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On the multiple existence of semi-positive solutions for a nonlinear Schrödinger system

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Abstract

The paper concerns multiplicity of vector solutions for nonlinear Schrödinger systems, in particular of semi-positive solutions. New variational techniques are developed to study the existence of this type of solutions. Asymptotic behaviors are examined in various parameter regimes including both attractive and repulsive cases.

0. Introduction

In this paper, we consider the following nonlinear Schrödinger systems:

$$\begin{aligned}
&-\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2 \quad \text{in } \Omega, \\
&-\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v \quad \text{in } \Omega, \\
&\quad u, v \in H_0^1(\Omega).
\end{aligned} \tag{$*$}$$

Here Ω is a bounded domain in $\mathbf{R}^n$ ($n \leqslant 3$) and $\lambda_i, \mu_i > 0$ for $i = 1, 2$. In this paper, we show the multiple existence of semi-positive solutions (u_k, v_k) for $(*)$. As there may be semi-trivial solutions (which are zero for some components) we call a solution non-trivial if every component is non-zero. Here we say a non-trivial solution (u, v) is a semi-positive solution for $(*)$ if and only if it satisfies $u > 0$ or $v > 0$ in Ω.

For positive solutions (which means $u > 0$ and $v > 0$ in Ω) of nonlinear Schrödinger systems, there has been extensive work in recent years (cf. [1–7,11,13,15–22,24,27–30] and their references). In particular, we refer to results of [13] which partially inspire our work of the current paper. Dancer, Wei and Weth [13] showed that the a priori bounds of positive solutions and the multiplicity of positive solutions of nonlinear Schrödinger systems are complementary to each other depending on the parameter regimes. They showed the existence of a priori bounds of positive solutions

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for some nonlinear Schrödinger systems which contain (∗). Applying their result to (∗), when $\beta > -\sqrt{\mu_1\mu_2}$, there exists a constant $C = C(\beta, \mu_1, \mu_2, \Omega)$ such that $\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)} \leqslant C$ for any positive solutions (u, v). On the other hand, when $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$ in (∗), they showed the multiple existence of positive solutions of (∗). More precisely, when $\beta \leqslant -1$, (∗) has an unbounded sequence of positive solutions $(u_k)_{k=1}^\infty$ such that

$$\|u_k\|_{L^\infty(\Omega)} + \|v_k\|_{L^\infty(\Omega)} \to \infty \quad \text{as } k \to \infty.$$

These positive solutions were given by minimax method from making use of a symmetry $\sigma(u, v) = (v, u)$. That is, the variational functional $I_\beta(u, v)$ associated with (∗) satisfies $I_\beta(\sigma(u, v)) = I_\beta(u, v)$ for $\sigma(u, v) = (v, u)$. This multiplicity result was recovered and generalized to the non-symmetric case of $\mu_1 \neq \mu_2$ by using a bifurcation method in [5] in which an unbounded sequence of positive solutions was established for $\beta \leqslant -\sqrt{\mu_1\mu_2}$ when the domain is radial.

For nonlinear Schrödinger systems (∗) with $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$, these results suggest that $\beta = -\sqrt{\mu_1\mu_2}$ is the threshold that divides the existence of a priori bounds of positive solutions and the existence of an unbounded sequence of positive solutions. In this paper, we consider the existence and multiplicity of semi-positive solution of (∗). A natural question is to examine the coupling constant β and to find the coupling value that separates the a priori bounds and infinitely many semi-positive solutions. Our results suggest that $\beta = 0$ is the threshold dividing the existence of a priori bounds of semi-positive solutions and the existence of an unbounded sequence of semi-positive solutions. This is the main motivation of the current work. We also study the asymptotic properties of semi-positive solutions when $\beta \to 0$ and $\beta \to \infty$, and establish multiplicity results of semi-positive solutions in these regimes.

When $\beta < 0$, we get infinite many semi-positive solutions of (∗) by the following theorem.

Theorem 0.1. *Let $\beta < 0$. Then (∗) has a sequence of solutions (u_k, v_k) such that*

$$u_k > 0, \quad \|u_k\|_{L^\infty(\Omega)} + \|v_k\|_{L^\infty(\Omega)} \to \infty \text{ as } k \to \infty.$$

Moreover, if $\beta \in (-\sqrt{\mu_1\mu_2}, 0)$, then v_k must change sign for large k.

When $\beta > 0$ is small, we get multiplicity of semi-positive solutions of (∗) as follows.

Theorem 0.2. *For given $k \in \mathbf{N}$, there exists $\beta_k > 0$ such that, for any $\beta \in (0, \beta_k)$, we have k semi-positive solutions (u_i, v_i) of (∗) with $u_i > 0$ in Ω $(i = 1, 2, \ldots, k)$.*

Roughly speaking, our semi-positive solutions are given by making use of a symmetry $\sigma(u, v) = (u, -v)$. That is, it is essential that the variational functional $I_\beta(u, v)$ satisfies $I_\beta(u, v) = I_\beta(u, -v)$. More generally, we develop an abstract framework in Section 2. We consider the following situation. Let H be a Hilbert space and suppose that $\sigma : H \to H$ satisfies

$$\sigma^2 = id_H, \tag{0.1}$$

$$\sigma \neq id_H. \tag{0.2}$$

Then, for C^1-manifold $M \subset H$ which does not contain fix points of σ and C^1-functional $J : M \to \mathbf{R}$ satisfying $J(\sigma(u)) = J(u)$ and some conditions, we can prove the multiple existence of the critical values of J. For details, see Section 2. We point out that generalizations and variants of the genus theory have been established recently in [9,10,26]. Refs. [9,10] were for existence of multiple vector solutions of some elliptic systems. Ref. [26] was on existence of multiple sign-changing vector solutions with each component sign-changing for systems like (∗) in the defocussing case (i.e., $\mu_j \leqslant 0$). In the general perspective we use partial symmetry for variants of the genus theory in this paper.

Next, we consider the asymptotic behavior of semi-positive solutions as $\beta \to 0$. To state our result about the asymptotic behavior, we need the following notations: for $J_2(v) = (4\mu_2\|v\|_{L^4(\Omega)}^4)^{-1}$: $\Sigma_2 = \{v \in H_0^1(\Omega) \mid \int_\Omega |\nabla v|^2 + \lambda_2|u|^2\,dx = 1\} \to \mathbf{R}$, we define symmetric mountain pass values b_n^2 $(n \in \mathbf{N} \cup \{0\})$ by

$$b_n^2 = \inf_{\gamma_2 \in \Gamma_n^2} \max_{\theta \in S^n} J_2(\gamma_2(\theta)),$$

$$\Gamma_n^2 = \{\gamma_2(\theta) \in C(S^n, \Sigma_2) \mid \gamma_2(-\theta) = -\gamma_2(\theta) \text{ for all } \theta \in S^n\},$$

where $S^n = \{\theta = (\theta_1, \ldots, \theta_{n+1}) \in \mathbf{R}^{n+1} \mid |\theta| = 1\}$. Now, we show the following theorem.

Theorem 0.3. *For given $k \in \mathbf{N}$, there exists $\beta'_k > 0$ such that, for any $\beta \in (-\beta'_k, \beta'_k)$, we have k solutions $(u_{i,\beta}, v_{i,\beta})$ of $(*)$ with $u_{i,\beta} > 0$ in Ω $(i = 1, 2, \ldots, k)$ and $(u_{i,\beta}, v_{i,\beta})$ satisfy the following: extracting a subsequence $\beta_j \to 0$, we have*

$$(u_{i,\beta_j}, v_{i,\beta_j}) \to (u_{i,0}, v_{i,0}) \quad \text{in } H^1_0(\Omega) \times H^1_0(\Omega).$$

Here $u_{i,0}$ is a positive least energy solution of

$$\begin{aligned} &-\Delta u + \lambda_1 u = \mu_1 u^3 \quad \text{in } \Omega, \\ &u \in H^1_0(\Omega). \end{aligned} \tag{0.3}$$

$v_{i,0}$ is a solution of

$$\begin{aligned} &-\Delta v + \lambda_2 v = \mu_2 v^3 \quad \text{in } \Omega, \\ &v \in H^1_0(\Omega). \end{aligned} \tag{0.4}$$

In particular, $v_{i,0}$ corresponds to the critical value b^2_i which is given by a symmetric mountain pass theorem.

Remark 0.4. The functional $J_2(v) : \Sigma_2 \to \mathbf{R}$ corresponds to (0.4). In fact, for a critical point v_0 of J_2, $(\sqrt{\mu_2}\|v_0\|^2_{L^4(\Omega)})^{-1} v_0$ is a non-trivial solution of (0.4).

Remark 0.5. The semi-positive solutions $(u_{i,\beta}, v_{i,\beta})$ in Theorem 0.3 may be different from the semi-positive solutions (u_i, v_i) in Theorem 0.1 or Theorem 0.2.

Next, we consider the semi-positive solutions for the case β is large. In [18], Liu and Wang showed that, for given $k \in \mathbf{N}$, there exists $\overline{\beta'}_k > 0$ such that, for any $\beta > \overline{\beta'}_k$, $(*)$ has at least k solutions. In this paper, we get multiplicity of semi-positive solutions of $(*)$ as follows.

Theorem 0.6. *For given $k \in \mathbf{N}$, there exists $\overline{\beta}_k > 0$ such that, for any $\beta > \overline{\beta}_k$, $(*)$ has at least k semi-positive solutions $(u_{i,\beta}, v_{i,\beta})$ with $u_{i,\beta} > 0$ in Ω $(i = 1, 2, \ldots, k)$.*

We study the asymptotic behavior as $\beta \to \infty$. For the solution $(u_{i,\beta}, v_{i,\beta})$ of Theorem 0.6, $(\sqrt{\beta} u_{i,\beta}, \sqrt{\beta} v_{i,\beta})$ is bounded in $H^1_0(\Omega) \times H^1_0(\Omega)$ as $\beta \to \infty$. (See Section 7.) Thus, extracting a subsequence $\beta_j \to \infty$, we expect that $(\sqrt{\beta_j} u_{i,\beta_j}, \sqrt{\beta_j} v_{i,\beta_j})$ approaches to a solution of

$$\begin{aligned} &-\Delta u + \lambda_1 u = u v^2 \quad \text{in } \Omega, \\ &-\Delta v + \lambda_2 v = u^2 v \quad \text{in } \Omega, \\ &u, v \in H^1_0(\Omega). \end{aligned} \tag{0.5}$$

Here, we remark that (0.5) does not have semi-trivial solutions. In fact, letting $(0, v)$ be a solution of (0.5), we also have $v = 0$ from the second equation of (0.5). For the limiting equation (0.5), we have the following:

Theorem 0.7. *Eq. (0.5) has infinitely many semi-positive solutions (u_k, v_k) such that $u_k > 0$ in Ω and*

$$\|u_k\|_{L^\infty(\Omega)} + \|v_k\|_{L^\infty(\Omega)} \to \infty \quad \text{as } k \to \infty. \tag{0.6}$$

Moreover, when $\lambda_1 = \lambda_2$, v_k must change sign for large $k \in \mathbf{N}$.

Remark 0.8. The solutions (u_k, v_k) of Theorem 0.7 are characterized by values $e_{k,\infty}$ which are defined as follows. Let $N = \{(u, v) \in H^1_0(\Omega) \times H^1_0(\Omega) \mid \int_{\mathbf{R}^N} |\nabla u|^2 + |\nabla v|^2 + \lambda_1 |u|^2 + \lambda_2 |v|^2\, dx = 1,\ u_+ v \not\equiv 0\}$, $\tilde{J}_\infty(u, v) = (8\|u_+ v\|^2_{L^2(\Omega)})^{-1}$. We define $e_{k,\infty}$ $(k \in \mathbf{N} \cup \{0\})$ by

$$e_{k,\infty} = \inf\bigl\{c \in \mathbf{R} \bigm| \gamma\bigl([\tilde{J}_\infty \leqslant c]_N\bigr) \geqslant k\bigr\}.$$

Here γ is a genus corresponding to $\sigma(u, v) = (u, -v)$ which is defined in Section 2.

Remark 0.9. When $\lambda_1 = \lambda_2 = \lambda > 0$, all positive solutions (u, v) of (0.5) must satisfy $u = v$. In fact, $u - v$ satisfies

$$-\Delta(u - v) + \lambda(u - v) = uv(v - u).$$

Multiplying $u - v$ and integrating over Ω the above equation, we have

$$\int_\Omega \left|\nabla(u - v)\right|^2 + \lambda(u - v)^2\,dx = -\int_\Omega uv(u - v)^2\,dx.$$

Thus we have $u = v$. We also remark that there exist a priori bounds of $-\Delta u + \lambda u = u^3$ in Ω and $u = 0$ on $\partial\Omega$. Therefore, when $\lambda_1 = \lambda_2 = \lambda > 0$, (0.6) implies that v_k is a sign-changing solution for large $k \in \mathbf{N}$. When $\lambda_1 \neq \lambda_2$ we do not know whether v_k changes sign.

Now, we get the following theorem about the asymptotic behavior as $\beta \to \infty$.

Theorem 0.10. *For given $k \in \mathbf{N}$, let $(u_{k,\beta}, v_{k,\beta})$ be a family of solutions of $(*)$ which are given in Theorem 0.6. Then there exist a subsequence $\beta_j \to \infty$ and $(u_{k,\infty}, v_{k,\infty}) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that*

$$\left(\sqrt{\beta_j}\,u_{k,\beta_j}, \sqrt{\beta_j}\,v_{k,\beta_j}\right) \to (u_{k,\infty}, v_{k,\infty}) \quad \text{in } H_0^1(\Omega) \times H_0^1(\Omega).$$

Here $(u_{k,\infty}, v_{k,\infty})$ is a solution of (0.5) and corresponds to critical value $e_{k,\infty}$.

We devote the next four sections to the proofs of our theorems. For the case $\beta \leqslant 0$ or the case $\beta > 0$ small, we reduce the functional $I_\beta(u, v)$ to a functional $J_\beta(u, v)$ defined on a subset of a torus $\Sigma_1 \times \Sigma_2$ in Section 1. On the other hand, for the case $\beta > 0$ is large, we reduce the functional $\tilde{I}_\beta(u, v)$ to a functional $\tilde{J}_\beta(u, v)$ defined on a subset of the sphere Σ in Section 6. In Section 2, we give an abstract theory for the multiple existence of the critical values of C^1-functional $J : M \to \mathbf{R}$ satisfying $J(\sigma(u)) = J(u)$. We will get most of our multiple existence of semi-positive solutions by using these abstract results. In Section 3, we will show Theorem 0.1 and Theorem 0.2. In Sections 4–5, we will prove Theorem 0.3. To show this, we apply the method from [25]. In Sections 6–7, we will show Theorems 0.6, 0.7 and 0.10.

1. The functional setting for the case $\beta \leqslant 0$ or the case $\beta > 0$ small

To prove the existence of semi-positive solutions (u, v) with $u > 0$, we seek critical points of the following functional

$$I_\beta(u, v) = \frac{1}{2}\left(|||u|||_{\lambda_1}^2 + |||v|||_{\lambda_2}^2\right) - \frac{1}{4}\left(\mu_1\|u_+\|_4^4 + \mu_2\|v\|_4^4\right) - \frac{\beta}{2}\|u_+ v\|_2^2 : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbf{R}.$$

Here we use notations $u_+ = \max\{u, 0\}$, $u_- = \min\{u, 0\}$ and

$$|||u|||_\lambda^2 = \int_\Omega |\nabla u|^2 + \lambda u^2\,dx, \qquad \|u\|_p^p = \int_\Omega |u|^p\,dx.$$

For a critical point (u, v) of $I_\beta(u, v)$, the positivity of u comes from the following proposition.

Proposition 1.1. *Let (u, v) be a critical point of $I_\beta(u, v)$ with $u \neq 0$. Then we have $u > 0$ in Ω.*

Proof. Let (u, v) be a critical point of $I_\beta(u, v)$. Then $\nabla I_\beta(u, v)(u_-, 0) = |||u_-|||_{\lambda_1}^2 = 0$. Thus we have $u_+ \equiv u \geqslant 0$. Now, for $\beta \leqslant 0$, u satisfies

$$-\Delta u + \left(\lambda_1 - \beta v^2\right)u = \mu_1 u^3 \geqslant 0.$$

For $\beta > 0$, u satisfies

$$-\Delta u + \lambda_1 u = \left(\mu_1 u^2 + \beta v^2\right)u \geqslant 0.$$

Since the maximum principle works for u in both cases, we have $u > 0$ in Ω. $\square$

We set $\Sigma_i = \{u \in H_0^1(\Omega) \mid |||u|||_{\lambda_i} = 1\}$ for $i = 1, 2$. We remark that there exists $C_1 > 0$ such that

$$\|u\|_4,\ \|v\|_4 < C_1 \quad \text{for all } (u, v) \in \Sigma_1 \times \Sigma_2. \tag{1.1}$$

To seek non-trivial critical points of $I_\beta(u, v)$, sometimes one may reduce $I_\beta(u, v)$ to a functional defined on a Nehari manifold with co-dimension 2. In this paper, we reduce $I_\beta(u, v)$ to a functional defined on an open subset of torus $\Sigma_1 \times \Sigma_2$. Since we also consider a perturbation problem for β (Theorem 0.3), it is easy to treat a domain which does not depend on β. This is the main reason to reduce the functional to one on the torus but not on a Nehari manifold.

1.1. The reduction to a functional on a torus

When $\beta \in \mathbf{R}$, we set

$$N_\beta = \left\{ (u, v) \in \Sigma_1 \times \Sigma_2 \;\middle|\; \begin{array}{l} g_1(u, v) := \mu_1\mu_2 \|u_+\|_4^4 \|v\|_4^4 - \beta^2 \|u_+ v\|_2^4 > 0, \\ g_2(u, v) := \mu_1 \|u_+\|_4^4 - \beta \|u_+ v\|_2^2 > 0, \\ g_3(u, v) := \mu_2 \|v\|_4^4 - \beta \|u_+ v\|_2^2 > 0 \end{array} \right\}.$$

From the Hölder inequality, we see that

$$N_\beta = \begin{cases} \{(u, v) \in \Sigma_1 \times \Sigma_2 \mid g_1(u, v) > 0\}, & \beta \in (-\infty, -\sqrt{\mu_1\mu_2}\,], \\ \{(u, v) \in \Sigma_1 \times \Sigma_2 \mid u_+ \not\equiv 0\}, & \beta \in (-\sqrt{\mu_1\mu_2}, 0], \\ \{(u, v) \in \Sigma_1 \times \Sigma_2 \mid g_2(u, v) > 0,\ g_3(u, v) > 0\}, & \beta \in (0, \infty). \end{cases}$$

We remark that, for all $\beta \in \mathbf{R}$, $(u, v) \in N_\beta$ implies $g_1(u, v) > 0$ and $u_+ \not\equiv 0$. We can define a functional $J_\beta(u, v)$ on N_β by the following proposition.

Proposition 1.2. *For any $(u, v) \in N_\beta$, a function*

$$(s, t) \mapsto I_\beta(su, tv) : \mathbf{R}_+^2 \to \mathbf{R}$$

has a unique maximum point $(s_\beta(u, v), t_\beta(u, v))$. Moreover, setting

$$J_\beta(u, v) = \sup_{s,t>0} I_\beta(su, tv),$$

we have

$$J_\beta(u, v) = \frac{1}{4}\big(s_\beta(u, v)^2 + t_\beta(u, v)^2\big) \tag{1.2}$$

$$= \frac{1}{4}\big(\mu_1 s_\beta(u, v)^4 \|u_+\|_4^4 + \mu_2 t_\beta(u, v)^4 \|v\|_4^4 + 2\beta s_\beta(u, v)^2 t_\beta(u, v)^2 \|u_+ v\|_2^2\big) \tag{1.3}$$

$$= \frac{1}{4} \cdot \frac{\mu_1 \|u_+\|_4^4 + \mu_2 \|v\|_4^4 - 2\beta \|u_+ v\|_2^2}{\mu_1\mu_2 \|u_+\|_4^4 \|v\|_4^4 - \beta^2 \|u_+ v\|_2^4} \tag{1.4}$$

and

(i) *$s_\beta(u, v)$, $t_\beta(u, v) : N \to \mathbf{R}_+$ are C^1-functions.*
(ii) *$J_\beta(u, v) : N_\beta \to \mathbf{R}$ is a C^1-function.*
(iii) *If $(u, v) \in N_\beta$ is a critical point of $J_\beta(u, v)$, then $(s_\beta(u, v)u, t_\beta(u, v)v)$ is a non-trivial critical point of $I_\beta(u, v)$.*
(iv) *$J_\beta(u, v)$ satisfies (PS)-condition.*

Proof. For any $(u, v) \in N_\beta$, we set

$$f(s, t) = I_\beta(su, tv) : \mathbf{R}_+^2 \to \mathbf{R}.$$

Differentiating $f(s, t)$, we have

$$\frac{\partial f}{\partial s}(s,t) = s - s^3\mu_1\|u_+\|_4^4 - st^2\beta\|u_+v\|_2^2,$$
$$\frac{\partial f}{\partial t}(s,t) = t - t^3\mu_2\|v\|_4^4 - s^2t\beta\|u_+v\|_2^2.$$

Thus critical points (s,t) of $f(s,t)$ satisfy

$$\begin{bmatrix} \mu_1\|u_+\|_4^4, & \beta\|u_+v\|_2^2 \\ \beta\|u_+v\|_2^2, & \mu_2\|v\|_4^4 \end{bmatrix}\begin{bmatrix} s^2 \\ t^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Here, noting $\mu_1\mu_2\|u_+\|_4^4\|v\|_4^4 - \beta^2\|u_+v\|_2^4 > 0$, we have

$$\begin{aligned} \begin{bmatrix} s^2 \\ t^2 \end{bmatrix} &= \frac{1}{\mu_1\mu_2\|u_+\|_4^4\|v\|_4^4 - \beta^2\|u_+v\|_2^4}\begin{bmatrix} \mu_2\|v\|_4^4, & -\beta\|u_+v\|_2^2 \\ -\beta\|uv\|_2^2, & \mu_1\|u_+\|_4^4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\mu_1\mu_2\|u_+\|_4^4\|v\|_4^4 - \beta^2\|u_+v\|_2^4}\begin{bmatrix} \mu_2\|v\|_4^4 - \beta\|u_+v\|_2^2 \\ \mu_1\|u_+\|_4^4 - \beta\|u_+v\|_2^2 \end{bmatrix}. \end{aligned} \tag{1.5}$$

Since $(u,v) \in N_\beta$, $f(s,t)$ has a unique critical point $(s_0,t_0) = (s_\beta(u,v), t_\beta(u,v))$. Next, to show (s_0,t_0) is a maximum point, we calculate the second derivatives of $f(s,t)$.

$$\frac{\partial^2 f}{\partial s^2}(s,t) = 1 - 3s^2\mu_1\|u_+\|_4^4 - t^2\beta\|u_+v\|_2^2 = \frac{1}{s}\frac{\partial f}{\partial s}(s,t) - 2s^2\mu_1\|u_+\|_4^4,$$
$$\frac{\partial^2 f}{\partial t\partial s}(s,t) = -2st\beta\|u_+v\|_2^2,$$
$$\frac{\partial^2 f}{\partial t^2}(s,t) = 1 - 3t^2\mu_2\|v\|_4^4 - s^2\beta\|u_+v\|_2^2 = \frac{1}{t}\frac{\partial f}{\partial t}(s,t) - 2t^2\mu_2\|v\|_4^4.$$

Therefore, we have

$$A = \frac{\partial^2 f}{\partial s^2}(s_0,t_0) = -2s_0^2\mu_1\|u_+\|_4^4,$$
$$B = \frac{\partial^2 f}{\partial t\partial s}(s_0,t_0) = -2\beta s_0t_0\|u_+v\|_2^2.$$
$$C = \frac{\partial^2 f}{\partial t^2}(s_0,t_0) = -2t_0^2\mu_2\|v\|_4^4.$$

Since $A < 0$ and $AC - B^2 = 4s_0^2t_0^2(\mu_1\mu_2\|u_+\|_4^4\|v\|_4^4 - \beta^2\|u_+v\|_2^4) > 0$, (s_0,t_0) is a maximum point of $f(s,t)$. Thus, by direct calculations, we get (1.2)–(1.4).

Next we show (i). To show (i), we use the implicit function theorem. We consider the following function:

$$\mathbf{F}(s,t,u,v) = \begin{bmatrix} F(s,t,u,v) \\ G(s,t,u,v) \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial s}(s,t) \\ \frac{\partial f}{\partial t}(s,t) \end{bmatrix} : \mathbf{R}_+^2 \times N_\beta \to \mathbf{R}^2.$$

Now, for any $(u,v) \in N_\beta$, we have

$$\mathbf{F}(s_0,t_0,u,v) = \mathbf{0},$$
$$\begin{bmatrix} \frac{\partial F}{\partial s}(s_0,t_0,u,v) & \frac{\partial F}{\partial t}(s_0,t_0,u,v) \\ \frac{\partial G}{\partial s}(s_0,t_0,u,v) & \frac{\partial F}{\partial t}(s_0,t_0,u,v) \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}.$$

Thus from the implicit function theorem, we can easily see the C^1-property of $(s_0,t_0) = (s_\beta(u,v), t_\beta(u,v))$.

We show (ii). Noting

$$J_\beta(u,v) = I_\beta\big(s_\beta(u,v)u, t_\beta(u,v)v\big),$$

we can easily find that $J_\beta(u,v)$ is a C^1-function. Moreover we have

$$\nabla_u J_\beta(u,v)\varphi = \nabla_u I_\beta\big(s_\beta(u,v)u, t_\beta(u,v)v\big)\big(\nabla_u s_\beta(u,v)\varphi u + s_\beta(u,v)\varphi\big) + \nabla_v I_\beta\big(s_\beta(u,v)u, t_\beta(u,v)v\big)\nabla_u t_\beta(u,v)\varphi v = \nabla_u I_\beta\big(s_\beta(u,v)u, t_\beta(u,v)v\big)s_\beta(u,v)\varphi, \tag{1.6}$$

$$\nabla_v J_\beta(u,v)\psi = \nabla_v I_\beta\big(s_\beta(u,v)u, t_\beta(u,v)v\big)t_\beta(u,v)\psi. \tag{1.7}$$

Thus, if $(u,v) \in N_\beta$ is a critical point of $J_\beta(u,v)$, then $(s_\beta(u,v)u, t_\beta(u,v)v)$ is a non-trivial critical point of $I_\beta(u,v)$ and we get (iii).

Finally, we show (iv). If $(u_n, v_n) \in N_\beta$ is a (PS)-sequence for J_β, then $J_\beta(u_n, v_n)$ are bounded and this means the boundedness of $(s_\beta(u_n,v_n), t_\beta(u_n,v_n))$ from (1.2). Thus from (1.6)–(1.7), $(s_\beta(u_n,v_n)u_n, t_\beta(u_n,v_n)v_n)$ is also a (PS)-sequence for I_β. Since $I_\beta(u,v)$ satisfies (PS)-condition, $J_\beta(u,v)$ also satisfies (PS)-condition. □

From (1.2), for all $\beta \in \mathbf{R}$, it is obvious that $J_\beta(u,v)$ is bounded from below. Moreover, we have the following proposition.

Proposition 1.3. *When $\beta < 0$, we have*

$$\liminf_{(u,v)\in N_\beta,\ \mathrm{dist}\{(u,v),\partial N_\beta\}\to 0} J_\beta(u,v) = \infty. \tag{1.8}$$

Proof. For any sequence $((u_n,v_n))_{n=1}^\infty \subset N_\beta$ with $g_1(u_n,v_n) \to 0$ $(n\to\infty)$, we need to show $J_\beta(u_n,v_n) \to \infty$ $(n\to\infty)$. Since $|||u_n|||_{\lambda_1} = |||v_n|||_{\lambda_2} = 1$, for some $u_0, v_0 \in H_0^1(\Omega)$, we may assume

$$u_n \to u_0, \qquad v_n \to v_0 \quad \text{strongly in } L^4(\Omega).$$

Here if $g_2(u_0,v_0) + g_3(u_0,v_0) > 0$, then it is obvious that (1.8) holds. Thus we assume $g_2(u_0,v_0) + g_3(u_0,v_0) = 0$. Since $\beta < 0$, we have $u_0 = v_0 = 0$ and we find $\|u_n\|_4^4 \to 0$, $\|v_n\|_4^4 \to 0$ as $n \to \infty$. Since $J_\beta(u,v)$ is written by (1.4), we get (1.8). □

Remark 1.4. From Proposition 1.3, when $\beta < 0$, the behavior of $J_\beta(u,v)$ in the neighborhood of ∂N_β does not disturb deformation arguments. When $\beta > 0$, it is complicated by the behavior of $J_\beta(u,v)$ in the neighborhood of ∂N_β and we cannot expect the property like (1.8). But for $\beta > 0$ small, $J_\beta(u,v)$ satisfies the property like (1.8) on a proper subset $M_\delta \subset N_\beta$. (See Proposition 1.9.)

1.2. The case $\beta > 0$ small

For $\delta > 0$, we set

$$M_\delta = \big\{(u,v) \in \Sigma_1 \times \Sigma_2 \bigm| \mu_1\|u_+\|_4^4 > \delta,\ \mu_2\|v\|_4^4 > \delta\big\}.$$

We remark that $M_\delta \neq \emptyset$ if $\delta < \frac{1}{4b_0}$ where b_0 is given by

$$b_0 = \min\big\{b_0^1, b_0^2\big\} > 0, \qquad b_0^1 = \inf_{u\in\Sigma_1} \frac{1}{4\mu_1\|u\|_4^4} > 0, \qquad b_0^2 = \inf_{v\in\Sigma_2} \frac{1}{4\mu_2\|v\|_4^4} > 0. \tag{1.9}$$

Here b_0^i $(i = 1, 2)$ is a least energy level of (1.15) and (1.17) respectively. (See Remark 1.8.) We also remark that M_δ is independent of β.

Lemma 1.5. *For any given $\delta \in (0, \frac{1}{4b_0})$, there exists $\beta_\delta \in (0, \sqrt{\mu_1\mu_2})$ such that*

$$M_\delta \subset N_\beta \quad \textit{for all } \beta \in (-\sqrt{\mu_1\mu_2}, \beta_\delta).$$

Proof. When $\beta \in (-\sqrt{\mu_1\mu_2}, 0)$, $M_\delta \subset N_\beta$ is obvious. For $\delta \in (0, \frac{1}{4b_0})$, we choose $\beta_\delta > 0$ satisfying $\delta > \beta_\delta C_1^4$. Here C_1 is a constant given in (1.1). Then it holds

$$\mu_1\|u_+\|_4^4 > \delta > \beta_\delta C_1^4 \geqslant \beta\|u_+ v\|_2^2 \quad \text{for all } (u,v) \in M_\delta,\ \beta \in [0, \beta_\delta).$$

By a similar way, we have $\mu_2\|v\|_4^4 > \beta\|u_+ v\|_2^2$. Thus we get $M_\delta \subset N_\beta$ for all $\beta \in (-\sqrt{\mu_1\mu_2}, \beta_\delta)$. □

From Lemma 1.5, $J_\beta(u,v)$ is defined on M_δ.

Lemma 1.6. *For any given $\delta \in (0, \frac{1}{4b_0})$, there exists a constant $C_\delta > 0$ which does not depend on β such that*

$$s_\beta(u,v) \leqslant C_\delta, \qquad t_\beta(u,v) \leqslant C_\delta \quad \text{for all } (u,v) \in M_\delta,\ \beta \in (-\beta_\delta, \beta_\delta). \tag{1.10}$$

Here β_δ was given in Lemma 1.5. Moreover it holds

$$\big(s_\beta(u,v), t_\beta(u,v)\big) \to \left(\frac{1}{\sqrt{\mu_1}\|u_+\|_4^2}, \frac{1}{\sqrt{\mu_2}\|v\|_4^2}\right) \quad \text{uniformly for } (u,v) \in M_\delta \text{ as } \beta \to 0. \tag{1.11}$$

Proof. Suppose $(u,v) \in M_\delta$, $\beta \in (-\beta_\delta, \beta_\delta)$. Since $s_\beta(u,v)$ was written by (1.5), we have

$$s_\beta(u,v)^2 = \frac{\mu_2\|v\|_4^4 - \beta\|u_+v\|_2^2}{\mu_1\mu_2\|u_+\|_4^4\|v\|_4^4 - \beta^2\|u_+v\|_2^2} \leqslant \frac{(\mu_2+\beta_\delta)C_1^4}{(\mu_1\mu_2 - \beta_\delta^2)\frac{\delta^2}{\mu_1\mu_2}}.$$

Here C_1 is a constant given in (1.1) and we have used the fact that $\mu_1\|u_+\|_4^4, \mu_2\|v\|_4^4 \geqslant \delta$ for all $(u,v) \in M_\delta$. And we also have

$$s_\beta(u,v)^2 \to \frac{1}{\mu_1\|u_+\|_4^4} \quad \text{uniformly for } (u,v) \in M_\delta \text{ as } \beta \to 0.$$

Since $t_\beta(u,v)$ also was similarly written by (1.5), we obtain (1.10) and (1.11). □

Proposition 1.7. *For any given $\delta \in (0, \frac{1}{4b_0})$, there exists a constant $c_\delta(\beta)$ with $c_\delta(\beta) \to 0$ (as $\beta \to 0$) such that $J_\beta(u,v)$ satisfies*

$$\big|J_\beta(u,v) - J_1(u) - J_2(v)\big| \leqslant c_\delta(\beta) \quad \text{for all } (u,v) \in M_\delta,\ \beta \in (-\beta_\delta, \beta_\delta), \tag{1.12}$$

$$\big\|\!\big|\nabla_u J_\beta(u,v) - \nabla J_1(u)\big|\!\big\|_{\lambda_1 *} \leqslant c_\delta(\beta) \quad \text{for all } (u,v) \in M_\delta,\ \beta \in (-\beta_\delta, \beta_\delta), \tag{1.13}$$

$$\big\|\!\big|\nabla_v J_\beta(u,v) - \nabla J_2(v)\big|\!\big\|_{\lambda_2 *} \leqslant c_\delta(\beta) \quad \text{for all } (u,v) \in M_\delta,\ \beta \in (-\beta_\delta, \beta_\delta), \tag{1.14}$$

where, for $i = 1,2$, $J_i(u) = \frac{1}{4\mu_i\|u\|_4^4}$, $T_u\Sigma_i = \{v \in H_0^1(\Omega) \mid \langle u,v\rangle_{\lambda_i} = 0\}$ and

$$\big\|\!\big|\nabla J_i(u)\big|\!\big\|_{\lambda_i *} = \sup_{v \in T_u\Sigma_i,\ \|\!|v|\!\|_{\lambda_i} = 1} \big|\nabla J_i(u)v\big|.$$

Remark 1.8. For any $u \in \Sigma_1$ with $u_+ \neq 0$, a function $s \mapsto I_1(su) = \frac{s^2}{2} - \frac{s^4}{4}\mu_1\|u_+\|_4^4$ has a maximum value at a unique maximum point $s = \frac{1}{\sqrt{\mu_1}\|u_+\|_4^2}$ and we can write as follows

$$J_1(u) = \sup_{s>0} I_1(su) = \frac{1}{4\mu_1\|u_+\|_4^4}, \tag{1.15}$$

$$\nabla J_1(u)\varphi = -\frac{1}{\mu_1\|u_+\|_4^8}\int_\Omega u_+^3\varphi\,dx \quad \text{for all } \varphi \in T_u\Sigma_1. \tag{1.16}$$

By a similar way, for any $u \in \Sigma_2$, a function $t \mapsto I_2(tu) = \frac{t^2}{2} - \frac{t^4}{4}\mu_2\|v\|_4^4$ has a unique maximum point and we have

$$J_2(u) = \sup_{t>0} I_2(tv) = \frac{1}{4\mu_2\|v\|_4^4}, \tag{1.17}$$

$$\nabla J_2(v)\psi = -\frac{1}{\mu_2\|v\|_4^8}\int_\Omega v^3\psi\,dx \quad \text{for all } \psi \in T_v\Sigma_2. \tag{1.18}$$

Proof of Proposition 1.7. From (1.4), (1.15) and (1.17), we can directly calculate $J_\beta(u,v)-J_1(u)-J_2(v)$ as follows:

$$J_\beta(u,v)-J_1(u)-J_2(v)=\frac{1}{4}\cdot\frac{\beta\|u_+v\|_2^2}{\mu_1\mu_2\|u_+\|_4^4\|v\|_4^4-\beta^2\|u_+v\|_2^4}\left(\frac{\beta\|u_+v\|_2^2}{\mu_1\|u_+\|_4^4}+\frac{\beta\|u_+v\|_2^2}{\mu_2\|v\|_4^4}-2\right).$$

For $(u,v)\in M_\delta$, $\beta\in(-\beta_\delta,\beta_\delta)$, we have

$$\left|J_\beta(u,v)-J_1(u)-J_2(v)\right|\leqslant\frac{C_1^4|\beta|}{4(\mu_1\mu_2-\beta^2)\frac{\delta^2}{\mu_1\mu_2}}\left(\frac{C_1^4|\beta|}{\delta}+\frac{C_1^4|\beta|}{\delta}+2\right). \tag{1.19}$$

Here C_1 is a constant given in (1.1) and we have used the fact that $\mu_1\|u_+\|_4^4, \mu_2\|v\|_4^4\geqslant\delta$ for all $(u,v)\in M_\delta$. From (1.19), we get (1.12). Next we calculate $\nabla_u J_\beta(u,v)\varphi-\nabla J_1(u)\varphi$ for any $\varphi\in T_u\Sigma_1$. From (1.6),

$$\nabla_u J_\beta(u,v)\varphi=-s_\beta(u,v)^4\mu_1\int_\Omega u_+^3\varphi\,dx-\beta s_\beta(u,v)^2t_\beta(u,v)^2\int_\Omega u_+v^2\varphi\,dx.$$

Combining (1.16), we have

$$\begin{aligned}\left|\nabla_u J_\beta(u,v)\varphi-\nabla J_1(u)\varphi\right|&\leqslant\left|s_\beta(u,v)^4-\frac{1}{\mu_1^2\|u_+\|_4^8}\right|\mu_1\int_\Omega u_+^3|\varphi|\,dx+|\beta|s_\beta(u,v)^2t_\beta(u,v)^2\int_\Omega u_+v^2|\varphi|\,dx\\&\leqslant\left|s_\beta(u,v)^4-\frac{1}{\mu_1^2\|u_+\|_4^8}\right|\mu_1C_1^4|||\varphi|||_{\lambda_1}+|\beta|C_\delta^4C_1^4|||\varphi|||_{\lambda_1}.\end{aligned}$$

We obtain (1.13) from the above inequality and Lemma 1.6. (1.14) also holds from a similar calculation. □

For small $\beta>0$, the following proposition plays a role similar to Proposition 1.3.

Proposition 1.9. *For any $\beta\in(-\beta_\delta,\beta_\delta)$, we have*

$$\sup_{(u,v)\in M_\delta}J_\beta(u,v)\leqslant\frac{1}{2\delta}+c_\delta(\beta), \tag{1.20}$$

$$\inf_{(u,v)\in\partial M_\delta}J_\beta(u,v)\geqslant\frac{1}{4\delta}+b_0-c_\delta(\beta). \tag{1.21}$$

Here b_0 was given in (1.9).

Proof. From Proposition 1.7, for $(u,v)\in M_\delta$, $\beta\in(-\beta_\delta,\beta_\delta)$, we have

$$J_1(u)+J_2(v)-c_\delta(\beta)\leqslant J_\beta(u,v)\leqslant J_1(u)+J_2(v)+c_\delta(\beta).$$

We remark that

$$\inf_{u\in\Sigma_1,u_+\not\equiv0}J_1(u)\geqslant b_0^1\geqslant b_0,\qquad\inf_{v\in\Sigma_2}J_2(v)\geqslant b_0^2\geqslant b_0.$$

Here $(u,v)\in\partial M_\delta$ implies $J_1(u)=\frac{1}{4\delta}$ or $J_2(v)=\frac{1}{4\delta}$ and $(u,v)\in M_\delta$ implies $J_1(u)\leqslant\frac{1}{4\delta}$ or $J_2(v)\leqslant\frac{1}{4\delta}$. Therefore we get (1.20) and (1.21). □

2. The multiplicity of critical values for σ-invariant functionals

In this section, we construct abstract theories to get the multiple existence of critical points of functionals having symmetry $J(\sigma(u))=J(u)$ where u is in a Hilbert space and σ satisfies (0.1)–(0.2). To do so, we construct a genus type index for the symmetry σ. In [23] or [13], the authors constructed the genus type index for $\sigma(-u)=u$ in the scaler case or $\sigma(u,v)=(v,u)$ in the vector case respectively.

In this section, let H be a Hilbert space and $\sigma:H\to H$ be a bounded linear operator satisfying (0.1)–(0.2). Setting $H_0=\{u\in H\mid\sigma(u)=u\}$, H_0 is a subspace composed of fixed points of σ. Here $H_0\neq H$ from (0.2). We also set $H_1=H_0^\perp\neq\{0\}$. For any $u\in H$, we uniquely write $u=u_0+u_1$, $(u_0,u_1)\in H_0\oplus H_1$. Then, from (0.1)–(0.2), we have

$$\sigma(u_0+u_1)=u_0-u_1\quad\text{for all } u=u_0+u_1\in H_0+H_1.$$

For this $\sigma : H \to H$, we define a genus as follows:

Definition 2.1. For any σ-invariant closed set $A \subset H \setminus H_0$, $\gamma(A)$ is the least integer n such that there exists a function $g \in C(A, \mathbf{R}^n \setminus \{0\})$ with

$$g\big(\sigma(u)\big) = -g(u) \quad \text{for all } u \in A. \tag{2.1}$$

If there is no such g, we define $\gamma(A) = \infty$. We also define $\gamma(\emptyset) = 0$.

Here, when g satisfies (2.1), we say g is a σ-odd function. When $J \in C(A, \mathbf{R})$ satisfies

$$J\big(\sigma(u)\big) = J(u) \quad \text{for all } u \in A,$$

we say J is a σ-invariant functional or a σ-even functional. When $h \in C(A, H)$ satisfies

$$h\big(\sigma(u)\big) = \sigma\big(h(u)\big) \quad \text{for all } u \in A,$$

we say h is σ-equivariant.

The following theorem is the main theorem in this section:

Theorem 2.2. *Let $M \subset H \setminus H_0$ be a σ-invariant C^1-manifold and $J : M \to \mathbf{R}$ be a σ-even C^1-functional satisfying (PS)-condition. Moreover, we assume that*

$$\inf_{u \in M} J(u) > -\infty, \tag{2.2}$$

$$\liminf_{u \in M,\, \operatorname{dist}\{u, \partial M\} \to 0} J(u) = \infty, \tag{2.3}$$

and, for any $k \in \mathbf{N}$, there exists $\psi \in C(S^k, M)$ with $\psi(-x) = \sigma(\psi(x))$. Then J has an unbounded nondecreasing sequence of critical values $(c_k)_{k=1}^{\infty}$. Here c_k is defined by

$$c_k = \inf\big\{c \in \mathbf{R} \,\big|\, \gamma\big([J \leqslant c]_M\big) \geqslant k\big\},$$
$$[J \leqslant c]_M = \big\{u \in M \,\big|\, J(u) \leqslant c\big\}. \tag{2.4}$$

Firstly we state the properties of our genus. These are similar to the properties of the genus type index constructed in [23] or [13].

Lemma 2.3. *Let $A, B \subset H \setminus H_0$ be σ-invariant closed sets. Then we have*:

(i) *If $A \subset B$, then $\gamma(A) \leqslant \gamma(B)$.*
(ii) *$\gamma(A \cup B) \leqslant \gamma(A) + \gamma(B)$.*
(iii) *If $h \in C(A, H \setminus H_0)$ satisfies $h(\sigma(u)) = \sigma(h(u))$, then $\gamma(A) \leqslant \gamma(h(A))$.*
(iv) *$\gamma(\overline{A \setminus B}) \geqslant \gamma(A) - \gamma(B)$.*
(v) *If $\gamma(A) > 1$, then A is an infinite set.*
(vi) *If A is a compact set, then $\gamma(A) < \infty$. Moreover there exists σ-invariant neighborhood of N of A in M such that $\gamma(A) = \gamma(\overline{N})$.*
(vii) *If $\psi \in C(S^n, H \setminus H_0)$ satisfies $\psi(-u) = \sigma(\psi(u))$, then $\gamma(\psi(S^n)) \geqslant n + 1$.*

Proof. First of all, we show (iii). If $\gamma(h(A)) = \infty$, (iii) is trivial. Supposing $\gamma(h(A)) = m < \infty$, there exists σ-odd function $g \in C(h(A), \mathbf{R}^m \setminus \{0\})$. Then $(g \circ h) \in C(A, \mathbf{R}^m \setminus \{0\})$ satisfies $(g \circ h)(\sigma(u)) = g(\sigma(h(u))) = -(g \circ h)(u)$. Thus we have $\gamma(A) \leqslant m = \gamma(h(A))$ and (iii) holds. We get (i), taking an inclusion map $id_A \in C(A, B)$ in (iii). Next, we show (v). When A is a finite set, A is written by $A = \{u_1, \dots, u_k, \sigma(u_1), \dots, \sigma(u_k)\}$ where $u_1, \dots, u_k, \sigma(u_1), \dots, \sigma(u_k)$ are different from each other. Then we have $g \in C(A, \mathbf{R} \setminus \{0\})$ such that $g(x_i) = 1$, $g(\sigma(x_i)) = -1$ for $i = 1, \dots, k$. Thus we find $\gamma(A) = 1$. This implies (v).

Next, we show (ii). Supposing $\gamma(A) = n < \infty$, $\gamma(B) = m < \infty$, there exist σ-odd functions $g \in C(A, \mathbf{R}^n \setminus \{0\})$ and $h \in C(B, \mathbf{R}^m \setminus \{0\})$. By the extension theorem of Tietze, we have $\hat{g}, \hat{h} \in C(H, H)$ such that $\hat{g}(u) = g(u)$ for all $u \in A$ and $\hat{h}(u) = h(u)$ for all $u \in B$. Here, set

$$\tilde{g}(u) = \frac{\hat{g}(u) - \hat{g}(\sigma(u))}{2}, \qquad \tilde{h}(u) = \frac{\hat{h}(u) - \hat{h}(\sigma(u))}{2}.$$

Then $\tilde{g}$ and $\tilde{h}$ are σ-odd and also an extension of g and h respectively. Since $f = (\tilde{g}|_{A\cup B}, \tilde{h}|_{A\cup B}) \in C(A \cup B, \mathbf{R}^{n+m} \setminus \{0\})$ also σ-odd, we get $\gamma(A \cup B) \leqslant n + m = \gamma(A) + \gamma(B)$. (iv) easily follows from (i) and (ii).

Next, we show (vi). For any $u \in A$, we set $T_u = B_{d_u/2}(u) \cup B_{d_u/2}(\sigma(u))$ where $d_u = \mathrm{dist}\{u, H_0\} > 0$. Then we have $\gamma(T_u) = 1$. Since A is compact and $\{T_u \mid u \in A\}$ are open covering of A, for finite $u_1, \ldots, u_k \in A$, we have $A \subset \bigcup_{i=1}^k T_{u_i}$. From (ii), we get $\gamma(A) \leqslant k$. Next, we show later part of (vi). We remark that letting $N_\delta(A)$ be δ-neighborhood of A in M, $N_\delta(A)$ is σ-invariant and $N_\delta(A) \subset H \setminus H_0$ for small $\delta > 0$. Supposing $\gamma(A) = n$, there exists a σ-odd function $g \in C(A, \mathbf{R}^n \setminus \{0\})$. By a similar way to show (iii), we have σ-odd function $\tilde{g} \in C(N_\delta(A), \mathbf{R}^n \setminus \{0\})$. Thus we get $\gamma(N_\delta(A)) \leqslant n = \gamma(A)$. On the other hand, $A \subset N_\delta(A)$ implies $\gamma(N_\delta(A)) \geqslant \gamma(A)$. Thus we get $\gamma(N_\delta(A)) = \gamma(A)$.

Finally we show (vii). By a contradiction, we assume $\gamma(\psi(S^n)) \leqslant n$. Then there exists a σ-odd function $g \in C(\psi(S^n)), \mathbf{R}^n \setminus \{0\})$. Here $g \circ \psi \in C(S^n, \mathbf{R}^n \setminus \{0\})$ is an odd function but this contradicts the Borsuk–Ulam theorem. Thus we obtain (vii). □

Proposition 2.4. *Let $M \subset H \setminus H_0$ be a σ-invariant C^1-manifold and $J : M \to \mathbf{R}$ be a σ-even C^1-functional satisfying (PS)-condition. Moreover, we assume that*

$$\liminf_{u\in M,\, \mathrm{dist}\{u,\partial M\}\to 0} J(u) = d \leqslant \infty. \tag{2.5}$$

Then, for any $c < d$ and $\delta > 0$, there exist $\epsilon > 0$ and $\eta : [0, 1] \times [J \leqslant c + \epsilon]_M \to [J \leqslant c + \epsilon]_M$ such that

$$\eta(0, u) = u \quad \text{for all } u \in [J \leqslant c_k + \epsilon]_M, \tag{2.6}$$

$$\eta(1, u) \in [J \leqslant c_k - \epsilon]_M \quad \text{for all } u \in [J \leqslant c_k + \epsilon]_M \setminus N_\delta(K_c), \tag{2.7}$$

$$\eta\big(1, \sigma(u)\big) = \sigma\big(\eta(1, u)\big) \quad \text{for all } u \in [J \leqslant c_k + \epsilon]_M. \tag{2.8}$$

Here $K_c = \{u \in M \mid J(u) = c,\ J'(u) = 0\}$ and $N_\delta(K_c)$ is δ-neighborhood of K_c in M.

Proof. For any $u \in M$, we uniquely write $u = u_0 + u_1 \in H_0 + H_1$ and $J(\sigma(u))$ is also uniquely written as $J(\sigma(u)) = J(u_0 - u_1)$. Since $J : M \to \mathbf{R}$ is σ-even, we also have

$$J(u_0 - u_1) = J(u_0 + u_1) \quad \text{for all } u = u_0 + u_1 \in H_0 + H_1.$$

Therefore, noting $\nabla_u = \nabla_{u_0} + \nabla_{u_1}$, we obtain

$$\nabla J\big(\sigma(u)\big)\varphi = \sigma\big(\nabla J(u)\big)\varphi = \nabla J(u)\sigma(\varphi). \tag{2.9}$$

Constructing a deformation flow $\eta : [0, 1] \times [J \leqslant c + \epsilon]_M \to [J \leqslant c + \epsilon]_M$ by a standard way, it is obvious that η satisfies (2.6)–(2.7). In addition, (2.8) holds from (2.9). □

Proof of Theorem 2.2. Firstly we show (i). By a contradiction, we suppose that c_k is not a critical point. From the definition of c_k, for any $\epsilon > 0$, we have $\gamma([J \leqslant c_k + \epsilon]_M) \geqslant k$. Applying Proposition 2.4 for $c = c_k$ and $K_{c_k} = \emptyset$, there exist $\epsilon > 0$ and $\eta : [0, 1] \times [J \leqslant c_k + \epsilon]_M \to [J \leqslant c_k + \epsilon]_M$ such that

$$\eta(0, u) = u \quad \text{for all } u \in [J \leqslant c_k + \epsilon]_M, \tag{2.10}$$

$$\eta(1, u) \in [J \leqslant c_k - \epsilon]_M \quad \text{for all } u \in [J \leqslant c_k + \epsilon]_M, \tag{2.11}$$

$$\eta\big(1, \sigma(u)\big) = \sigma\big(\eta(1, u)\big) \quad \text{for all } u \in [J \leqslant c_k + \epsilon]_M. \tag{2.12}$$

From (2.12) and (iii) of Lemma 2.3, we have

$$\gamma\big([J \leqslant c_k + \epsilon]_M\big) \leqslant \gamma\big(\eta\big(1, [J \leqslant c_k + \epsilon]_M\big)\big). \tag{2.13}$$

From (2.11) and (i) of Lemma 2.3, we have

$$\gamma\big(\eta\big(1,[J\leqslant c_k+\epsilon]_M\big)\big)\leqslant\gamma\big([J\leqslant c_k-\epsilon]_M\big). \tag{2.14}$$

Combining (2.13)–(2.14), we get $\gamma([J\leqslant c_k-\epsilon]_M)\geqslant\gamma([J\leqslant c_k+\epsilon]_M)\geqslant k$ and this contradicts the definition of c_k. Thus c_k is a critical point. (ii) is obvious from the definition of c_k. Next we show (iii). By a contradiction, we suppose that $c_k\to\bar c<\infty$ as $k\to\infty$. Since J satisfies (PS)-condition, $K_{\bar c}=\{u\in M\mid J(u)=\bar c,\ J'(u)=0\}$ is a compact set. Thus, from (vi) of Lemma 2.3, there exists a σ-invariant neighborhood of $N_\delta(K_{\bar c})$ such that $\gamma(K_{\bar c})=\gamma(\overline{N_\delta(K_{\bar c})})=q<\infty$. Applying Proposition 2.4 for $c=\bar c$ and $K_{\bar c}$, there exist $\epsilon>0$ and $\eta:[0,1]\times[J\leqslant\bar c+\epsilon]_M\to[J\leqslant\bar c+\epsilon]_M$ such that

$$\eta(0,u)=u\quad\text{for all } u\in[J\leqslant\bar c+\epsilon_0]_M, \tag{2.15}$$

$$\eta(1,u)\in[J\leqslant\bar c-\epsilon]_M\quad\text{for all } u\in[J\leqslant\bar c+\epsilon]_M\setminus N_\delta(K_{\bar c}), \tag{2.16}$$

$$\eta\big(1,\sigma(u)\big)=\sigma\big(\eta(1,u)\big)\quad\text{for all } u\in[J\leqslant\bar c+\epsilon]_M. \tag{2.17}$$

Since $c_k\to\bar c<\infty$ as $k\to\infty$, there exists k_0 such that

$$\bar c-\frac{\epsilon}{2}<c_k\leqslant\bar c\quad\text{for all } k\geqslant k_0.$$

From the definition c_{k_0+q}, we have $\gamma([J\leqslant c_{k_0+q}+\epsilon]_M)\geqslant k_0+q$. Then, using (i), (iii) and (iv) of Lemma 2.3, we have

$$\begin{aligned}\gamma\left(\left[J\leqslant c_{k_0}-\frac{\epsilon}{2}\right]_M\right)&\geqslant\gamma\big([J\leqslant\bar c-\epsilon]_M\big)\\&\geqslant\gamma\big(\eta\big(1,[J\leqslant c_{k_0+q}+\epsilon]_M\setminus N_\delta(K_{\bar c})\big)\big)\\&\geqslant\gamma\big(\eta\big(1,[J\leqslant c_{k_0+q}+\epsilon]_M\big)-\gamma\big(\overline{N_\delta(K_{\bar c})}\big)\big)\\&\geqslant(k_0+q)-q=k_0.\end{aligned}$$

This is a contradiction to the definition of c_{k_0}. Thus we see that $c_k\to\infty$ as $k\to\infty$. □

By a similar way to Theorem 2.2, we get the following theorem.

Theorem 2.5. *Let $M\subset H\setminus H_0$ be a σ-invariant C^1-manifold and $J:M\to\mathbf{R}$ be a σ-even C^1-functional satisfying (PS)-condition. Moreover, we assume that*

$$\inf_{u\in M}J(u)>-\infty, \tag{2.18}$$

$$\liminf_{u\in M,\,\mathrm{dist}\{u,\partial M\}\to 0}J(u)=d<\infty, \tag{2.19}$$

and, for some $k\in\mathbf{N}$, there exists $\psi\in C(S^k,M)$ with $\psi(-x)=\sigma(\psi(x))$ such that $\sup_{x\in S^k}J(\psi(x))<d$. Then $J(u)$ has at least k critical points.

Proof. We define c_i $(1\leqslant i\leqslant k)$ as (2.4). Then we see that $c_1\leqslant c_2\leqslant\cdots\leqslant c_k$ $(<d)$ are critical values of $J(u)$. Moreover, if $c_i=c_{i+1}=\cdots=c_{i+q}$ holds, then $\gamma(K_{c_i})\geqslant q+1$. This is shown by a similar way to show $c_k\to\infty$ in the proof of Lemma 2.3. From (v) of Lemma 2.3, $\gamma(K_{c_i})\geqslant q+1\geqslant 2$ implies K_{c_i} is an infinite set. Thus we get Theorem 2.5. □

3. Proofs of Theorem 0.1 and Theorem 0.2

In this section, we will give the proofs of Theorem 0.1 and Theorem 0.2 by using abstract theories for $\sigma(u,v)=(u,-v):H_0^1(\Omega)\times H_0^1(\Omega)\to H_0^1(\Omega)\times H_0^1(\Omega)$. To apply our abstract theory, we need the following lemma.

Lemma 3.1. *Suppose $\beta<0$. For any $k\in\mathbf{N}$, there exists $\psi\in C(S^k,N_\beta)$ such that $\psi(-v)=\sigma(\psi(v))$.*

Proof. We choose non-empty open sets $\Omega_1, \Omega_2 \subset \Omega$ with $\Omega_1 \cap \Omega_2 = \emptyset$. We also choose $u_0 \in H_0^1(\Omega_1)$ such that $|||u_0|||_{\lambda_1} = 1$ and $u_{0+} \not\equiv 0$. Let W_k be a k-dimensional subspace of $H_0^1(\Omega_2)$. Then it is obvious that $\mu_1\mu_2\|u_{0+}\|_4^4\|v\|_4^4 - \beta^2\|u_{0+}v\|_2^4 > 0$ for all $v \in S^k := \{v \in W_k \mid |||v|||_{\lambda_2} = 1\}$. Thus, setting $\psi(v) = (u_0, v)$, $\psi(v)$ satisfies $\psi(v) \in N_\beta$ for all $v \in S^k$ and $\psi(-v) = (u_0, -v) = \sigma(\psi(v))$. □

Here, we give the proof of Theorem 0.1.

Proof of Theorem 0.1. Suppose $\beta < 0$. We apply Theorem 2.2 for $H = H_0^1(\Omega) \times H_0^1(\Omega)$, $\sigma(u, v) = (u, -v)$, $M = N_\beta$, $J(u) = J_\beta(u, v)$. Firstly, we will check that the assumptions of Theorem 2.2 hold. From Proposition 1.3, we have

$$\liminf_{(u,v)\in N_\beta,\, \mathrm{dist}\{(u,v),\partial N_\beta\}\to 0} J_\beta(u, v) = \infty.$$

Moreover, from Lemma 3.1, for any $k \in \mathbf{N}$, there exists $\psi \in C(S^k, N_\beta)$ such that $\psi(-u) = \sigma(\psi(u))$. Therefore the assumptions of Theorem 2.2 hold and J_β has a sequence of critical values $(c_k)_{k=1}^\infty$ such that $c_k \to \infty$ as $k \to \infty$. Let (u_k, v_k) be a critical point of J_β corresponding to c_k and we set $(U_k, V_k) = (s_\beta(u_k, v_k)u_k, t_\beta(u_k, v_k)v_k)$. Then, from (iii) of Proposition 1.2, (U_k, V_k) is a non-trivial critical point of I_β. From Proposition 1.1, we see $U_k > 0$ in Ω. Moreover, from (1.3) and $\beta < 0$, we find

$$\big(\mu_1\|U_k\|_\infty^4 + \mu_2\|V_k\|_\infty^4\big)|\Omega| \geqslant \mu_1\|U_k\|_4^4 + \mu_2\|V_k\|_4^4 \geqslant 4c_k \to \infty \quad (k \to \infty).$$

Thus we get $\|U_k\|_\infty + \|V_k\|_\infty \to \infty$. On the other hand, when $\beta > -\sqrt{\mu_1\mu_2}$, there exists a priori bound of positive solution of (∗) by a result of [13]. Thus, when $\beta \in (-\sqrt{\mu_1\mu_2}, 0)$, V_k must change sign for large k. Now, the proof of Theorem 0.1 is complete. □

Next, we show Theorem 0.2. To prove Theorem 0.2, we need the following lemma.

Lemma 3.2. *For any given $k \in \mathbf{N}$, there exist $\delta_k > 0$, $\beta_k > 0$ and $\psi \in C(S^k, M_{\delta_k})$ with $\psi(-v) = \sigma(\psi(v))$ such that*

$$\sup_{v\in S^k} J_\beta\big(\psi(v)\big) \leqslant d = \inf_{(u,v)\in\partial M_{\delta_k}} J_\beta(u, v) \quad \text{for all } \beta \in (-\beta_k, \beta_k). \tag{3.1}$$

Proof. Let W_k be k-dimensional subspace of $H_0^1(\Omega)$ such that

$$W_1 \subset W_2 \subset \cdots \subset W_k \subset W_{k+1} \subset \cdots.$$

For any given $k \in \mathbf{N}$, we choose small $\delta_k > 0$ satisfying

$$\mu_2\|v\|_4^4 > 4\delta_k \quad \text{for all } v \in S^k := \big\{v \in W_k \bigm| |||v|||_{\lambda_2} = 1\big\}.$$

We remark that δ_k also satisfies $4\delta_k \in (0, \frac{1}{4b_0})$. For this $\delta_k > 0$, from Proposition 1.9 and Proposition 1.7, there exists $\beta_k = \beta_{\delta_k} > 0$ such that, for all $\beta \in (-\beta_k, \beta_k)$, we have

$$\sup_{(u,v)\in M_{4\delta_k}} J_\beta(u, v) \leqslant \frac{1}{8\delta_k} + c_{\delta_k}(\beta),$$

$$\inf_{(u,v)\in\partial M_{\delta_k}} J_\beta(u, v) \geqslant \frac{1}{4\delta_k} + b_0 - c_{\delta_k}(\beta),$$

$$\big|2c_{\delta_k}(\beta)\big| < \frac{1}{8\delta_k} - b_0.$$

Here we choose $u_0 \in H_0^1(\Omega)$ such that $|||u_0|||_{\lambda_1} = 1$ and $\|u_{0+}\|_4^4 \geqslant 4\delta_k$. Setting $\psi(v) = (u_0, v)$, $\psi(v)$ satisfies

$$\psi(v) \in M_{4\delta_k} \subset M_{\delta_k} \quad \text{for all } v \in S^k,$$
$$\psi(-v) = (u_0, -v) = \sigma\big(\psi(v)\big).$$

Then $\psi(v)$ satisfies (3.1) and we get Lemma 3.2. □

Now, we give the proof of Theorem 0.2.

Proof of Theorem 0.2. From Lemma 3.2, for any given $k \in \mathbf{N}$, there exist $\delta_k > 0$, $\beta_k > 0$ and $\psi \in C(S^k, M_{\delta_k})$ with $\psi(-v) = \sigma(\psi(v))$ such that

$$\sup_{v\in S^k} J_\beta\big(\psi(v)\big) \leqslant d = \inf_{(u,v)\in\partial M_{\delta_k}} J_\beta(u,v) \quad \text{for all } \beta \in (-\beta_k, \beta_k).$$

Here, setting $H = H_0^1(\Omega) \times H_0^1(\Omega)$, $\sigma(u,v) = (u,-v)$, $M = M_{\delta_k}$, $J(u) = J_\beta(u,v)$ $(0 < \beta < \beta_k)$, the assumptions of Theorem 2.5 hold. Thus J_β has at least k critical points. In conclusion from Proposition 1.2, we get Theorem 0.2. □

4. The asymptotic behavior of some critical values of J_β

In this section, for $J_\beta(u,v)$, we will define the mountain pass values corresponding to solutions in Theorem 0.3. Firstly, for $J_2(v)$, we define symmetric mountain pass values b_n^2 $(n \in \mathbf{N} \cup \{0\})$ by

$$b_n^2 = \inf_{\gamma_2\in\Gamma_n^2} \max_{\theta\in S^n} J_2\big(\gamma_2(\theta)\big),$$

$$\Gamma_n^2 = \big\{\gamma_2(\theta) \in C\big(S^n, \Sigma_2\big) \bigm| \gamma_2(-\theta) = -\gamma_2(\theta) \text{ for all } \theta \in S^n\big\},$$

where $S^n = \{\theta = (\theta_1, \ldots, \theta_{n+1}) \in \mathbf{R}^{n+1} \mid |\theta| = 1\}$. Then, from the symmetric mountain pass theory for J_2, b_n^2 satisfies the following:

(i) b_n^2 is a critical value of J_2. In particular, b_0^2 is a least energy level of J_2.
(ii) $b_0^2 < b_1^2 \leqslant b_2^2 \leqslant \cdots \leqslant b_n^2 \leqslant b_{n+1}^2 \leqslant \cdots$.
(iii) $b_n^2 \to \infty$ as $n \to \infty$.

Now, from Lemma 3.2, for any given $k \in N$, there exist $\delta_k > 0$, $\beta_k > 0$ and $\psi \in C(S^k, M_{\delta_k})$ with $\psi(-v) = \sigma(\psi(v))$ such that

$$\sup_{v\in S^k} J_\beta\big(\psi(v)\big) \leqslant d = \inf_{(u,v)\in\partial M_{\delta_k}} J_\beta(u,v) \quad \text{for all } \beta \in (-\beta_k, \beta_k).$$

We fix $k \in N$, $\delta_k > 0$ and $\beta_k > 0$ as above. Here, for $\beta \in (-\beta_k, \beta_k)$, we define minimax values $d_{i,\beta}$ $(i = 1, 2, \ldots, k)$ of $J_\beta(u,v)$ by the following:

$$d_{i,\beta} = \inf_{g\in\Gamma_i} \max_{\theta\in S^i} J_\beta\big(\gamma(\theta)\big),$$

$$\Gamma_i = \big\{\gamma(\theta) \in C\big(S^i, M_{\delta_k}\big) \bigm| \gamma(-\theta) = \sigma\big(\gamma(\theta)\big) \text{ for all } \theta \in S^i\big\}. \tag{4.1}$$

We remark that $\Gamma_i \neq \emptyset$ by the existence of ψ. We show that $d_{i,\beta}$ satisfies the following proposition.

Proposition 4.1. *For $i = 1, 2, \ldots, k$, we have*:

(i) *$d_{i,\beta}$ is a critical value of $J_\beta(u,v)$ for $\beta \in (-\beta_k, \beta_k)$.*
(ii) *$d_{i,\beta} \to b_0^1 + b_i^2$ as $\beta \to 0$.*

Proof. Firstly we show (i). By a contradiction, we suppose that $d_{i,\beta}$ is not a critical point. For $\epsilon_0 > 0$, there exists $\gamma \in \Gamma_i$ such that $\sup_{\theta\in S^i} J_\beta(\gamma(\theta)) \leqslant d_{i,\beta} + \epsilon_0$. Here, applying Proposition 2.4, we have small $\epsilon \in (0, \epsilon_0)$ and $\eta : [0,1] \times [J_\beta \leqslant d_{i,\beta} + \epsilon]_{M_{\delta_k}} \to [J_\beta \leqslant d_{i,\beta} + \epsilon]_{M_{\delta_k}}$ such that

$$\eta(0,u) = u \quad \text{for all } u \in [J_\beta \leqslant d_{i,\beta} + \epsilon]_{M_{\delta_k}}, \tag{4.2}$$

$$\eta(1,u) \in [J_\beta \leqslant d_{i,\beta} - \epsilon]_{M_{\delta_k}} \quad \text{for all } u \in [J_\beta \leqslant d_{i,\beta} + \epsilon]_{M_{\delta_k}}, \tag{4.3}$$

$$\eta\big(1, \sigma(u)\big) = \sigma\big(\eta(1,u)\big) \quad \text{for all } u \in [J_\beta \leqslant d_{i,\beta} + \epsilon]_{M_{\delta_k}}. \tag{4.4}$$

Setting $\tilde\gamma(\theta) = \eta(1, \gamma(\theta))$, we have $\tilde\gamma \in \Gamma_i$ and $\sup_{\theta\in S^i} J_\beta(\tilde\gamma(\theta)) \leqslant d_{i,\beta} - \epsilon$. This contradicts the definition of $d_{i,\beta}$. Thus $d_{i,\beta}$ is a critical point.

Next, we show (ii). From Proposition 1.7, we have

$$J_1(u) + J_2(v) - c_{\delta_k}(\beta) \leqslant J_\beta(u, v) \leqslant J_1(u) + J_2(v) + c_{\delta_k}(\beta)$$

for all $(u, v) \in M_{\delta_k}$ and $\beta \in (-\beta_k, \beta_k)$. For any $\epsilon > 0$, we choose $\gamma_2 \in \Gamma_i^2$ such that

$$\max_{\theta \in S^i} J_2\big(\gamma_2(\theta)\big) \leqslant b_i^2 + \epsilon.$$

Setting $\gamma(\theta) = (u_0, \gamma_2(\theta))$ where u_0 is a minimizer of $J_1(u)$, then we have $\gamma(\theta) \in \Gamma_i$ and

$$d_{i,\beta} \leqslant \max_{\theta \in S^i} J_\beta\big(\gamma(\theta)\big) \leqslant J_1(u_0) + \max_{\theta \in S^i} J_2\big(\gamma_2(\theta)\big) + c_{\delta_k}(\beta) \leqslant b_0^1 + b_i^2 + \epsilon + c_{\delta_k}(\beta). \tag{4.5}$$

On the other hand, we choose $\gamma \in \Gamma_i$ such that

$$\max_{\theta \in S^i} J_\beta\big(\gamma(\theta)\big) \leqslant d_{i,\beta} + \epsilon.$$

Writing $\gamma(\theta) = (\gamma_1(\theta), \gamma_2(\theta)) \in \Sigma_1 \times \Sigma_2$, we have $\gamma_2(\theta) \in \Gamma_i^2$ and

$$b_0^1 + b_i^2 \leqslant J_1\big(\gamma_1(\theta)\big) + \max_{\theta \in S^i} J_2\big(\gamma_2(\theta)\big) \leqslant \max_{\theta \in S^i} J_\beta\big(\gamma(\theta)\big) + c_{\delta_k}(\beta) \leqslant d_{i,\beta} + \epsilon + c_{\delta_k}(\beta). \tag{4.6}$$

From (4.5)–(4.6), we have

$$\big|d_{i,\beta} - \big(b_0^1 + b_i^2\big)\big| \leqslant \epsilon + c_{\delta_k}(\beta).$$

Since $\epsilon > 0$ is arbitrary and $c_{\delta_k}(\beta) \to 0$ as $\beta \to 0$, we obtain (ii). □

5. Proof of Theorem 0.3

In this section, we will complete the proof of Theorem 0.3. For $i \in \{1, 2, \dots, k\}$, we show the following proposition.

Proposition 5.1. *For any $\epsilon > 0$, there exists $\beta_k' > 0$ such that, for all $|\beta| < \beta_k'$, $J_\beta(u, v)$ has critical points in A_β^ϵ which are defined by*

$$A_\beta^\epsilon = \left\{ (u, v) \in M_{\delta_k} \;\middle|\; \begin{array}{l} db_0^1 \leqslant J_1(u) \leqslant b_0^1 + \epsilon, \\ b_0^1 + b_i^2 - \epsilon \leqslant J_\beta(u, v) \leqslant b_0^1 + b_i^2 + \epsilon \end{array} \right\}.$$

We remark that A_β^ϵ is an invariant set for $\sigma(u, v) = (u, -v)$ and $A_\beta^\epsilon \neq \emptyset$. If Proposition 5.1 holds, then we get Theorem 0.3 as follows:

Proof of Theorem 0.3. From Proposition 5.1, for all $\epsilon > 0$ and $|\beta| < \beta_k'$, there exists critical point $(u_{i,\beta}, v_{i,\beta})$ of $J_\beta(u, v)$ which satisfies

$$b_0^1 \leqslant J_1(u_{i,\beta}) \leqslant b_0^1 + \epsilon,$$
$$b_0^1 + b_i^2 - \epsilon \leqslant J_\beta(u_{i,\beta}, v_{i,\beta}) \leqslant b_0^1 + b_i^2 + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, from Proposition 1.7, we see that $u_{i,\beta}$, $v_{i,\beta}$ satisfy

$$J_1(u_{i,\beta}) \to b_0^1 \qquad J_1'(u_{i,\beta}) \to 0 \quad \text{as } \beta \to 0,$$
$$J_2(v_{i,\beta}) \to b_i^2 \qquad J_2'(v_{i,\beta}) \to 0 \quad \text{as } \beta \to 0.$$

Thus, after extracting subsequence $\beta_j \to 0$, there exist $u_{i,0} \in \Sigma_1$ and $v_{i,0} \in \Sigma_2$ which are critical points of $J_1(u)$ and $J_2(v)$ respectively, such that

$$u_{i,\beta_j} \to u_{i,0} \quad \text{as } \beta_j \to 0, \quad J_1(u_{i,0}) = b_0^1, \qquad J_1'(u_{i,0}) = 0, \tag{5.1}$$

$$v_{i,\beta_j} \to v_{i,0} \quad \text{as } \beta_j \to 0, \quad J_2(v_{i,0}) = b_i^2, \qquad J_2'(v_{i,0}) = 0. \tag{5.2}$$

From (5.1)–(5.2), Proposition 1.1 and Proposition 1.2, we get Theorem 0.3. □

In what follows, we will show Proposition 5.1 by a contradiction. If Proposition 5.1 does not hold, then there exist $\epsilon_0 > 0$ and a sequence $\beta_j \to 0$ such that $J_{\beta_j}(u,v)$ does not have critical points in $A^{\epsilon_0}_{\beta_j}$.

Here, we remark that a set of critical values of $J_1(u)$ is nowhere dense. Thus there exists $b^1_0 + \frac{1}{3}\epsilon_0 < a_0 < a_1 < b^1_0 + \epsilon_0$ such that $J_1(u)$ does not have critical points in $[a_0 \leqslant J_1 \leqslant a_1]_{\Sigma_1}$.

Remark 5.2. Fučík, Kučera, Nečas, Souček and Souček [14] gave a result for the Morse–Sard theorem in infinite dimensional setting. Since $\tilde{I}_1(u) = \frac{1}{2}\|u\|^2_{\lambda_1} - \frac{1}{4}\mu_1\|u\|^4_4 : H^1_0(\Omega) \to \mathbf{R}$ is analytic and satisfies (PS)-condition, the set of critical values of $\tilde{I}_1(u)$ is measure zero and closed. Thus the set of critical values of $\tilde{I}_1(u)$ is nowhere dense. This implies the nowhere denseness of the set of critical values of $J_1(u)$. Moreover there exist further results of Dancer [12] and Cao and Noussair [8] about when critical values of $I_1(u)$ are isolated.

Since there are not critical points of $J_1(u)$ in $[a_0 \leqslant J_1 \leqslant a_1]_{\Sigma_1}$, we set

$$\rho_0 = \inf_{u \in [a_0 \leqslant J_1 \leqslant a_1]_{\Sigma_1}} \big|\!\big|\!\big| \nabla J_1(u) \big|\!\big|\!\big|_{\lambda_1 *} > 0. \tag{5.3}$$

Then we have the following lemma.

Lemma 5.3. . *For sufficiently small $|\beta_j| > 0$, we have the following: for any $(u,v) \in A^{\epsilon_0}_{\beta_j}$ with $u \in [a_0 \leqslant J_1 \leqslant a_1]$, there exists $(X,Y) \in T_u\Sigma_1 \times T_v\Sigma_2$ such that*

$$\|X\|_{\lambda_1} = 1, \qquad Y = 0,$$
$$\nabla J_1(u)X \geqslant \frac{\rho_0}{2}, \qquad \nabla J_{\beta_j}(u,v)(X,Y) \geqslant \frac{\rho_0}{2}.$$

Proof. Let $(u,v) \in A^{\epsilon_0}_{\beta_j}$ with $u \in [a_0 \leqslant J_1 \leqslant a_1]_{\Sigma_1}$. From (5.3), we see that there exists $X \in T_u\Sigma_1$ such that

$$\nabla J_1(u)X \geqslant \frac{3\rho_0}{4}.$$

From Proposition 1.7, choosing small $|\beta_j| > 0$ such that $c_{\delta_k}(\beta_j) < \frac{\rho_0}{4}$, we have

$$\nabla J_{\beta_j}(u,v)(X,0) \geqslant \nabla J_1(u)X - c_{\delta_k}(\beta_j)|\!|\!|X|\!|\!|_{\lambda_1} \geqslant \frac{\rho_0}{2}.$$

Thus we get Lemma 5.3. □

Lemma 5.4. *For small $|\beta_j| > 0$, there exists a vector field $(u,v) \mapsto (X(u,v), Y(u,v)) : A^{\epsilon_0}_{\beta_j} \to T_u\Sigma_1 \times T_v\Sigma_2$ such that:*

(i) $|\!|\!|X(u,v)|\!|\!|^2_{\lambda_1} + |\!|\!|Y(u,v)|\!|\!|^2_{\lambda_2} = 1$ *and $(X(u,v), Y(u,v))$ are Lipschitz continuous.*
(ii) $(X(\sigma(u,v)), Y(\sigma(u,v))) = \sigma(X(u,v), Y(u,v))$.
(iii) *There exists $\mu_j > 0$ such that $\nabla J_{\beta_j}(u,v)(X(u,v), Y(u,v)) \geqslant \mu_j$ for all $(u,v) \in A^{\epsilon_0}_{\beta_j}$.*
(iv) *For any $(u,v) \in A^{\epsilon_0}_{\beta_j}$ with $u \in [a_0 \leqslant J_1 \leqslant a_1]_{\Sigma_1}$, we have $\nabla J_1(u)X(u,v) \geqslant \frac{\rho_0}{2}$ and $\nabla J_{\beta_j}(u,v)(X(u,v), Y(u,v)) \geqslant \frac{\rho_0}{2}$.*

Proof. Since $J_{\beta_j}(u,v)$ does not have critical points in $A^{\epsilon_0}_{\beta_j}$, there exists $\mu_j > 0$ such that

$$\mu_j = \inf_{(u,v) \in A^{\epsilon_0}_{\beta_j}} \big|\!\big|\!\big| \nabla J_{\beta_j}(u,v) \big|\!\big|\!\big|_* > 0. \tag{5.4}$$

We also remark that $\nabla J_{\beta_j}(\sigma(u,v)) = \sigma(\nabla J_{\beta_j}(u,v))$. Thus from (5.4) and Lemma 5.3, we can construct a vector field with desired properties. □

Here we consider the following ODE:

$$\frac{d\eta_1}{dt} = -X(\eta_1, \eta_2), \qquad \frac{d\eta_2}{dt} = -Y(\eta_1, \eta_2),$$
$$\eta_1(0; u_0, v_0) = u_0, \qquad \eta_2(0; u_0, v_0) = v_0.$$

Then deformation flow $\eta(t,(u,v)) = (\eta_1(t,(u,v)), \eta_1(t,(u,v)))$ satisfies the following:

(i) When $\eta(t,(u,v)) \in A_{\beta_j}^{\epsilon_0}$, we have $\frac{d}{dt} J_{\beta_j}(\eta(t,(u,v))) \leqslant -\mu_j$.
(ii) When $\eta(t,(u,v)) \in A_{\beta_j}^{\epsilon_0} \cap [a_0 \leqslant J_1 \leqslant a_1]_{\Sigma_1}$, we have $\frac{d}{dt} J_{\beta_j}(\eta(t,(u,v))) \leqslant -\frac{\rho_0}{2}$ and $\frac{d}{dt} J_1(\eta_1(t,(u,v))) \leqslant -\frac{\rho_0}{2}$.

From (ii), we see that, for $(u,v) \in A_{\beta_j}^{\epsilon_0}$ with $J_1(u) < b_0^1 + \frac{1}{3}\epsilon_0$, when $\eta(t,(u,v))$ passes through $\partial A_{\beta_j}^{\epsilon_0}$, $\eta(t,(u,v))$ must satisfy $J_{\beta_j}(\eta(t,(u,v))) = b_0^1 + b_i^2 - \epsilon_0$. Moreover, from (i), $\eta(t,(u,v))$ must pass through $\partial A_{\beta_j}^{\epsilon_0}$ for finite time. Now, we complete the proof of Proposition 5.1.

Completion of the proof of Proposition 5.1. By the definition of b_0^1 and b_i^2, we can choose $u_0 \in \Sigma_1$ and $\gamma_2(\theta) \in \Gamma_i^2$ such that

$$J_1(u_0) < b_0^1 + \frac{1}{3}\epsilon_0,$$
$$\max_{\theta \in S^i} J_2\big(\gamma_2(\theta)\big) < b_i^2 + \frac{1}{3}\epsilon_0.$$

We set

$$\gamma(\theta) = \big(u_0, \gamma_2(\theta)\big) \in \Gamma_i.$$

Since $d_{i,\beta_j} \to b_0^1 + b_i^2$ as $\beta_j \to 0$, for sufficiently small $|\beta_j| > 0$, we have

$$\max_{\theta \in S^i} J_{\beta_j}\big(\gamma(\theta)\big) < b_0^1 + b_i^2 + \epsilon_0.$$

Moreover $J_{\beta_j}(\gamma(\theta)) \geqslant b_0^1 + b_i^2 - \epsilon_0$ implies $\gamma(\theta) \in A_{\beta_j}^{\epsilon_0}$. For large $t > 0$, we set

$$\tilde{\gamma}(\theta) = \big(\eta_1\big(t; \gamma(\theta)\big), \eta_2\big(t; \gamma(\theta)\big)\big).$$

Then we have $\tilde{\gamma}(\theta) \in \Gamma_i$ and

$$\max_{\theta \in S^i} J_{\beta_j}\big(\tilde{\gamma}(\theta)\big) < b_0^1 + b_i^2 - \epsilon_0.$$

This is a contradiction for (4.1) and Proposition 4.1. Thus Proposition 5.1 holds and we complete the proofs of our theorems. □

6. The setting for large β and the proofs of Theorem 0.6 and Theorem 0.7

To prove Theorem 0.6 and Theorem 0.10, we seek critical points of the following functional

$$\tilde{I}_\beta(u,v) = \frac{1}{2}\big(\|\|u\|\|_{\lambda_1}^2 + \|\|v\|\|_{\lambda_2}^2\big) - \frac{1}{4\beta}\big(\mu_1\|u_+\|_4^4 + \mu_2\|v\|_4^4\big) - \frac{1}{2}\|u_+v\|_2^2 : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbf{R}.$$

Here, when $\beta = \infty$, we regard $\tilde{I}_\infty(u,v)$ as

$$\tilde{I}_\infty(u,v) = \frac{1}{2}\big(\|\|u\|\|_{\lambda_1}^2 + \|\|v\|\|_{\lambda_2}^2\big) - \frac{1}{2}\|u_+v\|_2^2 : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbf{R}.$$

We remark that if (u,v) is a critical point of $\tilde{I}_\beta(u,v)$ for $\beta \in (0,\infty)$ then $(u/\sqrt{\beta}, v/\sqrt{\beta})$ is a solution of $(*)$ and if $u \neq 0$ we have $u > 0$ in Ω from Proposition 1.1. Similarly, if (u,v) is a critical point of $\tilde{I}_\infty(u,v)$, then (u,v) is a solution of (0.5). We set

$$\Sigma = \{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \mid |||u|||_{\lambda_1}^2 + |||v|||_{\lambda_2}^2 = 1\},$$
$$\Sigma_+ = \{(u, v) \in \Sigma \mid |||u_-|||_{\lambda_1} < 1\},$$
$$N = \{(u, v) \in \Sigma \mid u_+ v \not\equiv 0\}.$$

For $\beta \in (0, \infty]$, we define a functional $\tilde{J}_\beta(u, v)$ as follows.

Proposition 6.1. *Suppose $\beta \in (0, \infty]$. For any $(u, v) \in \Sigma_+$ if $\beta < \infty$, $(u, v) \in N$ if $\beta = \infty$, a function*

$$t \mapsto \tilde{I}_\beta(tu, tv) : \mathbf{R}_+ \to \mathbf{R}$$

has a unique maximum point

$$\tilde{t}_\beta(u, v) = \begin{cases} \sqrt{\beta}(\mu_1\|u_+\|_4^4 + \mu_2\|v\|_4^4 + 2\beta\|u_+ v\|_2^2)^{-\frac{1}{2}}, & \beta \in (0, \infty), \\ (\sqrt{2}\|u_+ v\|_2)^{-1}, & \beta = \infty. \end{cases}$$

Moreover, setting

$$\tilde{J}_\beta(u, v) = \sup_{t>0} \tilde{I}_\beta(tu, tv) = \begin{cases} \frac{\beta}{4(\mu_1\|u_+\|_4^4 + \mu_2\|v\|_4^4 + 2\beta\|u_+ v\|_2^2)}, & \beta \in (0, \infty), \\ \frac{1}{8}\|u_+ v\|_2^2, & \beta = \infty, \end{cases}$$

we have:

(i) *$\tilde{t}_\beta(u, v) : \Sigma_+ \to \mathbf{R}_+$ is a C^1-function.*
(ii) *$\tilde{J}_\beta(u, v) : \Sigma_+ \to \mathbf{R}$ is a C^1-function.*
(iii) *If $(u, v) \in \Sigma_+$ is a critical point of $\tilde{J}_\beta(u, v)$, then $(\tilde{t}_\beta(u, v)u, \tilde{t}_\beta(u, v)v)$ is a non-trivial critical point of $\tilde{I}_\beta(u, v)$.*
(iv) *$\tilde{J}_\beta(u, v)$ satisfies PS-condition.*

Proof. For $\beta \in (0, \infty]$, from direct calculations, we can write $\tilde{t}_\beta(u, v)$ and $\tilde{J}_\beta(u, v)$ explicitly. Thus, from those representations, we see that (i)–(ii) hold. (iii)–(iv) also are very standard. □

We seek critical points of $\tilde{J}_\beta$ in $N = \{(u, v) \in \Sigma, \mid u_+ v \not\equiv 0\} \subset \Sigma_+$.

Proposition 6.2. *When $\beta \in (0, \infty]$, we have*

$$\liminf_{(u,v)\in N,\, \mathrm{dist}\{(u,v), \partial N\} \to 0} \tilde{J}_\beta(u, v) = \beta b_0. \tag{6.1}$$

Here b_0 was given in (1.9) and, when $\beta = \infty$, we regard βb_0 as ∞. In particular, $\tilde{J}_\beta(u, v) < \beta b_0$ implies $u_+ v \neq 0$.

Proof. When $\beta = \infty$, (6.1) clearly holds. Thus we suppose $\beta \in (0, \infty)$. For any sequence $((u_n, v_n))_{n=1}^\infty \subset N$ with $\|u_{n+} v_n\|_2 \to 0$ $(n \to \infty)$, we should show $\liminf_{n\to\infty} \tilde{J}_\beta(u_n, v_n) \geqslant \beta b_0$. Since $|||u_n|||_{\lambda_1} + |||v_n|||_{\lambda_2} = 1$, there exist subsequence $n_j \to \infty$ and some $u_0, v_0 \in H_0^1(\Omega)$ such that

$$u_{n_j +} \to 0, \qquad v_{n_j} \to v_0 \quad \text{strongly in } L^4(\Omega).$$

Here if $u_0 = v_0 = 0$, then it is obvious that $\lim_{n_j\to\infty} \tilde{J}_\beta(u_{n_j}, v_{n_j}) = \infty$. On the other hand, if $u_0 = 0$, $v_0 \neq 0$, we have

$$\lim_{n_j\to\infty} \tilde{J}_\beta(u_{n_j}, v_{n_j}) = \frac{\beta}{4\mu_2\|v_0\|_4^4} \geqslant \beta b_0.$$

Thus we assume $(u_0, v_0) \in V = \{u, v \in \Sigma \mid u_+ \neq 0, v \neq 0, uv = 0\}$. Then we can also show

$$\inf_{(u,v)\in V} \tilde{J}_\beta(u, v) = \inf_{(u,v)\in \partial V} \tilde{J}_\beta(u, v) \geqslant \beta b_0. \tag{6.2}$$

In fact, letting $(u_*, v_*) \in V$ be a minimizer of $\inf_{(u,v)\in V} \tilde{J}_\beta(u, v)$, then (u_*, v_*) is a solution of $(*)$ with $\beta = 0$ and $\tilde{J}_\beta(u_*, v_*) \geqslant 2\beta b_0$. Thus $\inf_{(u,v)\in V} \tilde{J}_\beta(u, v)$ does not have minimizers in V and we get (6.2). Thus we get Proposition 6.2. □

Next, we give the proofs of Theorem 0.6 and Theorem 0.7. To show these theorems, we need the following lemma.

Lemma 6.3. *For any given $k \in \mathbf{N}$, there exist $\overline{\beta}_k \in (0, \infty)$ and $\psi \in C(S^k, N)$ with $\psi(-u) = \sigma(\psi(u))$ such that*

$$\sup_{u \in S^k} \tilde{J}_\beta\big(\psi(u)\big) \leqslant \beta b_0 \quad \text{for all } \beta \in (\overline{\beta}_k, \infty]. \tag{6.3}$$

Proof. Let W_k be k-dimensional subspaces of $H_0^1(\Omega)$ such that $W_1 \subset W_2 \subset \cdots \subset W_k \subset W_{k+1} \subset \cdots$. For any given $k \in \mathbf{N}$, we set $S^k := \{u \in W_k \mid |||u|||_{\lambda_2} = 1\}$ and define $\psi(u) : S^k \to N$ by

$$\psi(u) = \left(\frac{|u|}{\sqrt{2}|||u|||_{\lambda_1}}, \frac{u}{\sqrt{2}} \right).$$

Here we choose $\overline{\beta}_k$ satisfying

$$\overline{\beta}_k b_0 \geqslant \sup_{u \in S^k} \frac{|||u|||_{\lambda_1}^2}{2\|u\|_4^4}.$$

Then $\psi(u)$ satisfies $\psi(-u) = \sigma(\psi(u))$ and

$$\tilde{J}_\beta\big(\psi(u)\big) \leqslant \frac{|||u|||_{\lambda_1}^2}{2\|(|u|u)\|_2^2} \leqslant \overline{\beta}_k b_0 \quad \text{for all } \beta \in (\overline{\beta}_k, \infty].$$

Thus we get Lemma 6.3. □

Now, we show Theorem 0.6.

Proof of Theorem 0.6. From Lemma 6.3, for any given $k \in \mathbf{N}$, there exist $\overline{\beta}_k > 0$ and $\psi \in C(S^k, N)$ with $\psi(-v) = \sigma(\psi(v))$ such that

$$\sup_{v \in S^k} J_\beta\big(\psi(v)\big) \leqslant \beta b_0 \quad \text{for all } \beta > \overline{\beta}_k.$$

Thus, from Theorem 2.5, $\tilde{J}_\beta$ has at least k critical values $e_{1,\beta} \leqslant e_{2,\beta} \leqslant \cdots \leqslant e_{k,\beta}$. Here $e_{i,\beta}$ is defined as follows:

$$e_{i,\beta} = \inf\big\{c \in \mathbf{R} \bigm| \gamma\big([\tilde{J}_\beta \leqslant c]_N\big) \geqslant i\big\}. \tag{6.4}$$

Let $(u_{i,\beta}, v_{i,\beta})$ be a critical point corresponding to critical value $e_{i,\beta}$ of $\tilde{J}_\beta(u, v)$. We set $(U_{i,\beta}, V_{i,\beta}) = (\frac{1}{\sqrt{\beta}} t_\beta(u_{i,\beta}, v_{i,\beta}) u_{i,\beta}, \frac{1}{\sqrt{\beta}} t_\beta(u_{i,\beta}, v_{i,\beta}) v_{i,\beta})$. Then $(U_{i,\beta}, V_{i,\beta})$ are solutions of $(*)$ and we get Theorem 0.6. □

Finally, we give the proof of Theorem 0.7.

Proof of Theorem 0.7. Firstly we remark that $\liminf_{(u,v) \in N, \operatorname{dist}\{(u,v), \partial N\} \to 0} \tilde{J}_\infty(u, v) = \infty$ from Proposition 6.2. And, from Lemma 6.3, for any $k \in \mathbf{N}$, there exists $\psi \in C(S^k, N)$ with $\psi(-v) = \sigma(\psi(v))$. Thus, from Theorem 2.2, $\tilde{J}_\infty$ has a sequence of critical values $(e_{k,\infty})_{k=1}^\infty$ such that $e_{k,\infty} \to \infty$ as $k \to \infty$. Here $e_{k,\infty}$ is defined by

$$e_{k,\infty} = \inf\big\{c \in \mathbf{R} \bigm| \gamma\big([\tilde{J}_\infty \leqslant c]_N\big) \geqslant k\big\}. \tag{6.5}$$

Let (u_k, v_k) be a critical point of $\tilde{J}_\infty$ corresponding to $e_{k,\infty}$ and we set $(U_k, V_k) = (\tilde{t}_\infty(u_k, v_k)u_k, \tilde{t}_\infty(u_k, v_k)u_k)$. Then (U_k, V_k) is a solution of (0.5) and we find

$$\begin{aligned} \|U_{k+} V_k\|_\infty^2 |\Omega| \geqslant \|U_{k+} V_k\|_2^2 &= \tilde{t}_\infty(u_k, v_k)^4 \|u_{k+} v_k\|_2^2 \\ &= 8\tilde{J}_\infty(u_k, v_k) = 8e_{k,\infty} \to \infty \quad (k \to \infty). \end{aligned}$$

From the above inequality, we get $\|U_k\|_\infty + \|V_k\|_\infty \to \infty$. Moreover, from observation in Remark 0.9, when $\lambda_1 = \lambda_2$, v_k must change sign for large k. From the above results, the proof of Theorem 0.7 is complete. □

7. The asymptotic behavior as $\beta \to \infty$

In this section, we consider the asymptotic behavior of solutions which were given in Theorem 0.6 as $\beta \to \infty$. In what follows, we fix a $k \in \mathbf{N}$ and let $(u_{k,\beta}, v_{k,\beta})$ be a family of critical points of $\tilde{J}_\beta(u, v)$ corresponding to critical value $e_{k,\beta}$. Here $e_{k,\beta}$ was defined in (6.4). The following theorem is the main theorem in this section.

Theorem 7.1. *There exists a subsequence $\beta_j \to \infty$ such that*

$$(u_{k,\beta_j}, v_{k,\beta_j}) \to (u_{k,\infty}, v_{k,\infty}) \quad \text{in } H_0^1(\Omega) \times H_0^1(\Omega).$$

Here $(u_{k,\infty}, v_{k,\infty})$ is a critical point of $\tilde{J}_\infty(u, v)$ corresponding to the critical value $e_{k,\infty}$. Here $e_{k,\infty}$ was defined in (6.5).

We remark that Theorem 0.10 easily follows from Theorem 7.1.

Proof of Theorem 0.10. For the $(u_{k,\beta_j}, v_{k,\beta_j})$ and $(u_{k,\infty}, v_{k,\infty})$ in Theorem 7.1, we set

$$(U_{k,\beta_j}, V_{k,\beta_j}) = \left(\frac{1}{\sqrt{\beta_j}} t_{\beta_j}(u_{k,\beta_j}, v_{k,\beta_j}) u_{k,\beta_j}, \frac{1}{\sqrt{\beta_j}} t_{\beta_j}(u_{k,\beta_j}, v_{k,\beta_j}) v_{k,\beta_j} \right),$$
$$(U_{k,\infty}, V_{k,\infty}) = \left(t_\infty(u_{k,\infty}, v_{k,\infty}) u_{k,\infty}, t_\infty(u_{k,\infty}, v_{k,\infty}) v_{k,\infty} \right).$$

Then $(U_{k,\beta_j}, V_{k,\beta_j})$ are solutions of $(*)$ obtained in Theorem 0.6 and $(\sqrt{\beta_j} U_{k,\beta_j}, \sqrt{\beta_j} V_{k,\beta_j})$ converges to $(U_{k,\infty}, V_{k,\infty})$ which is a solution of (0.5) corresponding to critical value $e_{k,\infty}$. These complete the proof of Theorem 0.10. □

In the rest of this section, we will show Theorem 7.1. We need the following lemmas.

Lemma 7.2. *For any $M > 0$, we have*

$$\frac{1}{8M} - \frac{1}{2\beta}(\mu_1 + \mu_2) C_1^4 \leqslant \|u_+ v\|_2^2 \leqslant C_1^4 \quad \text{for all } (u, v) \in [J_\beta \leqslant M]_N. \tag{7.1}$$

Here C_1 was given in (1.1).

Proof. Since $J_\beta(u, v) \leqslant M$ is equivalent to $\frac{1}{8M} \leqslant \frac{1}{\beta}(\mu_1 \|u_+\|_4^4 + \mu_2 \|v\|_4^4) + 2\|u_+ v\|_2^2$, we easily get Lemma 7.2. □

Lemma 7.3. *For any $M > 0$ and $\epsilon > 0$, there exists $\beta''_M = \beta''_M(\epsilon) > 0$ such that, for all $\beta \geqslant \beta''_M$, we have*

$$J_\beta(u, v) < J_\infty(u, v) \leqslant J_\beta(u, v) + \epsilon \quad \text{for all } (u, v) \in [J_\beta \leqslant M]_N.$$

Proof. By a direct computation, we have

$$\tilde{J}_\beta(u, v) = \tilde{J}_\infty(u, v) - \frac{\mu_1 \|u_+\|_4^4 + \mu_2 \|v\|_4^4}{8(\mu_1 \|u_+\|_4^4 + \mu_2 \|v\|_4^4 + 2\beta \|u_+ v\|_2^2) \|u_+ v\|_2^2}. \tag{7.2}$$

Thus it is trivial that $\tilde{J}_\beta(u, v) < \tilde{J}_\infty(u, v)$. For any $M > 0$, from Lemma 7.2, $J_\beta(u, v) \leqslant M$ implies (7.1). From (7.1) and (7.2), we get Lemma 7.3. □

To show Theorem 7.1, the following proposition is essential.

Proposition 7.4. *We have $e_{k,\beta} \leqslant e_{k,\infty}$ and*

$$e_{k,\beta} \to e_{k,\infty} \quad \text{as } \beta \to \infty.$$

Here $e_{k,\beta}$ and $e_{k,\infty}$ were defined in (6.4) and (6.5) respectively.

Proof. Firstly, we show $e_{k,\beta} \leqslant e_{k,\infty}$. Since $\tilde{J}_\beta(u,v) < \tilde{J}_\infty(u,v)$, we have $[J_\infty \leqslant c]_N \subset [J_\beta \leqslant c]_N$ for any $c \in \mathbf{R}$. From the definitions of $e_{k,\infty}$ and (i) of Lemma 2.3, for any $\epsilon > 0$, we have

$$\gamma\big([J_\beta \leqslant e_{k,\infty} + \epsilon]_N\big) \geqslant \gamma\big([J_\infty \leqslant e_{k,\infty} + \epsilon]_N\big) \geqslant k.$$

This implies $e_{k,\beta} \leqslant e_{k,\infty} + \epsilon$ and, since $\epsilon > 0$ is arbitrary, we get $e_{k,\beta} \leqslant e_{k,\infty}$. Next we show $e_{k,\beta} \to e_{k,\infty}$ as $\beta \to \infty$. From Lemma 7.3, for $M = e_{k,\infty} + 1$ and any $\epsilon \in (0, \frac{1}{2})$, there exists $\beta''_M > 0$ such that for all $\beta \geqslant \beta''_M$ we have $[J_\beta \leqslant c_{k,\beta} + \epsilon]_N \subset [J_\infty \leqslant c_{k,\beta} + 2\epsilon]_N$. From the definitions of $e_{k,\beta}$ and (i) of Lemma 2.3, we get

$$\gamma\big([J_\infty \leqslant e_{k,\beta} + 2\epsilon]_N\big) \geqslant \gamma\big([J_\beta \leqslant e_{k,\beta} + \epsilon]_N\big) \geqslant k.$$

Thus we have $e_{k,\infty} \leqslant e_{k,\beta} + 2\epsilon$ for all $\beta \geqslant \beta''_M$. Combining $e_{k,\beta} \leqslant e_{k,\infty}$, we get $e_{k,\beta} \to e_{k,\infty}$ as $\beta \to \infty$. □

Now we give the proof of Theorem 7.1.

Proof of Theorem 7.1. Let $(u_{k,\beta}, v_{k,\beta})$ be a family of critical points of $\tilde{J}_\beta(u,v)$ corresponding to critical value $e_{k,\beta}$. Then there exist $u_{k,\infty}, v_{k,\infty} \in H_0^1(\Omega)$ and a subsequence $(\beta_j)_{j=1}^\infty$ such that

$$u_{k,\beta_j} \to u_{k,\infty}, \qquad v_{k,\beta_j} \to v_{k,\infty} \quad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^4(\Omega) \text{ as } \beta_j \to \infty.$$

Here, from Lemma 7.2, we see that $(u_{k,\infty})_+ \neq 0, v_{k,\infty} \neq 0$. Thus $t_{\beta_j}(u_{k,\beta_j}, v_{k,\beta_j})$ also converges to $t_\infty = (\sqrt{2}\|(u_{k,\infty})_+ v_{k,\infty}\|_2^2)^{-1}$ and $(t_\infty u_{k,\infty}, t_\infty v_{k,\infty})$ is a critical point of $\tilde{I}_\infty(u,v)$. Now, if we show

$$|||u_{k,\infty}|||_{\lambda_1} + |||v_{k,\infty}|||_{\lambda_2} = 1, \tag{7.3}$$

then our proof is complete. In fact, if (7.3) holds, then u_{k,β_j} and v_{k,β_j} strongly converge to $u_{k,\infty}$ and $v_{k,\infty}$ in $H_0^1(\Omega)$, respectively. Moreover, from Proposition 7.4, we have $\tilde{J}_\infty(u_{k,\infty}, v_{k,\infty}) = e_{k,\infty}$. Thus Theorem 7.1 obviously holds.

We will show (7.3). Since $(t_\infty u_{k,\infty}, t_\infty v_{k,\infty})$ is a critical point of $\tilde{I}_\infty(u,v)$, we have

$$|||u_{k,\infty}|||_{\lambda_1} + |||v_{k,\infty}|||_{\lambda_2} = 2t_\infty^2 \big\|(u_{k,\infty})_+ v_{k,\infty}\big\|_2^2. \tag{7.4}$$

On the other hand, from the representation of $t_{\beta_j}(u_{k,\beta_j}, v_{k,\beta_j})$ in Proposition 6.1, we have

$$1 = t_{\beta_j}(u_{k,\beta_j}, v_{k,\beta_j})^2 \left[\frac{1}{\beta}\big(\mu_1\|u_{k,\beta_j}\|_4^4 + \mu_2\|v_{k,\beta_j}\|_4^4\big) + 2\big\|(u_{k,\beta_j})_+ v_{k,\beta_j}\big\|_2^2\right]. \tag{7.5}$$

From (7.4) and (7.5), we get (7.3) and Theorem 7.1 holds. □

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