the "zooming" procedure in the proof of Theorem 12. Note for further use that since X_0 is compact, [4, Lemma 2.2] (e.g.) gives that the inclusion $\mathcal{H}_p^{t,t_-} \subset \mathcal{H}_p^{t',t'_-}$ for $t' \leqslant t$ and $t'_- \leqslant t_-$ is compact if $t' < t$.

To study H_p^{t,t_-}, we will mainly study $H_p^{t,0}$ and H_p^{0,t_-} and use interpolation (via Proposition 19). It is therefore useful to recall some classical properties of these spaces.

When $t \geqslant 0$, the space H_p^t is the classical Sobolev space. By [26, Theorem I.4.1], it satisfies a Fubini property: if u is a function on $\mathbb{R}^d$, define a function u_j on $\mathbb{R}^{d-1}$ as follows: $u_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$ is the $H_p^t(\mathbb{R})$-norm of the restriction of u to the line $\{(x_1, \dots, x_{j-1}, x, x_{j+1}, \dots, x_d) \mid x \in \mathbb{R}\}$. Then u belongs to $H_p^t(\mathbb{R}^d)$ if and only if each u_j belongs to $L_p(\mathbb{R}^{d-1})$, and the norms $\|u\|_{H_p^t}$ and $\sum_{j=1}^d \|u_j\|_{L_p}$ are equivalent. (This is true for any set of coordinates, but for simplicity we shall use a fixed system of coordinates.) This makes it often possible to study only the one-dimensional situation, and extend it readily to d dimensions.

For $t_- > 0$, the space H_p^{0,t_-} also has a Fubini-type property: the norm $\|u\|_{H_p^{0,t_-}}$ is equivalent to $\sum_{j=d_u+1}^{d} \|u_j\|_{L_p}$ where u_j is the $H_p^{t_-}(\mathbb{R})$-norm of a restriction of u as above (the proof of [26, Theorem I.4.1] directly applies, we may take any coordinates on $\mathbb{R}^d$ which preserve the stable leaves of the original coordinate system used to define H_p^{0,t_-}, for simplicity we shall fix this original coordinate system). In particular, the study of H_p^{0,t_-} reduces to the study of the usual Sobolev space in one dimension.

Finally, for $t_- \in \mathbb{R}$, the space H_p^{0,t_-} also has a slightly different Fubini-type property. Let u be a function on $\mathbb{R}^d$, and define a function v on $\mathbb{R}^{d_u}$ as follows: $v(x)$ is the $H_p^{t_-}(\mathbb{R}^{d_s})$-norm of the restriction of u to $\{x\} \times \mathbb{R}^{d_s}$. Then $\|u\|_{H_p^{0,t_-}(\mathbb{R}^d)} = \|v\|_{L_p(\mathbb{R}^{d_u})}$: this follows from the fact that the function $(1+|\eta|^2)^{t_-/2}$ does not depend on the variable ξ, which makes it possible to integrate away the variable x using the Fourier inversion formula (see [26, p. 1045] for details).

We will refer to these properties respectively as the one-dimensional and the d_s-dimensional Fubini properties of H_p^{0,t_-}.

4.1. Multiplication by functions

Lemma 22. *Let $t > 0$, $t_- < 0$ and $\alpha > 0$ be real numbers with $t + |t_-| < \alpha$. For any $p \in (1, \infty)$, there exists a constant $C_\#$ such that for any C^α function $g : \mathbb{R}^d \to \mathbb{C}$, for any distribution $u \in H_p^{t,t_-}$, the distribution gu also belongs to H_p^{t,t_-} and satisfies*

$$\|g \cdot u\|_{H_p^{t,t_-}} \leqslant C_\# \|g\|_{C^\alpha} \|u\|_{H_p^{t,t_-}}.$$

The assertion $gu \in H_p^{t,t_-}$ should be interpreted as explained after Theorem 12.

Proof. Let $t^0 = t + |t_-|$, $t_-^0 = -t^0$ and $\theta = t/t^0$, so that $(t, t_-) = (\theta t^0, (1-\theta)t_-^0)$ and $\max(t^0, |t_-^0|) < \alpha$. We will write H_p^{t,t_-} as an interpolation space with parameter θ between $H_p^{t^0}$ and H_p^{0,t_-^0}, thereby reducing the proof to the study of $H_p^{t^0}$ and H_p^{0,t_-^0}.

First, since $H_p^{t^0}$ is the classical Sobolev space, [29, Corollary 4.2.2] shows that

$$\|gu\|_{H_p^{t^0}} \leqslant C_\# \|g\|_{C^\alpha} \|u\|_{H_p^{t^0}}, \tag{21}$$

where $C_\#$ depends only on t^0 and α, whenever $|t^0| < \alpha$.

Together with the d_s-dimensional Fubini-type property of H_p^{0,t_-^0}, this readily implies

$$\|gu\|_{H_p^{0,t_-^0}} \leqslant C_\# \|g\|_{C^\alpha} \|u\|_{H_p^{0,t_-^0}} \tag{22}$$

whenever $|t_-^0| < \alpha$.

Interpolating between (21) and (22) via Proposition 19, we get the conclusion of the lemma. □

The following extension of a classical result of Strichartz is the key to our results:

Lemma 23. *Let $1 < p < \infty$ and $1/p - 1 < t_- \leqslant 0 \leqslant t < 1/p$. There exists a constant $C_\#$ satisfying the following property. Let $N \geqslant 1$, and let O be a set in $\mathbb{R}^d$ whose intersection with almost every line parallel to some coordinate axis has at most N connected components. Then, for any $u \in H_p^{t,t_-}$, the distribution $1_O u$ also belongs to H_p^{t,t_-}, and satisfies*

$$\|1_O u\|_{H_p^{t,t_-}} \leqslant C_\# N \|u\|_{H_p^{t,t_-}}. \tag{23}$$

Proof. If $t \in [0, 1/p)$ and the dimension is 1, a result of Strichartz [26, Corollary II.4.2] shows that, for any interval I of $\mathbb{R}$ and any function $u \in H_p^t(\mathbb{R})$, then $1_I u$ belongs to $H_p^t(\mathbb{R})$ and its norm is bounded by $C_\# \|u\|_{H_p^t(\mathbb{R})}$, for some universal constant $C_\#$ depending only on t, p. If O is a union of N intervals $I_1, \ldots, I_N$, this yields

$$\|1_O u\|_{H_p^t(\mathbb{R})} \leqslant \sum \|1_{I_i} u\|_{H_p^t(\mathbb{R})} \leqslant C_\# N \|u\|_{H_p^t(\mathbb{R})}. \tag{24}$$

For $d > 1$, the Fubini property of $H_p^t(\mathbb{R}^d)$ (described at the beginning of Section 4) shows that the previous property extends from $\mathbb{R}$ to $\mathbb{R}^d$: if a set O intersects almost every line parallel to a coordinate axis along at most N connected components, then $\|1_O u\|_{H_p^t} \leqslant C_\# N \|u\|_{H_p^t}$. See also [24, §4.6.3] for alternative sufficient conditions on O and p, t ensuring that 1_O is a multiplier of H_p^t.

Assume now that $t = 0$ and $t_- \in (0, 1/p)$. Since the space H_p^{0,t_-} also has the Fubini property, the previous argument still applies, and gives $\|1_O u\|_{H_p^{0,t_-}} \leqslant C_\# N \|u\|_{H_p^{0,t_-}}$. If $t = 0$ and $t_- \in (1/p - 1, 0)$, the same result follows by duality.

Interpolating via Proposition 19, the set of parameters $(1/p, t, t_-)$ for which the conclusion of the lemma holds is convex. It therefore contains the convex hull of $\{(1/p, t, 0) \mid 0 \leqslant t < 1/p\}$ and $\{(1/p, 0, t_-) \mid 1/p - 1 < t_- \leqslant 0\}$, which coincides with the set $\{(1/p, t, t_-) \mid 1/p - 1 < t_- \leqslant 0 \leqslant t < 1/p\}$. □

4.2. Composition with smooth maps preserving the stable leaves

In this paragraph, we study the behavior of H_p^{t,t_-} under the composition with smooth maps preserving the stable leaves.

Let us start with a very rough and easy to prove lemma.

Lemma 24. *Let $1 < p < \infty$, and t, t_- be real numbers with $|t| + |t_-| \leqslant 1$. There exists a constant $C_\#$ such that, for any invertible matrix A on $\mathbb{R}^d$, sending $\{0\} \times \mathbb{R}^{d_s}$ to itself, and for any $u \in H_p^{t,t_-}$,*

$$\|u \circ A\|_{H_p^{t,t_-}} \leqslant C_\# |\det A|^{-1/p} \max\left(\|A\|, \left\|A^{-1}\right\|\right) \|u\|_{H_p^{t,t_-}}. \tag{25}$$

Proof. By [24, Proposition 2.1.2(iv)+(vii)], the H_p^1-norm is equivalent to the norm $\|u\|_{L_p} + \|Du\|_{L_p}$. Hence, $\|u \circ A\|_{H_p^{1,0}} \leqslant C_\# |\det A|^{-\frac{1}{p}} \max(\|A\|, 1) \|u\|_{H_p^{1,0}}$. Similarly, $\||\det A|^{-1} u \circ A^{-1}\|_{H_{p'}^{0,1}} \leqslant C_\# |\det A|^{-1+\frac{1}{p'}} \max(\|A^{-1}\|, 1) \times \|u\|_{H_{p'}^{0,1}}$, by a d_s-dimensional Fubini-type argument. Since the adjoint of $u \mapsto |\det A|^{-1} u \circ A^{-1}$ is $u \mapsto u \circ A$, the general case follows by duality (Lemma 20) and interpolation (Proposition 19). □

Lemma 25. *Let $\alpha \in (0, 1)$, let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a $C^{1+\alpha}$ diffeomorphism sending stable leaves to stable leaves, and let A be a matrix such that, for all $z \in \mathbb{R}^d$, $\|A^{-1} \circ DF(z)\| \leqslant 2$ and $\|DF(z)^{-1} \circ A\| \leqslant 2$.*

Assume moreover that A can be written as $M_0^{-1} \begin{pmatrix} A^u & 0 \\ 0 & A^s \end{pmatrix} M_1$, where M_0 and M_1 are matrices sending stable leaves to stable leaves, and $\mu_u := \|A^u\| \leqslant 1$, $\mu_s := \|(A^s)^{-1}\|^{-1} \geqslant 1$.[4]

Then, for all $t > 0$ and $t_- < 0$ with $t + |t_-| < \alpha$ and $t + t_- < 0$, for all $p \in (1, \infty)$, there exists a constant $C_\#$ depending only on $\max(\|M_0\|, \|M_0^{-1}\|, \|M_1\|, \|M_1^{-1}\|)$ and t, t_-, p, and a constant $C(A, F)$ such that, for all $u \in H_p^{t,t_-}$,

$$\|u \circ F\|_{H_p^{t,t_-}} \leqslant C_\# \| \det A / \det DF\|_{C^\alpha} |\det A|^{-1/p} \max\left(\mu_u^t, \mu_s^{t+t_-}\right) \|u\|_{H_p^{t,t_-}} + C\|u\|_{H_p^{0,t_-}}.$$

In the applications to transfer operators, F will be the local *inverse* of some iterate T^n of a piecewise hyperbolic map. Since T^n is contracting along E^s and expanding along E^u, the map F will therefore satisfy the assumptions of the lemma regarding μ_s and μ_u.

Proof of Lemma 25. We will write $u \circ F = u \circ A \circ A^{-1} \circ F$. Hence, we need to study the composition with A and $A^{-1} \circ F$. We claim that

$$\|u \circ A\|_{H_p^{t,t_-}} \leqslant |\det A|^{-1/p} C_\# \max\left(\mu_u^t, \mu_s^{t+t_-}\right) \|u\|_{H_p^{t,t_-}} + C\|u\|_{H_p^{0,t_-}} \tag{26}$$

and

[4] The matrix norms are the operator norms with respect to the usual euclidean metric on $\mathbb{R}^d$, so that the norm of a matrix equals the norm of its transpose.

$$\|u \circ A^{-1} \circ F\|_{H_p^{t,t_-}} \leqslant C_\# \|\det A / \det DF\|_{C^\alpha} \|u\|_{H_p^{t,t_-}}. \tag{27}$$

Together, these equations prove the lemma.

First step. Let us prove (26). This is a special case of [4, Lemma 2.10] (replacing $(0, t_-)$ by $(t - 1/2, t_-)$). We will give the proof for the convenience of the reader, since it is at the same time very simple and at the heart of our argument. Lemma 24 deals with the composition with M_0^{-1} and M_1, hence we can assume that $M_0 = M_1 = \mathrm{Id}$.

We want to estimate $\|u \circ A\|_{H_p^{t,t_-}} = \|\mathcal{F}^{-1}(a_{t,t_-}\mathcal{F}(u \circ A))\|_{L_p}$, where a_{t,t_-} is defined in (19). A change of variables readily gives $\mathcal{F}^{-1}(a_{t,t_-}\mathcal{F}(u \circ A)) = \mathcal{F}^{-1}(a_{t,t_-} \circ {}^tA \cdot \mathcal{F}u) \circ A$. Hence, we have to show that

$$\|\mathcal{F}^{-1}(a_{t,t_-} \circ {}^tA \cdot \mathcal{F}u)\|_{L_p} \leqslant C_\# \max(\mu_u^t, \mu_s^{t+t_-}) \|u\|_{H_p^{t,t_-}} + C\|u\|_{H_p^{0,t_-}}. \tag{28}$$

Write ${}^tA = \begin{pmatrix} U & 0 \\ 0 & S \end{pmatrix}$ with $|U\xi| \leqslant \mu_u|\xi|$ and $|S\eta| \geqslant \mu_s|\eta|$ by definition of μ_u, μ_s. Let

$$b(\xi,\eta) = a_{t,t_-} \circ {}^tA(\xi,\eta) = (1 + |U\xi|^2 + |S\eta|^2)^{t/2}(1 + |S\eta|^2)^{t_-/2}. \tag{29}$$

Let us prove that, if C is large enough, we have

$$b \leqslant C_\# \max(\mu_u^t, \mu_s^{t+t_-}) a_{t,t_-} + C a_{0,t_-}. \tag{30}$$

Assume that we can prove this equation, as well as the corresponding estimates for the successive derivatives of b, i.e., $|\zeta^\gamma D^\gamma(b/(C_\# \max(\mu_u^t, \mu_s^{t+t_-})a_{t,t_-} + Ca_{0,t_-}))| \leqslant C_\#$ for any $\gamma = (\gamma_1, \dots, \gamma_d)$ with $\gamma_j \in \{0,1\}$, and any $\zeta \in \mathbb{R}^d$. Then Theorem 21 applied to $b/(C_\# \max(\mu_u^t, \mu_s^{t+t_-})a_{t,t_-} + Ca_{0,t_-})$ gives

$$\|\mathcal{F}^{-1}(b\mathcal{F}u)\|_{L_p} \leqslant C_\# \|\mathcal{F}^{-1}((C_\# \max(\mu_u^t, \mu_s^{t+t_-})a_{t,t_-} + Ca_{0,t_-})\mathcal{F}u)\|_{L_p}, \tag{31}$$

which yields (28).

Let us now prove (30) (the proof for the derivatives of b is similar). We will freely use the following trivial inequalities: for $x \geqslant 1$ and $\lambda \geqslant 1$,

$$\frac{1}{\lambda}(1 + \lambda x) \leqslant 1 + x \leqslant \frac{2}{\lambda}(1 + \lambda x). \tag{32}$$

Assume first $|U\xi|^2 \leqslant |S\eta|^2$ and $|S\eta|^2 \geqslant 1$. Then, since $t > 0$ and $t + t_- < 0$,

$$\begin{aligned} b(\xi,\eta) &\leqslant (1 + 2|S\eta|^2)^{t/2}(1 + |S\eta|^2)^{t_-/2} \leqslant 2^{t/2}(1 + |S\eta|^2)^{t/2}(1 + |S\eta|^2)^{t_-/2} \\ &\leqslant 2^{t/2}(1 + \mu_s^2|\eta|^2)^{(t+t_-)/2} \leqslant 2^{t/2}(\mu_s^2/2)^{(t+t_-)/2}(1 + |\eta|^2)^{(t+t_-)/2} \\ &\leqslant 2^{-t_-/2}\mu_s^{(t+t_-)}a_{t,t_-}(\xi,\eta). \end{aligned}$$

If $|U\xi|^2 \geqslant |S\eta|^2$ and $|U\xi|^2 \geqslant 1$, then

$$\begin{aligned} b(\xi,\eta) &\leqslant (1 + 2|U\xi|^2)^{t/2}(1 + |S\eta|^2)^{t_-/2} \leqslant 2^{t/2}(1 + |U\xi|^2)^{t/2}(1 + \mu_s^2|\eta|^2)^{t_-/2} \\ &\leqslant 2^{t/2}(1 + \mu_u^2|\xi|^2)^{t/2}(1 + |\eta|^2)^{t_-/2} \leqslant 2^{t/2}(2\mu_u^2)^{t/2}(1 + |\xi|^2)^{t/2}(1 + |\eta|^2)^{t_-/2} \\ &\leqslant 2^t\mu_u^t a_{t,t_-}(\xi,\eta). \end{aligned}$$

In the remaining case, ξ and η are uniformly bounded, and (30) follows by choosing C large enough. This concludes the proof of (26).

Second step. Let us now prove (27). We will write $\tilde{F} = A^{-1} \circ F$. As in the proof of Lemmas 22, 23, and 24, we will study simpler spaces before concluding by interpolation. We thus write $(t, t_-) = (\theta t^0, (1-\theta)t_-^0)$ for some $0 < \theta < 1$ and $t^0, -t_-^0 \in (0, \alpha)$. Let us note that the derivatives of $\tilde{F}$ and $\tilde{F}^{-1}$ are everywhere bounded by 2, hence their determinants are bounded by 2^d.

By [24, Proposition 2.1.2(iv)+(vii)], the H_p^1-norm is equivalent to the norm $\|u\|_{L_p} + \|Du\|_{L_p}$. Since the derivative of $\tilde{F}$ has norm everywhere bounded by 2 and $|\det D\tilde{F}| \leqslant 2^d$ by assumption, we get after a change of variables $\|u \circ \tilde{F}\|_{H_p^1} \leqslant C_\#\|u\|_{H_p^1}$. Since $\|u \circ \tilde{F}\|_{L_p} \leqslant C_\#\|u\|_{L_p}$, the interpolation inequality (16) between H_p^1 and $L_p = H_p^0$ gives

$$\|u \circ \tilde{F}\|_{H_p^{t^0}} \leqslant C_\#\|u\|_{H_p^{t^0}}. \tag{33}$$

Applying the same argument via Fubini to $\tilde F^{-1}$ on each leaf of the vertical direction, we also have $\|u\circ\tilde F^{-1}\|_{H^{0,1}_{p'}}\leqslant C_\#\|u\|_{H^{0,1}_{p'}}$. The adjoint of the composition by $\tilde F^{-1}$ is given by $\mathcal P(u)=\det D\tilde F\cdot u\circ\tilde F$. Hence, duality yields $\|\mathcal P u\|_{H^{0,-1}_p}\leqslant C_\#\|u\|_{H^{0,-1}_p}$. Since $\mathcal P$ is bounded by $C_\#$ on L_p, we get by interpolation between $H^{0,-1}_p$ and $L_p=H^{0,0}_p$

$$\|\det D\tilde F\cdot u\circ\tilde F\|_{H_p^{0,t_-^0}}\leqslant C_\#\|u\|_{H_p^{0,t_-^0}}. \tag{34}$$

Together with (22), we obtain

$$\|u\circ\tilde F\|_{H_p^{0,t_-^0}}\leqslant C_\#\|1/\det D\tilde F\|_{C^\alpha}\|\det D\tilde F\cdot u\circ\tilde F\|_{H_p^{0,t_-^0}}\leqslant C_\#\|1/\det D\tilde F\|_{C^\alpha}\|u\|_{H_p^{0,t_-^0}}. \tag{35}$$

Interpolating between (33) and (35), we get

$$\|u\circ\tilde F\|_{H_p^{t,t_-}}\leqslant C_\#\|1/\det D\tilde F\|_{C^\alpha}^{1-\theta}\|u\|_{H_p^{t,t_-}}. \tag{36}$$

Finally, $1/\det D\tilde F=\det A/\det DF$ is bounded from below, and (27) follows. □

Remark 26 *(Invariance).* The arguments in the second step of the proof of Lemma 25 (with $A=\mathrm{Id}$) also imply that, whenever $t>0$ and $t_-<0$ satisfy $t+|t_-|<\alpha$, then the space H_p^{t,t_-} is invariant under the composition with $C^{1+\alpha}$ diffeomorphisms of $\mathbb R^d$ sending stable leaves to stable leaves.

Remark 27 *(Extending [4] to $C^{1+\alpha}$ Anosov diffeomorphisms).* If $0<\alpha<1$ we can apply Lemma 25. If $\alpha\geqslant 1$ and $t>0$, $t+t_-<0$ satisfy $t+|t_-|<\alpha$, letting m be the smallest integer $\geqslant t+|t_-|$, [24, Proposition 2.1.2(iv)+(vii)], implies that the H_p^m-norm is equivalent to the norm $\sum_{|\gamma|\leqslant m}\|\partial^\gamma u\|_{L_p}$. Thus, replacing the matrix A in Lemma 25 by a C^∞ diffeomorphism A preserving stable leaves, with least expansion $\mu_s\geqslant 1$ on the verticals, and whose inverse preserves horizontal cones with least expansion $\mu_s^{-1}\geqslant 1$, and such that $\|DA^{-1}\circ DF\|_{C^{m-1}}\leqslant 2$ and $\|DF^{-1}\circ DA\|_{C^{m-1}}\leqslant 2$, we get, by applying [4, Lemma 2.10] to prove the analogue of (26), that

$$\|u\circ F\|_{H_p^{t,t_-}}\leqslant C_\#\|\det DA/\det DF\|_{C^\alpha}|\det DA|^{-1/p}\max\left(\mu_u^t,\mu_s^{t+t_-}\right)\|u\|_{H_p^{t,t_-}}+C\|u\|_{H_p^{t-1/2,t_-}}.$$

The proof of Theorem 12 then applies to any $C^{1+\alpha}$ Anosov diffeomorphism T with $C^{1+\alpha}$ stable distribution, and to any C^α weight g, with $\alpha>0$.

4.3. Localization

Lemma 28 *(Localization principle). Let $\eta:\mathbb R^d\to[0,1]$ be a C^∞ function with compact support and write $\eta_m(x)=\eta(x+m)$. For any $p\in(1,\infty)$ and t, $t_-\in\mathbb R$, there exists $C_\#>0$ so that for each $u\in H_p^{t,t_-}$*

$$\left(\sum_{m\in\mathbb Z^d}\|\eta_m u\|^p_{H_p^{t,t_-}}\right)^{1/p}\leqslant C_\#\|u\|_{H_p^{t,t_-}}. \tag{37}$$

Remark 29. If, in addition to the assumptions of Lemma 28, one supposes that $\sum_{m\in\mathbb Z^d}\eta_m(x)=1$ for all x, then one can show that there is $C_\#$ so that for each u such that $\eta_m u\in H_p^{t,t_-}$ for all m we have

$$\|u\|_{H_p^{t,t_-}}\leqslant C_\#\left(\sum_{m\in\mathbb Z^d}\|\eta_m u\|^p_{H_p^{t,t_-}}\right)^{1/p}.$$

(We shall not need the above bound.)

Proof of Lemma 28. For $t_-=0$ and arbitrary t, Lemma 28 is a result of Triebel [29, Theorem 2.4.7] based on a Paley–Littlewood-type decomposition. Moreover, the constant $C_\#$ depends only on the size of the support of η, and its C^k-norm for some large enough k.

To handle $t_-\in\mathbb R$, we will (again) start from the result for the classical Sobolev space and use Fubini and interpolation, as follows.

Let us prove the lemma for $t=0$ and $t_- \in \mathbb{R}$, using a d_s-dimensional Fubini argument. We have

$$\sum_{m\in\mathbb{Z}^d} \|\eta_m u\|^p_{H_p^{0,t_-}(\mathbb{R}^d)} = \sum_{m\in\mathbb{Z}^d} \int_{x\in\mathbb{R}^{d_u}} \|\eta_m u\|^p_{H_p^{t_-}(\{x\}\times\mathbb{R}^{d_s})} \, \mathrm{d}x. \tag{38}$$

For each $x \in \mathbb{R}^{d_u}$, the values of $m \in \mathbb{Z}^d$ for which the restriction of $\eta_m u$ to $\{x\}\times\mathbb{R}^{d_s}$ is nonzero are contained in a set $M(x)\times\mathbb{Z}^{d_s}$, where $\operatorname{Card} M(x)$ is bounded independently of x. Using the result of Triebel for the Sobolev space $H_p^{t_-}(\mathbb{R}^{d_s})$, we get

$$\sum_{m\in\mathbb{Z}^d} \|\eta_m u\|^p_{H_p^{t_-}(\{x\}\times\mathbb{R}^{d_s})} \leqslant C_\# \|u\|^p_{H_p^{t_-}(\{x\}\times\mathbb{R}^{d_s})}. \tag{39}$$

Integrating over $x \in \mathbb{R}^{d_u}$ and using the Fubini equality

$$\int_{x\in\mathbb{R}^{d_u}} \|u\|^p_{H_p^{t_-}(\{x\}\times\mathbb{R}^{d_s})} \, \mathrm{d}x = \|u\|^p_{H_p^{0,t_-}}, \tag{40}$$

we obtain the lemma for $t=0$ and $t_- \in \mathbb{R}$.

Consider the map $u \mapsto (\eta_m u)_{m\in\mathbb{Z}^d}$. We have shown that it sends continuously H_p^t to $\ell_p(H_p^t)$ and H_p^{0,t_-} to $\ell_p(H_p^{0,t_-})$. By interpolation, for any $\theta\in(0,1)$, it sends $[H_p^t, H_p^{0,t_-}]_\theta$ to $[\ell_p(H_p^t), \ell_p(H_p^{0,t_-})]_\theta$. By Proposition 19, the first space is $H_p^{(1-\theta)t,\theta t_-}$. Moreover, [28, Theorem 1.18.1] shows that, for any pair of Banach spaces A, B, one has $[\ell_p(A), \ell_p(B)]_\theta = \ell_p([A,B]_\theta)$. Hence, again by Proposition 19, the second space is $\ell_p(H_p^{(1-\theta)t,\theta t_-})$. This proves the lemma. □

5. Proof of the main theorem

In this section, we prove Theorem 12. Let us fix once and for all a piecewise $C^{1+\alpha}$ hyperbolic map T and a C^α function g, satisfying the assumptions of this theorem. We will denote by $C_\#$ constants that depend only on p, t, t_- and T.

We recall that the norm on $\mathcal{H}_p^{t,t_-}$ has been defined in (5) using a partition of unity $\rho_1,\dots,\rho_J$ and charts $\kappa_1,\dots,\kappa_J$ subordinated to this partition of unity.

In the following arguments, when working on a set $\overline{O_\mathbf{i}}$ or in a neighborhood of this set (with $\mathbf{i}$ of length n), then T^n will implicitly mean $T_\mathbf{i}$. In the same way, $g^{(n)}$ will rather be a smooth extension of $g^{(n)}|_{O_\mathbf{i}}$ to a neighborhood of $\overline{O_\mathbf{i}}$. This should not cause any confusion.

To study $\mathcal{L}_g^n$, we will need, in addition to the estimates from Section 4, to iterate the inverse branches $T_\mathbf{i}^{-1}$, to truncate the functions and to use partitions of unity. To do this, we will use the three following lemmas.

Lemma 30. *There exists a constant $C_\#$ such that, for any n and $\mathbf{i}=(i_0,\dots,i_{n-1})$, for any $x\in\overline{O_\mathbf{i}}$, for any $j,k\in[1,J]$ such that $x\in\operatorname{supp}\rho_j$ and $y=T_\mathbf{i}x\in\operatorname{supp}\rho_k$, there exists a neighborhood O of y and a $C^{1+\alpha}$ diffeomorphism F of $\mathbb{R}^d$, coinciding with $\kappa_j\circ T_\mathbf{i}^{-1}\circ\kappa_k^{-1}$ on $\kappa_k(O)$, and satisfying the assumptions of Lemma 25 with $\mu_u\leqslant C_\#\lambda_{u,n}^{-1}(x)$ and $\mu_s\geqslant C_\#^{-1}\lambda_{s,n}^{-1}(x)$, and*

$$\max\big(\|M_0\|, \big\|M_0^{-1}\big\|, \|M_1\|, \big\|M_1^{-1}\big\|\big)\leqslant C_\#.$$

Proof. Let $F_0=\kappa_j\circ T_\mathbf{i}^{-1}\circ\kappa_k^{-1}$, it is defined on a neighborhood of $\kappa_k(y)$. Moreover, let P be a d_u-dimensional subspace of the unstable cone at x, and let M_0, M_1 be invertible matrices (with bounded norms) sending respectively $D\kappa_j(x)P$ and $D\kappa_k(y)DT_\mathbf{i}(x)P$ to $\mathbb{R}^{d_u}\times\{0\}$, and stable leaves to stable leaves. Such matrices exist since the unstable cone is uniformly bounded away from the stable direction.

Let $A=DF_0(\kappa_k(y))$, then $M_0AM_1^{-1}$ sends $\mathbb{R}^{d_u}\times\{0\}$ to itself, and $\{0\}\times\mathbb{R}^{d_s}$ to itself, i.e., it is block-diagonal. Hence, the matrix A satisfies the assumptions of Lemma 25. Let F be a $C^{1+\alpha}$ diffeomorphism of $\mathbb{R}^d$ coinciding with F_0 on a neighborhood of $\kappa_k(y)$ and such that $DF(z)$ is everywhere close to A. Up to taking a smaller neighborhood O of y (depending on n), the claims of Lemma 30 hold for F. □

Lemma 31. *There exists $C_\#$ such that, for any n, for any $\mathbf{i} = (i_0, \dots, i_{n-1})$, for any $x \in \overline{O_{\mathbf{i}}}$, for any j such that $x \in \operatorname{supp} \rho_j$, there exists a neighborhood O' of x and a matrix M sending stable leaves to stable leaves, with*

$$\max\bigl(\|M\|, \|M^{-1}\|\bigr) \leqslant C_\#,$$

such that $M\kappa_j(O' \cap O_{\mathbf{i}})$ intersects almost any line parallel to a coordinate axis along at most $C_\# n$ connected components.

Proof. Let L be as in Definition 5. Fix $\mathbf{i} = (i_0, \dots, i_{n-1})$ and $x \in \overline{O_{\mathbf{i}}}$. Let $a_1, \dots, a_d$ be a basis of $\mathcal{T}_x X$, which is close to an orthonormal basis, such that its last d_s vectors form a basis of $E^s(x)$. We can ensure that, for any $\ell < n$, $DT^\ell(x)a_k$ is L-generic with respect to ∂O_{i_j}, for $d_u < k \leqslant d$. This is indeed a consequence of the definition of weak transversality. Moving slightly the vectors a_k for $1 \leqslant k \leqslant d_u$, we can also ensure that $DT^\ell(x)a_k$ is transversal to the hypersurfaces defining ∂O_{i_j} at $T^\ell x$ for any $\ell < n$.

Let $b_k = D\kappa_j(x) \cdot a_k$, so that $b_1, \dots, b_d$ is a basis of $\mathbb{R}^d$. Multiplying a_k by a scalar, we can ensure that b_k has norm 1. If O' is a small enough neighborhood of x, then $\kappa_j(O' \cap (T_{i_{\ell-1}} \dots T_{i_0})^{-1} O_{i_\ell})$ intersects almost any line oriented by one of the vectors b_k, $d_u < k \leqslant d$, along at most L connected components, by definition of L-genericity. If $A_1, \dots, A_n$ are subsets of $\mathbb{R}$, each of which is the union of at most L intervals, then $\bigcap A_i$ is a union of at most nL intervals. Therefore, $\kappa_j(O' \cap O_{\mathbf{i}})$ intersects almost any line oriented by one of the vectors b_k, $d_u < k \leqslant d$, along at most nL connected components.

Moreover, it intersects any line oriented by one of the vectors b_k, $1 \leqslant k \leqslant d_u$, along at most one connected component by construction.

Let M be the matrix sending $b_1, \dots, b_d$ to the canonical basis of $\mathbb{R}^d$, it satisfies the requirements of the lemma. □

If $L = 1$, we can replace in the previous lemma the bound $C_\# n$ by a bound $C_\#$, since the intersection of n intervals is always an interval, but this is not true in general.

The following lemma on partitions of unity is similar to [5, Lemma 7.1].

Lemma 32. *Let t and t_- be arbitrary real numbers. There exists a constant $C_\#$ such that, for any distributions $v_1, \dots, v_l$ with compact support in $\mathbb{R}^d$, belonging to H_p^{t,t_-}, there exists a constant C depending only on the supports of the distributions v_i with*

$$\left\| \sum_{i=1}^{l} v_i \right\|_{H_p^{t,t_-}}^p \leqslant C_\# m^{p-1} \sum_{i=1}^{l} \|v_i\|_{H_p^{t,t_-}}^p + C \sum_{i=1}^{l} \|v_i\|_{H_p^{t-1,t_-}}^p, \tag{41}$$

where m is the intersection multiplicity of the supports of the v_i's, i.e., $m = \sup_{x \in \mathbb{R}^d} \operatorname{Card}\{i \mid x \in \operatorname{supp}(v_i)\}$.

Proof. Let A be the operator acting on distributions by $Av = \mathcal{F}^{-1}((1 + |\xi|^2 + |\eta|^2)^{t/2}(1 + |\eta|^2)^{t_-/2}\mathcal{F}v)$, so that $\|v\|_{\mathcal{H}_p^{t,t_-}} = \|Av\|_{L_p}$.

By [4, Lemma 2.7], for any distribution v with compact support K and any neighborhood K' of this support, there exist $C > 0$ and a function $\Psi : \mathbb{R}^d \to [0, 1]$ equal to 1 on K and vanishing on the complement of K', with

$$\|\Psi Av - Av\|_{L_p} \leqslant C \|v\|_{H_p^{t-1,t_-}}. \tag{42}$$

Let $v_1, \dots, v_l$ be distributions with compact supports whose intersection multiplicity is m. Choose neighborhoods $K_1', \dots, K_l'$ of the supports of the v_is whose intersection multiplicity is also m, and functions $\Psi_1, \dots, \Psi_l$ as above. Then

$$\left\| \sum_i v_i \right\|_{H_p^{t,t_-}}^p = \left\| \sum_i Av_i \right\|_{L_p}^p \leqslant \left\| \sum_i \Psi_i Av_i \right\|_{L_p}^p + C \sum_i \|v_i\|_{H_p^{t-1,t_-}}^p. \tag{43}$$

By convexity, the inequality $(x_1 + \dots + x_m)^p \leqslant m^{p-1} \sum x_i^p$ holds for any nonnegative numbers $x_1, \dots, x_m$. Since the multiplicity of the K_i's is at most m, this yields

$$\left| \sum_i \Psi_i Av_i \right|^p \leqslant m^{p-1} \sum_i |Av_i|^p. \tag{44}$$

Integrating this inequality and using (43), we get the lemma. □

Proof of Theorem 12. Let p, t and t_- be as in the assumptions of the theorem. Let $n>0$, and let $r_n>1$ (the precise value of r_n will be chosen later). We define a dilation R_n on $\mathbb{R}^d$ by $R_n(z)=r_n z$. Let $\|u\|_n$ be another norm on $\mathcal{H}_p^{t,t_-}$, given by

$$\|u\|_n=\sum_{j=1}^{J}\bigl\|(\rho_j u)\circ\kappa_j^{-1}\circ R_n^{-1}\bigr\|_{H_p^{t,t_-}}. \tag{45}$$

The norm $\|u\|_n$ is of course equivalent to the usual norm on $\mathcal{H}_p^{t,t_-}$, but we look at the space X_0 at a smaller scale. Functions are much flatter at this new scale, so that estimates involving their C^α norm, such as Lemma 22 or Lemma 25, will not cause problems. This will also enable us to use partitions of unity with very small supports without spoiling the estimates. The use of this "zooming" norm is similar to the good choice of ϵ_0 in [25], or the use of weighted norms in [18].

We will prove that, if n is fixed and r_n is large enough, then

$$\bigl\|\mathcal{L}_g^n u\bigr\|_n^p\leqslant C\|u\|^p_{\mathcal{H}_p^{0,t_-}}+C_\# n^p D_n^b\bigl(D_n^e\bigr)^{p-1}\bigl\|\bigl|\det DT^n\bigr|\max\bigl(\lambda_{u,n}^{-t},\lambda_{s,n}^{-(t+t_-)}\bigr)^p\bigl|g^{(n)}\bigr|^p\bigr\|_{L_\infty}\|u\|_n^p. \tag{46}$$

The injection of $\mathcal{H}_p^{t,t_-}$ into $\mathcal{H}_p^{0,t_-}$ is compact. Hence, by Hennion's theorem [22], the essential spectral radius of $\mathcal{L}_g^n$ acting on $\mathcal{H}_p^{t,t_-}$ (for either $\|u\|_{\mathcal{H}_p^{t,t_-}}$ or $\|u\|_n$, since these norms are equivalent) is at most

$$\bigl[C_\# n^p D_n^b\bigl(D_n^e\bigr)^{p-1}\bigl\|\bigl|\det DT^n\bigr|\max\bigl(\lambda_{u,n}^{-t},\lambda_{s,n}^{-(t+t_-)}\bigr)^p\bigl|g^{(n)}\bigr|^p\bigr\|_{L_\infty}\bigr]^{1/p}. \tag{47}$$

Taking the power $1/n$ and letting n tend to ∞, we obtain Theorem 12 since the quantity $(C_\# n^p)^{1/pn}$ converges to 1 (here, it is essential that $C_\#$ does not depend on n).

It remains to prove (46), for large enough r_n. The estimate will be subdivided into three steps:

(1) Decomposing u into a sum of distributions $v_{j,m}$ with small supports and well controlled $\|\cdot\|_n$ norms.
(2) Estimating each term $(1_{O_{\mathbf{i}}} g^{(n)} v_{j,m})\circ T_{\mathbf{i}}^{-1}$, for $\mathbf{i}$ of length n.
(3) Adding all terms to obtain $\mathcal{L}_g^n u$.

First step. For $1\leqslant j\leqslant J$ and $m\in\mathbb{Z}^d$, let $\tilde v_{j,m}=\eta_m\cdot(\rho_j u)\circ\kappa_j^{-1}\circ R_n^{-1}$, where $\eta_m(x)=\eta(x+m)$, with $\eta:\mathbb{R}^d\to[0,1]$ a compactly supported C^∞ function so that $\sum_{m\in\mathbb{Z}^d}\eta_m=1$. Since the intersection multiplicity of the supports of the functions η_m is bounded, this is also the case for the $\tilde v_{j,m}$. Moreover, if j is fixed, we get using Lemma 28

$$\begin{aligned}\sum_{m\in\mathbb{Z}^d}\|\tilde v_{j,m}\|^p_{H_p^{t,t_-}}&=\sum_{m\in\mathbb{Z}^d}\bigl\|\eta_m\cdot(\rho_j u)\circ\kappa_j^{-1}\circ R_n^{-1}\bigr\|^p_{H_p^{t,t_-}}\\&\leqslant C_\#\bigl\|(\rho_j u)\circ\kappa_j^{-1}\circ R_n^{-1}\bigr\|^p_{H_p^{t,t_-}}\leqslant C_\#\|u\|_n^p.\end{aligned} \tag{48}$$

Since R_n expands the distances by a factor r_n while the size of the supports of the functions η_m is uniformly bounded, the supports of the distributions

$$v_{j,m}=\tilde v_{j,m}\circ R_n\circ\kappa_j=\eta_m\circ R_n\circ\kappa_j\cdot(\rho_j u)$$

are arbitrarily small if r_n is large enough. Finally

$$u=\sum_j\rho_j u=\sum_{j,m}v_{j,m}. \tag{49}$$

Second step. Fix $j,k\in\{1,\dots,J\}$, $m\in\mathbb{Z}^d$ and $\mathbf{i}=(i_0,\dots,i_{n-1})$. We will prove that

$$\begin{aligned}&\bigl\|\bigl(\rho_k\bigl(g^{(n)}1_{O_{\mathbf{i}}}v_{j,m}\bigr)\circ T_{\mathbf{i}}^{-1}\bigr)\circ\kappa_k^{-1}\circ R_n^{-1}\bigr\|_{H_p^{t,t_-}}\\&\quad\leqslant C\|u\|_{\mathcal{H}_p^{0,t_-}}+C_\# n\bigl\|\bigl|\det DT^n\bigr|^{1/p}g^{(n)}\max\bigl(\lambda_{u,n}^{-t},\lambda_{s,n}^{-(t+t_-)}\bigr)\bigr\|_{L_\infty}\|\tilde v_{j,m}\|_{H_p^{t,t_-}}.\end{aligned} \tag{50}$$

First, if the support of $v_{j,m}$ is small enough (which can be ensured by taking r_n large enough), there exists a neighborhood O of this support and a matrix M satisfying the conclusion of Lemma 31: this follows from Lemma 31 and the compactness of X_0. Therefore, the intersection of $R_n(M(\kappa_j(O \cap O_{\mathbf{i}})))$ with almost any line parallel to a coordinate axis contains at most $C_\# n$ connected components. Hence, Lemma 23 implies that the multiplication by $1_{O\cap O_{\mathbf{i}}} \circ \kappa_j^{-1} \circ M^{-1} \circ R_n^{-1}$ sends H_p^{t,t_-} into itself, with a norm bounded by $C_\# n$. Using the fact that M and R_n commute, the properties of M, and Lemma 24, we get

$$\left\| 1_{O_{\mathbf{i}}} \circ \kappa_j^{-1} \circ R_n^{-1} \cdot \tilde{v}_{j,m} \right\|_{H_p^{t,t_-}} \leqslant C_\# n \|\tilde{v}_{j,m}\|_{H_p^{t,t_-}}. \tag{51}$$

(Recall that $v_{j,m}$ is supported inside O.) Next, let

$$\tilde{v}_{j,k,m} = \left((\rho_k \circ T_{\mathbf{i}}) 1_{O_{\mathbf{i}}}\right) \circ \kappa_j^{-1} \circ R_n^{-1} \cdot \tilde{v}_{j,m}$$

(we suppress $\mathbf{i}$ from the notation for simplicity). Let also χ be a C^∞ function supported in the neighborhood O of the support of $v_{j,m}$ with $\chi \equiv 1$ on this support. Up to taking larger r_n we may ensure that

$$\left\| \left(\chi (\rho_k \circ T_{\mathbf{i}})\right) \circ \kappa_j^{-1} \circ R_n^{-1} \right\|_{C^\alpha} \leqslant C_\#.$$

Then Lemma 22 and (51) imply

$$\|\tilde{v}_{j,k,m}\|_{H_p^{t,t_-}} \leqslant C_\# n \|\tilde{v}_{j,m}\|_{H_p^{t,t_-}}. \tag{52}$$

In addition, we have

$$\begin{aligned} \left((\rho_k \circ T_{\mathbf{i}}) 1_{O_{\mathbf{i}}} v_{j,m}\right) \circ T_{\mathbf{i}}^{-1} \circ \kappa_k^{-1} \circ R_n^{-1} &= \tilde{v}_{j,k,m} \circ R_n \circ \kappa_j \circ T_{\mathbf{i}}^{-1} \circ \kappa_k^{-1} \circ R_n^{-1} \\ &= \tilde{v}_{j,k,m} \circ R_n \circ F \circ R_n^{-1}, \end{aligned} \tag{53}$$

where F is given by Lemma 30 (we use the fact that the support of $v_{j,m} \circ T_{\mathbf{i}}^{-1}$ is contained in a very small neighborhood O' if r_n is large enough, and again the compactness of X_0). The diffeomorphism F satisfies the assumptions of Lemma 25. Since the dilations R_n commute with any matrix, this is also the case of the diffeomorphism $G = R_n \circ F \circ R_n^{-1}$. Applying Lemma 25 to G, we get (for some point x in the support of $v_{j,m}$, and some matrix A of the form $DF(R_n^{-1}(z))$ for some z)

$$\begin{aligned} &\left\| \tilde{v}_{j,k,m} \circ R_n \circ F \circ R_n^{-1} \right\|_{H_p^{t,t_-}} \\ &\quad \leqslant C\|u\|_{\mathcal{H}_p^{0,t_-}} + C_\# \left\| \frac{\det A}{\det DG} \right\|_{C^\alpha} |\det A|^{-1/p} \max\left(\lambda_{u,n}(x)^{-t}, \lambda_{s,n}(x)^{-(t+t_-)}\right) \|\tilde{v}_{j,k,m}\|_{H_p^{t,t_-}}. \end{aligned} \tag{54}$$

The factor $\det A$ is close to $\det DT_{\mathbf{i}}(x)^{-1}$. Moreover, $\det DG = (\det DF) \circ R_n^{-1}$. By choosing r_n large enough, we can make sure that the C^α norm of $\det DG$ is controlled by its sup norm, to ensure that $\|\det A / \det DG\|_{C^\alpha}$ is uniformly bounded.

Let χ' be a C^∞ function supported in O' with $\chi' \equiv 1$ on the support of $v_{j,m} \circ T_{\mathbf{i}}^{-1}$. For $\delta > 0$, we can ensure by increasing r_n that the C^α norm of $(\chi' \cdot g^{(n)} \circ T_{\mathbf{i}}^{-1}) \circ \kappa_k^{-1} \circ R_n^{-1}$ is bounded by $|g^{(n)}(x)| + \delta$ for some x in the support of $v_{j,m}$. Choosing $\delta > 0$ small enough, we deduce from (54), Lemma 22 and (52)

$$\begin{aligned} &\left\| \left(\rho_k \left(g^{(n)} 1_{O_{\mathbf{i}}} v_{j,m}\right) \circ T_{\mathbf{i}}^{-1}\right) \circ \kappa_k^{-1} \circ R_n^{-1} \right\|_{H_p^{t,t_-}} \\ &\quad \leqslant C\|u\|_{\mathcal{H}_p^{0,t_-}} + C_\# n \left\| \left|\det DT^n\right|^{1/p} g^{(n)} \max\left(\lambda_{u,n}^{-t}, \lambda_{s,n}^{-(t+t_-)}\right) \right\|_{L_\infty} \|\tilde{v}_{j,m}\|_{H_p^{t,t_-}}. \end{aligned}$$

This proves (50).

Third step. We have $\mathcal{L}_g^n u = \sum_{j,m} \sum_{\mathbf{i}} (1_{O_{\mathbf{i}}} g^{(n)} v_{j,m}) \circ T_{\mathbf{i}}^{-1}$. (Note that only finitely many terms in this sum are nonzero by compactness of the support of each ρ_j.) We claim that the intersection multiplicity of the supports of the functions $(1_{O_{\mathbf{i}}} g^{(n)} v_{j,m}) \circ T_{\mathbf{i}}^{-1}$ is bounded by $C_\# D_n^e$. Indeed, this follows from the fact that any point $x \in X_0$ belongs to at most D_n^e sets $\overline{T_{\mathbf{i}}(O_{\mathbf{i}})}$, and that the intersection multiplicity of the supports of the functions $v_{j,m}$ is bounded.

To estimate $\|\mathcal{L}_g^n u\|_n$, we have to bound each term $\|(\rho_k \mathcal{L}_g^n u) \circ \kappa_k^{-1} \circ R_n^{-1}\|_{H_p^{t,t_-}}$, for $1 \leqslant k \leqslant J$. Let us fix such a k. By Lemma 32, we have

$$\left\|\left(\rho_k \mathcal{L}_g^n u\right) \circ \kappa_k^{-1} \circ R_n^{-1}\right\|_{H_p^{t,t_-}}^p \leqslant C\|u\|_{\mathcal{H}_p^{0,t_-}}^p + C_\#\left(C_\# D_n^e\right)^{p-1} \sum_{j,m,\mathbf{i}} \left\|\left(\rho_k\left(1_{O_\mathbf{i}} g^{(n)} v_{j,m}\right) \circ T_\mathbf{i}^{-1}\right) \circ \kappa_k^{-1} \circ R_n^{-1}\right\|_{H_p^{t,t_-}}^p.$$

We can bound each term in the sum using (50) and the convexity inequality $(a+b)^p \leqslant 2^{p-1}(a^p+b^p)$. Moreover, for any (j,m), the number of parameters $\mathbf{i}$ for which the corresponding term is nonzero is bounded by the number of sets $\overline{O_\mathbf{i}}$ intersecting the support of $v_{j,m}$. Choosing r_n large enough, we can ensure that the supports of the $v_{j,m}$ are small enough so that this number is bounded by D_n^b. Together with (48), this concludes the proof of (46), and of Theorem 12. □

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Appendix A. Corrigendum to [4, Lemma 2.8] – About interpolation

During the preparation of this paper, we realized that the statement of a lemma in [4] is not correct. This has no consequence on the other claims in [4], and plays no role in the present paper, but let us nevertheless give the correct statement as well as a sketch of proof, since it is related to some topics of the present paper.

The statement of [4, Lemma 2.8] should be replaced by: letting $n = [|t|] + [|t+t_-|] + d + 4$, if g is C^n, then

$$\|gu\|_{\mathcal{H}_p^{t,t_-}} \leqslant C_\# \|g\|_{C^{n-1}(C_s^1)} \|u\|_{\mathcal{H}_p^{t,t_-}} + C\|u\|_{\mathcal{H}_p^{t-1,t_-}}, \tag{55}$$

where $\|g\|_{C^{n-1}(C_s^1)}$ is the maximum between $\|g\|_{L_\infty}$ and the C^{n-1} norm of the first derivatives of g along E^s. It was mistakenly claimed in [4, Lemma 2.8] that it is enough to take $n=3$. The sentence “This can be shown by a straightforward … oscillatory integral argument” in the proof there should be replaced by “This can be shown by integrating by parts $[|p|] + [|q|] + d + 1$ times in total with respect to (u,v), noting that

$$\begin{aligned}&\left(1+|\eta - s\theta|^2 + |\xi - s\omega|^2\right)^{p/2}\left(1+|\xi - s\omega|^2\right)^{q/2}\left(1+|\eta|^2+|\xi|^2\right)^{-p/2}\left(1+|\xi|^2\right)^{-q/2}\\ &\quad\leqslant 16\left(1+|s\omega|^2\right)^{|q|/2}\left(1+|s\theta|^2+|s\omega|^2\right)^{|p|/2}.\end{aligned}$$

Since $\partial^{\gamma''+\gamma'} h$ has been differentiated up to 3 times including $|\gamma'| \in \{1,2\}$ times along x-directions, we get at most $[|p|] + [|q|] + d + 4$ derivatives in total”. In particular [4, Lemma 2.8] only holds if g is sufficiently differentiable.

We derive via interpolation in Lemma 22 a simpler Leibniz-type bound which takes the place of [4, Lemma 2.8] and is valid for $g \in C^\alpha$ for any $\alpha > 0$. The “zooming” norm (45) then allows us to replace $\|g\|_{C^\alpha}$ by a sup-norm type estimate for arbitrary g.

The interpolation estimates also yield a chain-rule-type bound (Lemma 25 and Remark 27) which extends [4, Lemma 2.10] to arbitrary differentiability: the proof of [4, Lemma 2.10] uses that T is C^∞ implicitly in several places (when referring to arguments of [1]), although a modification of this proof along the lines given above gives the claim for C^k dynamics, with $k(d)$ large if d is large.

Appendix B. Properties of physical measures

In this section, we prove Theorem 14. In fact, we will prove a more general result in a more abstract context. Let X be a manifold, X_0 a compact subset of X with positive (and finite) Lebesgue measure, and $T: X_0 \to X_0$ a measurable transformation for which Lebesgue measure is nonsingular. We will denote in this appendix by $\mathcal{L}$ the corresponding transfer operator, defined by duality on $L_1(\mathrm{Leb})$ by $\int_{X_0} \mathcal{L} f \cdot g \,\mathrm{dLeb} = \int_{X_0} f \cdot g \circ T \,\mathrm{dLeb}$ whenever g is bounded and measurable.

Theorem 33. *Let H be a Banach space of distributions supported on X_0. Assume that*

(1) *There exist $\alpha > 0$ and $C > 0$ such that, for any $u \in H \cap L_\infty(\mathrm{Leb})$ and $f \in C^\alpha(X)$, then $fu \in H$ and $\|fu\|_H \leqslant C\|f\|_{C^\alpha}\|u\|_H$.*

(2) *The space $H \cap L_\infty(\mathrm{Leb})$ is dense in H.*
(3) *The transfer operator $\mathcal{L}$ associated to T sends continuously $H \cap L_\infty(\mathrm{Leb})$ into itself and satisfies, for any $u \in H \cap L_\infty(\mathrm{Leb})$, the inequality $\|\mathcal{L}u\|_H \leqslant C\|u\|_H$. Therefore, $\mathcal{L}$ admits a continuous extension to H (still denoted by $\mathcal{L}$). We assume that the essential spectral radius of this extension is < 1.*
(4) *There exist $f_0 \in H \cap L_\infty(\mathrm{Leb})$ taking its values in $[0,1]$ and $N_0 > 0$ such that, for any $\phi \in L_\infty(\mathrm{Leb})$, then $f_0 = 1$ on the support of $\mathcal{L}^{N_0}\phi$.*
(5) *For any $u \in H$ which is a limit (in the H topology) of nonnegative functions $u_n \in H \cap L_\infty(\mathrm{Leb})$ and for which there exists a measure μ_u such that*[5] *$\langle u, g\,\mathrm{dLeb}\rangle = \int g\,\mathrm{d}\mu_u$ for any C^∞ function g, then the measure μ_u gives zero mass to the discontinuity set of T.*

Then there exist a finite number of probability measures $\mu_1, \ldots, \mu_l$ which are T-invariant and ergodic, and disjoint sets $A_1, \ldots, A_l$ such that $\mu_i(A_i) = 1$, $\mathrm{Leb}(A_i) > 0$, $\mathrm{Leb}(X_0 \setminus \bigcup_{i=1}^{l} A_i) = 0$ and, for every $x \in A_i$ and every function $f \in \overline{C^0(X_0) \cap H}$ (the closure of $C^0(X_0) \cap H$ in $C^0(X_0)$), then $\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) \to \int f\,\mathrm{d}\mu_i$.

Moreover, for every i, there exist an integer k_i and a decomposition $\mu_i = \mu_{i,1} + \cdots + \mu_{i,k_i}$ such that T sends $\mu_{i,j}$ to $\mu_{i,j+1}$ for $j \in \mathbb{Z}/k_i\mathbb{Z}$, and the probability measures $k_i\mu_{i,j}$ are exponentially mixing for T^{k_i} and C^α test functions.

The proof will also describe a direct relationship between the eigenfunctions of $\mathcal{L}$ for eigenvalues of modulus 1, and the physical measures of T. The first part of the proof is directly borrowed from [8].

The first, second and fourth conditions say that the space H is sufficiently large. They are satisfied in the setting of this paper (taking $f_0 = 1_{X_0}$, which belongs to $\mathcal{H}_p^{t,t-}$), but also in the case of an attractor, when $T(X_0)$ is contained in the interior of X_0 (the function f_0 can be taken C^∞, compactly supported in the interior of X_0, equal to 1 on $T(X_0)$).

The fifth condition is necessary, as shown by Example 1 in Section 2: taking for H the space of distributions in the Sobolev space H_2^{-1} supported in $[-1,1] \times \{0,1\}$, then all the assumptions of the theorem but the fifth one are satisfied, and the conclusion of the theorem does not hold.

Proof of Theorem 33. Let us first prove the existence of $C > 0$ such that, for any $n \in \mathbb{N}$,

$$\left\|\mathcal{L}^n\right\|_{H \to H} \leqslant C. \tag{56}$$

Otherwise, $\mathcal{L}$ has an eigenvalue of modulus > 1, or a nontrivial Jordan block for an eigenvalue of modulus 1. Let λ be an eigenvalue of $\mathcal{L}$ of maximal modulus, with a Jordan block of maximal size d. Since $L_\infty \cap H$ is dense in H, its image under the eigenprojections is dense in the eigenspaces, which are finite dimensional. Hence, it coincides with the full eigenspaces. Therefore, there exists a bounded function f such that $n^{-d}\sum_{i=0}^{n-1} \lambda^{-i}\mathcal{L}^i f$ converges to a nonzero limit u. For any C^∞ function g,

$$\langle u, g\,\mathrm{dLeb}\rangle = \lim \frac{1}{n^d}\sum_{i=0}^{n-1} \lambda^{-i}\left\langle \mathcal{L}^i f, g\,\mathrm{dLeb}\right\rangle = \lim \frac{1}{n^d}\sum_{i=0}^{n-1} \lambda^{-i}\int f \cdot g \circ T^i\,\mathrm{dLeb}.$$

If $|\lambda| > 1$ or $d \geqslant 2$, this quantity converges to 0 when $n \to \infty$ since $\int f \cdot g \circ T^i\,\mathrm{dLeb}$ is uniformly bounded. This contradicts the fact that u is nonzero, and proves (56).

For $|\lambda| = 1$, let E_λ denote the corresponding eigenspace, and $\Pi_\lambda : H \to E_\lambda$ the corresponding eigenprojection. It is given by

$$\Pi_\lambda f = \lim \frac{1}{n}\sum_{i=0}^{n-1} \lambda^{-i}\mathcal{L}^i f, \tag{57}$$

where the convergence holds in H. Since $L_\infty(\mathrm{Leb}) \cap H$ is dense in H, $E_\lambda = \Pi_\lambda(L_\infty(\mathrm{Leb}) \cap H)$. For any $f \in L_\infty(\mathrm{Leb}) \cap H$ and $g \in C^\infty$,

$$\left|\langle \Pi_\lambda f, g\,\mathrm{dLeb}\rangle\right| \leqslant \lim \frac{1}{n}\sum_{i=0}^{n-1}\left|\int f \cdot g \circ T^i\,\mathrm{dLeb}\right| \leqslant C\|f\|_{L_\infty}\|g\|_{C^0}. \tag{58}$$

[5] We write $\langle u, g\,\mathrm{dLeb}\rangle$ and not $\langle u, g\rangle$, in accordance with the convention stated in the footnote 3, viewing distributions as generalized functions which can only be integrated against smooth densities.

By the Riesz representation theorem on the compact space X_0, this implies that, for any $u \in E_\lambda$, there exists a finite measure μ_u on X_0 such that $\langle u, g\,\mathrm{dLeb}\rangle = \int g\,\mathrm{d}\mu_u$. Moreover, for any $i \geqslant N_0$ and any bounded measurable function $g \geqslant 0$,

$$\left|\int f \cdot g \circ T^i\,\mathrm{dLeb}\right| = \left|\int \mathcal{L}^{N_0} f \cdot g \circ T^{i-N_0}\,\mathrm{dLeb}\right| = \left|\int \mathcal{L}^{N_0} f \cdot f_0 \cdot g \circ T^{i-N_0}\,\mathrm{dLeb}\right|$$
$$\leqslant C\|f\|_{L_\infty} \int f_0 \cdot g \circ T^{i-N_0}\,\mathrm{dLeb} = C\|f\|_{L_\infty} \int \mathcal{L}^{i-N_0} f_0 \cdot g\,\mathrm{dLeb}.$$

Averaging and taking the limit, we obtain

$$\left|\int g\,\mathrm{d}\mu_{\Pi_\lambda f}\right| \leqslant C\|f\|_{L_\infty} \int g\,\mathrm{d}\mu_{\Pi_1 f_0}. \tag{59}$$

This means that the measures μ_u are all absolutely continuous with respect to the reference measure $\mu := \mu_{\Pi_1 f_0}$, with bounded density.

Let us show that the measure μ is invariant. This is formally trivial from the computation

$$\int g\,\mathrm{d}\mu = \langle \Pi_1 f_0, g\,\mathrm{dLeb}\rangle = \langle \mathcal{L}\Pi_1 f_0, g\,\mathrm{dLeb}\rangle = \langle \Pi_1 f_0, g \circ T\,\mathrm{dLeb}\rangle = \int g \circ T\,\mathrm{d}\mu.$$

However, this argument is not correct since $\langle \Pi_1 f_0, g \circ T\,\mathrm{dLeb}\rangle$ is not well defined since g is not smooth. More importantly, even if we could define it, the equality between $\langle \Pi_1 f_0, g \circ T\,\mathrm{dLeb}\rangle$ and $\int g \circ T\,\mathrm{d}\mu$ would not be trivial since the relationship between $\Pi_1 f_0$ and $\mathrm{d}\mu$ is established only for continuous functions.

The rigorous proof relies on the fifth assumption of the theorem. By definition, if g is C^∞, then $\int g\,\mathrm{d}\mu = \lim \int g\,\mathrm{d}(\frac{1}{n}\sum_{i=0}^{n-1} T_*^i(f_0\,\mathrm{Leb}))$. By density, this equality extends to C^0 functions, hence μ is the weak limit of the sequence of measures $\frac{1}{n}\sum_{i=0}^{n-1} T_*^i(f_0\,\mathrm{Leb})$. In turn, a classical property of weak convergence [7, Theorem 5.2(iii)] implies that, for any function h whose discontinuity set has zero measure for μ,

$$\int h\,\mathrm{d}\mu = \lim \int h\,\mathrm{d}\left(\frac{1}{n}\sum_{i=0}^{n-1} T_*^i(f_0\,\mathrm{Leb})\right). \tag{60}$$

If g is a continuous function, then $g \circ T$ is continuous except on the discontinuity set of T. The fifth assumption of the theorem shows that this set has zero measure for μ. Hence, (60) applies to $g \circ T$. It also applies to g. Since the right-hand side for g and $g \circ T$ coincide up to $O(1/n)$, this yields $\int g \circ T\,\mathrm{d}\mu = \int g\,\mathrm{d}\mu$ and concludes the proof of the invariance of μ.

In the following, we shall encounter several instances of similar equations that are formally trivial but need a rigorous justification. Let us give a last justification of this type, and leave the remaining ones to the reader. We claim that, if $\phi \in C^\alpha$ and $g \in C^\infty$,

$$\left\langle \mathcal{L}^i(\phi \Pi_1 f_0), g\,\mathrm{dLeb}\right\rangle = \int \phi \cdot g \circ T^i\,\mathrm{d}\mu. \tag{61}$$

Indeed, $\mathcal{L}^i(\phi \Pi_1 f_0)$ is the limit in H of $\mathcal{L}^i(\phi \frac{1}{n}\sum_{j=0}^{n-1} \mathcal{L}^j f_0)$, hence

$$\left\langle \mathcal{L}^i(\phi \Pi_1 f_0), g\,\mathrm{dLeb}\right\rangle = \lim \frac{1}{n}\sum_{j=0}^{n-1}\left\langle \mathcal{L}^i\left(\phi \mathcal{L}^j f_0\right), g\,\mathrm{dLeb}\right\rangle = \lim \frac{1}{n}\sum_{j=0}^{n-1}\int \phi \mathcal{L}^j f_0 \cdot g \circ T^i\,\mathrm{dLeb}$$
$$= \lim \int \phi \cdot g \circ T^i\,\mathrm{d}\left(\frac{1}{n}\sum_{j=0}^{n-1} T_*^j(f_0\,\mathrm{Leb})\right).$$

The measure μ gives zero mass to the discontinuities of $g \circ T^i$ (since it is invariant and gives zero mass to the discontinuities of T). Hence, (60) holds for $\phi \cdot g \circ T^i$. This concludes the proof of (61).

For any $u \in E_\lambda$, write $\mu_u = \phi_u \mu$ where $\phi_u \in L_\infty(\mu)$ is defined μ-almost everywhere. The equation $\mathcal{L}u = \lambda u$ translates into $T_*(\phi_u \mu) = \lambda \phi_u \mu$. Hence, since μ is invariant,

$$\int \left|\phi_u \circ T - \lambda^{-1}\phi_u\right|^2 \mathrm{d}\mu = \int |\phi_u|^2 \circ T \,\mathrm{d}\mu + \int |\phi_u|^2 - 2\Re \int \overline{\phi_u} \circ T \lambda^{-1}\phi_u \,\mathrm{d}\mu$$
$$= 2\int |\phi_u|^2 \,\mathrm{d}\mu - 2\Re \int \lambda^{-1}\overline{\phi_u}\,\mathrm{d}T_*(\phi_u\mu) = 0.$$

Let $F_\lambda = \{\phi \in L_\infty(\mu) \mid \phi \circ T = \lambda^{-1}\phi\}$ (this is a space of equivalence classes of functions), then the map $\Phi_\lambda : u \mapsto \phi_u$ sends (injectively) E_λ to F_λ. Let us show that it is also surjective.

Let $\phi \in F_\lambda$. By Lusin's theorem, the measurable function ϕ can be approximated in $L_1(\mu)$ by a continuous function, which itself can be uniformly approximated by a C^∞ function. Therefore, there exists a sequence of C^α functions ϕ_p with $\|\phi - \phi_p\|_{L_1(\mu)} \leqslant 1/p$. Let $u_p = \Pi_\lambda(\phi_p \Pi_1 f_0)$, and let $\mu_p = \mu_{u_p}$. Let us prove that the total mass of the measure $\phi\mathrm{d}\mu - \mathrm{d}\mu_p$ converges to 0. If g is a C^∞ function,

$$\int g\,\mathrm{d}\mu_p = \langle u_p, g\,\mathrm{dLeb}\rangle = \lim \frac{1}{n}\sum_{i=0}^{n-1}\lambda^{-i}\left\langle \mathcal{L}^i(\phi_p \Pi_1 f_0), g\,\mathrm{dLeb}\right\rangle = \lim \frac{1}{n}\sum_{i=0}^{n-1}\lambda^{-i}\int \phi_p \cdot g \circ T^i \,\mathrm{d}\mu,$$

by (61). On the other hand, for any n, since μ is invariant and $\phi \circ T = \lambda^{-1}\phi$,

$$\int g\phi\,\mathrm{d}\mu = \frac{1}{n}\sum_{i=0}^{n-1}\int g \circ T^i \phi \circ T^i \,\mathrm{d}\mu = \frac{1}{n}\sum_{i=0}^{n-1}\lambda^{-i}\int g \circ T^i \phi \,\mathrm{d}\mu.$$

Subtracting the two previous equations, we get

$$\left|\int g\phi\,\mathrm{d}\mu - \int g\,\mathrm{d}\mu_p\right| \leqslant \|\phi - \phi_p\|_{L_1(\mu)}\|g\|_{C^0}, \tag{62}$$

which proves that the total mass of $\phi\,\mathrm{d}\mu - \mathrm{d}\mu_p$ converges to 0.

The sequence u_p belongs to the finite dimensional space E_λ, and the elements of E_λ are separated by the linear forms given by the integration along C^∞ densities (since H is a space of distributions). Since $\langle u_p, g\,\mathrm{dLeb}\rangle$ converges for any g, the sequence u_p is therefore converging to a limit u_∞. By construction, $\Phi_\lambda(u_\infty) = \phi$. This concludes the proof of the surjectivity of Φ_λ.

The eigenvalues of $\mathcal{L}$ of modulus 1 are exactly the λ such that F_λ is not reduced to 0. This set is a group, since $\phi_\lambda\phi_{\lambda'} \in F_{\lambda\lambda'}$ whenever $\phi_\lambda \in F_\lambda$ and $\phi_{\lambda'} \in F_{\lambda'}$. Since $\mathcal{L}$ only has a finite number of eigenvalues of modulus 1, this implies that these eigenvalues are roots of unity. In particular, there exists $N > 0$ such that $\lambda^N = 1$ for any eigenvalue λ.

Let us now assume that 1 is the only eigenvalue of $\mathcal{L}$ of modulus 1 (in the general case, this will be true for $\mathcal{L}^N$, so we will be able to deduce the general case from this particular case). Under this assumption, for any $u \in H$, $\mathcal{L}^n u$ converges to $\Pi_1 u$.

Consider the subset of F_1 (the bounded measurable T-invariant functions) given by the nonnegative functions with integral 1. It is nonempty, since it contains the function 1. It is a convex cone in F_1, whose extremal points are of the form 1_B for some minimal invariant set B. Such extremal points are automatically linearly independent. Since F_1 is finite dimensional, there is only a finite number of them, say $1_{B_1}, \dots, 1_{B_l}$, and a function belongs to F_1 if and only if it can be written as $\phi = \sum \alpha_i 1_{B_i}$ for some scalars $\alpha_1, \dots, \alpha_l$. The decomposition of the function $1 \in F_1$ is given by $1 = \sum 1_{B_i}$, hence the sets B_i cover the whole space up to a set of zero measure for μ. Moreover, since B_i is minimal, the measure $\mu_i := \frac{1_{B_i}\mu}{\mu(B_i)}$ is an invariant ergodic probability measure.

Let $u_i = \Phi_1^{-1}(1_{B_i}) \in H$, then any element of E_1 is a linear combination of the u_i. In particular, this applies to $\Pi_1(f u_i)$ for any $f \in C^\alpha$. Let us show that

$$\Pi_1(f u_i) = \left(\int f\,\mathrm{d}\mu_i\right)u_i. \tag{63}$$

We can write $\Pi_1(f u_i) = \sum_j a_{ij}(f)u_j$. Let us fix once and for all l sequences of C^α functions $\phi_{j,p}$ taking values in $[0, 1]$ and such that $\phi_{j,p}$ converges in $L_1(\mu)$ to 1_{B_j}. Since $\langle u_j, \phi_{j',p}\,\mathrm{dLeb}\rangle = \int_{B_j} \phi_{j',p}\,\mathrm{d}\mu \to \delta_{jj'}\mu(B_j)$, we have $a_{ij}(f) = \frac{1}{\mu(B_j)}\lim_{p\to\infty}\langle \Pi_1(f u_i), \phi_{j,p}\,\mathrm{dLeb}\rangle$. Moreover, if p is fixed,

$$\left\langle \Pi_1(f u_i), \phi_{j,p}\,\mathrm{dLeb}\right\rangle = \lim_{n\to\infty}\left\langle \mathcal{L}^n(f u_i), \phi_{j,p}\,\mathrm{dLeb}\right\rangle = \lim_{n\to\infty}\int_{B_i} f\phi_{j,p} \circ T^n \,\mathrm{d}\mu.$$

Writing $\phi_{j,p}\circ T^n = 1_{B_j}\circ T^n + (\phi_{j,p}-1_{B_j})\circ T^n$ and using $1_{B_j}\circ T^n = 1_{B_j}$ and $\|(\phi_{j,p}-1_{B_j})\circ T^n\|_{L_1(\mu)} = \|\phi_{j,p} - 1_{B_j}\|_{L_1(\mu)} \to_{p\to\infty} 0$, we obtain (63).

This enables us to deduce that each measure μ_i is exponentially mixing, as follows. Let $\delta < 1$ be such that $\|\mathcal{L}^n - \Pi_1\|_{H\to H} = \mathrm{O}(\delta^n)$. Then, if f, g are C^α functions,

$$\begin{aligned}\int f\cdot g\circ T^n \,\mathrm{d}\mu_i &= \frac{1}{\mu(B_i)}\big\langle \mathcal{L}^n(fu_i), g\,\mathrm{dLeb}\big\rangle \\ &= \frac{1}{\mu(B_i)}\big\langle \Pi_1(fu_i), g\,\mathrm{dLeb}\big\rangle + \mathrm{O}\big(\delta^n\big) \\ &= \left(\int f\,\mathrm{d}\mu_i\right)\frac{1}{\mu(B_i)}\langle u_i, g\,\mathrm{dLeb}\rangle + \mathrm{O}\big(\delta^n\big) \\ &= \left(\int f\,\mathrm{d}\mu_i\right)\left(\int g\,\mathrm{d}\mu_i\right) + \mathrm{O}\big(\delta^n\big).\end{aligned}$$

We now turn to the relationships between Lebesgue measure and the measures μ_i. For any function $f \in L_\infty(\mathrm{Leb}) \cap H$, let us write

$$\Pi_1(f) = \sum_{i=1}^{l} b_i(f)u_i. \tag{64}$$

We will need to describe the coefficients $b_i(f)$. Let n_p be a sequence tending fast enough to ∞ so that $\|\mathcal{L}^{n_p} - \Pi_1\|_{H\to H}\|\phi_{i,p}\|_{C^\alpha} \to_{p\to\infty} 0$. If f belongs to $L_\infty(\mathrm{Leb})\cap H$,

$$\begin{aligned}\int f\cdot\phi_{i,p}\circ T^{n_p}\,\mathrm{dLeb} &= \big\langle \mathcal{L}^{n_p} f, \phi_{i,p}\,\mathrm{dLeb}\big\rangle \\ &= \big\langle \phi_{i,p}\big(\mathcal{L}^{n_p} - \Pi_1\big)f, \mathrm{dLeb}\big\rangle + \langle \Pi_1 f, \phi_{i,p}\,\mathrm{dLeb}\rangle \\ &= \mathrm{o}(1) + \sum_{j=1}^{l} b_j(f)\int_{B_j}\phi_{i,p}\,\mathrm{d}\mu = \mathrm{o}(1) + b_i(f)\mu(B_i).\end{aligned}$$

The same argument even shows that $\int f\cdot(\frac{1}{n_p}\sum_{n=n_p}^{2n_p-1}\phi_{i,p}\circ T^n)\,\mathrm{dLeb} \to \mu(B_i)b_i(f)$.

Moreover, the sequence $\frac{1}{n_p}\sum_{n=n_p}^{2n_p-1}\phi_{i,p}\circ T^n$ is bounded by 1 (since $\phi_{i,p}$ takes its values in [0, 1]), and asymptotically invariant. Let $h_i : X \to [0,1]$ be one of its weak limits in $L^2(\mathrm{Leb})$, it is invariant and satisfies

$$b_i(f) = \frac{1}{\mu(B_i)}\int f h_i\,\mathrm{dLeb}. \tag{65}$$

Since $b_i(f_0) = 1$, we have $\int h_i f_0\,\mathrm{dLeb} = \mu(B_i)$.

Let us now compute $\int h_i h_j f_0\,\mathrm{dLeb}$. We have

$$\mu(B_j)b_j\big(\phi_{i,p}\mathcal{L}^n f_0\big) = \int \phi_{i,p}\mathcal{L}^n f_0 h_j\,\mathrm{dLeb} = \int f_0\phi_{i,p}\circ T^n h_j\circ T^n\,\mathrm{dLeb} = \int \phi_{i,p}\circ T^n h_j f_0\,\mathrm{dLeb}.$$

Taking the average and the weak-limit, we obtain

$$\int h_i h_j f_0\,\mathrm{dLeb} = \mu(B_j)\lim_{p\to\infty}\frac{1}{n_p}\sum_{n=n_p}^{2n_p-1} b_j\big(\phi_{i,p}\mathcal{L}^n f_0\big). \tag{66}$$

Moreover, if $n \geqslant n_p$,

$$\phi_{i,p}\mathcal{L}^n f_0 = \phi_{i,p}\big(\mathcal{L}^n - \Pi_1\big)f_0 + \phi_{i,p}\Pi_1 f_0. \tag{67}$$

The first term converges to 0 in H, and the computation made in (62) shows that $\Pi_1(\phi_{i,p}\Pi_1 f_0)$ converges to u_i. This implies that $b_j(\phi_{i,p}\mathcal{L}^n f_0)$ converges to δ_{ij}. This yields

$$\int h_i h_j f_0\,\mathrm{dLeb} = \mu(B_j)\delta_{ij}. \tag{68}$$

Let $X_1 = \{x \mid f_0(x) > 0\}$. Taking $i = j$, we get $\int h_i^2 f_0 \,\mathrm{dLeb} = \mu(B_i) = \int h_i f_0 \,\mathrm{dLeb}$. Since h_i takes its values in [0, 1], this shows that there exists a subset C_i^0 of X_1 such that $h_i 1_{X_1} = 1_{C_i^0}$, with $\int_{C_i^0} f_0 \,\mathrm{dLeb} = \mu(B_i)$. Moreover, (68) shows that $\mathrm{Leb}(C_i^0 \cap C_j^0) = 0$ if $i \neq j$. Let $C_i = T^{-N_0} C_i^0$, then these sets are disjoint. For any function $f \in L_\infty(\mathrm{Leb}) \cap H$, since $\mathcal{L}^{N_0} f$ is supported in X_1,

$$b_i(f) = b_i\big(\mathcal{L}^{N_0} f\big) = \frac{1}{\mu(B_i)} \int \mathcal{L}^{N_0} f h_i \,\mathrm{dLeb} = \frac{1}{\mu(B_i)} \int \mathcal{L}^{N_0} f \cdot 1_{C_i^0} \,\mathrm{dLeb}$$
$$= \frac{1}{\mu(B_i)} \int f \cdot 1_{C_i^0} \circ T^{N_0} \,\mathrm{dLeb} = \frac{1}{\mu(B_i)} \int_{C_i} f \,\mathrm{dLeb}.$$

Moreover, since $\mathcal{L}^{N_0} 1$ is supported on the sets C_i^0,

$$\mathrm{Leb}(X_0) = \int 1 \,\mathrm{dLeb} = \int \mathcal{L}^{N_0} 1 \,\mathrm{dLeb} = \int \mathcal{L}^{N_0} 1 \cdot 1_{\bigcup C_i^0} \,\mathrm{dLeb} = \int 1_{\bigcup C_i^0} \circ T^{N_0} \,\mathrm{dLeb} = \int 1_{\bigcup C_i} \,\mathrm{dLeb}.$$

This shows that the sets C_i form a partition of the space modulo a set of zero Lebesgue measure. We have proved that

$$\Pi_1(f) = \sum_{i=1}^{l} \frac{\int_{C_i} f \,\mathrm{dLeb}}{\mu(B_i)} u_i. \tag{69}$$

Let us now turn to the convergence of $\frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j$, for $f \in L_\infty(\mathrm{Leb}) \cap H$. Let $S_n f = \sum_{j=0}^{n-1} f \circ T^j$, we will estimate $\int |S_n f/n - S_m f/m|^2 f_0 \,\mathrm{dLeb}$. For $i, j \geqslant 0$, we have

$$\int f \circ T^i \cdot f \circ T^{i+j} f_0 \,\mathrm{dLeb} = \int f \mathcal{L}^i (f_0) \cdot f \circ T^j \,\mathrm{dLeb}$$
$$= \int \mathcal{L}^j \big(f \mathcal{L}^i f_0\big) f \,\mathrm{dLeb} = \big\langle \mathcal{L}^j \big(f \mathcal{L}^i f_0\big), f \big\rangle$$
$$= \big\langle \mathcal{L}^j (f \Pi_1 f_0), f \big\rangle + \mathrm{O}\big(\delta^i\big) = \big\langle \Pi_1 (f \Pi_1 f_0), f \big\rangle + \mathrm{O}\big(\delta^i\big) + \mathrm{O}\big(\delta^j\big),$$

where $\delta < 1$ is given by the spectral gap of the operator $\mathcal{L}$. Hence, for $n, m > 0$,

$$\int S_n f \cdot S_m f f_0 \,\mathrm{dLeb} = nm \big\langle \Pi_1 (f \Pi_1 f_0), f \big\rangle + \sum_{\substack{0 \leqslant i \leqslant n-1 \\ 0 \leqslant j \leqslant m-1-i}} \mathrm{O}\big(\delta^i\big) + \mathrm{O}\big(\delta^j\big) + \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 < j \leqslant n-1-i}} \mathrm{O}\big(\delta^i\big) + \mathrm{O}\big(\delta^j\big)$$
$$= nm \big\langle \Pi_1 (f \Pi_1 f_0), f \big\rangle + \mathrm{O}(n) + \mathrm{O}(m).$$

Expanding the square in $|S_n f/n - S_m f/m|^2$, we get using the previous equation

$$\int |S_n f/n - S_m f/m|^2 f_0 \,\mathrm{dLeb} = \frac{1}{n^2} \int S_n f \cdot S_n f f_0 \,\mathrm{dLeb} + \frac{1}{m^2} \int S_m f \cdot S_m f f_0 \,\mathrm{dLeb}$$
$$- \frac{2}{nm} \int S_n f \cdot S_m f f_0 \,\mathrm{dLeb}$$
$$= \mathrm{O}(1/n) + \mathrm{O}(1/m).$$

The functions $g_p = S_{p^4} f/p^4$ therefore satisfy $\|g_{p+1} - g_p\|_{L_2(f_0 \,\mathrm{dLeb})} = \mathrm{O}(1/p^2)$, which is summable. This implies that g_p converges in $L_2(f_0 \,\mathrm{dLeb})$ and almost everywhere for this measure. For a general $n \in \mathbb{N}$, let p be such that $p^4 \leqslant n < (p+1)^4$, then $S_n f/n - S_{p^4} f/p^4$ is uniformly small if n is large. Hence, $S_n f/n$ converges almost everywhere and in $L_2(f_0 \,\mathrm{dLeb})$, to a function $\phi_f \in L_2(f_0 \,\mathrm{dLeb})$.

Let us now identify the function ϕ_f. For any smooth function ϕ,

$$\int \phi \cdot f \circ T^n f_0 \,\mathrm{dLeb} = \big\langle \mathcal{L}^n (\phi f_0), f \,\mathrm{dLeb} \big\rangle \to \big\langle \Pi_1 (\phi f_0), f \,\mathrm{dLeb} \big\rangle = \sum_{i=1}^{l} b_i (\phi f_0) \int_{B_i} f \,\mathrm{d}\mu$$
$$= \sum_{i=1}^{l} \frac{\int_{C_i} \phi f_0 \,\mathrm{dLeb}}{\mu(B_i)} \int_{B_i} f \,\mathrm{d}\mu = \int \left(\sum_{i=1}^{l} 1_{C_i} \frac{\int_{B_i} f \,\mathrm{d}\mu}{\mu(B_i)} \right) \phi f_0 \,\mathrm{dLeb}.$$

This shows that, with respect to the measure $f_0\,\mathrm{dLeb}$, the sequence of functions $f \circ T^n$ converges weakly to the function $\tilde\phi_f := \sum_{i=1}^l 1_{C_i}(\int f\,\mathrm{d}\mu_i)$. In turn, $S_n f/n$ converges weakly to $\tilde\phi_f$. However, $S_n f/n$ converges strongly to ϕ_f, hence $\phi_f = \tilde\phi_f$ almost everywhere for $f_0\,\mathrm{dLeb}$, and in particular on almost all $\bigcup_{i=1}^l C_i^0$.

Let A_i^f be the set of points for which $S_n f/n$ converges to $\int f\,\mathrm{d}\mu_i$. We have shown that A_i^f contains a full Lebesgue measure subset of C_i^0. However, A_i^f is T-invariant, hence it contains a full Lebesgue measure subset of C_i. Since the sets C_i cover Lebesgue almost all the space, $\mathrm{Leb}(X \setminus \bigcup_{i=1}^l A_i^f) = 0$. By the Birkhoff ergodic theorem, A_i^f is also a full μ measure subset of B_i. Let f_n be a countable sequence of functions in $C^0(X_0) \cap H$, which is C^0-dense in $\overline{C^0(X_0) \cap H}$, and set $A_i = \bigcap_{n\in\mathbb{N}} A_i^{f_n}$. These sets satisfy the conclusion of the theorem.

This concludes the proof of the theorem when 1 is the only eigenvalue of modulus 1 of $\mathcal{L}$. If $\mathcal{L}$ has other eigenvalues of modulus 1, let N be such that $\lambda^N = 1$ for all these eigenvalues λ. The above result applies to T^N, and gives sets $A_1, \dots, A_l$ and probability measures $\mu_1, \dots, \mu_l$. The map T induces a permutation of the sets A_i (modulo sets of 0 measure for μ), say $T(A_i) = A_{\sigma(i)}$ mod 0 for some permutation σ of $\{1, \dots, l\}$. For any orbit $(i_1, \dots, i_k)$ of σ, the measure $\frac{1}{k}(\mu_{i_1} + \cdots + \mu_{i_k})$ is T-invariant, and its basin of attraction contains $\bigcap_{j=0}^{N-1} T^{-j}(A_{i_1} \cup \cdots \cup A_{i_k})$. These measures are the measures of the statement of the theorem, and their properties readily follow from the corresponding properties for T^N. □

To deduce Theorem 14 from Theorem 33, we just have to check the fifth condition of Theorem 33 since the other ones are trivially satisfied. Working locally in a chart, it is sufficient to prove the following lemma:

Lemma 34. *Let K be a compact smooth hypersurface with boundary in $\mathbb{R}^d$, whose intersection with almost every line parallel to a coordinate axis has at most $L < \infty$ points. Let $1/p - 1 < t_- \leqslant 0 \leqslant t < 1/p$, and let $u \in H_p^{t,t_-}$ be such that*

- *there exists a sequence of nonnegative functions $u_n \in H_p^{t,t_-} \cap L_\infty(\mathrm{Leb})$ converging in H_p^{t,t_-} to u;*
- *there exists a measure μ with $\langle u, g\,\mathrm{dLeb}\rangle = \int g\,\mathrm{d}\mu$ for any C^∞ function g;*
- *the support of u does not intersect ∂K.*

Then $\mu(K) = 0$.

Proof. Let us first prove that there exists a sequence of neighborhoods K_n of $K \cap \operatorname{supp} u$, whose intersection with almost every line parallel to a coordinate axis has at most $L' < \infty$ connected components, and with $\mathrm{Leb}(K_n) \to 0$.

Working locally, we can assume that K is transversal to a coordinate direction, say the last one. Hence, we can assume that u is supported in $[-1/2, 1/2]^{d-1} \times \mathbb{R}$, and that K can be written as the graph of a smooth function f,

$$K = \{(x_1, \dots, x_{d-1}, f(x_1, \dots, x_{d-1})) \mid (x_1, \dots, x_{d-1}) \in [-1, 1]^{d-1}\}. \tag{70}$$

Let $K_n = \{(x_1, \dots, x_{d-1}, f(x_1, \dots, x_{d-1}) + y) \mid (x_1, \dots, x_{d-1}) \in [-1, 1]^{d-1}, |y| < 1/n\}$. It is a neighborhood of $K \cap \operatorname{supp} u$. It intersects any line parallel to the last coordinate axis along one connected component. Consider now another coordinate axis, say the first one. Fix $(x_2, \dots, x_{d-1})$. Then the boundary of $K_n \cap (\mathbb{R} \times \{(x_2, \dots, x_{d-1})\} \times \mathbb{R})$ is formed of two vertical segments and two translates of the graph of the function $x \mapsto f(x, x_2, \dots, x_d)$. For almost every $(x_2, \dots, x_d)$, this graph intersects almost every horizontal line along at most L points. Hence, the intersection of almost every horizontal line with the boundary of $K_n \cap (\mathbb{R} \times \{(x_2, \dots, x_{d-1})\} \times \mathbb{R})$ has at most $2L + 2$ points. In particular, K_n intersects almost every horizontal line along at most $2L + 1$ connected components. This concludes the construction of K_n.

By Lemma 23, there exists a constant C such that, for any $n \in \mathbb{N}$, the multiplication by 1_{K_n} sends H_p^{t,t_-} into itself, with a norm bounded by C. In particular, 1_{K_n} belongs to H_p^{t,t_-} and is bounded in this space.

Let us show that 1_{K_n} tends to 0 in H_p^{t,t_-}. Let $t' \in (t, 1/p)$. Then 1_{K_n} is also bounded in H_p^{t',t_-} by the same argument. Since the injection of H_p^{t',t_-} in H_p^{t,t_-} is compact, the sequence 1_{K_n} is therefore relatively compact in H_p^{t,t_-}. Let v be one of its cluster values. For any smooth function g,

$$\langle v, g\,\mathrm{dLeb}\rangle = \lim \langle 1_{K_n}, g\,\mathrm{dLeb}\rangle = \lim \int 1_{K_n} g\,\mathrm{dLeb} = 0, \tag{71}$$

since $\mathrm{Leb}(K_n)$ tends to 0. Hence, v is the zero distribution. The sequence 1_{K_n} is relatively compact in H_p^{t,t_-} and its only cluster value is zero, hence it converges to 0.

Let us now show that, for any $v \in H_p^{t,t_-}$,

$$\|1_{K_n} v\|_{H_p^{t,t_-}} \to 0. \tag{72}$$

Choose a C^∞ function ϕ with $\|v-\phi\|_{H_p^{t,t_-}} \leqslant \epsilon$, then

$$\|1_{K_n} v\|_{H_p^{t,t_-}} \leqslant \big\|1_{K_n}(v-\phi)\big\|_{H_p^{t,t_-}} + \|1_{K_n}\phi\|_{H_p^{t,t_-}} \leqslant C\|v-\phi\|_{H_p^{t,t_-}} + \|\phi\|_{C^1}\|1_{K_n}\|_{H_p^{t,t_-}} \leqslant C\epsilon + o(1).$$

This proves (72).

Let g be a C^∞ function supported in K_n, taking its values in [0, 1], equal to 1 on K. We claim that

$$\int g\,\mathrm{d}\mu \leqslant \langle 1_{K_n} u, \mathrm{dLeb}\rangle. \tag{73}$$

Indeed, write $u = \lim u_m$ where u_m is a nonnegative function belonging to $L_\infty(\mathrm{Leb}) \cap H_p^{t,t_-}$. Then $\langle u_m, g\,\mathrm{dLeb}\rangle = \int g u_m\,\mathrm{dLeb} \leqslant \int 1_{K_n} u_m\,\mathrm{dLeb} = \langle 1_{K_n} u_m, \mathrm{dLeb}\rangle$. Taking the limit over m, we get (73).

We can now conclude the proof: by (73), we have $\mu(K) \leqslant C\|1_{K_n} u\|_{H_p^{t,t_-}}$. This quantity converges to 0 by (72). □

Remark 35. The proof of the previous lemma implies that Dirac masses cannot belong to H_p^{t,t_-} if $1/p - 1 < t_- \leqslant 0 \leqslant t < 1/p$: assume for a contradiction that δ_0, the Dirac mass at 0, belongs to H_p^{t,t_-}. Take K_n the ball of radius $1/n$ centered at 0. Then $\delta_0 = 1_{K_n}\delta_0$ for each n, but $1_{K_n}\delta_0$ tends to zero in H_p^{t,t_-} as $n \to \infty$, a contradiction.

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