

Non-collapsing in fully non-linear curvature flows [☆]

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Abstract

We consider compact, embedded hypersurfaces of Euclidean spaces evolving by fully non-linear flows in which the normal speed of motion is a homogeneous degree one, concave or convex function of the principal curvatures, and prove a non-collapsing estimate: Precisely, the function which gives the curvature of the largest interior ball touching the hypersurface at each point is a subsolution of the linearized flow equation if the speed is concave. If the speed is convex then there is an analogous statement for exterior balls. In particular, if the hypersurface moves with positive speed and the speed is concave in the principal curvatures, the curvature of the largest touching interior ball is bounded by a multiple of the speed as long as the solution exists. The proof uses a maximum principle applied to a function of two points on the evolving hypersurface. We illustrate the techniques required for dealing with such functions in a proof of the known containment principle for flows of hypersurfaces.

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1. Introduction

Let M^n be a connected, compact manifold, and $X : M^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$ a family of smooth embeddings evolving by a curvature flow

$$\frac{\partial X}{\partial t} = -F\nu, \quad (1)$$

where ν is the unit normal, and the speed F is produced from the second fundamental form and the metric by evaluating a smooth, symmetric, homogeneous degree one, monotone increasing function of the principal curvatures. We require that this function be defined on a symmetric, convex cone $\Gamma \subset \mathbb{R}^n$. Thus F is determined completely by the components h_{ij} of the second fundamental form with respect to any orthonormal frame. In fact,¹ as a function of the

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¹ See [10] and the references therein.

matrix h_{ij} , F is smooth, homogeneous of degree one, strictly monotone increasing, and independent of the chosen frame. Noting that the embedding $X(\cdot, t)$ separates \mathbb{R}^{n+1} into an open, precompact enclosed region, Ω_t , and a non-compact exterior, we specify that the unit normal ν be chosen to point out of Ω_t . Having fixed the orientation in this way, we will assume below that F is either concave or convex with respect to the component matrix of the second fundamental form. The purpose of the paper is to prove a non-collapsing result for such flows, analogous to the result proved for the mean curvature flow by the first author in [3]. We expect that this will provide a key step towards understanding the singular behaviour of such flows for non-convex solutions: In the case of the mean curvature flow, the monotonicity formula of Huisken [16] provides a lot of information about the structure of singularities, and this is complemented by the asymptotic convexity results of Huisken and Sinestrari [18,19], and the differential Harnack or Li–Yau–Hamilton type inequality proved by Hamilton [12]. The latter is available for a large class of flows [2], but there are no analogues of the monotonicity formula or the asymptotic convexity result. The non-collapsing estimate does not precisely replace either of these, but seems a useful tool which may be used in their stead.

The δ -non-collapsing estimate proved for the mean curvature flow in [3] amounts to the statement that every point of the evolving hypersurface is touched by interior and exterior balls with radii equal to a constant δ divided by the mean curvature H . It was shown there that this non-collapsing for interior balls is equivalent to the inequality

$$\|X(x, t) - X(y, t)\|^2 \geq \frac{2\delta}{H(x, t)} \langle X(x, t) - X(y, t), \nu(x, t) \rangle$$

for all $x, y \in M$. Equivalently, this amounts to the inequality

$$Z(x, y, t) := \frac{2\langle X(x, t) - X(y, t), \nu(x, t) \rangle}{\|X(x, t) - X(y, t)\|^2} \leq \frac{H(x, t)}{\delta} \tag{2}$$

for all $(x, y) \in (M \times M) \setminus D$, where D is the diagonal $D = \{(x, x) : x \in M\}$. Note that the supremum of the left-hand side of (2) over y gives the geodesic curvature of the largest interior ball which touches at x (see Section 2 where this statement is made precise). Below we will formulate a non-collapsing result for more general curvature flows in terms of this quantity.

Definition 1. The *interior ball curvature* $\bar{Z}(x, t)$ at the point (x, t) is defined by $\bar{Z}(x, t) = \sup\{Z(x, y, t) : y \in M, y \neq x\}$. The *exterior ball curvature* $\underline{Z}(x, t)$ at the point (x, t) is defined by $\underline{Z}(x, t) = \inf\{Z(x, y, t) : y \in M, y \neq x\}$.

In the results to be described, an important role will be played by an equation we call the *linearized flow*. To motivate this consider a smooth family of solutions $X : M \times [0, T) \times (-a, a) \rightarrow \mathbb{R}^{n+1}$, and define $f : M \times [0, T) \rightarrow \mathbb{R}$ by $f(x, t) = \langle \frac{\partial}{\partial s}(X(x, t, s))|_{s=0}, \nu(x, t) \rangle$. Then f satisfies the equation

$$\frac{\partial f}{\partial t} = \dot{F}^{kl} \nabla_k \nabla_l f + \dot{F}^{kl} h_k^p h_{pl} f. \tag{3}$$

Here \dot{F}^{kl} is the derivative of F with respect to the components h_{kl} of the second fundamental form, defined by $\dot{F}^{kl}|_A B_{kl} = \frac{d}{ds}(F(A + sB))|_{s=0}$ for any symmetric B . Particular solutions of (3) include the speed F (see [1, Theorem 3.7]), corresponding to time translation $X(x, t, s) = X(x, t + s)$, the functions $\langle \nu(x, t), \vec{e} \rangle$ for $\vec{e} \in \mathbb{R}^{n+1}$ fixed, corresponding to spatial translations $X(x, t, s) = X(x, t) + s\vec{e}$, and the function $\langle \nu(x, t), X(x, t) \rangle + 2tF(x, t)$ (see [25] or [10, Theorem 14]), corresponding to the scalings $X(x, s, t) = (1 + s)X(x, (1 + s)^{-2}t)$.

To formulate our main result we need to recall the notion of viscosity subsolution or supersolution for parabolic equations: If M is a manifold with (possibly time-dependent) connection ∇ and $v : M \times [0, T) \rightarrow \mathbb{R}$ is continuous, then v is a viscosity subsolution of the equation $\frac{\partial u}{\partial t} = G(x, t, u, \nabla u, \nabla^2 u)$ if for every $(x_0, t_0) \in M \times [0, T)$ and every C^2 function ϕ on $M \times [0, T)$ such that $\phi(x_0, t_0) = v(x_0, t_0)$ and $\phi(x, t) \geq v(x, t)$ for x in a neighbourhood of x_0 and for $t \leq t_0$ sufficiently close to t_0 , it is true that $\frac{\partial \phi}{\partial t} \leq G(x, t, \phi, \nabla \phi, \nabla^2 \phi)$ at the point (x_0, t_0) . The function v is a viscosity supersolution if the same holds with both inequalities for ϕ reversed.

Our main result is the following:

Theorem 2. Assume that M is connected and $X : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ is an embedded solution of (1). If F is convex then \underline{Z} is a viscosity supersolution of the linearized flow (3). If F is concave then \bar{Z} is a viscosity subsolution of (3).

Note that in many cases the assumption of embeddedness need only be made on the initial hypersurface (see the remarks at the end of Section 3). Before we prove Theorem 2, we mention an important consequence:

Corollary 3. *If F is convex and X is an embedded solution of the curvature flow (1) with positive F , then $\inf_M \frac{Z(x,t)}{F(x,t)}$ is non-decreasing in t . If F is concave and X is an embedded solution of (1) with positive F , then $\sup_M \frac{\bar{Z}(x,t)}{F(x,t)}$ is non-increasing in t .*

Proof. Since F satisfies Eq. (3) the result reduces to a simple comparison property of viscosity subsolutions and supersolutions. We include the argument here for completeness: Assume F is convex, and for each t let $\phi(t) = \inf_{x \in M} \frac{Z(x,t)}{F(x,t)}$. We must show that ϕ is non-decreasing in t . We will accomplish this by proving that $\underline{Z}(x, t) - (\phi(t_0) - \varepsilon e^{t-t_0})F(x, t) \geq 0$ for any $t_0 \in [0, T)$, $t \in [t_0, T)$ and $\varepsilon > 0$. Taking the limit $\varepsilon \rightarrow 0$ then gives $\underline{Z}(x, t) \geq \phi(t_0)F(x, t)$ and hence $\phi(t) \geq \phi(t_0)$ for $t \geq t_0$.

Fix $t_0 \in [0, T)$ and $\varepsilon > 0$. Then $\underline{Z}(x, t_0) - (\phi(t_0) - \varepsilon)F(x, t_0) \geq \varepsilon F(x, t_0) > 0$ for all x , so if $\underline{Z} - (\phi(t_0) - \varepsilon e^{t-t_0})F$ does not remain positive for $t > t_0$ then there exists a time $t_1 > t_0$ and a point $x_1 \in M$ such that $\underline{Z} - (\phi(t_0) - \varepsilon e^{t-t_0})F$ is non-negative on $M \times [t_0, t_1]$, but $\underline{Z}(x_1, t_1) - (\phi(t_0) - \varepsilon e^{t_1-t_0})F(x_1, t_1) = 0$. Since \underline{Z} is a supersolution of Eq. (3), we have at this point

$$\begin{aligned} 0 &\leq \frac{\partial}{\partial t} ((\phi(t_0) - \varepsilon e^{t-t_0})F) - \dot{F}^{kl} \nabla_k \nabla_l ((\phi(t_0) - \varepsilon e^{t-t_0})F) - (\phi(t_0) - \varepsilon e^{t-t_0})F \dot{F}^{kl} h_k^p h_{pl} \\ &= -\varepsilon e^{t-t_0} F + (\phi(t_0) - \varepsilon e^{t-t_0}) (\dot{F}^{kl} \nabla_k \nabla_l F + \dot{F}^{kl} h_k^p h_{pl}) \\ &\quad - \dot{F}^{kl} \nabla_k \nabla_l ((\phi(t_0) - \varepsilon e^{t-t_0})F) - (\phi(t_0) - \varepsilon e^{t-t_0})F \dot{F}^{kl} h_k^p h_{pl} \\ &= -\varepsilon e^{t-t_0} F \\ &< 0, \end{aligned}$$

a contradiction proving that $\underline{Z} - (\phi(t_0) - \varepsilon e^{t-t_0})F$ remains positive. The argument for F concave is similar. \square

Corollary 3 is equivalent to the statement that the interior (for F concave) or exterior (for F convex) of the evolving hypersurfaces remains δ -non-collapsed on the scale of F , in the sense of [3].

We remark here that the interpretation of the non-collapsing estimate via subsolutions and supersolutions of the linearized flow (3) gives a new perspective even for the mean curvature flow. Indeed, our proof is quite different from that in [3], and rather more transparent.

2. The geometry of Z

We make precise here the meaning of upper and lower bounds on Z for fixed $x \in M$:

Proposition 4. *Fix $t \in [0, T)$ and $x \in M$. Then $\bar{Z}(x, t) \leq k$ if and only if there exists an open subset B of Ω_t with connected smooth boundary having all principal curvatures equal to k with respect to the outer normal, and such that $X(x, t) \in \partial B$. Similarly, $\underline{Z}(x, t) \geq k$ if and only if there exists an open set B with connected smooth boundary having all principal curvatures equal to k with respect to the outer normal, and such that $\Omega_t \subset B$ and $X(x, t) \in \partial B$.*

Thus a positive upper bound $\bar{Z}(x, t) \leq k$ amounts to an enclosed ball of radius $1/k$ touching at $X(x, t)$; a negative lower bound $\underline{Z}(x, t) \geq -k$ is equivalent to an exterior ball of radius $1/k$ touching at $X(x, t)$; a zero lower bound $\underline{Z}(x, t) \geq 0$ is equivalent to the statement that $X(M, t)$ is contained in a half-space touching at $X(x, t)$; and a positive lower bound $\underline{Z}(x, t)$ is equivalent to the statement that $X(M, t)$ is contained in a ball of radius $1/k$ which touches at $X(x, t)$.

Proof of Proposition 4. An upper bound $\bar{Z}(x, t) \leq k$ is equivalent to the statement $Z(x, y, t) \leq k$ for all $y \neq x$. By definition of Z this gives

$$2\langle X(x, t) - X(y, t), \nu(x, t) \rangle \leq k \|X(x, t) - X(y, t)\|^2$$

for all $y \neq x$. Dividing by k and completing the square this gives

$$\left\| X(y, t) - \left(X(x, t) - \frac{1}{k}v(x, t) \right) \right\|^2 \geq \frac{1}{k^2}$$

for all $y \neq x$. Setting $p = X(x, t) - \frac{1}{k}v(x, t)$, this says precisely that $X(M, t) \cap B_{1/k}(p) = \emptyset$ and $X(x, t) \in \partial B_{1/k}(p)$. That is, there is an enclosed ball of curvature k touching at $X(x, t)$. For the converse we note that the exterior normal vector of an enclosed ball touching at $X(x, t)$ agrees with $v(x, t)$, and work backwards though the above argument.

If $Z(x, t) \geq 0$ then $X(x, y, t) \geq 0$ for all $y \neq x$, so that $\langle X(x, t) - X(y, t), v(x, t) \rangle \geq 0$ for all $y \neq x$. Thus $X(M, t)$ lies in the closed half-space $H = \{z: z \cdot e \leq c\}$ where $e = v(x, t)$ and $c = X(x, t) \cdot v(x, t)$. Again $X(x, t)$ lies in the boundary of H .

The remaining cases are similar. \square

3. Interlude: the containment principle

The proof of the main theorem uses computations of the second derivatives of the function Z over the product $M \times M$, and involves a careful choice of coefficients particularly in the mixed second derivatives. We note that there are many precedents for computations of this sort: Kruřkov [23] applied maximum principles to the difference of values at two points for solutions of parabolic equations in one space variable; for elliptic problems quantities such as this were used by Korevaar [22], Kennington [21] and Kawohl [20] to derive a variety of convexity properties of solutions. For parabolic equations estimates on the modulus of continuity have been developed in [7,8] and were applied in [9,24] to eigenfunctions and heat kernels. In geometric flow problems related ideas appear in work on the curve-shortening problem by Huisken [17] and Hamilton [13] and on Ricci flow by Hamilton [14]. More recent refinements of these techniques appear in [4–6].

Before proving the main result, we illustrate some of the techniques involved in a simpler problem: The containment principle for solutions of fully non-linear curvature flows of hypersurfaces. For this problem we can consider speeds F which need not be homogeneous of degree one, and need not be either convex or concave:

Theorem 5. *Assume that F is an odd, increasing, symmetric function of the principal curvatures defined on $\Gamma \cup (-\Gamma)$, where $\Gamma \subset \mathbb{R}^n$ is a symmetric cone containing the positive cone, and $-\Gamma = \{-A: A \in \Gamma\}$. Let $X_i : M_i \times [0, T) \rightarrow \mathbb{R}^{n+1}$, $i = 1, 2$, be two compact solutions to (1) with $X_1(M_1, 0) \cap X_2(M_2, 0) = \emptyset$. Then the distance from $X_1(M_1, t)$ to $X_2(M_2, t)$ is non-decreasing, and in particular $X_1(M_1, t) \cap X_2(M_2, t) = \emptyset$ for $t \in [0, T)$.*

Proof. Define $d : M_1 \times M_2 \times [0, T) \rightarrow \mathbb{R}$ by

$$d(x, y, t) = \|X_1(x, t) - X_2(y, t)\|.$$

We show

$$\min_{M_1 \times M_2} d(\cdot, t) \geq \min_{M_1 \times M_2} d(\cdot, 0),$$

which is positive, since the initial hypersurfaces are disjoint. As notation we will also set

$$w(x, y, t) = \frac{X_1(x, t) - X_2(y, t)}{d(x, y, t)}$$

and write $\partial_i^x = \frac{\partial X_1}{\partial x_i}$ and $\partial_i^y = \frac{\partial X_2}{\partial y_j}$.

The function d evolves under (1) by

$$\frac{\partial}{\partial t} d = \langle w, -F_x v_x + F_y v_y \rangle. \tag{4}$$

Suppose there is a spatial minimum of d at (x_0, y_0, t_0) . Then at this point,

$$\nabla^{M_1 \times M_2} d = 0 \quad \text{and} \quad \text{Hess}^{M_1 \times M_2} d \geq 0.$$

Choosing local orthonormal coordinates on $M_1 \times M_2$ at (x_0, y_0, t_0) , that is, orthonormal coordinates $\{x^i\}$ at x_0 and orthonormal coordinates $\{y^i\}$ at y_0 we have

$$\nabla_j^{M_1} d = \langle \partial_j^x, w \rangle \quad \text{and} \quad \nabla_j^{M_2} d = -\langle \partial_j^y, w \rangle.$$

Since we assumed that F is odd, the flow is invariant under change of orientation and we can choose $v_x = v_y = w$. In view of the definition of w , we have at (x_0, y_0, t_0) that

$$\nabla_j^{M_1} w = \frac{1}{d} \partial_j^x \quad \text{and} \quad \nabla_j^{M_2} w = -\frac{1}{d} \partial_j^y. \tag{5}$$

For the second spatial derivatives of d we have

$$\begin{aligned} \nabla_i^{M_1} \nabla_j^{M_1} d &= \langle \nabla_i^{M_1} \nabla_j^{M_1} X_1, w \rangle + \langle \partial_j^x, \nabla_i^{M_1} w \rangle, \\ \nabla_i^{M_2} \nabla_j^{M_1} d &= \langle \partial_j^x, \nabla_i^{M_2} w \rangle \quad \text{and} \\ \nabla_i^{M_2} \nabla_j^{M_2} d &= -\langle \nabla_i^{M_2} \nabla_j^{M_2} X_2, w \rangle - \langle \partial_j^y, \nabla_i^{M_2} w \rangle. \end{aligned}$$

Using (5), at (x_0, y_0, t_0) these become

$$\begin{aligned} \nabla_i^{M_1} \nabla_j^{M_1} d &= \langle \nabla_i^{M_1} \nabla_j^{M_1} X_1, w \rangle + \frac{1}{d} g_{ij}^{M_1}, \\ \nabla_i^{M_2} \nabla_j^{M_1} d &= -\frac{1}{d} \langle \partial_j^x, \partial_i^y \rangle \quad \text{and} \\ \nabla_i^{M_2} \nabla_j^{M_2} d &= -\langle \nabla_i^{M_2} \nabla_j^{M_2} X_2, w \rangle + \frac{1}{d} g_{ij}^{M_2}. \end{aligned}$$

We derive the following at (x_0, y_0, t_0) : For any vector v we have

$$\begin{aligned} 0 &\leq v^i v^j (\nabla_i^{M_1} \nabla_j^{M_1} d + 2\nabla_i^{M_2} \nabla_j^{M_1} d + \nabla_i^{M_2} \nabla_j^{M_2} d) \\ &= -h^x_{ij} v^i v^j \langle v_x, w \rangle + \frac{1}{d} g_{ij}^{M_1} v^i v^j + h^y_{ij} v^i v^j \langle v_y, w \rangle + \frac{1}{d} g_{ij}^{M_2} v^i v^j - \frac{2}{d} v^i v^j \langle \partial_i^x, \partial_j^y \rangle. \end{aligned}$$

Since $w = v_x = v_y$, the local coordinates near x and y may be chosen such that $\partial_i^x = \partial_i^y$ for all i and $g_{ij}^{M_1} = g_{ij}^{M_2} = \delta_{ij}$. The above becomes

$$h^x_{ij} v^i v^j \leq h^y_{ij} v^i v^j,$$

or since v is arbitrary, $h^x_{ij} \leq h^y_{ij}$. Finally, since F is monotone, we have $F_x \leq F_y$, and hence by (4) we have

$$\frac{\partial d}{\partial t} = -F_x + F_y \geq 0. \quad \square$$

Remarks. (1) If F is as in Theorem 5, it can be shown using a similar argument as above that for compact solutions of (1) with embedded initial hypersurface, the evolving hypersurfaces remain embedded while the curvature remains bounded: Defining $d_{\mathbb{R}^{n+1}}(x, y, t) = \|X(x, t) - X(y, t)\|$, a curvature bound implies that there is a neighbourhood E of $D = \{(x, x) : x \in M\}$ in $M \times M$ such that

$$d_{\mathbb{R}^{n+1}}(x, y, \cdot) \geq C d_M(x, y).$$

We may then apply the argument for the containment principle on $(M \times M) \setminus E$ to conclude that embeddedness is preserved.

(2) In the containment principle the assumption that F is odd can be relaxed if we make an additional topological assumption on the hypersurfaces to guarantee the correct orientation: If we assume F is defined on an arbitrary symmetric cone Γ containing the positive cone, and $M_1 = \partial\Omega_1$ and $M_2 = \partial\Omega_2$ with $\Omega_1 \subset \Omega_2$, and require that the unit normal to M_i points out of Ω_i for $i = 1, 2$, then the above argument goes through with minor changes. Without such a condition disjointness may not be preserved: For example if $n = 2$ and $F = H + |A|$, with the cone $\Gamma = \{(\kappa_1, \kappa_2) : \max\{\kappa_1, \kappa_2\} > 0\}$, then surfaces with opposite orientation will move closer together (and can cross). In this example it is also true that embedded initial surfaces can evolve smoothly to become non-embedded.

4. Proof of the main theorem

We now prove Theorem 2, namely, that \underline{Z} (\bar{Z}) is a viscosity supersolution (subsolution) of the linearized flow (3) when if F is convex (concave).

As in the previous section, the proof involves computation with the second derivatives over the product $M \times M$. However, the computation here has an unexpected feature: In all the previous computations of this type mentioned above, the two points x and y have appeared in a symmetric way, so that the choice of coefficients in the second derivatives is determined by information at both points. This has been a serious obstacle to applications of the methods to fully non-linear flows, since the coefficients of the equation at different points would involve the second derivatives (or second fundamental form) at different points, and there is insufficient control on these to allow a useful comparison. However, in the present computation x and y play very different roles, and in particular the function Z only depends on x at the level of the highest derivatives. Accordingly we are able to use a choice of coefficients in the second derivatives which depends on x but not on y , thus removing any need to compare the second fundamental form at different points. The key observation that makes this choice work is given in Lemma 7.

Proof of Theorem 2. The definitions of $\bar{Z}(x, t)$ and $\underline{Z}(x, t)$ involve extrema of Z over the non-compact set $\{y \in M: y \neq x\}$. Accordingly we begin by extending Z to a continuous function on a suitable compactification.

The diagonal D is a compact submanifold of dimension and codimension n in $M \times M$. The normal subspace $N_{(x,x)}D$ of D at (x, x) is the subspace $\{(u, -u): u \in T_x M\} \subset T_{(x,x)}(M \times M)$. The tubular neighbourhood theorem provides $r > 0$ such that the exponential map is a diffeomorphism on $\{(x, x, u, -u) \in TM \times TM: 0 < \|u\| < r\}$. We ‘blow up’ along D to define a manifold with boundary \hat{M} which compactifies $(M \times M) \setminus D$ as follows: As a set, \hat{M} is the disjoint union of $(M \times M) \setminus D$ with the unit sphere bundle $SM = \{(x, v) \in TM: \|v\| = 1\}$. The manifold-with-boundary structure is defined by the atlas generated by all charts for $(M \times M) \setminus D$, together with the charts \hat{Y} from $SM \times (0, r)$ defined by taking a chart Y for SM , and setting $\hat{Y}(z, s) := (\exp(sY(z)), \exp(-sY(z)))$.

We extend the function Z to $\hat{M} \times [0, T]$ as follows: For $(x, y) \in (M \times M) \setminus D$ and $t \in [0, T]$ we define

$$Z(x, y, t) = \frac{2\langle X(x, t) - X(y, t), \nu(x, t) \rangle}{\|X(x, t) - X(y, t)\|^2}.$$

For $(x, v) \in SM$ we define

$$Z(x, v, t) = h_{(x,t)}(v, v),$$

where $h_{(x,t)}$ is the second fundamental form of M_t at x . Since X is an embedding, Z is continuous on $(M \times M) \setminus D$. A straightforward computation shows that the above extension of Z to \hat{M} is also continuous. It follows that $\bar{Z}(x, t)$ is attained on \hat{M} , in the sense that either there exists $y \in M \setminus \{x\}$ such that $\bar{Z}(x, t) = Z(x, y, t)$, or there exists $v \in T_x M$ with $\|v\| = 1$ such that $\bar{Z}(x, t) = Z(x, v, t)$. Also, since the supremum over $M \setminus \{x\}$ equals the supremum over \hat{M} , and this is no less than the supremum over the boundary SM , we have that $\bar{Z}(x, t)$ is no less than the maximum principal curvature $\kappa_{\max}(x, t)$. Similarly, $\underline{Z}(x, t)$ is attained on \hat{M} and is no greater than the minimum principal curvature $\kappa_{\min}(x, t)$.

To prove that \bar{Z} is a subsolution if F is concave, we consider, for an arbitrary point, (x_0, t_0) , an arbitrary C^2 function ϕ which lies above \bar{Z} on a neighbourhood of (x_0, t_0) in $M \times [0, t_0]$, with equality at (x_0, t_0) , and prove a differential inequality for ϕ at (x_0, t_0) .

Observe that for all x close to x_0 , and all $t \leq t_0$ close to t_0 we have $Z(x, y, t) \leq \bar{Z}(x, t) \leq \phi(x, t)$ for each $y \neq x$ in M , and $Z(x, v, t) \leq \bar{Z}(x, t) \leq \phi(x, t)$ for all $v \in S_x M$. Furthermore equality holds in the last inequality in both cases when $(x, t) = (x_0, t_0)$. By definition of \bar{Z} we either have $Z(x_0, y_0, t_0) = \bar{Z}(x_0, t_0)$ for some $y_0 \neq x_0$, or we have $Z(x_0, \xi_0, t_0) = \bar{Z}(x_0, t_0)$ for some $\xi_0 \in S_{x_0} M$.

We consider the latter case first: Define a smooth unit vector field ξ near (x_0, t_0) by choosing $\xi(x_0, t_0) = \xi_0$, extending to (x, t_0) for x close to x_0 by parallel translation along geodesics, and extending in the time direction by solving $\frac{\partial \xi}{\partial t} = F\mathcal{W}(\xi)$, where \mathcal{W} is the Weingarten map. This construction implies that $\nabla \xi(x_0, t_0) = 0$ and $\nabla^2 \xi(x_0, t_0) = 0$, and from the evolution equation for the second fundamental form we find that

$$\frac{\partial}{\partial t}(h(\xi, \xi)) = \dot{F}^{kl} \nabla_k \nabla_l (h(\xi, \xi)) + \dot{F}^{kl,pq} \nabla_\xi h_{kl} \nabla_\xi h_{pq} + h(\xi, \xi) \dot{F}^{kl} h_k^p h_{pl}$$

at the point (x_0, t_0) . The second term on the right is non-positive by the concavity of F . At the point (x_0, t_0) we also have $\phi = h(\xi, \xi)$, and since $\phi \geq h(\xi, \xi)$ at nearby points and earlier times we also have $\frac{\partial \phi}{\partial t} \leq \frac{\partial}{\partial t}(h(\xi, \xi))$ and $\nabla^2 \phi \geq \nabla^2(h(\xi, \xi))$ at this point. Combining these inequalities gives $\frac{\partial \phi}{\partial t} \leq \dot{F}^{kl} \nabla_k \nabla_l \phi + \phi \dot{F}^{kl} h_k^p h_{pl}$ at (x_0, t_0) as required.

Next we consider the case where $Z(x_0, y_0, t_0) = \phi(x_0, t_0)$ for some $y_0 \neq x_0$, and $\phi(x, t) \geq Z(x, y, t)$ for all points x near x_0 , times $t \leq t_0$ near t_0 , and arbitrary $y \neq x$ in M . This implies that $\frac{\partial \phi}{\partial t}(x_0, t_0) \leq \frac{\partial Z}{\partial t}(x_0, y_0, t_0)$, that the first spatial derivatives of $\phi - Z$ in x and y vanish at (x_0, y_0, t_0) and that the second spatial derivatives of $\phi - Z$ are non-negative at (x_0, y_0, t_0) . We compute these derivatives, working in local normal coordinates $\{x^i\}$ near x and $\{y^j\}$ near y . To simplify notation we define $d = |X(x, t) - X(y, t)|$ and $w = \frac{X(x,t) - X(y,t)}{d}$ and write $\partial_i^x = \frac{\partial X}{\partial x^i}$. We first compute the first spatial derivatives with respect to y :

$$\frac{\partial}{\partial y^i}(\phi - Z) = \frac{2}{d^2}(\partial_i^y \langle v_x - dZw \rangle). \tag{6}$$

This determines the tangent plane at y . In fact the following stronger statement holds:

Lemma 6. *At the point (x_0, y_0, t_0) , $v_y = v_x - dZw$.*

Proof. By Proposition 4, there is an interior ball B of radius $1/Z$ touching at $X(x, t)$ and $X(y, t)$. The outward normal to B at these points agrees with the outward normal to the hypersurface $X(M, t)$. In particular $v_y = Z(X(y, t) - (X(x, t) - 1/Zv_x)) = v_x - dZw$. \square

The first derivatives with respect to x are slightly more complicated:

$$\frac{\partial}{\partial x^i}(\phi - Z) = \frac{\partial \phi}{\partial x^i} - \frac{2}{d}(h^{xp}_i \langle w, \partial_p^x \rangle - Z \langle w, \partial_i^x \rangle). \tag{7}$$

The left, and therefore right, sides of Eqs. (6) and (7) vanish at (x_0, y_0, t_0) .

Now we differentiate further to find the second derivatives: Using the fact that the first derivatives of Z with respect to y vanish, we find

$$\begin{aligned} \frac{\partial^2}{\partial y^i \partial y^j}(\phi - Z) &= \frac{2}{d^2} \{ \langle h^y_{ij} v_y, dZw - v_x \rangle + Z \langle \partial_i^y, \partial_j^y \rangle \} \\ &= \frac{2}{d^2} (Z \delta_{ij} - h^y_{ij}). \end{aligned} \tag{8}$$

Differentiating (6) with respect to the x coordinates gives the mixed partial derivatives:

$$\frac{\partial^2}{\partial x^j \partial y^i}(\phi - Z) = -\frac{2}{d^2} (Z \delta_j^p - h^{xp}_j) \langle \partial_i^y, \partial_p^x \rangle - \frac{2}{d} \frac{\partial \phi}{\partial x^j} \langle w, \partial_i^y \rangle. \tag{9}$$

Differentiating (7) with respect to the x coordinates gives

$$\begin{aligned} \frac{\partial^2}{\partial x^i \partial x^j}(\phi - Z) &= \frac{2}{d^2} (Z \delta_{ij} - h^x_{ij}) + Zh^x_{jp} \delta^{pq} h^x_{qi} - \frac{2}{d} \nabla_p h^x_{ij} \delta^{pq} \langle w, \partial_q^x \rangle \\ &\quad - Z^2 h^x_{ij} + \frac{2}{d} \frac{\partial \phi}{\partial x^j} \langle w, \partial_i^x \rangle + \frac{2}{d} \frac{\partial \phi}{\partial x^i} \langle w, \partial_j^x \rangle + \frac{\partial^2 \phi}{\partial x^i \partial x^j}. \end{aligned} \tag{10}$$

Finally we compute the time derivative:

$$\begin{aligned} \frac{\partial}{\partial t}(\phi - Z) &= \frac{\partial \phi}{\partial t} + \frac{2F_x}{d^2} - \frac{2F_y}{d^2} \langle v_y, v_x - dZw \rangle - \frac{2}{d} \langle w, \nabla F_x \rangle - Z^2 F_x \\ &= \frac{\partial \phi}{\partial t} + \frac{2F_x}{d^2} - \frac{2F_y}{d^2} - \frac{2}{d} \langle w, \nabla F_x \rangle - Z^2 F_x. \end{aligned} \tag{11}$$

Combining Eqs. (8)–(11) and the inequalities at (x_0, y_0, t_0) we obtain

$$\begin{aligned}
 0 &\leq -\frac{\partial}{\partial t}(\phi - Z) + \dot{F}_x^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j}(\phi - Z) + 2\frac{\partial^2}{\partial x^i \partial y^j}(\phi - Z) + \frac{\partial^2}{\partial y^i \partial y^j}(\phi - Z) \right) \\
 &= -\frac{\partial \phi}{\partial t} + \dot{F}_x^{ij} \nabla_i \nabla_j \phi + \phi \dot{F}_x^{ij} h^x_{ip} \delta^{pq} h^x_{qj} - \frac{4F_x}{d^2} + \frac{4}{d^2} \dot{F}_x^{ij} h^x_{iq} \delta^{qp} \langle \partial_j^y, \partial_p^x \rangle \\
 &\quad + \frac{2F_y}{d^2} - \frac{2}{d^2} \dot{F}_x^{ij} h^y_{ij} + \frac{4Z}{d^2} \dot{F}_x^{ij} \delta_{ij} - \frac{4Z}{d^2} \dot{F}_x^{ij} \langle \partial_i^x, \partial_j^y \rangle + \frac{4}{d} \dot{F}_x^{ij} \frac{\partial \phi}{\partial x^i} \langle w, \partial_j^x - \partial_j^y \rangle.
 \end{aligned} \tag{12}$$

Now note that, by the homogeneity of F , $F_x = \dot{F}_x^{ij} h^x_{ij}$, so that

$$-\frac{4F_x}{d^2} + \frac{4}{d^2} \dot{F}_x^{ij} h^x_{iq} \delta^{qp} \langle \partial_j^y, \partial_p^x \rangle = -\frac{4}{d^2} \dot{F}_x^{ij} h^x_{iq} \delta^{qp} (\delta_{jp} - \langle \partial_j^y, \partial_p^x \rangle).$$

We can also write

$$\frac{4Z}{d^2} \dot{F}_x^{ij} \delta_{ij} - \frac{4Z}{d^2} \dot{F}_x^{ij} \langle \partial_i^x, \partial_j^y \rangle = \frac{4Z}{d^2} \dot{F}_x^{ij} (\delta_{ij} - \langle \partial_j^y, \partial_i^x \rangle).$$

To control the first two terms on the second line of (12) we use the following observation:

Lemma 7. *If F is concave, then for any $y \neq x$ we have*

$$\dot{F}_x^{ij} h^y_{ij} \geq F_y.$$

If F is convex, then the reverse inequality holds.

Proof. Let $A = h^x$ and $B = h^y$. Then concavity of F gives

$$F(B) \leq F(A) + \dot{F}_A(B - A) = F(A) + \dot{F}_A(B) - \dot{F}_A(A).$$

The homogeneity of F gives by the Euler relation that $\dot{F}_A(A) = F(A)$, yielding

$$F(B) \leq \dot{F}_A(B)$$

as claimed. The inequality is reversed for F convex. \square

Using these observations, together with the identity for $\frac{\partial \phi}{\partial x^i}$ coming from the vanishing of $\frac{\partial}{\partial x^i}(\phi - Z)$ in Eq. (7), we find

$$\begin{aligned}
 0 &\leq -\frac{\partial \phi}{\partial t} + \dot{F}_x^{ij} \nabla_i \nabla_j \phi + \phi \dot{F}_x^{ij} h^x_{ip} \delta^{pq} h^x_{qj} \\
 &\quad + \frac{4}{d^2} \dot{F}_x^{ij} (Z \delta_{ip} - h^x_{ip}) \delta^{pq} (\delta_{qj} - \langle \partial_j^y, \partial_q^x \rangle) + 2 \langle w, \partial_q^x \rangle \langle w, \partial_j^y - \partial_j^x \rangle.
 \end{aligned}$$

We now prove that the final term is non-positive, that is,

Lemma 8. *The term $\dot{F}_x^{ij} (Z \delta_{ip} - h^x_{ip}) \delta^{pq} (\delta_{qj} - \langle \partial_j^y, \partial_q^x \rangle) + 2 \langle w, \partial_q^x \rangle \langle w, \partial_j^y - \partial_j^x \rangle$ is non-positive.*

Proof. We now choose the local coordinates $\{x^i\}$ and $\{y^i\}$ more carefully. Throughout we continue to compute at the minimum (x_0, y_0, t_0) . Then we may choose ∂_n^y and ∂_n^x to be coplanar with w , and $\partial_i^y = \partial_i^x$ for $i = 1, \dots, n - 1$. This ensures that $\delta_{qj} - \langle \partial_j^y, \partial_q^x \rangle + 2 \langle w, \partial_q^x \rangle \langle w, \partial_j^y - \partial_j^x \rangle$ is non-zero only when $j = q = n$.

By Proposition 4, there is an interior ball B which touches the hypersurface at $X(x, t)$ and $X(y, t)$ and has exterior normal v_x at $X(x, t)$. By convexity of B we have $\langle X(x, t) - X(y, t), v(x, t) \rangle \geq 0$, and hence² $\langle w, v_x \rangle \geq 0$. Define $\alpha \in [0, \pi/2)$ by $\langle w, v_x \rangle = \sin \alpha$. Note that we have one final degree of freedom in the coordinates: the directions of

² In fact, if the normal is reversed – so that $\langle w, v_x \rangle \leq 0$ – a similar argument yields the same conclusion, namely inequality (14). This is an important observation, required in the proof of exterior non-collapsing for convex speeds below.

∂_n^x and ∂_n^y . Direct ∂_n^x such that $\langle w, \partial_n^x \rangle = -\cos \alpha$. Now define $\theta \in [0, \pi/2)$ and the orientation of ∂_n^y by the conditions $\langle \partial_n^y, \partial_n^x \rangle = -\cos 2\theta$ and $\langle \partial_n^y, \nu_x \rangle = \sin 2\theta$. Then the vanishing of $\partial_{y_n}(\phi - Z)$ implies

$$\langle \partial_n^y, \nu_x \rangle = 2\langle w, \nu_x \rangle \langle \partial_n^y, w \rangle \implies \sin 2\theta \cos 2\alpha = \sin 2\alpha \cos 2\theta. \tag{13}$$

That is, $\sin(2\theta - 2\alpha) = 0$ and we find $\theta = \alpha$. The identity (13) now implies that $\langle \partial_n^y, w \rangle = \cos \theta$ and we may compute

$$\begin{aligned} \delta_{qj} - \langle \partial_j^y, \partial_q^x \rangle + 2\langle w, \partial_q^x \rangle \langle w, \partial_j^y - \partial_j^x \rangle &= 1 + \cos(2\theta) + 2\cos \theta(-\cos \theta - \cos \theta) \\ &= 2\cos^2 \theta - 4\cos^2 \theta = -2\cos^2 \theta \leq 0. \end{aligned} \tag{14}$$

Now consider the coefficient matrix $\dot{F}_x^{ij}(Z\delta_{ip} - h^x_{ip})\delta^{pq}$ of the preceding term. Since $Z(x_0, y_0, t_0) = \bar{Z}(x_0, t_0) \leq \kappa_{\max}(x_0, t_0) := \max_{\xi \in S_{x_0} M} h(\xi, \xi)$, we find that the matrix $Z\delta_{ij} - h^x_{ij}$ is non-negative definite at (x_0, y_0, t_0) . In a frame which diagonalizes the second fundamental form at x , \dot{F}_x is also diagonal, so we see that $\dot{F}_x^{ij}(Z\delta_{ip} - h^x_{ip})\delta^{pq}$ is symmetric and non-negative definite. The result follows. \square

We can now conclude that

$$0 \leq -\frac{\partial \phi}{\partial t} + \dot{F}_x^{ij} \nabla_i \nabla_j \phi + \phi \dot{F}_x^{ij} h^x_{ip} g_x^{pq} h^x_{qj},$$

which completes the proof that \bar{Z} is a viscosity subsolution of (3).

Now consider the case that F is convex with respect to h_{ij} . Then the flow (1) is equivalent to the flow

$$\frac{\partial X}{\partial t} = -F_*(\lambda_{*1}, \dots, \lambda_{*n})\nu_*,$$

where ν_* is the inward pointing unit normal, $\lambda_{*i} = -\lambda_i$ are the principal curvatures with respect to ν_* , and F_* is defined by $F_*(z) := -F(-z)$. We therefore have

$$\begin{aligned} \dot{F}_*^{ij}(\lambda_{*1}, \dots, \lambda_{*n}) &= \dot{F}^{ij}(\lambda_1, \dots, \lambda_n) \quad \text{and} \\ \ddot{F}_*^{pq,rs}(\lambda_{*1}, \dots, \lambda_{*n}) &= -\ddot{F}^{pq,rs}(\lambda_1, \dots, \lambda_n). \end{aligned}$$

So F_* is concave with respect to the ‘inward pointing’ second fundamental form. Now consider

$$Z_* := \frac{\langle w, \nu_* \rangle}{d} = -Z.$$

Then the claim follows if we can show that $\underline{Z} = -\bar{Z}_*$ is a viscosity supersolution of (3). But this already follows from the calculations above, since all of the viscosity inequalities are reversed, as are the inequalities of Lemma 7 and Lemma 8; the latter following from the sign reversal of the term $Z\delta_{ip} - h^x_{ip}$. \square

5. Conclusions and remarks

We make some final remarks and mention here some immediate implications of the non-collapsing result:

- (1) Interior non-collapsing for concave F rules out blow-up limits such as the product of the grim reaper with \mathbb{R}^{n-1} (if the initial hypersurface has positive F), since this has the interior ball curvature \bar{Z} asymptotically constant while the speed F approaches zero, violating Corollary 3. The exterior non-collapsing does not appear to rule out this possibility. Note that without the assumption of embeddedness, such singularities do indeed occur, even in mean curvature flow.
- (2) In the case of mean curvature flow where both interior and exterior non-collapsing hold, we are able to deduce directly that for uniformly convex hypersurfaces all principal curvatures are comparable, implying a simple proof of the Huisken and Gage–Hamilton theorems on the asymptotic behaviour for convex solutions [15,11]. If only one-sided non-collapsing holds then we cannot immediately conclude such a strong result, but nevertheless the convergence arguments in the convex case become rather easy: For example, in the case where F is convex, we have $\underline{Z}(x, t) \geq \varepsilon F(x, t) \geq \varepsilon \kappa_{\max}(x, t)$, from which it follows that the circumradius (bounded by the reciprocal of $\underline{Z}(x, t)$ for any x) is bounded by ε^{-1} times the inradius. No such result holds in the case where F is concave,

however – this should not be surprising since there are examples of concave, homogeneous degree one functions F such that convex hypersurfaces can evolve to be non-convex under Eq. (1) (see [10, Example 1]).

- (3) As in the case of mean curvature flow, analogues of Corollary 3 hold with F replaced by any positive solution of the linearized flow (3). In particular we can allow star-shaped initial hypersurfaces even if F is not positive, by using the solution $\langle X, \nu \rangle + 2tF$ of (3).
- (4) If the assumption that M is connected is dropped, then some caution is required (similar considerations apply as in the containment principle of Section 3): If the hypersurface $X(M, 0)$ is the boundary of a region Ω_0 and the normal is everywhere pointing out of Ω , then the proof goes through unaltered. However the result may not apply if two connected components of the hypersurface bound a common region, with the normals pointing into the region in one component and out on the other.

References

- [1] Ben Andrews, Contraction of convex hypersurfaces in Euclidean space, *Calc. Var. Partial Differential Equations* 2 (2) (1994) 151–171.
- [2] Ben Andrews, Harnack inequalities for evolving hypersurfaces, *Math. Z.* 217 (2) (1994) 179–197.
- [3] Ben Andrews, Non-collapsing in mean-convex mean curvature flow, *Geom. Topol.*, in press, available at <http://arxiv.org/abs/1108.0247>.
- [4] Ben Andrews, Paul Bryan, Curvature bound for curve shortening flow via distance comparison and a direct proof of Grayson’s theorem, *J. Reine Angew. Math.* 653 (2011) 179–187.
- [5] Ben Andrews, Paul Bryan, Curvature bounds by isoperimetric comparison for normalized Ricci flow on the two-sphere, *Calc. Var. Partial Differential Equations* 39 (3–4) (2010) 419–428.
- [6] Ben Andrews, Paul Bryan, A comparison theorem for the isoperimetric profile under curve-shortening flow, *Comm. Anal. Geom.* 19 (3) (2011) 503–539.
- [7] Ben Andrews, Julie Clutterbuck, Lipschitz bounds for solutions of quasilinear parabolic equations in one space variable, *J. Differential Equations* 246 (11) (2009) 4268–4283.
- [8] Ben Andrews, Julie Clutterbuck, Time-interior gradient estimates for quasilinear parabolic equations, *Indiana Univ. Math. J.* 58 (1) (2009) 351–380.
- [9] Ben Andrews, Julie Clutterbuck, Proof of the fundamental gap conjecture, *J. Amer. Math. Soc.* 24 (2011) 899–916.
- [10] Ben Andrews, James McCoy, Yu Zheng, Contracting convex hypersurfaces by curvature, *Calc. Var. Partial Differential Equations* (2012), <http://dx.doi.org/10.1007/s00526-012-0530-3>, in press, available at <http://arxiv.org/abs/1104.0756>.
- [11] M. Gage, R.S. Hamilton, The heat equation shrinking convex plane curves, *J. Differential Geom.* 23 (1) (1986) 69–96.
- [12] Richard S. Hamilton, Harnack estimate for the mean curvature flow, *J. Differential Geom.* 41 (1) (1995) 215–226.
- [13] Richard S. Hamilton, Isoperimetric estimates for the curve shrinking flow in the plane, in: *Modern Methods in Complex Analysis*, Princeton, NJ, 1992, in: *Ann. of Math. Stud.*, vol. 137, Princeton Univ. Press, Princeton, NJ, 1995, pp. 201–222.
- [14] Richard S. Hamilton, An isoperimetric estimate for the Ricci flow on the two-sphere, in: *Modern Methods in Complex Analysis*, Princeton, NJ, 1992, in: *Ann. of Math. Stud.*, vol. 137, Princeton Univ. Press, Princeton, NJ, 1995, pp. 191–200.
- [15] Gerhard Huisken, Flow by mean curvature of convex surfaces into spheres, *J. Differential Geom.* 20 (1) (1984) 237–266.
- [16] Gerhard Huisken, Asymptotic behavior for singularities of the mean curvature flow, *J. Differential Geom.* 31 (1) (1990) 285–299.
- [17] Gerhard Huisken, A distance comparison principle for evolving curves, *Asian J. Math.* 2 (1) (1998) 127–133.
- [18] Gerhard Huisken, Carlo Sinestrari, Mean curvature flow singularities for mean convex surfaces, *Calc. Var. Partial Differential Equations* 8 (1) (1999) 1–14.
- [19] Gerhard Huisken, Carlo Sinestrari, Convexity estimates for mean curvature flow and singularities of mean convex surfaces, *Acta Math.* 183 (1) (1999) 45–70.
- [20] Bernhard Kawohl, Rearrangements and Convexity of Level Sets in PDE, *Lecture Notes in Math.*, vol. 1150, Springer-Verlag, Berlin, 1985.
- [21] Alan U. Kennington, Power concavity and boundary value problems, *Indiana Univ. Math. J.* 34 (3) (1985) 687–704.
- [22] Nicholas J. Korevaar, Convex solutions to non-linear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J.* 32 (4) (1983) 603–614.
- [23] S.N. Kružkov, Quasilinear parabolic equations and systems with two independent variables, *Trudy Sem. Petrovsk.* 5 (1979) 217–272 (in Russian).
- [24] Lei Ni, Estimates on the modulus of expansion for vector fields solving nonlinear equations, *J. Math. Pures Appl.* (2012), <http://dx.doi.org/10.1016/j.matpur.2012.05.009>, in press, available at <http://arxiv.org/abs/1107.2351>.
- [25] Knut Smoczyk, Starshaped hypersurfaces and the mean curvature flow, *Manuscripta Math.* 95 (2) (1998) 225–236.