Stability of multipeakons

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Abstract

The Camassa–Holm equation possesses well-known peaked solitary waves that are called peakons. Their orbital stability has been established by Constantin and Strauss in [A. Constantin, W. Strauss, Stability of peakons, Comm. Pure Appl. Math. 53 (2000) 603–610]. We prove here the stability of ordered trains of peakons. We also establish a result on the stability of multipeakons.
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1. Introduction

The Camassa–Holm equation $(C–H)_\kappa$, $\kappa \geqslant 0$,

$$u_t - u_{txx} = -2\kappa u_x - 3u u_x + 2u_x u_{xx} + u u_{xxx}, \quad (t, x) \in \mathbb{R}^2, \tag{1}$$

can be derived as a model for the propagation of unidirectional shalow water waves over a flat bottom by writing the Green–Naghdi equations in Lie–Poisson Hamiltonian form and then making an asymptotic expansion which keeps the Hamiltonian structure [5,23]. It was also found independently by Dai [15] as a model for nonlinear waves in cylindrical hyperelastic rods and was, in fact, first discovered by the method of recursive operator by Fokas and Fuchssteiner [20] as an example of bi-Hamiltonian equation.

$(C–H)_\kappa$ is completely integrable (see [5–7,10]). It possesses among others the following invariants

$$E(v) = \int_{\mathbb{R}} v^2(x) + v_x^2(x)\, dx \quad \text{and} \quad F(v) = \int_{\mathbb{R}} v^3(x) + v(x)v_x^2(x) + 2\kappa v^2(x)\, dx \tag{2}$$

and can be written in Hamiltonian form as

$$\partial_t E'(u) = -\partial_x F'(u). \tag{3}$$

For $\kappa > 0$ it possesses smooth positive solitary waves $\varphi_{\kappa,c}$ with speed $c > 2\kappa$, their orbital stability has been proved in [12] by applying the classical spectral method initiated by Benjamin [2] (see also [21]). In [19], following the general method developed in [24] (see also [18]), the authors proved the stability of ordered trains of such solitary waves.

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It is worth recalling that this general method requires principally two ingredients: A property of almost monotonicity which says that for a solution close to $\varphi_{\kappa,c}$, the part of the energy traveling at the right of $\varphi_{\kappa,c}(\cdot - ct)$ is almost time decreasing; A dynamical proof of the stability of the solitary wave using the spectral approach (as in [2] or [21] for instance).

In this paper we consider the Camassa–Holm equation in the case $\kappa = 0$, that is

$$u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx}, \quad (t, x) \in \mathbb{R}^2. \tag{4}$$

It is worth noticing that (4) has to be rewritten as

$$u_t - u_{txx} = -\frac{3}{2}\partial_x(u^2) - \frac{1}{2}\partial_x(u_x^2) + \frac{1}{2}\partial_x^3(u^2) \tag{5}$$

or

$$u_t + uu_x + (1 - \partial_x^2)^{-1}\partial_x(u^2 + u_x^2/2) = 0 \tag{6}$$

to give a meaning to some non-smooth solutions. Henceforth, we refer to (5) as the Camassa–Holm equation (C–H). (5) possesses solitary waves but they are non-smooth and are called peakons. They are given by

$$u(t, x) = \varphi_c(x - ct) = c\varphi(x - ct) = ce^{-|x-ct|}, \quad c \in \mathbb{R}, \tag{7}$$

and are solutions of (5) in the distribution sense. Let us point out that the presence of a peak at the wave crest is analogous to the case of exact traveling wave solutions of the governing equations for water waves representing waves of greatest height (cf. [8] and [9]).

The stability of the peakons seems not to enter the general framework mentioned above (see the beginning of Section 3 for further commentaries on this aspect). However, Constantin and Strauss [11] succeeded in proving their orbital stability by a direct approach. In this work, following the general strategy initiated in [24] (note that due to the reasons mentioned above, the general method of [24] is not directly applicable here), we combine the monotonicity result proved in [18] with localized versions of the estimates established in [11] to derive the stability of the trains of peakons.

Before stating the main result we have to introduce the function space where will live our class of solutions to the equation. For I a finite or infinite interval of $\mathbb{R}$, we denote by $Y(I)$ the function space[1]

$$Y(I) := \{u \in C(I; H^1(\mathbb{R})) \cap L^\infty(I; W^{1,1}(\mathbb{R})), \ u_x \in L^\infty(I; BV(\mathbb{R}))\}. \tag{8}$$

We are now ready to state our main result.

Theorem 1.1. *Let be given N velocities $c_1, \ldots, c_N$ such that $0 < c_1 < c_2 < \cdots < c_N$. There exist γ_0, $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that if $u \in Y([0, T[)$, with $0 < T \leqslant \infty$, is a solution of (C–H) satisfying*

$$\left\| u_0 - \sum_{j=1}^N \varphi_{c_j}(\cdot - z_j^0) \right\|_{H^1} \leqslant \varepsilon^2 \tag{9}$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_j^0 - z_{j-1}^0 \geqslant L$, with $L > L_0$, then there exist $x_1(t), \ldots, x_N(t)$ such that

$$\sup_{[0,T[} \left\| u(t, \cdot) - \sum_{j=1}^N \varphi_{c_j}(\cdot - x_j(t)) \right\|_{H^1} \leqslant A(\sqrt{\varepsilon} + L^{-1/8}) \tag{10}$$

and

$$x_j(t) - x_{j-1}(t) > L/2, \quad \forall t \in [0, T[. \tag{11}$$

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[1] $W^{1,1}(\mathbb{R})$ is the space of $L^1(\mathbb{R})$ functions with derivatives in $L^1(\mathbb{R})$ and $BV(\mathbb{R})$ is the space of function with bounded variation.

Remark 1.1. We do not know how to prove the monotonicity result in Lemma 4.2, and thus Theorem 1.1, for solutions that are only in $C([0, T[; H^1(\mathbb{R}))$ which is the hypothesis required for the stability of a single peakon (cf. [11]). Note anyway that there exists no well-posedness result in the class $C([0, T[; H^1(\mathbb{R}))$ for general initial data in $H^1(\mathbb{R})$. On the other hand, according to Theorem 2.1 below, $u \in Y([0, T[)$ as soon as $u_0 \in H^1(\mathbb{R})$ and $(1 - \partial_x^2)u_0$ is a Radon measure with bounded variations.

Remark 1.2. Note that, since (5) is invariant by the change of unknown: $u(t, x) \mapsto -u(t, -x)$, Theorem 1.1 remains true, up to obvious changes, in the case where $c_N < \cdots < c_2 < c_1 < 0$, i.e. there are only anti-peakons. On the other hand, the proof of Theorem 1.1 cannot be directly adapted in the case where all the velocities do not have the same sign, i.e. there are peakons and anti-peakons. However, we strongly believe that such configuration can be treated by adding supplementary arguments.

As discovered by Camassa and Holm [5], (C–H) possesses also special solutions called multipeakons given by

$$u(t, x) = \sum_{i=1}^{N} p_j(t) e^{-|x-q_j(t)|},$$

where $(p_j(t), q_j(t))$ satisfy the differential system (61). In [1] (see also [5]), the asymptotic behavior of the multi-peakons is studied. In particular, the limits as t tends to $+\infty$ and $-\infty$ of $p_i(t)$ and $\dot{q}_i(t)$ are determined. Combining these asymptotics with the preceding theorem we get the following result on the stability of the variety $\mathcal{N}$ of $H^1(\mathbb{R})$ defined by

$$\mathcal{N} := \left\{ v = \sum_{i=1}^{N} p_j e^{-|\cdot - q_j|}, \ (p_1, \ldots, p_N) \in (\mathbb{R}_+)^N, \ q_1 < q_2 < \cdots < q_N \right\}.$$

Corollary 1.1. *Let be given N positive real numbers $p_1^0, \ldots, p_N^0$ and N real numbers $q_1^0 < \cdots < q_N^0$. For any $B > 0$ and any $\gamma > 0$ there exists $\alpha > 0$ such that if $u_0 \in H^1(\mathbb{R})$ satisfies[2] $m_0 := u_0 - u_{0,xx} \in \mathcal{M}_+(\mathbb{R})$ with*

$$\|m_0\|_{\mathcal{M}} \leqslant B \quad and \quad \left\| u_0 - \sum_{j=1}^{N} p_j^0 \exp(\cdot - q_j^0) \right\|_{H^1} \leqslant \alpha \tag{12}$$

then

$$\forall t \in \mathbb{R}, \quad \inf_{P \in (\mathbb{R}_+)^N, Q \in \mathbb{R}^N} \left\| u(t, \cdot) - \sum_{j=1}^{N} p_j \exp(\cdot - q_j) \right\|_{H^1} \leqslant \gamma. \tag{13}$$

Moreover, there exists $T > 0$ such that

$$\forall t \geqslant T, \quad \inf_{Q \in \mathcal{G}} \left\| u(t, \cdot) - \sum_{j=1}^{N} \lambda_j \exp(\cdot - q_j) \right\|_{H^1} \leqslant \gamma \tag{14}$$

and

$$\forall t \leqslant -T, \quad \inf_{Q \in \mathcal{G}} \left\| u(t, \cdot) - \sum_{j=1}^{N} \lambda_{N+1-j} \exp(\cdot - q_j) \right\|_{H^1} \leqslant \gamma, \tag{15}$$

where $\mathcal{G} := \{Q \in \mathbb{R}^N, \ q_1 < q_2 < \cdots < q_N\}$ and $0 < \lambda_1 < \cdots < \lambda_N$ are the eigenvalues of the matrix $(p_j^0 e^{-|q_i^0 - q_j^0|/2})_{1 \leqslant i, j \leqslant N}$.

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[2] $\mathcal{M}(\mathbb{R})$ is the space of Radon measures on $\mathbb{R}$ with bounded total variation and $\mathcal{M}_+(\mathbb{R})$ is the subset of non-negative measures.

Remark 1.3. From Theorem 2.1 below, note that $u \in Y([0, T])$ as soon as $(1 - \partial_x^2)u_0$ is a positive (or negative) Radon measure with finite variations. This is of course the case for any initial data of the form $\sum_{i=1}^{N} c_i e^{|x-q_j|}$ with $c_i > 0$, $i = 1, \ldots, N$.

This paper is organized as follows. In Section 2 we state a well-posedness result for (C–H) established in [13] and [16]. This allows us to work in the function space $Y([0, T])$ that contains the peakons. Next, in Section 3 we present the result and the proof of Constantin and Strauss on the stability of peakons. Section 4 is devoted to the proof of Theorem 1.1. It is divided into four subsections. First we use a modulation argument in order to control the distance between the different bumps of the solution we consider. Then we state a monotonicity result that was established in [18]. In Section 4.3 we establish a local version of an estimate involved in the stability of a single peakon. The proof of Theorem 1.1 is completed in Section 4.4. In Section 5 we recall some properties of the multipeakons and prove Corollary 1.1. Finally in Appendix A we give the proof of the monotonicity result for sake of completeness.

As mentioned above, the proof of the stability of trains of peakons does not enter the general framework [24,18, 25,19] on orbital stability of ordered trains of solitary waves. However, the strategy of combining the orbital stability of a single solitary wave with a monotonicity result seems to be quite robust.

2. Well-posedness result

Recall that the peakons do not belong to $H^{3/2}(\mathbb{R})$. In [13,16] (see also [26]) the following existence and uniqueness result is derived.

Theorem 2.1. *Let $u_0 \in H^1(\mathbb{R})$ with $m_0 := u_0 - u_{0,xx} \in \mathcal{M}(\mathbb{R})$ then there exists $T = T(\|m_0\|_{\mathcal{M}}) > 0$ and a unique solution $u \in Y([-T, T])$ to (C–H) with initial data u_0. The functionals $E(\cdot)$ and $F(\cdot)$ are constant along the trajectory and if m_0 has a definite sign then u is global in time.*

Moreover, let $\{u_{0,n}\} \subset H^1(\mathbb{R})$ with $\{u_{0,n} - \partial_x^2 u_{0,n}\}$ bounded in $\mathcal{M}_+(\mathbb{R})$ such that $u_{0,n} \to u_0$ in $H^1(\mathbb{R})$. Then, for all $T > 0$,

$$u_n \to u \text{ in } C\big([-T, T]; H^1(\mathbb{R})\big). \tag{16}$$

Let us note that the last assertion of the above theorem is not explicitly contained in the works mentioned above. However, following the same arguments as those developed in these works (see for instance Section 5 of [26]), one can prove that there exists a subsequence $\{u_{n_k}\}$ of solutions of (4) that converges in $C([-T, T]; H^1(\mathbb{R}))$ to some solution v of (4) belonging to $Y(-T, T)$. Since u_{0,n_k} converges to u_0 in H^1, it follows that $v(0) = u_0$ and thus $v = u$ by uniqueness. This ensures that the whole sequence $\{u_n\}$ converges to u in $C([-T, T]; H^1(\mathbb{R}))$ and concludes the proof of the last assertion.

It is worth pointing out that recently, in [3] and [4], Bressan and Constantin have constructed global conservative and dissipative solutions of the Camassa–Holm equation for any initial data in $H^1(\mathbb{R})$. However, even if for the conservative solutions, $E(\cdot)$ and $F(\cdot)$ are conserved quantities, these solutions are not known to be continuous with values in $H^1(\mathbb{R})$. Therefore, as noticed in Remark 1.1, even one single peakon is not known to be orbitally stable with respect to this class of solutions. For this reason we will work in the class of solutions constructed in Theorem 2.1.

3. Stability of a single peakon

Recall that the classical proof of orbital stability (see [2,21]), successfully used in the case $\kappa > 0$ in [12], is based on the spectral properties of the second differential operator of the invariant functional $L_c(\cdot) := cE(\cdot) - F(\cdot)$ evaluated at the solitary wave $\varphi_{\kappa,c}$ of (1). Indeed, using a Liouville substitution (cf. [12]), it can be shown that the spectrum of the L^2-self-adjoint operator

$$H_c := L_c''(\varphi_{\kappa,c}) = -\partial_x\big((2c - 2\varphi_{\kappa,c})\partial_x\big) - 6\varphi_{\kappa,c} - 2\partial_x^2 \varphi_{\kappa,c} + 2(c - 2\kappa)$$

contains a unique negative eigenvalue which is simple and that 0 is a simple eigenvalue associated with $\partial_x\varphi_{\kappa,c}$. The rest of the spectrum consists of a finite number of positive eigenvalues and of the essential spectrum $[2c - 4\kappa, +\infty[$. Therefore, controlling the negative direction by modulating the velocity c and using that $\langle E'(\varphi_{\kappa,c}), u - \varphi_{\kappa,c}\rangle \sim 0$ (since

$E(\cdot)$ is conserved) and the kernel direction by choosing a suitable translation $\varphi_{\kappa,c}(\cdot - r)$ of $\varphi_{\kappa,c}$, the orbital stability is proven by writing the Taylor expansion of $cE(\cdot) - F(\cdot)$ at $\varphi_{\kappa,c}$, recalling that $cE'(\varphi_{\kappa,c}) - F'(\varphi_{\kappa,c})$ vanishes.

Now, in the case $\kappa = 0$, H_c is degenerate since, in view of (7), $\varphi_c(0) = c$ and the Liouville substitution is no more well-defined. However, Constantin and Strauss (cf. [11]) succeeded in proving the orbital stability by a direct approach (see also [14] for another stability result, using Cazenave–Lions method, under additional assumptions on the initial data). Actually, a by-product of their proof is the following very rigid property: for any function v in some H^1-neighborhood of φ_c it holds

$$\left\| v - \varphi_c(\cdot - \xi) \right\|_{H^1}^2 \lesssim \left| E(v) - E(\varphi_c) \right| + \sqrt{c \left| L_c(v) - L_c(\varphi_c) \right|}$$

where $v(\xi) = \max_{\mathbb{R}} v$. Since $E(\cdot)$ and $F(\cdot)$ are conserved and are continuous functional on $H^1(\mathbb{R})$, this clearly leads to the orbital stability.

Since we will use similar considerations, we present here a sketch of the proof of the stability of peakons (Theorem 3.1) proved by Constantin and Strauss in [11].

Theorem 3.1. *Let be given $c > 0$. There exist $C > 0$ and $\varepsilon_0 > 0$ such that if $u \in C([0, T[; H^1(\mathbb{R}))$ is a solution of (4) such that $E(u(t))$ and $F(u(t))$ are conserved quantities on $[0, T[$ and $\|u(0) - \varphi_c\|_{H^1} \leqslant \varepsilon^2$ for some $0 < \varepsilon \leqslant \varepsilon_0$, then*

$$\left\| u(t, \cdot) - \varphi_c(\cdot - r(t)) \right\|_{H^1} \leqslant C\sqrt{\varepsilon}, \quad \forall t \in [0, T[, \tag{17}$$

where $r(t) \in \mathbb{R}$ is any point where the function $u(t, \cdot)$ attains its maximum.

The proof of this theorem is principally based on the following lemma of [11].

Lemma 3.1. *For any $u \in H^1(\mathbb{R})$ and $\xi \in \mathbb{R}$,*

$$E(u) - E(\varphi_c) = \left\| u - \varphi_c(\cdot - \xi) \right\|_{H^1}^2 + 4c\left(u(\xi) - c \right). \tag{18}$$

For any $u \in H^1(\mathbb{R})$, let $M = \max_{x \in \mathbb{R}}\{u(x)\}$, then

$$F(u) \leqslant M E(u) - \frac{2}{3}M^3. \tag{19}$$

Remark 3.1. It is worth noticing that (18) ensures that, for $c > 0$, the minimum of the H^1-distance between u and $\{\varphi_c(\cdot - \xi), \ \xi \in \mathbb{R}\}$ is exactly reached at any point ξ where u attains its maximum on $\mathbb{R}$.

Proof of Theorem 3.1. Let $u \in C([0, T[; H^1(\mathbb{R}))$ be a solution of (4) with $\|u(0) - \varphi_c\|_{H^1} \leqslant \varepsilon^2$ and let $\xi(t) \in \mathbb{R}$ be such that $u(t, \xi(t)) = \max_{\mathbb{R}} u(t, \cdot)$. By the remark above, $t \mapsto \|u(t) - \varphi_c(\cdot - \xi(t))\|_{H^1}$ is continuous on $[0, T[$ and $\|u(0) - \varphi_c(\cdot - \xi(0))\|_{H^1} \leqslant \varepsilon^2$. Moreover, as shown in [11], it is no to hard to check that for any $v \in H^1(\mathbb{R})$ such that $\|v - \varphi_c\|_{H^1} < \gamma$ for some $\gamma < c/2$, it holds

$$\left| E(v) - E(\varphi_c) \right| < 4c\gamma \quad \text{and} \quad \left| F(v) - F(\varphi_c) \right| < 10c^2\gamma. \tag{20}$$

From the conservation laws it follows that for any $t \in [0, T[$,

$$\left| E\left(u(t) \right) - E(\varphi_c) \right| < 4c\varepsilon^2 \quad \text{and} \quad \left| F\left(u(t) \right) - F(\varphi_c) \right| < 10c^2\varepsilon^2. \tag{21}$$

Therefore, by a classical continuity argument, it suffices to prove that for any $v \in H^1(\mathbb{R})$ satisfying (21) and $\|v - \varphi_c(\cdot - \xi)\|_{H^1} \leqslant \varepsilon^{1/4}$, with $v(\xi) = \max_{\mathbb{R}} v$, it holds actually

$$\left\| v - \varphi_c(\cdot - \xi) \right\|_{H^1} \lesssim \sqrt{\varepsilon}.$$

Setting $M = v(\xi)$ and $\delta = c - M = c - v(\xi)$, we notice that (18) ensures that for $\delta \leqslant 0$,

$$\left\| v - \varphi_c(\cdot - \xi) \right\|_{H^1}^2 \leqslant E(u_0) - E(\varphi_c) \lesssim \varepsilon^2.$$

Hence to prove the stability it remains to examine the case $\delta > 0$, that is the maximum of the function u is less than the maximum of the peakon φ_c. Substituting M by $c - \delta$ in (19), using (21) and that

$$E(\varphi_c) = 2c^2 \quad \text{and} \quad F(\varphi_c) = \frac{4}{3}c^3,$$ (22)

one can easily check that

$$\frac{4}{3}c^3 - O(\varepsilon^2) \leqslant (c - \delta)(2c^2 + O(\varepsilon^2)) - \frac{2}{3}(c - \delta)^3$$

which leads to

$$\delta^2(c - \delta/3) \leqslant O(\varepsilon^2).$$ (23)

On the other hand, on account of the hypothesis $\|v - \varphi_c(\cdot - \xi)\|_{H^1} \leqslant \varepsilon^{1/4}$ and of the continuous embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$, it holds $\delta < c/2$ for ε small enough. Therefore (23) ensures that $\delta \leqslant C\varepsilon$, the constant C depending only on c. This estimate on δ combining with (18) and (21) concludes the proof of Theorem 3.1. $\quad\square$

4. Stability of multipeakons

For $\alpha > 0$ and $L > 0$ we define the following neighborhood of all the sums of N peakons of speed $c_1, \ldots, c_N$ with spatial shifts x_j that satisfied $x_j - x_{j-1} \geqslant L$,

$$U(\alpha, L) = \left\{ u \in H^1(\mathbb{R}), \quad \inf_{x_j - x_{j-1} > L} \left\| u - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - x_j) \right\|_{H^1} < \alpha \right\}.$$ (24)

By the continuity of the map $t \mapsto u(t)$ from $[0, T[$ into $H^1(\mathbb{R})$, to prove Theorem 1.1 it suffices to prove that there exist $A > 0$, $\varepsilon_0 > 0$ and $L_0 > 0$ such that $\forall L > L_0$ and $0 < \varepsilon < \varepsilon_0$, if u_0 satisfies (9) and if for some $0 < t_0 < T$,

$$u(t) \in U\left(A(\sqrt{\varepsilon} + L^{-1/8}), L/2\right) \quad \text{for all } t \in [0, t_0],$$ (25)

then

$$u(t_0) \in U\left(\frac{A}{2}(\sqrt{\varepsilon} + L^{-1/8}), \frac{2L}{3}\right).$$ (26)

Therefore, in the sequel of this section we will assume (25) for some $0 < \varepsilon < \varepsilon_0$ and $L > L_0$, with A, ε_0 and L_0 to be specified later, and we will prove (26).

4.1. Control of the distance between the peakons

In this subsection we want to prove that the different bumps of u that are individually close to a peakon get away from each others as time is increasing. This is crucial in our analysis since we do not know how to manage strong interactions.

Lemma 4.1. *Let u_0 satisfying* (9). *There exist $\alpha_0 > 0$, $L_0 > 0$ and $C_0 > 0$ such that for all $0 < \alpha < \alpha_0$ and $0 < L_0 < L$ if $u(t) \in U(\alpha, L/2)$ on $[0, t_0]$ for some $0 < t_0 < T$ then there exist C^1-functions $\tilde{x}_1, \ldots, \tilde{x}_N$ defined on $[0, t_0]$ such that*

$$\frac{d}{dt}\tilde{x}_i = c_i + O(\sqrt{\alpha}) + O(L^{-1}), \quad i = 1, \ldots, N,$$ (27)

$$\left\| u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t)) \right\|_{H^1} = O(\sqrt{\alpha}),$$ (28)

$$\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geqslant 3L/4 + (c_i - c_{i-1})t/2, \quad i = 2, \ldots, N.$$ (29)

Moreover, setting $J_i := [y_i(t), y_{i+1}(t)]$, $i = 1, \ldots, N$, with

$$y_1 = -\infty, \qquad y_{N+1} = +\infty \quad \text{and} \quad y_i(t) = \frac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2}, \quad i = 2, \ldots, N,$$ (30)

it holds

$$\left| x_i(t) - \tilde{x}_i(t) \right| \leqslant L/12, \quad i = 1, \dots, N, \tag{31}$$

where $x_1(t), \dots, x_N(t)$ are any point such that

$$u\bigl(t, x_i(t)\bigr) = \max_{J_i(t)} u(t), \quad i = 1, \dots, N. \tag{32}$$

Proof. To prove this lemma we use a modulation argument. The strategy is to construct N C^1-functions $\tilde{x}_1, \dots, \tilde{x}_N$ on $[0, t_0]$ satisfying a suitable orthogonality condition, see (37). Thanks to this orthogonality condition we will be able to prove that the speed of the $\tilde{x}_i$ stays close to c_i on $[0, t_0]$. $\quad \square$

Remark 4.1. It is crucial to note that in the previous works on stability of sum of solitary waves [24,18,19] one needs similar modulation to ensure (among other things) that v remains in a subspace of codimension two of $H^1(\mathbb{R})$ where the operator H_c (see the beginning of this section) is positive. Here, as already mentioned, we do not use such operator in the proof of orbital stability of peakons but we still need a modulation to ensure that the different bumps of u get away from each others.

For $Z = (z_1, \dots, z_N) \in \mathbb{R}^N$ fixed such that $z_i - z_{i-1} > L/2$, we set

$$R_Z(\cdot) = \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i).$$

For $0 < \alpha < \alpha_0$ we define the function

$$Y : (-\alpha, \alpha)^N \times B_{H^1}(R_Z, \alpha) \to \mathbb{R}^n,$$
$$(y_1, \dots, y_N, u) \mapsto \bigl(Y^1(y_1, \dots, y_N, u), \dots, Y^N(y_1, \dots, y_N, u)\bigr)$$

with

$$Y^i(y_1, \dots, y_N, u) = \int_{\mathbb{R}} \left(u - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - z_j - y_j) \right) \partial_x \varphi_{c_i}(\cdot - z_i - y_i).$$

Y is clearly of class C^1. For $i = 1, \dots, N$,

$$\frac{\partial Y^i}{\partial y_i}(y_1, \dots, y_N, u) = \int_{\mathbb{R}} \left(u_x - \sum_{j=1, \, j \neq i}^{N} \int_{\mathbb{R}} \partial_x \varphi_{c_j}(\cdot - z_j - y_j) \right) \partial_x \varphi_{c_i}(\cdot - z_i - y_i) \, dx, \tag{33}$$

and $\forall j \neq i$,

$$\frac{\partial Y^i}{\partial y_j}(y_1, \dots, y_N, u) = \int_{\mathbb{R}} \partial_x \varphi_{c_j}(\cdot - z_j - y_j) \partial_x \varphi_{c_i}(\cdot - z_i - y_i) \, dx.$$

Hence,

$$\frac{\partial Y^i}{\partial y_i}(0, \dots, 0, R_Z) = \| \partial_x \varphi_{c_i} \|_{L^2}^2 \geqslant c_1^2, \tag{34}$$

and, for $j \neq i$, using the exponential decay of φ_c and that $z_i - z_{i-1} > L$ we infer that for L_0 large enough (recall that $L > L_0$),

$$\frac{\partial Y^i}{\partial y_j}(0, \dots, 0, R_Z) = \int_{R} \partial_x \varphi_{c_j}(\cdot - z_j) \partial_x \varphi_{c_i}(\cdot - z_i) \, dx \leqslant O\bigl(e^{-L/4}\bigr).$$

We deduce that, for $L > 0$ large enough, $D_{(y_1, \dots, y_N)} Y(0, \dots, 0, R_Z) = D + P$ where D is an invertible diagonal matrix with $\|D^{-1}\| \leqslant (c_1)^{-2}$ and $\|P\| \leqslant O(e^{-L/4})$. Hence there exists $L_0 > 0$ such that for $L > L_0$,

$D_{(y_1,\ldots,y_N)}Y(0,\ldots,0,R_Z)$ is invertible with an inverse matrix of norm smaller than $2\,(c_1)^{-2}$. From the implicit function theorem we deduce that there exists $\beta_0 > 0$ and C^1 functions $(y_1,\ldots,y_N)$ from $B(R_Z,\beta_0)$ to a neighborhood of $(0,\ldots,0)$ which are uniquely determined such that

$$Y(y_1,\ldots,y_N,u) = 0 \quad \text{for all } u \in B(R_Z,\beta_0).$$

In particular, there exits $C_0 > 0$ such that if $u \in B(R_Z,\beta)$, with $0 < \beta \leqslant \beta_0$, then

$$\sum_{i=1}^N |y_i(u)| \leqslant C_0\beta. \tag{35}$$

Note that β_0 and C_0 only depend on c_1 and L_0 and not on the point $(z_1,\ldots,z_N)$. For $u \in B(R_Z,\beta_0)$ we set $\tilde{x}_i(u) = z_i + y_i(u)$. Assuming that $\beta_0 \leqslant \frac{L_0}{8C_0}$, $(\tilde{x}_1,\ldots,\tilde{x}_N)$ are thus C^1-functions on $B(R_Z,\beta)$ satisfying

$$\tilde{x}_j(u) - \tilde{x}_{j-1}(u) > L/2 - 2C_0\beta > L/4. \tag{36}$$

For $L \geqslant L_0$ and $0 < \alpha < \alpha_0 < \beta_0/2$ to be chosen later, we define the modulation of $u \in U(\alpha, L/2)$ in the following way: we cover the trajectory of u by a finite number of open balls in the following way:

$$\{u(t),\ t \in [0,t_0]\} \subset \bigcup_{k=1,\ldots,M} B(R_{Z^k}, 2\alpha).$$

It is worth noticing that, since $0 < \alpha < \alpha_0 < \beta_0/2$, the functions $\tilde{x}_j(u)$ are uniquely determined for $u \in B(R_{Z^k}), 2\alpha) \cap B(R_{Z^{k'}}, 2\alpha)$. We can thus define the functions $t \mapsto \tilde{x}_j(t)$ on $[0,t_0]$ by setting $\tilde{x}_j(t) = \tilde{x}_j(u(t))$. By construction

$$\int_{\mathbb{R}} \left(u(t,\cdot) - \sum_{j=1}^N \varphi_{c_j}\big(\cdot - \tilde{x}_j(t)\big) \right) \partial_x \varphi_{c_i}\big(\cdot - \tilde{x}_i(t)\big)\,dx = 0. \tag{37}$$

Moreover, on account of (35) and the fact that φ_c'' is the sum of an L^1 function and a Dirac mass it holds

$$\big\| u(t) - R_{\tilde{X}(t)} \big\|_{H^1} \lesssim C_0\sqrt{\alpha}, \quad \forall t \in [0,t_0]. \tag{38}$$

Let us now prove that the speed of $\tilde{x}_i$ stays close to c_i. We set

$$R_j(t) = \varphi_{c_j}\big(\cdot - \tilde{x}_j(t)\big) \quad \text{and} \quad v(t) = u(t) - \sum_{i=1}^N R_j(t) = u(t,\cdot) - R_{\tilde{X}(t)}.$$

Differentiating (37) with respect to time we get

$$\int_{\mathbb{R}} v_t \partial_x R_i = \dot{\tilde{x}}_i \big\langle \partial_x^2 R_i, v \big\rangle_{H^{-1},H^1},$$

and thus

$$\left| \int_{\mathbb{R}} v_t \partial_x R_i \right| \leqslant |\dot{\tilde{x}}_i| O\big(\|v\|_{H^1}\big) \leqslant |\dot{\tilde{x}}_i - c_i| O\big(\|v\|_{H^1}\big) + O\big(\|v\|_{H^1}\big). \tag{39}$$

Substituting u by $v + \sum_{j=1}^N R_j$ in (6) and using that R_j satisfies

$$\partial_t R_j + \big(\dot{\tilde{x}}_j - c_j\big)\partial_x R_j + R_j \partial_x R_j + \big(1 - \partial_x^2\big)^{-1}\partial_x\big[R_j^2 + \partial_x R_j^2/2\big] = 0,$$

we infer that v satisfies on $[0,t_0]$,

$$v_t - \sum_{j=1}^N (\dot{\tilde{x}}_j - c_j)\partial_x R_j = -\frac{1}{2}\partial_x\left[\left(v + \sum_{j=1}^N R_j\right)^2 - \sum_{j=1}^N R_j^2 \right] - \big(1 - \partial_x^2\big)^{-1}\partial_x\left[\left(v + \sum_{j=1}^N R_j\right)^2 \right.$$

$$\left. - \sum_{j=1}^N R_j^2 + \frac{1}{2}\left(v_x + \sum_{j=1}^N \partial_x R_j\right)^2 - \frac{1}{2}\sum_{j=1}^N (\partial_x R_j)^2 \right].$$

Taking the L^2-scalar product with $\partial_x R_i$, integrating by parts, using the decay of R_j and its first derivative, (38), (39) and (36), we find

$$\left|\dot{\tilde{x}}_i - c_i\right|\left(\|\partial_x R_i\|_{L^2}^2 + O(\sqrt{\alpha})\right) \leqslant O(\sqrt{\alpha}) + O\left(e^{-L/8}\right). \tag{40}$$

Taking α_0 small enough and L_0 large enough we get $|\dot{\tilde{x}}_i - c_i| \leqslant (c_i - c_{i-1})/4$ and thus for all $0 < \alpha < \alpha_0$ and $L \geqslant L_0 > 3C_0\varepsilon$, it follows from (9), (35) and (40) that

$$\tilde{x}_j(t) - \tilde{x}_{j-1}(t) > L - C_0\varepsilon + (c_j - c_{j-1})t/2, \quad \forall t \in [0, t_0], \tag{41}$$

which yields (29).

Finally from (38) and the continuous embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$, we infer that

$$u(x) = R_{\tilde{X}}(x) + O(\sqrt{\alpha}), \quad \forall x \in \mathbb{R}.$$

Applying this formula with $x = x_i = \max_{J_i(t)} u(t)$ and taking advantage of (29), we obtain

$$u(x_i) = c_i + O(\sqrt{\alpha}) + O\left(e^{-L/4}\right) \geqslant 2c_i/3.$$

On the other hand, for $x \in J_i \setminus]\tilde{x}_i - L/12, \tilde{x}_i + L/12[$, we get

$$u(x) \leqslant c_i e^{-L/12} + O(\sqrt{\alpha}) + O\left(e^{-L/4}\right) \leqslant c_i/2.$$

This ensures that x_i belongs to $[\tilde{x}_i - L/12, \tilde{x}_i + L/12]$.

4.2. Monotonicity property

Thanks to the preceding lemma, for $\varepsilon_0 > 0$ small enough and $L_0 > 0$ large enough, one can construct C^1-functions $\tilde{x}_1, \ldots, \tilde{x}_N$ defined on $[0, t_0]$ such that (27)–(31) are satisfied. In this subsection we state the almost monotonicity of functionals that are very close to the energy at the right of the ith bump, $i = 1, \ldots, N-1$ of u. The proof is similar to the one of Lemma 4.2 in [19]. We give it in Appendix A for sake of completeness.

Let Ψ be a C^∞ function such that $0 < \Psi \leqslant 1$, $\Psi' > 0$ on $\mathbb{R}$, $|\Psi'''| \leqslant 10|\Psi'|$ on $[-1, 1]$,

$$\Psi(x) = \begin{cases} e^{-|x|}, & x < -1, \\ 1 - e^{-|x|}, & x > 1. \end{cases}$$

Setting $\Psi_K = \Psi(\cdot/K)$, we introduce for $j \in \{2, \ldots, N\}$,

$$I_{j,K}(t) = I_{j,K}(t, u(t)) = \int_{\mathbb{R}} \left(u^2(t) + u_x^2(t)\right) \Psi_{j,K}(t)\, dx,$$

where $\Psi_{j,K}(t, x) = \Psi_K(x - y_j(t))$ with $y_j(t)$, $j = 2, \ldots, N$, defined in (30). Note that $I_{j,K}(t)$ is close to $\|u(t)\|_{H^1(x > y_j(t))}$ and thus measures the energy at the right of the $(j-1)$th bump of u. Finally, we set

$$\sigma_0 = \frac{1}{4} \min(c_1, c_2 - c_1, \ldots, c_N - c_{N-1}). \tag{42}$$

In [19], using some tricks developed in [17], the following monotonicity result is derived.

Lemma 4.2. *Let $u \in Y([0, T[)$ be a solution of* (C–H) *satisfying* (28) *on* $[0, t_0]$. *There exist $\alpha_0 > 0$ and $L_0 > 0$ only depending on c_1 such that if $0 < \alpha < \alpha_0$ and $L \geqslant L_0$ then for any $4 \leqslant K \lesssim L^{1/2}$,*

$$I_{j,K}(t) - I_{j,K}(0) \leqslant O\left(e^{-\frac{\sigma_0 L}{8K}}\right), \quad \forall j \in \{2, \ldots, N\},\ \forall t \in [0, t_0]. \tag{43}$$

4.3. A localized and a global estimate

We define the function $\Phi_i = \Phi_i(t, x)$ by $\Phi_1 = 1 - \Psi_{2,K} = 1 - \Psi_K(\cdot - y_2(t))$, $\Phi_N = \Psi_{N,K} = \Psi_K(\cdot - y_N(t))$ and for $i = 2, \ldots, N - 1$,

$$\Phi_i = \Psi_{i,K} - \Psi_{i+1,K} = \Psi_K(\cdot - y_i(t)) - \Psi_K(\cdot - y_{i+1}(t)),$$

where Ψ_K and the y_i's are defined in Section 4.2. It is easy to check that $\sum_{i=1}^N \Phi_{i,K} \equiv 1$. We take $L > 0$ and $L/K > 0$ large enough so that Φ_i satisfies

$$|1 - \Phi_{i,K}| \leqslant 4e^{-\frac{L}{4K}} \quad \text{on } [\tilde{x}_i - L/4, \tilde{x}_i + L/4], \tag{44}$$

and

$$|\Phi_{i,K}| \leqslant 4e^{-\frac{L}{4K}} \quad \text{on } [\tilde{x}_j - L/4, \tilde{x}_j + L/4] \text{ whenever } j \neq i. \tag{45}$$

We will use the following localized version of E and F defined for $i \in \{1, \ldots, N\}$, by

$$E_i^t(u) = \int_{\mathbb{R}} \Phi_i(t)(u^2 + u_x^2) \quad \text{and} \quad F_i^t(u) = \int_{\mathbb{R}} \Phi_i(t)(u^3 + uu_x^2). \tag{46}$$

Please note that henceforth we take $K = L^{1/2}/8$.

The following lemma gives a localized version of (19). Note that the functionals E_i and F_i do not depend on time in the statement below since we fix $\tilde{x}_1 < \cdots < \tilde{x}_N$.

Lemma 4.3. *Let be given N real numbers $\tilde{x}_1 < \cdots < \tilde{x}_N$ with $\tilde{x}_i - \tilde{x}_{i-1} \geqslant 2L/3$. Define the J_i's as in (30) and assume that, for $i = 1, \ldots, N$, there exists $x_i \in J_i$ such that $|x_i - \tilde{x}_i| \leqslant L/12$ and $u(x_i) = \max_{J_i} u := M_i$. Then, for any $u \in H^1(\mathbb{R})$, it holds*

$$F_i(u) \leqslant M_i E_i(u) - \frac{2}{3}M_i^3 + \|u_0\|_{H^1}^3 O(L^{-1/2}), \quad i \in \{1, \ldots, N\}. \tag{47}$$

Proof. Let $i \in \{1, \ldots, N\}$ be fixed. Following [11], we introduce the function g defined by

$$g(x) = \begin{cases} u(x) - u_x(x) & \text{for } x < x_i, \\ u(x) + u_x(x) & \text{for } x > x_i. \end{cases}$$

Integrating by parts we compute

$$\int ug^2 \Phi_i = \int_{-\infty}^{x_i} (u^3 + uu_x^2 - 2u^2u_x)\Phi_i + \int_{x_i}^{+\infty} (u^3 + uu_x^2 + 2u^2u_x)\Phi_i$$

$$= F_i(u) - \frac{4}{3}u(x_i)^3\Phi_i(x_i) + \frac{2}{3}\int_{-\infty}^{x_i} u^3\Phi_i' - \frac{2}{3}\int_{x_i}^{+\infty} u^3\Phi_i'. \tag{48}$$

Recall that we take $K = \sqrt{L}/8$ and thus $|\Phi'| \leqslant C/K = O(L^{-1/2})$. Moreover, since $|x_i - \tilde{x}_i| \leqslant L/12$, it follows from (44) that $\Phi_i(x_i) = 1 + O(e^{-L^{1/2}})$ and thus

$$\int ug^2 \Phi_i = F_i(u) - \frac{4}{3}M_i^3 + \|u_0\|_{H^1}^3 O(L^{-1/2}). \tag{49}$$

On the other hand,

$$\int ug^2 \Phi_i \leqslant M_i \int g^2 \Phi_i \leqslant M_i \left(E_i(u) - 2\int_{-\infty}^{x_i} uu_x\Phi_i + 2\int_{x_i}^{+\infty} uu_x\Phi_i \right)$$

$$\leqslant M_i E_i(u) - 2M_i^3 + \|u_0\|_{H^1}^3 O(L^{-1/2}). \tag{50}$$

This proves (47). $\quad\square$

Now let us state a global identity related to (18).

Lemma 4.4. *For any $Z \in \mathbb{R}^N$ such that $|z_i - z_{i-1}| \geqslant L/2$ and any $u \in H^1$ it holds*

$$E(u) - \sum_{i=1}^{N} E(\varphi_{c_i}) = \|u - R_Z\|_{H^1}^2 + 4 \sum_{i=1}^{N} c_i \big(u(z_i) - c_i\big) + O\big(e^{-L/4}\big). \tag{51}$$

Proof. Using the relation between φ an its derivative and integrating by parts, we get

$$E(u - R_Z) = E(u) + E(R_Z) - 2 \sum_{i=1}^{N} \int u \varphi_{c_i}(\cdot - z_i) + u_x \partial_x \varphi_{c_i}(\cdot - z_i)$$

$$= E(u) + E(R_Z) - 2 \sum_{i=1}^{N} \int u \varphi_{c_i}(\cdot - z_i) + 2 \sum_{i=1}^{N} \int_{z_i}^{+\infty} u_x \varphi_{c_i}(\cdot - z_i) - 2 \sum_{i=1}^{N} \int_{-\infty}^{z_i} u_x \varphi_{c_i}(\cdot - z_i)$$

$$= E(u) + E(R_Z) - 4 \sum_{i=1}^{N} c_i u(z_i).$$

On the other hand, since $|z_i - z_{i-1}| \geqslant L/2$, it is not too hard to check that

$$E(R_Z) = \sum_{i=1}^{N} E(\varphi_{c_i}) + O\big(e^{-L/4}\big) = 2 \sum_{i=1}^{N} c_i^2 + O\big(e^{-L/4}\big).$$

Combining these two identity, the desired result follows. □

As a consequence of this lemma, we obtain an estimate on the H^1 distance between $u(t)$ and $R_{X(t)}$.

Lemma 4.5. *Under the same hypotheses as in Lemma 4.1, the function $X = (x_1, \ldots, x_N)$ constructed in Lemma 4.1 satisfies on $[0, t_0]$,*

$$\|u(t) - R_{X(t)}\|_{H^1} \leqslant O(\alpha) + O\big(e^{-L/8}\big). \tag{52}$$

Proof. Since $u(t) \in U(\alpha, L/2)$ for $t \in [0, t_0]$, on account of Lemma 4.1 for any $t \in [0, t_0]$ there exists $Z = (z_1, \ldots, z_n)$ with $z_i \in J_i(t)$ such that $E(u(t) - R_Z) = O(\alpha^2)$. Recalling that $u(t, x_i(t)) = \max_{J_i(t)} u(t)$, we deduce (52) from (51). □

4.4. End of the proof of Theorem 1.1

Recall that $\sum_{i=1}^{N} E_i(v) = E(v)$ for any $v \in H^1(\mathbb{R})$. From (9) it is easy to check that

$$E\big(u(t)\big) = E(u_0) = \sum_{j=1}^{N} E(\varphi_{c_j}) + O\big(\varepsilon^2\big) + O\big(e^{-L/4}\big), \quad \forall t \in [0, T]. \tag{53}$$

Let us set $M_i = u(t_0, x_i(t_0))$ and $\delta_i = c_i - M_i$. To conclude the proof, it thus suffices to prove that there exists $C > 0$ which does not depend on A such that

$$\delta_i \leqslant C\big(\varepsilon + L^{-1/4}\big) \quad \text{for all } i. \tag{54}$$

Indeed, in this case (53) and (51), with $Z = X(t_0)$, ensure the existence of $C > 0$ independent of A such that

$$\left\| u - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - x_j) \right\|_{H^1} < C\big(\varepsilon^{1/2} + L^{-1/8}\big),$$

so that one can take $A = 2C$ to conclude the proof (recall that we already know from (29)–(31) that $x_i - x_{i-1} \geqslant 2L/3$ for $i \in \{2, \ldots, N\}$). Let us prove (54). From (47) by taking the sum over i one gets:

$$F\big(u(t_0)\big) = \sum_{i=1}^{N} F_i\big(u(t_0)\big) \leqslant \sum_{i=1}^{N} M_i E_i\big(u(t_0)\big) - \frac{2}{3} \sum_{i=1}^{N} M_i^3 + O\big(L^{-1/2}\big).$$

Setting $\Delta_0^{t_0} F(u) = F(u(t_0)) - F(u(0))$ and $\Delta_0^{t_0} E(u) = E(u(t_0)) - E(u(0))$, this implies

$$0 = \Delta_0^{t_0} F(u) = \sum_{i=1}^{N} \Delta_0^{t_0} F_i(u) \leqslant \sum_{i=1}^{N} M_i \Delta_0^t E_i(u) - 2/3 \sum_{i=1}^{N} M_i^3 + \sum_{i=1}^{N} \big(-F_i(u_0) + M_i E_i(u_0)\big) + O\big(L^{-1/2}\big). \tag{55}$$

By (9), the exponential decay of the φ_{c_i}'s and the Φ_i's, and the definition of E_i and F_i, it is easy to check that

$$\big|E_i(u_0) - E(\varphi_{c_i})\big| + \big|F_i(u_0) - F(\varphi_{c_i})\big| \leqslant O\big(\varepsilon^2\big) + O\big(e^{-\sqrt{L}}\big), \quad \forall i \in \{1, \ldots, N\}.$$

Setting $M_0 = 0$ and using (22), one thus finds after having substituted M_i by $c_i - \delta_i$ that

$$\sum_{i=1}^{N} \big(-F_i(u_0) + M_i E_i(u_0) - 2/3 M_i^3\big) = 2 \sum_{i=1}^{N} \left(-c_i \delta_i^2 + \frac{1}{3} \delta_i^3\right) + O\big(\varepsilon^2\big) + O\big(e^{-\sqrt{L}}\big). \tag{56}$$

Note that by (52) and the continuous embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$, $M_i = c_i + O(\alpha) + O(e^{-L/8})$, and thus

$$0 < M_1 < \cdots < M_N \quad \text{and} \quad \delta_i < c_i/2 \tag{57}$$

for $\alpha_0 = A(\sqrt{\varepsilon_0} + L_0^{-1/8})$ small enough. Using the Abel transformation and the monotonicity estimates (43), we thus get

$$\sum_{i=1}^{N} M_i \Delta_0^t E_i(u) = \sum_{i=1}^{N} (M_i - M_{i-1}) \Delta_0^t I_i \leqslant O\big(\varepsilon^2 + e^{-\sqrt{L}}\big). \tag{58}$$

Injecting (56) and (58) in (55) we obtain

$$\sum_{i=1}^{N} \left(c_i \delta_i^2 - \frac{1}{3} \delta_i(t)^3\right) = \sum_{i=1}^{N} \delta_i^2 \left(c_i - \frac{1}{3} \delta_i\right) \leqslant O\big(\varepsilon^2 + L^{-1/2}\big). \tag{59}$$

(57) and (59) yield (54) and concludes the proof of the theorem.

5. Proof of Corollary 1.1

As written in the introduction, Camassa and Holm discovered that (4) possesses special solutions given by

$$u(t, x) = \sum_{i=1}^{N} p_i(t) e^{|x - q_i(t)|} \tag{60}$$

where the $(p_i, q_i) \in (\mathbb{R}^2)$ satisfy the Hamiltonian system

$$\begin{cases} \dot{q}_i = \displaystyle\sum_{j=1}^{N} p_j e^{-|q_i - q_j|}, \\ \dot{p}_i = \displaystyle\sum_{j=1}^{N} p_i p_j \, \mathrm{sgn}(q_i - q_j) e^{-|q_i - q_j|}. \end{cases} \tag{61}$$

It is easy to check that the local solution of this differential system can be uniquely extended as soon as the q_i's stay distinct from each other. Indeed, if for some time $q_i = q_j$, with $i \neq j$, then uniqueness fails and one has to

distinguish between conservative and dissipative solutions (cf. [3,4]). In [22], Holden and Raynaud proved that the q_i's stay distinct from each other if the p_i's are all positive at time $t = 0$, i.e. there are only peakons (the case with only anti-peakons works also but in the case with peakon and anti-peakon this is no longer true). More precisely, they proved that if at time $t = 0$,

$$p_1, \ldots, p_N > 0 \quad \text{and} \quad q_1 < q_2 < q_N \tag{62}$$

then (62) remains true for all time. In particular, under these hypotheses the different peakons never overlap each others. For example, if a larger peakon follows a smaller one, it will come close to this last one and then transfer part of its energy to it. In this way, the smaller one will become the larger one and the two peakons will be well ordered. In [1] (see also [5]), using the integrability of (4), Beals et al. established a formula for the asymptotics of the q_i's and the p_i's. In particular, they prove the following limits for the p_i and $\dot{q}_i$, $i \in \{1, \ldots, N\}$,

$$\lim_{t \to +\infty} p_i(t) = \lim_{t \to +\infty} \dot{q}_i(t) = \lambda_i \tag{63}$$

and

$$\lim_{t \to -\infty} p_i(t) = \lim_{t \to -\infty} \dot{q}_i(t) = \lambda_{N+1-i}, \tag{64}$$

where $0 < \lambda_1 < \cdots < \lambda_N$ are the eigenvalues of the matrix $(p_j(0)e^{-|q_i(0)-q_j(0)|/2})_{i,j}$.

Remark 5.1. The matrix $A_N := (p_j e^{-|q_i-q_j|/2})_{1 \leqslant i,j \leqslant N}$ is obtained by substituting the multipeakon solution (60) in the isospectral problem

$$\Psi_{xx} = \left(\frac{1}{4} - \frac{m(t, \cdot)}{2\lambda} \right) \Psi, \quad \text{with } m = u - u_{xx}, \tag{65}$$

associated with the Camassa–Holm equation. More precisely, any solution of (65) with $m = 2 \sum_{i=1}^{N} p_i \delta_{q_i}$, that vanishes at $\mp\infty$, is completely determined by its values at the q_j's and satisfies

$$\lambda \Psi(q_i) = \sum_{j=1}^{N} p_j e^{-|q_i-q_j|/2} \Psi(q_j), \quad \forall i \in \{1, \ldots, N\}. \tag{66}$$

In [1], (65) is transformed into a density problem on $[-1, 1]$ by applying a Liouville transformation. The corresponding N-multipeakon matrix is then proved to possess N distinct positive eigenvalues. The arguments of [1] hold also clearly for A_N. Indeed, first since for any fixed λ, (65) has clearly at most one solution (up to multiplication by a scalar) that vanishes at $\mp\infty$, it follows that the eigenvalues of A_N are all of geometric multiplicity one. Next, setting $D = \text{diag}(p_i)$ and $\Lambda_{i,j} = e^{-|q_i-q_j|/2}$, A_N can be rewritten as $D\Lambda$. Since Λ is symmetric with $\Lambda_{ii} = 1$ and $|\Lambda_{ij}| < 1$ for $i \neq j$, Λ is actually positively defined. Therefore there exists B a symmetric positively defined matrix such that $\Lambda = B^2$. It is then easy to check that A_N and BDB have got the same spectrum and since BDB is symmetric positively defined, this ensures that A_N possesses N distinct positive eigenvalues.

Now, let be given $(p_i(0), q_i(0))$ satisfying (62) and $\gamma > 0$. From the asymptotics above there exists $T > 0$ such that

$$q_i(T) - q_{i-1}(T) > L \quad \text{and} \quad q_i(-T) - q_{i-1}(-T) > L \tag{67}$$

with

$$L > \max\left(L_0, \left(\frac{\gamma}{2A} \right)^8 \right). \tag{68}$$

From the last assertion of Theorem 2.1, for any given $B > 0$, there exists $\alpha > 0$ such that if u_0 satisfies (12) then for all $t \in [-T, T]$,

$$\left\| u(t) - \sum_{i=1}^{N} p_i(t) e^{|x-q_i(t)|} \right\|_{H^1} \leqslant \left(\frac{\gamma}{2A} \right)^4. \tag{69}$$

At this stage, it is crucial to remark that since (4) is invariant under the transformation $(t, x) \mapsto (-t, -x)$, Theorem 1.1 remains true when replacing t by $-t$, z_j^0 by $-z_j^0$ and $x_j(t)$ by $-x_j(-t)$. This gives a stability result in the past for trains of peakons that are ordered in the inverse order with respect to Theorem 1.1.

Combining (67), (69), Theorem 1.1 and the remark above, the first part of the corollary follows.

Finally, from (63)–(64), we can also assume that

$$\left| p_i(T) - \lambda_i \right| \leqslant \frac{1}{100N} \left(\frac{\gamma}{2A} \right)^4 \quad \text{and} \quad \left| p_i(-T) - \lambda_{N-i} \right| \leqslant \frac{1}{100N} \left(\frac{\gamma}{2A} \right)^4$$

so that

$$\left\| u(T) - \sum_{i=1}^{N} \lambda_i e^{|x - q_i(T)|} \right\|_{H^1} \leqslant \left(\frac{\gamma}{2A} \right)^4 \quad \text{and} \quad \left\| u(-T) - \sum_{i=1}^{N} \lambda_{N-i} e^{|x - q_i(-T)|} \right\|_{H^1} \leqslant \left(\frac{\gamma}{2A} \right)^4.$$

This completes the proof of the corollary.

Appendix A

Proof of Lemma 4.2. Let us assume that u is smooth since the case $u \in Y([0, T[)$ follows by modifying slightly the arguments (see Remark 3.2 of [19]). From (5), it is not too hard to check that for any smooth space function g, the following differential identity on the weighted energy holds:

$$\frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) g \, dx = \int_{\mathbb{R}} (u^3 + 4uu_x^2) g' \, dx - \int_{\mathbb{R}} u^3 g''' \, dx - \int_{\mathbb{R}} u g' (1 - \partial_x^2)^{-1} (2u^2 + u_x^2) \, dx. \tag{70}$$

Applying (70) with $g = \Psi_{j,K}$ one gets

$$\frac{d}{dt} \int_{\mathbb{R}} \Psi_{j,K} (u^2 + u_x^2) \, dx = -\dot{y}_j \int_{\mathbb{R}} \Psi'_{j,K} (u^2 + u_x^2) + \int_{\mathbb{R}} \Psi'_{j,K} (u^3 + 4uu_x^2) \, dx$$

$$- \int_{\mathbb{R}} \Psi'''_{j,K} u^3 \, dx - \int_{\mathbb{R}} \Psi'_{j,K} u (1 - \partial_x^2)^{-1} (2u^2 + u_x^2) \, dx$$

$$\leqslant -\frac{c_1}{2} \int_{\mathbb{R}} \Psi'_{j,K} (u^2 + u_x^2) + J_1 + J_2 + J_3. \tag{71}$$

We claim that for $i \in \{1, 2, 3\}$, it holds

$$J_i \leqslant \frac{c_1}{8} \int_{\mathbb{R}} \Psi'_{j,K} (u^2 + u_x^2) + \frac{C}{K} e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \tag{72}$$

To handle with J_1 we divide $\mathbb{R}$ into two regions D_j and D_j^c with

$$D_j = \left[\tilde{x}_{j-1}(t) + L/4, \tilde{x}_j(t) - L/4 \right].$$

First since from (29), for $x \in D_j^c$,

$$\left| x - y_j(t) \right| \geqslant \frac{\tilde{x}_j(t) - \tilde{x}_{j-1}(t)}{2} - L/4 \geqslant \frac{c_j - c_{j-1}}{2} t + L/8,$$

we infer from the definition of Ψ in Section 4.2 that

$$\int_{D_j^c} \Psi'_{j,K} (u^3 + 4uu_x^2) \leqslant \frac{C}{K} \|u_0\|_{H^1}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

On the other hand, on D_j we notice, according to (28), that

$$\|u(t)\|_{L^\infty_{D_j}} \leqslant \sum_{i=1}^{N} \|\varphi_{c_i}\big(\cdot - \tilde{x}_i(t)\big)\|_{L^\infty}(D_j) + \left\|u - \sum_{i=1}^{N} \varphi_{c_i}\big(\cdot - \tilde{x}_i(t)\big)\right\|_{L^\infty(D_j)} \leqslant Ce^{-L/8} + O(\sqrt{\alpha}). \qquad (73)$$

Therefore, for α small enough and L large enough it holds

$$J_1 \leqslant \frac{c_1}{8} \int_{\mathbb{R}} \Psi'_{j,K}\big(u^2 + u_x^2\big) + \frac{C}{K} e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

Since J_2 can be handled in exactly the same way, it remains to treat J_3. For this, we first notice as above that

$$-\int_{D_j^c} u\Psi'_{j,K}\big(1 - \partial_x^2\big)^{-1}\big(2u^2 + u_x^2\big) \leqslant 2\|u\|_\infty \sup_{x \in D_j^c} \big|\Psi'_{j,K}\big(x - y_j(t)\big)\big| \int_{\mathbb{R}} e^{-|x|} * \big(u^2 + u_x^2\big)\, dx$$

$$\leqslant \frac{C}{K} \|u_0\|_{H^1}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}, \qquad (74)$$

since

$$\forall f \in L^1(\mathbb{R}), \quad \big(1 - \partial_x^2\big)^{-1} f = \frac{1}{2} e^{-|x|} * f. \qquad (75)$$

Now in the region D_j, noticing that $\Psi'_{j,K}$ and $u^2 + u_x^2/2$ are non-negative, we get

$$-\int_{D_j} u\Psi'_{j,K}\big(1 - \partial_x^2\big)^{-1}\big(2u^2 + u_x^2\big) \leqslant \|u(t)\|_{L^\infty(D_j)} \int_{D_j} \Psi'_{j,K}\big(1 - \partial_x^2\big)^{-1}\big(2u^2 + u_x^2\big)$$

$$\leqslant \|u(t)\|_{L^\infty(D_j)} \int_{\mathbb{R}} \big(2u^2 + u_x^2\big)\big(1 - \partial_x^2\big)^{-1}\Psi'_{j,K}. \qquad (76)$$

On the other hand, from the definition of Ψ in Section 4.2 and (75) we infer that for $K \geqslant 4$,

$$\big(1 - \partial_x^2\big)\Psi'_{j,K} \geqslant \left(1 - \frac{10}{K^2}\right)\Psi'_{j,K} \quad \Rightarrow \quad \big(1 - \partial_x^2\big)^{-1}\Psi'_{j,K} \leqslant \left(1 - \frac{10}{K^2}\right)^{-1}\Psi'_{j,K}.$$

Therefore, taking $K \geqslant 4$ and using (73) we deduce for α small enough and L large enough that

$$-\int_{D_j} u\Psi'_K\big(1 - \partial_x^2\big)^{-1}\big(2u^2 + u_x^2\big) \leqslant \frac{c_1}{8} \int_{\mathbb{R}} \big(u^2 + u_x^2\big)\Psi'_K. \qquad (77)$$

This completes the proof of (72). Gathering (71) and (72) we infer that

$$\frac{d}{dt} \int_{\mathbb{R}} \Psi_{j,K}\big(u^2 + u_x^2\big)\, dx \leqslant -\frac{c_1}{8} \int_{\mathbb{R}} \Psi'_{j,K}\big(u^2 + u_x^2\big) + \frac{C}{K} \|u_0\|_{H^1}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

Integrating this inequality between 0 and t, (43) follows. $\quad\square$

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