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# Flat chains of finite size in metric spaces

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**Abstract**

In this paper we investigate the notion of flat current in the metric spaces setting, and in particular we provide a definition of size of a flat current with possibly infinite mass. Exploiting the special nature of the 0-dimensional slices and the theory of metric-space valued $BV$ functions we prove that a $k$-current with finite size $T$ sits on a countably $\mathscr{H}^k$-rectifiable set, denoted by $set(T)$. Moreover we relate the size measure of $T$ to the geometry of the tangent space $\mathrm{Tan}^{(k)}(set(T), x)$.


## 1. Introduction

When dealing with problems in the calculus of variations involving geometric objects as curves and surfaces, suitable spaces of currents proved to be the right weak ambient space where compactness and semicontinuity can be obtained at the same time, and still homological constraints make sense. This framework proved to be successful in solving Plateau's problem, namely finding a mass-minimizing current satisfying suitable boundary conditions, as documented in the seminal papers [18,17] and in the classical book [16]. Following an intuition of De Giorgi [11] the theory of currents has been extended to nonsmooth spaces by Ambrosio and Kirchheim in [7], where the duality with smooth differential forms is replaced by the duality with Lipschitz functions (see also [22] for a friendly exposition and a local variant of the theory). This framework, available in a fairly general class of metric spaces, allows to prove again existence of solutions to Plateau's problem for integral currents, and more generally the existence of mass-minimizing currents in a fixed homology class (see [27]).

If we move from normal currents to flat currents with finite mass, other remarkable extensions of the classical theory have been obtained among others by Fleming in [19], and White in [28,29], dealing with Euclidean spaces and general group coefficients, and by De Pauw and Hardt [15], dealing with general spaces and general group coefficients at the same time. In this connection see also [9,8], where coefficients in $\mathbf{Z}_p$ are dealt with also in metric spaces, using the idea of taking the quotients of integral currents.

In [12], De Lellis proved in the metric framework the rectifiability of the "lowest dimensional part" of a flat chain with finite mass and real coefficients. As an example, one might consider the distributional derivative $Du$ of a $BV$

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function in $\mathbf{R}^n$, that can canonically be viewed as a flat $(n-1)$-dimensional current with finite mass. In this case only the restriction of $Du$ to the so-called jump set of $u$ provides an $(n-1)$-rectifiable measure, while the remaining part of $Du$ is diffuse.

The main goal of the paper is to provide an analogous rectifiability result also for flat chains with possibly infinite mass. We are looking, in the same spirit of [12], only to the rectifiability of the lowest dimensional part of $T$, defined in a suitable sense (different from [12], since the mass need not to be finite). For us, the main motivation for the study of flat chains with infinite mass has been the generalization of the theory of Mumford–Shah type functionals and special functions of bounded variation (see [3] and the references therein) to codimension higher than one. In this extension, that will be developed in [20], in order to get good semicontinuity and compactness properties it is useful to model the Jacobian current $Ju$ as the sum of a current with finite mass (describing the absolutely continuous part) and a current with finite size (describing the singularities).

The paper is organized as follows. In Section 2 we will carry out a thorough presentation of the space of flat $k$-currents $\mathbf{F}_k(E)$ in the metric context: in particular we will remark how many useful properties enjoyed by normal currents extend to this larger space.

Section 3 is devoted to the definition of the concentration measure $\mu_T$ for flat currents and the size functional $\mathbf{S}(T)=\mu_T(E)$, the main objects of our investigation. They are defined through an integral-geometric approach that involves only the 0-dimensional slices of the current which are required, in the case of finite size, to have a finite support (and, as a consequence, finite mass, according to Theorem 3.3). Then, we prove lower semicontinuity of size with respect to flat convergence, obtaining in particular a closure property for sequences of currents with equibounded size. In the Euclidean setting different notions of size, constructed for instance by relaxation or implementing the techniques of multiple valued functions, have been studied by several authors, and can be found in [1,14,23,13] together with the properties of currents satisfying suitable size bounds.

In Section 4 we introduce a quantity $\mathcal{G}(T,T')$, called hybrid distance, in the class $\mathbf{B}_0(E)$ of flat boundaries with finite mass: it takes into account all representations $T-T'=\partial(X+R)$, with $X$ having finite mass and $R$ having finite size. This results in a smaller distance, compared to the classical one where no $R$ term is present, which allows to extend the $BV$ estimates for the slice operator from currents with finite mass to currents with finite size. Here we use the flexibility of these $BV$ estimates, namely the possibility to adapt them to several classes of "geometric" distances (see for instance [24]). The distance $\mathcal{G}$, though weaker than the classical flat distance, will be proved to be still sufficiently strong to control the oscillations of the atoms of the slices. In order to show the separability of $(\mathbf{B}_0(E),\mathcal{G})$, we will use some results from the theory of optimal transportation in geodesic spaces, see for instance [4].

Since we aim to prove a rectifiability result, we recall in Section 5 the concept of rectifiable set and the main features of the theory of functions of bounded variation taking values in metric spaces introduced in [5]. In particular we will extensively use the concept of approximate upper limit of the difference quotient as a tool to measure the slope of a function: along the lines of [9,28] we can turn pointwise control of this slope into Lipschitz estimates on a family of sets which exhaust almost all the domain (see Theorem 5.3 for the precise statement).

The main result of our paper is described in the following rectifiability theorem, proved in Section 6:

**Theorem 6.1** *(Rectifiability of currents of finite size). For every flat current $T\in\mathbf{F}_k(E)$ with finite size the measure $\mu_T$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set. The least one, up to $\mathscr{H}^k$-null sets, is given by*

$$set(T):=\left\{x\in E:\ \limsup_{r\downarrow 0}\frac{\mu_T(B_r(x))}{r^k}>0\right\}.$$

This result is established first for 1-dimensional currents, and then extended to the general case via an iterated slicing procedure, along the lines of [9,28] but using the distance $\mathcal{G}$ adapted to our problem.

In the last Section 7 we compare $\mu_T$ to $\mathscr{H}^k \llcorner set(T)$. Along the lines of [7,6], we are able to describe the density $\lambda(x)$ of $\mu_T$ w.r.t. $\mathscr{H}^k \llcorner set(T)$ in terms of the geometry of the approximate tangent space $\mathrm{Tan}^{(k)}(set(T),x)$. In the Euclidean case, the factor $\lambda$ is equal to 1.

## 2. Notation and basic properties of flat chains

### 2.1. Flat currents in a metric space

Our ambient space will be a boundedly compact geodesic space $(E, d)$ (i.e. bounded closed sets are compact). Notice that any such space is separable. We will use the standard notation $B_r(x)$ for the open balls in $E$, $\mathrm{Lip}(E)$ for the space of real-valued Lipschitz functions and $\mathrm{Lip}_b(E)$ for bounded Lipschitz functions. In this paper we will adopt the concept of current in a metric space developed in [7,6,8,22]: a metric $k$-current $T$ is a map

$$T : \mathrm{Lip}_b(E) \times \left[\mathrm{Lip}(E)\right]^k \to \mathbf{R}$$

defined on $(k+1)$-tuples $(f, \pi_1, \dots, \pi_k)$ satisfying the following properties of multilinearity, continuity and locality introduced in [7]:

(i) $T$ is multilinear in $(f, \pi_1, \dots, \pi_k)$,
(ii) $\lim_i T(f, \pi_1^i, \dots, \pi_k^i) = T(f, \pi_1, \dots, \pi_k)$ whenever $\pi_k^i \to \pi^i$ pointwise in $E$ with $\mathrm{Lip}(\pi_j^i) \leqslant C$,
(iii) $T(f, \pi_1, \dots, \pi_k) = 0$ if for some $i \in \{1, \dots, k\}$ the function $\pi_i$ is constant in a neighborhood of $\{f \neq 0\}$.

It can be proved that these three properties imply that the map $T$ is alternating in the $(\pi_1, \dots, \pi_k)$ variables, hence we use the more expressive notation $f\,d\pi_1 \wedge \dots \wedge d\pi_k$ for the generic argument. The concepts of boundary operator $\partial$, mass $\mathbf{M}$ and push forward of a current in the metric context are taken for granted, and as customary we denote by $\mathbf{M}_k(E)$ the space of finite mass $k$-dimensional currents with real coefficients and by $\mathbf{N}_k(E)$ the subspace of normal currents. We recall that the action of currents with finite mass can canonically be extended to $f\,d\pi$ with $f$ bounded Borel and $\pi_1, \dots, \pi_k$ Lipschitz.

For every $k$-current $T$ we let

$$\mathbf{F}(T) = \inf\left\{\mathbf{M}(T - \partial Y) + \mathbf{M}(Y)\colon\ Y \in \mathbf{M}_{k+1}(E)\right\} \tag{2.1}$$

be its flat norm. It is a straightforward calculation to show that $\mathbf{F}$ is a norm on $\mathbf{M}_k(E)$, and that

$$\mathbf{F}(\partial T) \leqslant \mathbf{F}(T) \leqslant \mathbf{M}(T). \tag{2.2}$$

Our primary space of currents is the following:

**Definition 2.1.** We define the space of flat currents $\mathbf{F}_k(E)$ as the $\mathbf{F}$-completion of the space of normal currents:

$$\mathbf{F}_k(E) = \widehat{\mathbf{N}_k(E)}^{\mathbf{F}}.$$

The first inequality in (2.2) immediately implies that if $T \in \mathbf{F}_k(E)$ then $\partial T \in \mathbf{F}_{k-1}(E)$. Recall also that any flat current $T$ of finite mass can be approximated by a sequence $(Z_h)$ of normal currents in mass norm. In fact, by definition there exist currents $(T_h) \subset \mathbf{N}_k(E)$ and $(Y_h) \subset \mathbf{M}_{k+1}(E)$ such that

$$\mathbf{M}(T - T_h - \partial Y_h) + \mathbf{M}(Y_h) \to 0.$$

The hypothesis $\mathbf{M}(T) < \infty$ yields $\mathbf{M}(\partial Y_h) < \infty$, hence the currents $Z_h = T_h + \partial Y_h$ are normal and clearly $\mathbf{M}(T - Z_h) \to 0$. As we will see later on, many properties of the space of normal currents behave nicely under convergence in the flat norm (2.1) and therefore can be extended to the completion. On the other hand, every definition involving a completion procedure somehow hides the true nature of the objects under consideration. The following proposition partially overcomes this inconvenience:

**Proposition 2.2.** *(See [16, 4.1.24].) The space of flat k-currents can be characterized as*

$$\mathbf{F}_k(E) = \left\{X + \partial Y\colon\ X \in \mathbf{F}_k(E),\ Y \in \mathbf{F}_{k+1}(E),\ \mathbf{M}(X) + \mathbf{M}(Y) < \infty\right\}. \tag{2.3}$$

**Proof.** We need only to show that $\mathbf{F}_k(E)$ is contained in the right-hand side, as the opposite inclusion follows by additivity and stability of flat currents under the boundary operator. Let $(T_h) \subset \mathbf{N}_k(E)$ be a sequence of normal currents fastly converging towards $T \in \mathbf{F}_k(E)$: $\sum_h \mathbf{F}(T_{h+1} - T_h) < \infty$. There exist normal currents $X_h$ and $Y_h$ such that

$$T_{h+1} - T_h = X_h + \partial Y_h \quad \text{and} \quad \mathbf{M}(X_h) + \mathbf{M}(Y_h) < 2\mathbf{F}(T_{h+1} - T_h).$$

The $\mathbf{M}$-converging series $\sum_h X_h$ and $\sum_h Y_h$ define two flat currents, respectively $X \in \mathbf{F}_k(E)$ and $Y \in \mathbf{F}_{k+1}(E)$, of finite mass such that $T - T_0 = X + \partial Y$. □

### 2.2. Restriction and slicing

Let us recall the definition of slicing: given $T \in \mathbf{N}_k(E)$ and $u \in \mathrm{Lip}(E)$, the slicing of $T$ via $u$ is defined as

$$\langle T, u, r\rangle = \partial\big(T \llcorner \{u < r\}\big) - (\partial T) \llcorner \{u < r\}$$

and belongs to $\mathbf{N}_{k-1}(E)$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$. We will sometimes write $T_r = \langle T, u, r\rangle$ to shorten the writing and to emphasize the dependence of the slice on the variable $r$. The slices $\langle T, u, r\rangle$ are uniquely determined, up to Lebesgue negligible sets, by the following two properties (see [7, Theorem 5.7]):

$$\langle T, u, r\rangle \text{ is concentrated on } u^{-1}(r) \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbf{R}; \tag{2.4}$$

$$\int_{\mathbf{R}} \phi(r)\langle T, u, r\rangle\, dr = T \llcorner (\phi \circ u)\, du \quad \text{for all } \phi \text{ bounded Borel.} \tag{2.5}$$

In [8,6] (and [16, 4.2.1] for the classical case in Euclidean space), it is proved that these families of currents enjoy the following estimates:

$$\int_a^{*b} \mathbf{F}\big(T \llcorner \{u < r\}\big)\, dr \leqslant \big(b - a + \mathrm{Lip}(u)\big)\mathbf{F}(T), \tag{2.6}$$

$$\int_a^{*b} \mathbf{F}\big(\langle T, u, r\rangle\big)\, dr \leqslant \mathrm{Lip}(u)\mathbf{F}(T), \tag{2.7}$$

where $\int^*$ denotes the outer integral.

**Proposition 2.3.** *The operations of restriction and slicing via a Lipschitz map can be extended to the space of flat currents in such a way that* $\sum_h \mathbf{F}(T_h - T) < \infty$ *implies*

$$\mathbf{F}\big(\langle T_h, u, r\rangle - \langle T, u, r\rangle\big) \to 0 \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbf{R}.$$

*Moreover, inequalities* (2.6) *and* (2.7) *hold for a generic* $T \in \mathbf{F}_k(E)$.

**Proof.** Let $T \in \mathbf{F}_k(E)$ and let $(T_h)$ be a sequence of normal currents rapidly converging to $T$: $\sum_h \mathbf{F}(T_h - T) < \infty$. Thanks to the subadditivity of the outer integral it is fairly easy to show that for $\mathscr{L}^1$-a.e. $r$ both sequences $(T_h \llcorner \{u < r\})$ and $(\langle T_h, u, r\rangle)$ are $\mathbf{F}$-Cauchy, hence they admit a limit. Note that these limits do not depend on the particular $(T_h)$ we choose: if $(T'_h)$ were another sequence rapidly converging to $T$, we could merge it with $(T_h)$ setting $T''_{2h} = T_h$, $T''_{2h+1} = T'_h$. Then $(T''_h)$ would have converging restrictions and slices for almost every $r$. Therefore the limits

$$\lim_h T_h \llcorner \{u < r\} \quad \text{and} \quad \lim_h T'_h \llcorner \{u < r\}$$

must agree for a set of values $r$ of full measure; similarly for the sequence of slices $(\langle T_h, u, r\rangle)$. Finally we write $T$ as an $\mathbf{F}$-convergent sum of normal currents

$$T = T_N + \sum_{h=N}^{\infty} (T_{h+1} - T_h) \quad \text{with} \quad \sum_{h=N}^{\infty} \mathbf{F}(T_{h+1} - T_h) < \varepsilon.$$

Hence, since $\mathbf{F}(T_N) \leqslant \mathbf{F}(T) + \varepsilon$, applying (2.6) and the subadditivity of the upper integral

$$\int_a^{*b} \mathbf{F}\big(T \llcorner \{u<r\}\big)\,dr \leqslant \int_a^{*b} \mathbf{F}\big(T_N \llcorner \{u<r\}\big)\,dr + \sum_{h=N}^{\infty} \int_a^{*b} \mathbf{F}\big((T_{h+1}-T_h) \llcorner \{u<r\}\big)\,dr \overset{(2.6)}{\leqslant} \big(b-a+\mathrm{Lip}(u)\big)\big(\mathbf{F}(T)+2\varepsilon\big)$$

we prove the thesis. The statement for (2.7) can be proved in the same way. □

Proposition 2.3 allows us to extend many properties of slicing and restriction from normal currents to flat currents by density.

First of all, given $\ell \leqslant k$ the slicing of a current $T \in \mathbf{F}_k(E)$ by a vector-valued map $\pi = (\pi^1, \dots, \pi^\ell) \in \mathrm{Lip}(E, \mathbf{R}^\ell)$ can be defined inductively:

$$\langle T, \pi, x\rangle = \big\langle \big\langle T, \big(\pi^1, \dots, \pi^{\ell-1}\big), (x_1, \dots, x_{l-1})\big\rangle, \pi^\ell, x_\ell\big\rangle.$$

Fubini's theorem ensures that these iterated slices are meaningful for $\mathscr{L}^\ell$-a.e. $x \in \mathbf{R}^\ell$, and it is easy to show by induction that $\partial\langle T, u, r\rangle = (-1)^\ell \langle \partial T, u, r\rangle$. In particular, for every $u \in \mathrm{Lip}(E)$ slicing and boundary operator commute via the relation

$$\partial\langle T, u, r\rangle = -\langle \partial T, u, r\rangle \quad \text{for } \mathscr{L}^1\text{-a.e. } r \in \mathbf{R}. \tag{2.8}$$

**Lemma 2.4** *(Slice and restriction commute). Let $T \in \mathbf{F}_k(E)$, $\pi, u \in \mathrm{Lip}(E)$. Then*

$$\langle T, \pi, r\rangle \llcorner \{u<s\} = \big\langle T \llcorner \{u<s\}, \pi, r\big\rangle \quad \text{for } \mathscr{L}^2\text{-a.e. } (r,s) \in \mathbf{R}^2. \tag{2.9}$$

**Proof.** We start with $T \in \mathbf{N}_k(E)$. It is immediate to check that, for $s$ fixed, the currents in the left-hand side of (2.9) fulfill (2.4) and (2.5) relative to $T \llcorner \{u<s\}$, therefore they coincide with $\langle T \llcorner \{u<s\}, \pi, r\rangle$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$. Let now $T$ be flat and let $(T_h) \subset \mathbf{N}_k(E)$ with $\sum_h \mathbf{F}(T_h - T) < \infty$: we want to pass to the limit in the identity

$$\langle T_h, \pi, r\rangle \llcorner \{u<s\} = \big\langle T_h \llcorner \{u<s\}, \pi, r\big\rangle \quad \text{for } \mathscr{L}^2\text{-a.e. } (r,s) \in \mathbf{R}^2. \tag{2.10}$$

We know that $\sum_h \mathbf{F}(T_h \llcorner \{u<s\} - T \llcorner \{u<s\}) < \infty$ for $\mathscr{L}^1$-a.e. $s \in \mathbf{R}$; for any such $s$ by Proposition 2.3 we can plug $(T_h - T) \llcorner \{u<r\}$ into inequality (2.7) and infer that the right-hand sides in (2.10) converge to $\langle T \llcorner \{u<s\}, \pi, r\rangle$ with respect to $\mathbf{F}$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$. On the other hand, we also know that $\sum_h \mathbf{F}(\langle T_h, \pi, r\rangle - \langle T, \pi, r\rangle) < \infty$ for $\mathscr{L}^1$-a.e. $r \in \mathbf{R}$; for any $r$ for which this property holds the left-hand sides in (2.10) converge with respect to $\mathbf{F}$ to $\langle T_h, \pi, r\rangle \llcorner \{u<s\}$ for $\mathscr{L}^1$-a.e. $s \in \mathbf{R}$, again by Proposition 2.3 and Eq. (2.6). Therefore, passing to the limit as $h \to \infty$ in (2.10), using Fubini's theorem, we conclude. □

**Lemma 2.5** *(Set additivity of restrictions). Let $T \in \mathbf{F}_k(E)$, $\pi_1, \pi_2 \in \mathrm{Lip}(E)$ and $\bar t \in \mathbf{R}$ such that the sets $\{\pi_1 < \bar t\}$ and $\{\pi_2 < \bar t\}$ have positive distance. Let $\pi := \min\{\pi_1, \pi_2\}$. Then*

$$T \llcorner \{\pi < t\} = T \llcorner \{\pi_1 < t\} + T \llcorner \{\pi_2 < t\} \quad \text{for a.e. } t < \bar t. \tag{2.11}$$

**Proof.** Let $t < \bar t$. Since $\{\pi_1 < t\}$ and $\{\pi_2 < t\}$ are distant the function

$$\psi(x) = \frac{d(x, \{\pi_1 < t\})}{d(x, \{\pi_1 < t\}) + d(x, \{\pi_2 < t\})}$$

is Lipschitz and equals 0 in $\{\pi_1 < t\}$ and 1 in $\{\pi_2 < t\}$. Let $(T_h)$ be a sequence of normal currents rapidly converging to $T$ such that

$$\sum_h \mathbf{F}\big(T_{h+1} \llcorner \{\pi < t\} - T_h \llcorner \{\pi < t\}\big) < \infty.$$

Then the sequence $S_h = \psi T_h \llcorner \{\pi < t\} = T_h \llcorner \{\pi_2 < t\}$ satisfies

$$\mathbf{F}(S_{h+1} - S_h) \leqslant \max\big\{\|\psi\|_\infty, \mathrm{Lip}(\psi)\big\}\mathbf{F}\big(T_{h+1} \llcorner \{\pi < t\} - T_h \llcorner \{\pi < t\}\big),$$

hence $S_h$ converges to $T \llcorner \{\pi_2 < t\}$ in the flat norm. Similarly for $T \llcorner \{\pi_1 < t\}$. Eq. (2.11) holds for normal currents, and since the same sequence $(T_h)$ is used to define the three restrictions, set additivity is straightforward by passing to the limit. □

*2.3. Support and push forward*

We adopt (see also [2]) as definition of support of a flat current $T$ the set:

$$\operatorname{spt}(T) = \{x \in E:\ T \llcorner B_r(x) \neq 0 \text{ for } \mathscr{L}^1\text{-a.e. } r > 0\}. \tag{2.12}$$

Observe that the a.e. in the definition is motivated by the fact that slices exist only up to $\mathscr{L}^1$-negligible sets, and that $\operatorname{spt}(T) = \operatorname{spt}\|T\|$ whenever $T \in \mathbf{M}_k(E)$.

**Proposition 2.6.** $\operatorname{spt}(T)$ *is a closed set and* $x \notin \operatorname{spt}(T)$ *implies* $T \llcorner B_r(x) = 0$ *for a.e.* $r \in (0, \operatorname{dist}(x, \operatorname{spt}(T)))$.

**Proof.** Let $x \notin \operatorname{spt}(T)$: there must be a set $A$ of radii of positive $\mathscr{L}^1$-measure such that $T \llcorner B_r(x) = 0$ for $\mathscr{L}^1$-a.e. $r \in A$. If $(T_h) \subset \mathbf{N}_k(E)$ is a sequence of normal currents rapidly converging to $T$, by (2.6) we obtain that for almost every $r \in A$

$$T_h \llcorner B_r(x) \to T \llcorner B_r(x) = 0 \tag{2.13}$$

rapidly. Fix now $r > 0$ with this property, $y \in B_r(x)$ and $\rho < r - d(x,y)$: we want to prove that $T \llcorner B_\rho(y) = 0$ for a.e. $\rho \in (0, r - d(x,y))$. Since $T_h$ has finite mass we have $(T_h \llcorner B_r(x)) \llcorner B_\rho(y) = T_h \llcorner B_\rho(y)$, and since the convergence in (2.13) is rapid, again for almost every $\rho$ in $(0, r - d(x,y))$

$$T_h \llcorner B_\rho(y) = \big(T_h \llcorner B_r(x)\big) \llcorner B_\rho(y) \to \big(T \llcorner B_r(x)\big) \llcorner B_\rho(y) = 0 \llcorner B_\rho(y) = 0.$$

Hence $B_r(x) \cap \operatorname{spt}(T) = \emptyset$. □

**Proposition 2.7.** *For all* $T \in \mathbf{F}_k(E)$ *the following properties hold*:

(i) *If* $\operatorname{spt}(f)$ *is compact then*

$$\operatorname{spt}(f) \cap \operatorname{spt}(T) = \emptyset \quad \Longrightarrow \quad T(f\,d\pi) = 0 \quad \forall \pi \in \operatorname{Lip}\big(E, \mathbf{R}^k\big). \tag{2.14}$$

(ii) *For all* $u \in \operatorname{Lip}(E)$

$$\operatorname{spt}\big(T \llcorner \{u < t\}\big) \subset \operatorname{spt}(T) \cap \{u \leqslant t\} \quad \textit{for } \mathscr{L}^1\textit{-a.e. } t \in \mathbf{R}. \tag{2.15}$$

**Proof.** (i) In this proof only, let us conventionally say that $T \llcorner B_r(x_0) = 0$ if there exist normal currents $T_n$ such that $\mathbf{F}(T_n - T) \to 0$ and $\mathbf{F}(T_n \llcorner B_r(x_0)) \to 0$. In the proof of (2.14), we assume first that $\operatorname{spt}(f)$ is contained in a ball $B_r(x)$ such that $T \llcorner B_r(x) = 0$. By assumption $T_h \llcorner B_r(x_0)(f) \to 0$ for suitable approximating normal currents $T_h$; on the other hand, $T_h \llcorner B_r(x_0)(f) = T_h(f) \to T(f)$.

Now, let us consider the general case. Since the space is boundedly compact, we can find an open bounded neighborhood $U$ of $\operatorname{spt}(f)$ such that $\overline{U} \cap \operatorname{spt}(T) = \emptyset$. By Proposition 2.6, any $x \in \overline{U}$ is the center of a ball $B_x$ such that $T \llcorner B_x = 0$: we can therefore extract a finite subcover $\{B_j\}$ and build a partition of unity $\{\chi_j\}$ made of nonnegative Lipschitz functions such that $\sum_j \chi_j = 1$ in $\operatorname{spt}(f)$ and $\operatorname{spt}(\chi_j) \subset B_j$. Hence $f = \sum_j f\chi_j$ and the previous step yields

$$T(f\,d\pi) = \sum_j T(f\chi_j\,d\pi) = 0.$$

(ii) There exist $(T_h) \subset \mathbf{N}_k(E)$ such that for almost every $t$

$$\sum_h \mathbf{F}\big(T_h \llcorner \{u < t\} - T \llcorner \{u < t\}\big) < \infty.$$

We fix $t$ with these properties and $x \notin \operatorname{spt}(T) \cap \{u \leqslant t\}$, so that $\operatorname{spt}(T) \cap \{u \leqslant t\} \cap B_r(x) = \emptyset$ for $r \in (0, \bar r)$ for some $\bar r > 0$. We obtain that

$$0 = \big(T_h \llcorner \{u < t\}\big) \llcorner B_r(x) \to \big(T \llcorner \{u < t\}\big) \llcorner B_r(x)$$

for a.e. $r \in (0, \bar r)$, hence $x \notin \operatorname{spt}(T \llcorner \{u < t\})$. □

Given a Lipschitz map $\Phi : E \to F$ between two metric spaces and given $T \in \mathbf{F}_k(E)$ we let

$$(\Phi_\# T)(f\,d\pi) = T\big(f \circ \Phi\,d(\pi \circ \Phi)\big)$$

be the push forward of $T$ via the map $\Phi$.

**Proposition 2.8.** *For every $T \in \mathbf{F}_k(E)$ it holds $\Phi_\# T \in \mathbf{F}_k(F)$ and*

$$\mathbf{F}_F(\Phi_\# T) \leqslant \big[\mathrm{Lip}(\Phi)\big]^k \mathbf{F}_E(T).$$

*In particular, $\Phi_\# T \in \mathbf{F}(F)$. Furthermore, the push forward and the boundary operator commute.*

**Proof.** Since $\mathbf{F}_F(\Phi_\# S) \leqslant [\mathrm{Lip}(\Phi)]^k \mathbf{F}_E(S)$ holds for $S$ normal, the current $\Phi_\# T$ is flat and the estimate holds also for flat currents. The relation $\partial \Phi_\# T = \Phi_\# \partial T$ simply comes from the definition. □

Let us recall now the definition of supremum of a family of measures.

**Definition 2.9.** Let $\{\mu_i\}_{i \in I}$ be a family of Borel positive measures on $E$. Then, for every Borel subset of $E$, we define

$$\bigvee_{i \in I} \mu_i(B) = \sup\left\{ \sum_{i \in J} \mu_i(B_i):\ B_i \text{ pairwise disjoint and Borel}, \bigcup_{i \in J} B_i = B \right\}, \tag{2.16}$$

where $J$ runs through all countable subsets of $I$.

The set function $\bigvee_{i \in I} \mu_i$ is a Borel measure, and it is finite if and only if there exists a finite Borel measure $\sigma \geqslant \mu_i$ for any $i$. Notice that in (2.16) it would be equivalent to consider finite partitions of $B$ into Borel sets $B_1, \ldots, B_N$.

## 3. Concentration measure for a flat chain

In this section we introduce the notion of concentration measure for a flat current, possibly with infinite mass.

**Definition 3.1** *(Concentration measure).* We say that a positive Borel measure $\mu$ is a concentration measure for $T \in \mathbf{F}_k(E)$ if $\mathscr{H}^0 \llcorner \mathrm{spt}(T) \leqslant \mu$ in the case $k = 0$, and if

$$\mu_{T,\pi} := \int_{\mathbf{R}^k} \mathscr{H}^0 \llcorner \mathrm{spt}\langle T, \pi, x\rangle \, d\mathscr{L}^k(x) \leqslant \mu \quad \forall \pi \in \mathrm{Lip}_1\big(E, \mathbf{R}^k\big) \tag{3.1}$$

for $k \geqslant 1$. The choice of $\mu$ can be optimized by choosing the least upper bound of the family $\{\mu_{T,\pi}\}$ in the lattice of nonnegative measures:

$$\mu_T := \bigvee_{\pi \in \mathrm{Lip}_1(E, \mathbf{R}^k)} \mu_{T,\pi}.$$

We shall call $\mu_T$ the concentration measure of $\mu$.

Notice that this definition has been given in terms of the supports of the slices of $T$, rather than the whole support of $T$. This choice is motivated by the special behavior of 0-dimensional flat chains illustrated in Section 3.1.

**Definition 3.2** *(Size).* We say that $T \in \mathbf{F}_k(E)$ has finite size if $\mu_T$ has finite mass. In this case we define

$$\mathbf{S}(T) := \mu_T(E).$$

### *3.1. Zero dimensional flat currents*

For zero dimensional flat currents some special properties hold: it is a well-known result in the theory of distributions that any distribution supported in a finite set is a finite sum of derivatives of Dirac masses. Here we present an analogous result for flat currents of finite size, which is also similar to the representation theorem for zero dimensional flat $G$-chains of finite mass obtained in [28] and to the result on integer-valued currents in [25].

**Theorem 3.3.** *Every $T \in \mathbf{F}_0(E)$ with finite size can be represented as*

$$T = \sum_{i=1}^{\mathbf{S}(T)} a_i [\![ x_i ]\!] \tag{3.2}$$

*where* $\mathrm{spt}(T) = \{x_i\colon\ i = 1, \dots, \mathbf{S}(T)\}$ *and* $a_i \in \mathbf{R}$. *In particular $T$ has finite mass.*

**Proof.** We will prove the theorem through several steps.

**Step 1.** First of all we claim that it is sufficient to prove the representation formula $T(f) = \sum_i a_i f(x_i)$ for functions $f \in \mathrm{Lip}_b(E)$ such that

(1) $f$ has compact support,
(2) $f$ is locally constant in a neighborhood of $\mathrm{spt}(T)$.

In fact, since bounded closed sets of $E$ are compact and $\mathrm{spt}(T)$ is finite, we can easily approximate any $f \in \mathrm{Lip}_b(E)$ by functions $f_n$ with uniformly bounded Lipschitz functions satisfying (1), (2) and pointwise convergent to $f$. We can then use the continuity axiom of currents to pass to the limit.

**Step 2.** Let us fix $f \in \mathrm{Lip}_b(E)$ satisfying conditions (1) and (2) above. Set $\gamma(x) = \min\{d(x, x_i), x_i \in \mathrm{spt}(T)\}$: for almost every $r < r_0 := \frac{1}{2}\min_{i \neq j}\{d(x_i, x_j)\}$ the current $T \llcorner \{\gamma < r\}$ is well-defined, and by Lemma 2.5 it equals the sum of the $T \llcorner B_r(x_i)$'s:

$$T = T \llcorner \{\gamma \geqslant r\} + \sum_{i=1}^{N} T \llcorner B_r(x_i).$$

The first term is null on $f$: in fact, by Eq. (2.15)

$$\mathrm{spt}\big(T \llcorner \{\gamma \geqslant r\}\big) \subset \mathrm{spt}(T) \cap \{\gamma \geqslant r\} = \emptyset,$$

hence by (2.14) $T \llcorner \{\gamma \geqslant r\}(f) = 0$. As a result $T(f) = \sum_i T \llcorner B_r(x_i)(f)$, independently of $r$.

**Step 3.** We reduced our problem to the characterization of $T \llcorner B_r(x_i)(f)$, whose support is $\{x_i\}$ by (2.15). For each $j$ we let $g_j$ be a Lipschitz function equal to 1 on $B_{r_0/2}(x_i)$ and equal to 0 on $E \setminus B_{r_0}(x_i)$: if $0 < s < r$ are radii such that the restrictions of $T$ to $B_r(x_i)$ and $B_s(x_i)$ exist, the difference $T \llcorner B_r(x_i) - T \llcorner B_s(x_i)$ satisfies

$$\mathrm{spt}\big(T \llcorner B_r(x_i) - T \llcorner B_s(x_i)\big) \overset{(2.15)}{\subset} \overline{B_r(x_i) \setminus B_s(x_i)} \cap \{x_i\} = \emptyset.$$

Therefore again by (2.14) $T \llcorner B_r(x_i)(h)$ is (essentially) constant in $r$ for any bounded Lipschitz function $h$. In particular (2.14) implies that:

- $T \llcorner B_r(x_i)(g_j)$ does not depend on $r < r_0$, and actually (2.14) gives $T \llcorner B_r(x_i)(g_j) = 0$ for $i \neq j$;
- since $f = \sum_j f(x_j) g_j$ in a neighborhood of $\mathrm{spt}(T)$,

$$T \llcorner B_r(x_i)(f) = f(x_i) T \llcorner B_r(x_i)(g_i).$$

Letting $a_i = T \llcorner B_r(x_i)(g_i)$ we obtain the thesis. □

As a consequence of Theorem 3.3 we obtain a closure theorem for sequences of flat currents with equibounded sizes:

**Theorem 3.4** *(Lower semicontinuity of size). Let $(T_h) \subset \mathbf{F}_k(E)$ be a sequence of currents with equibounded sizes and converging to $T$ in the flat norm*:

$$\mathbf{S}(T_h) \leqslant C < \infty, \qquad \lim_h \mathbf{F}(T_h - T) = 0.$$

*Then $T$ has finite size and*

$$\mathbf{S}(T) \leqslant \liminf_h \mathbf{S}(T_h). \tag{3.3}$$

**Proof.** Possibly extracting a subsequence we can assume that

$$\sum_h \mathbf{F}(T_h - T) < \infty \quad \text{and} \quad \lim_h \mathbf{S}(T_h) = \liminf_h \mathbf{S}(T_h). \tag{3.4}$$

If $k = 0$ we prove a slightly more general implication: for any open set $A \subset E$, $\mathbf{F}(T_h - T) \to 0$ and $\liminf_h \mathscr{H}^0(A \cap \operatorname{spt} T_h) < \infty$ implies

$$\mathscr{H}^0\big(A \cap \operatorname{spt}(T)\big) \leqslant \liminf_h \mathscr{H}^0\big(A \cap \operatorname{spt}(T_h)\big). \tag{3.5}$$

Indeed, consider $x \in \operatorname{spt}(T) \cap A$. Then by definition (2.12) and inequality (2.6) for every $\varepsilon > 0$ there exists $r < \varepsilon$ such that

(i) $B(x, r) \subset A$,
(ii) $\lim_h \mathbf{F}(T_h \llcorner B(x, r) - T \llcorner B(x, r)) = 0$,
(iii) $T \llcorner B(x, r) \neq 0$.

Point (ii) implies that for $h \geqslant h(x, \varepsilon)$ $T_h \llcorner B(x, r) \neq 0$, and since by Theorem 3.3 $T_h$ is a finite sum of Dirac deltas

$$T_h = \sum_{j=1}^{\mathbf{S}(T_h)} a_{j,h} [\![ y_{j,h} ]\!]$$

at least one of the points $y_{j,h}$ must belong to $B(x, r)$. If $A \cap \operatorname{spt}(T)$ contains $m$ distinct points $\{x_1, \ldots, x_m\}$, we can take $\varepsilon$ sufficiently small such that the family of balls $\{B(x_i, \varepsilon)\colon i = 1, \ldots, m\}$ is disjoint. Therefore there exist radii $r_i$ as above such that for every $h \geqslant \max_i h(x_i, \varepsilon)$ each ball $B(x_i, r_i)$ contains at least one point $y_{j,h'}$. Hence $m \leqslant \mu_{T_h}(A)$ and (3.5) follows.

In the case $k \geqslant 1$ we fix a projection $\pi \in \operatorname{Lip}_1(E, \mathbf{R}^k)$. Thanks to (3.4) we know that for $\mathscr{L}^k$-almost every $x \in \mathbf{R}^k$ the slices $T_{h,x}$ converge to $T_x$; moreover Fatou's lemma implies that

$$\int_{\mathbf{R}^k} \liminf_h \mathbf{S}(T_{h,x})\, dx \leqslant \liminf_h \int_{\mathbf{R}^k} \mathbf{S}(T_{h,x})\, dx = \liminf_h \mu_{T_h,\pi}(E) \leqslant \lim_h \mathbf{S}(T_h) \leqslant C < \infty.$$

The same argument can be applied for an open set $A \subset E$ and using (3.5) we get

$$\begin{aligned} \mu_{T,\pi}(A) &= \int_{\mathbf{R}^k} \mathscr{H}^0(A \cap \operatorname{spt} T_x)\, dx \leqslant \int_{\mathbf{R}^k} \liminf_h \mathscr{H}^0(A \cap \operatorname{spt} T_{h,x})\, dx \\ &\leqslant \liminf_h \int_{\mathbf{R}^k} \mathscr{H}^0(A \cap \operatorname{spt} T_{h,x})\, dx = \liminf_h \mu_{T_h,\pi}(A). \end{aligned} \tag{3.6}$$

The map $A \mapsto \nu(A) = \liminf_h \mu_{T_h}(A)$ is a finitely superadditive set-function, with $\nu(E)$ bounded from above by $\liminf_h \mathbf{S}(T_h)$; the chain of inequalities (3.6) simply expresses that $\mu_{T,\pi}(A) \leqslant \nu(A)$ on open sets $A$.

If $B_1, \ldots, B_N$ are pairwise disjoint Borel sets and $K_i \subset B_i$ are compact, we can find pairwise disjoint open sets $A_i$ containing $K_i$ and apply the superadditivity to get

$$\sum_{i=1}^N \mu_{T,\pi_i}(K_i) \leqslant \sum_{i=1}^N \nu(A_i) \leqslant \nu(E).$$

Since $K_i$ are arbitrary, the same inequality holds with $B_i$ in place of $K_i$. Since $B_i$, $\pi_i$ and $N$ are arbitrary, it follows that $\mu_T$ is a finite Borel measure and $\mu_T(E) \leqslant \nu(E)$. $\square$

## 4. A hybrid distance on zero dimensional flat boundaries

We let $\mathbf{B}_0(E) = \mathbf{M}_0(E) \cap \partial \mathbf{F}_1(E)$ be the space of finite mass boundaries of flat chains. We endow $\mathbf{B}_0(E)$ with the following distance of interpolation type:

**Definition 4.1** *(Hybrid distance).* For every $Q, Q' \in \mathbf{B}_0(E)$ we set

$$\mathcal{G}(Q, Q') = \inf\{\mathbf{S}(R) + \mathbf{M}(S):\ Q - Q' = \partial(R + S),\ R, S \in \mathbf{F}_1(E)\}.$$

It is plain that the triangle inequality holds, by the subadditivity of mass and size. It is also immediate to check that $\mathcal{G}$ is finite: indeed, since $Q = \partial T$ with $T \in \mathbf{F}_1(E)$, we may write $T = X + \partial Y$ with $X, Y$ flat and $\mathbf{M}(X) + \mathbf{M}(Y) < \infty$. Therefore $Q = \partial X$ and so $\mathcal{G}(0, Q) \leqslant \mathbf{M}(X) < \infty$. Occasionally we shall abbreviate $\mathcal{G}(Q) = \mathcal{G}(0, Q)$.

The proof of nondegeneracy of $\mathcal{G}$ is based on an "elimination argument".

**Proposition 4.2.** *Let $Q \in \mathbf{B}_0(E)$ satisfy $\mathcal{G}(Q) = 0$. Then $Q = 0$.*

**Proof.** Suppose $Q$ is not null and take an open set $A$ such that $|Q(A)| > 0$. Since $Q$ is a finite measure, by monotone approximation from the interior we can guarantee that the open set satisfies $\|Q\|(\partial A) = 0$. Therefore we can choose a small $\delta > 0$ such that

$$\left|Q(A)\right| > 4\|Q\|\left(A^\delta \setminus A\right), \tag{4.1}$$

where $A^\delta$ is the $\delta$-neighborhood of $A$. Let $\varepsilon > 0$: by hypothesis we can find flat currents $R, S$ with $Q = \partial(R + S)$ satisfying

$$\mathbf{S}(R) < \frac{\delta}{6} \quad \text{and} \quad \mathbf{M}(S) < \frac{\varepsilon\delta}{6}.$$

If $\rho_1$ and $\rho_2$ are two positive numbers in $(0, \delta)$ with $\rho_1 < \rho_2$, we let $\pi(x) = \operatorname{dist}(x, A)$ and $\Sigma_{\rho_1,\rho_2} = \{\rho_1 \leqslant \operatorname{dist}(\cdot, A) < \rho_2\}$. Using $\pi$ we can formally set the currents $R$ and $S$ to be zero within the ring $\Sigma_{\rho_1,\rho_2}$ through the following relation:

$$\begin{aligned} Q \llcorner (E \setminus \Sigma_{\rho_1,\rho_2}) &= (\partial R) \llcorner (E \setminus \Sigma_{\rho_1,\rho_2}) + (\partial S) \llcorner (E \setminus \Sigma_{\rho_1,\rho_2}) \\ &= \partial\left(R \llcorner (E \setminus \Sigma_{\rho_1,\rho_2})\right) + \partial\left(S \llcorner (E \setminus \Sigma_{\rho_1,\rho_2})\right) + S_{\rho_2} - S_{\rho_1} + R_{\rho_2} - R_{\rho_1}. \end{aligned} \tag{4.2}$$

Note that (4.2) actually holds if $\rho_1$ and $\rho_2$ belong to a subset of $(0, \delta)$ of full measure, since slices and restrictions of the currents $R$ and $S$ exist only almost everywhere. Inequality (3.1) gives

$$\int_0^{\frac{\delta}{3}} \mathscr{H}^0(\operatorname{spt} R_\rho)\, d\rho \leqslant \mu_{R,\pi}(E) \leqslant \mathbf{S}(R) < \frac{\delta}{6}$$

and since $\mathscr{H}^0(\operatorname{spt} R_\rho)$ is an integer, there must be a set of radii $\rho$ in $(0, \frac{\delta}{3})$ of length strictly greater than $\frac{\delta}{6}$ such that $R_\rho = 0$. Moreover

$$\int_0^{\frac{\delta}{3}} \mathbf{M}(S_\rho)\, d\rho \leqslant \mathbf{M}(S) < \frac{\varepsilon\delta}{6}$$

and therefore $\mathbf{M}(S_\rho) < \varepsilon$ in a subset of $(0, \frac{\delta}{3})$ of measure strictly greater than $\frac{\delta}{6}$. For the same reason we can find another set of positive measure contained in $(\frac{2\delta}{3}, \delta)$ where the same requirements hold. Putting together these two results we can pick two radii $\rho_1 \in (0, \frac{\delta}{3})$ and $\rho_2 \in (\frac{2\delta}{3}, \delta)$ such that Eq. (4.2) holds, $R_{\rho_1} = R_{\rho_2} = 0$ and $\mathbf{M}(S_{\rho_1}) + \mathbf{M}(S_{\rho_2}) < 2\varepsilon$. Take now a Lipschitz function $\psi$ such that:

- $0 \leqslant \psi \leqslant 1$,
- $\psi = 1$ in $A^{\frac{\delta}{3}}$ and $\psi = 0$ outside $A^{2\delta/3}$,
- $\operatorname{Lip}(\psi) \leqslant \frac{3}{\delta}$

and test it on the current $Q' = Q + S_{\rho_1} - S_{\rho_2}$. Since (4.1) gives $Q' = Q \llcorner \Sigma_{\rho_1,\rho_2} + \partial Y$, with $Y$ supported in the complement of $\Sigma_{\rho_1,\rho_2}$, and since $\psi$ is constant inside the ring $\Sigma_{\rho_1,\rho_2}$, by (4.1) we have $|Q'(\psi)| < |Q(A)|/4$. Hence

$$\left|Q(\psi)\right| \leqslant \frac{1}{4}\left|Q(A)\right| + \mathbf{M}(S_{\rho_1}) + \mathbf{M}(S_{\rho_2}) < \frac{1}{4}\left|Q(A)\right| + 2\varepsilon.$$

On the other hand, Eq. (4.1) yields

$$\left|Q(A)\right| - \left|Q(\psi)\right| \leqslant \left|Q(A) - Q(\psi)\right| \leqslant \|Q\|\left(A^\delta \setminus A\right) < \frac{|Q(A)|}{4}$$

and so $|Q(A)| < \frac{4}{3}|Q(\psi)| \leqslant \frac{1}{3}|Q(A)| + \frac{8}{3}\varepsilon$. Since $|Q(A)| > 0$, by choosing $\varepsilon$ sufficiently small we have a contradiction. So $Q^+(A) = Q^-(A)$ on every open set $A$. Since the family of open sets is stable by intersection and generates the $\sigma$-algebra of Borel sets, we get $Q^+ \equiv Q^-$, hence $Q = 0$. □

In order to apply the theory of functions of metric bounded variation developed in [5,7], we need to ensure that the space $(\mathbf{B}_0(E), \mathcal{G})$ is separable. Let us first relate the space of 0-currents to the theory of Optimal Transportation. Recall that a finite nonnegative Borel measure $\mu$ has finite first moments if $d(\cdot, x_0)$ belongs to $L^1(\mu)$ for some, and thus all, $x_0 \in X$. Given two such measures $\mu$ and $\nu$ with finite first moments and equal total mass ($\mu(E) = \nu(E)$) we let

$$W_1(\mu,\nu) = \inf\left\{ \int_{E\times E} d(x,y)\, d\sigma(x,y):\ \sigma \in \mathcal{M}_+(E\times E),\ \pi_{1\#}\sigma = \mu,\ \pi_{2\#}\sigma = \nu \right\} \tag{4.3}$$

be their 1-Wasserstein distance, where $\pi_1$ is the projection on the first coordinate and $\pi_2$ is the projection on the second one. For the many properties and applications of this distance we refer to the monograph [26]. Among them, we recall that the infimum (4.3) is attained by at least one nonnegative Borel measure $\sigma$, which we call optimal plan. Since $E$ is a geodesic space the Wasserstein distance can be lifted to the space of geodesics $Geo(E)$ of constant speed geodesics parametrized on [0, 1]:

$$W_1(\mu,\nu) = \inf\left\{ \int_{Geo(E)} d\big(\gamma(0),\gamma(1)\big)\, d\lambda(\gamma),\ \lambda \in \mathcal{M}_+\big(Geo(E)\big),\ (e_0,e_1)_\#\lambda = (\mu,\nu) \right\}. \tag{4.4}$$

Here $e_t(\gamma) = \gamma(t)$ denoted the evaluation map at time $t$. This allows us to make the following observation:

**Lemma 4.3.** *Let $Q \in \mathbf{M}_0(E)$ be such that $Q(1) = 0$ and the total variation measure $\|Q\|$ has finite first moment. Then $Q$ is representable as $\partial Y$ for some $Y \in \mathbf{F}_1(E)$ with $\mathbf{M}(Y) \leqslant W_1(Q^+, Q^-)$. In particular $\mathcal{G}(Q) \leqslant W_1(Q^+, Q^-)$.*

**Proof.** The two measures $Q^+$ and $Q^-$ given by Hahn decomposition theorem have finite first moments and have the same mass. We let $\lambda \in \mathcal{M}_+(Geo(E))$ be an optimal measure in problem (4.4) and we build

$$Y = \int_{Geo(E)} \gamma_\# [\![0,1]\!]\, d\lambda(\gamma).$$

Since $\partial\gamma_\#[\![0,1]\!] = \delta_{\gamma(1)} - \delta_{\gamma(0)}$, it is easily proved that $Y$ is actually a normal current with $Q = \partial Y$ and that

$$\mathbf{M}(Y) \leqslant \int_{Geo(E)} \mathbf{M}\big(\gamma_\#[\![0,1]\!]\big)\, d\lambda(\gamma) = \int_{Geo(E)} d(x,y)\, d\lambda(x,y) = W_1\big(Q^+, Q^-\big). \qquad \square$$

**Proposition 4.4.** *The metric space $(\mathbf{B}_0(E), \mathcal{G})$ is separable.*

**Proof.** We first show that the class of currents with bounded support is dense. In fact, let us fix a basepoint $x_0 \in E$ and $Q = \partial(R + S)$ with $\mathbf{S}(R) < \infty$ and $\mathbf{M}(S) < \infty$: as in Proposition 4.2, there are arbitrarily big radii $r$ such that $R_r = 0$, $\mathbf{M}(S_r)$ is finite and $\operatorname{spt}(S_r) \subset \partial B_r(x_0)$. As in (4.2), for a.e. $r > 0$ we obtain

$$Q \llcorner \big(E \setminus B_r(x_0)\big) = \partial\big(R \llcorner \big(E \setminus B_r(x_0)\big)\big) + \partial\big(S \llcorner \big(E \setminus B_r(x_0)\big)\big) + S_r,$$

so that $Q \llcorner B_r(x_0) + S_r$ belongs to $\mathbf{B}_0(E)$. Clearly $Q \llcorner B_r(x_0) + S_r$ is supported in $\overline{B}_r(x_0)$, and its $\mathcal{G}$-distance from $Q$ can be estimated by

$$\mathcal{G}\big(Q, Q \llcorner B_r(x_0) + S_r\big) = \mathcal{G}\big(Q \llcorner \big(E \setminus B_r(x_0)\big) - S_r\big) \leqslant \|S\|\big(E \setminus B_r(x_0)\big) + \mathbf{S}\big(E \setminus B_r(x_0)\big)$$

which is arbitrarily small provided we take $r$ sufficiently large.

Now, if $Q \in \mathbf{B}_0(E)$ has bounded support we may represent $Q = \partial Y$ for some normal current $Y$, so that

$$\mathcal{G}(Q, aQ) \leqslant |1-a|\mathbf{M}(Y).$$

This inequality can be used to show that the class of $Q$'s with bounded support such that $c(Q) = Q^+(E) = Q^-(E)$ is a rational number is dense.

Now, recall that the space of Borel probability measures in $E$ endowed with the $W_1$ distance is separable (see for instance [4, Proposition 7.1.5]) and let us denote by $\mathcal{D}$ a countable dense subset. If $Q \in \mathbf{B}_0(E)$ and $c = Q^+(E) = Q^-(E) \in \mathbf{Q}$, we may consider families $\nu_n$, $\mu_n$ contained in $\mathcal{D}$ converging respectively to $Q^+/c$ and $Q^-/c$ in Wasserstein distance and use the inequality

$$\mathcal{G}(Q, c\mu_n - c\nu_n) \leqslant \mathcal{G}\big(Q^+, c\mu_n\big) + \mathcal{G}\big(Q^-, c\nu_n\big) \leqslant W_1\big(Q^+, c\mu_n\big) + W_1\big(Q^-, c\nu_n\big)$$

to get $\mathcal{G}(Q, c\mu_n - c\nu_n) \to 0$. This proves the separability of $(\mathbf{B}_0(E), \mathcal{G})$. □

## 5. Rectifiable sets and functions of metric bounded variation

This section is devoted to the presentation of some technical tools that allow to study the rectifiability of certain subsets of a metric space.

### 5.1. Rectifiable sets

We begin by recalling the definition of countably $\mathscr{H}^k$-rectifiable set:

**Definition 5.1.** (See [16].) An $\mathscr{H}^k$-measurable set $\Sigma \subset E$ is called countably $\mathscr{H}^k$-rectifiable if there exist countably many sets $A_j \subset \mathbf{R}^k$ and Lipschitz maps $f_j : A_j \to E$ such that

$$\mathscr{H}^k\Big(\Sigma \setminus \bigcup_j f_j(A_j)\Big) = 0. \tag{5.1}$$

For $k = 0$ we define a countably $\mathscr{H}^0$-rectifiable set to be a finite or countable set.

We recall that since $E$ is complete and boundedly compact, the sets $A_i$ can be assumed to be closed or compact; moreover one can suppose that the images $f_j(A_j)$ are pairwise disjoint (see [21, Lemma 4]).

In order to prove a rectifiability result it is often necessary to prove that a certain parameterization function is Lipschitz. Among the many ways to measure the slope of a function, the following notion is quite flexible, since it is local and behaves well under slicing:

**Definition 5.2.** Let $A \subset \mathbf{R}^k$ be Borel and $f : A \to E$ be a Borel map. For $x \in A$ we define $\delta_x f$ as the smallest $N \geqslant 0$ such that

$$\lim_{r \downarrow 0} r^{-k} \mathscr{L}^k\left(\left\{y \in A \cap B_r(x) \colon \frac{d(f(y), f(x))}{r} > N\right\}\right) = 0.$$

This definition is a simplified version of Federer's definition of approximate upper limit of the difference quotients (we replaced $|y - x|$ by $r$ in the denominator). The next theorem, proved in [9, Theorem 5.1] (actually a simplified version of [16, Theorem 3.1.4]), is the weak version of the total differential theorem that implements the local slope $\delta_x f$ defined above instead of the classical differential:

**Theorem 5.3.** *Let $A \subset \mathbf{R}^k$ be Borel and $f : \mathbf{R}^k \to E$ be Borel.*

(i) *Let $k = n + m$, $x = (z, y)$, and assume that there exist Borel subsets $A_1$, $A_2$ of $A$ such that $\delta_z(f(\cdot, y)) < \infty$ for all $(z, y) \in A_1$ and $\delta_y(f(z, \cdot)) < \infty$ for all $(z, y) \in A_2$. Then $\delta_x f < \infty$ for $\mathscr{L}^k$-a.e. $x \in A_1 \cap A_2$;*
(ii) *if $\delta_x f < \infty$ for $\mathscr{L}^k$-a.e. $x \in A$ there exists a sequence of Borel sets $B_n \subset A$ such that $\mathscr{L}^k(A \setminus \bigcup_n B_n) = 0$ and the restriction of $f$ to $B_n$ is Lipschitz for all $n$.*

The following simple proposition, proved in [9, Proposition 5.2], relates 1-dimensional rectifiable sets and projections:

**Proposition 5.4.** *Let $K \subset \Gamma \subset E$, with $K$ countably $\mathscr{H}^1$-rectifiable, and let $\pi \in \mathrm{Lip}(E)$ be injective on $\Gamma$. Then $\delta(\pi|_\Gamma)^{-1}$ is finite $\mathscr{L}^1$-a.e. on $\pi(K)$.*

*5.2. Functions of metric bounded variation*

It is a well-known fact that, in absence of a linear structure, as in a generic metric space $(M, d_M)$, Lipschitz functions play the role of coordinates. Bearing in mind this idea we begin with a definition:

**Definition 5.5.** A metric space $(M, d_M)$ is called weakly separable if there exists a countable family $(\varphi_h)_{h\in\mathbf{N}} \subset \mathrm{Lip}_1(M) \cap \mathrm{Lip}_b(M)$ such that

$$d_M(x, y) = \sup_h \left|\varphi_h(x) - \varphi_h(y)\right| \quad \forall x, y \in M. \tag{WS}$$

Notice that separable metric spaces are particular cases of the class defined above, as it is sufficient to take as $\varphi_h$ truncations of the functions $d_M(\cdot, x_h)$ where $x_h$ run in a dense subset of $M$. In particular, Proposition 4.4 ensures that $(\mathbf{B}_0(E), \mathcal{G})$ is a weakly separable metric space. Observe also that given a Borel function $u : \mathbf{R}^k \to M$ we have that $\mathscr{L}^k$-a.e. $x \in \mathbf{R}^k$ is an approximate continuity point, namely

$$\left\{y:\ d_M\big(u(y), z\big) > \varepsilon\right\}$$

has 0-density at $x$ for all $\varepsilon > 0$, for some $z \in M$. The point $z$ is unique and we will denote it by $\tilde{u}(x)$. We shall denote, as in [3,5], by $S_u$ the set of approximate discontinuity points: it is a Lebesgue negligible Borel set and $\tilde{u} = u$ $\mathscr{L}^n$-a.e. in $\mathbf{R}^k$.

The oscillations of a function $u : \mathbf{R}^k \to M$ are detected through the composition with each $\varphi_h$. In analogy with the case where $M = \mathbf{R}^N$ is a Euclidean space, a natural definition of metric space valued $BV_{\mathrm{loc}}$ function would require that

$$\text{the distribution } D(\varphi_h \circ u) \text{ is a locally finite measure for every } h. \tag{D}$$

Although this condition easily characterizes the space $BV_{\mathrm{loc}}(\mathbf{R}^k, \mathbf{R}^N)$ if we take among the functions $\varphi_h$ (truncates of) coordinate projections, in the general context of metric spaces a uniformity among the measures $\{|D(\varphi_h \circ u)|\}_h$ is not a byproduct of condition (D). Therefore, as in [5, Definition 2.1], we define:

**Definition 5.6** *(Metric bounded variation).* Let $(M, d_M)$ be a weakly separable metric space and let $u : \mathbf{R}^k \to M$ be a Borel function. We say that $u$ has metric bounded variation if there exists a finite measure $\sigma$ in $\mathbf{R}^k$ such that

$$\left|D(\varphi_h \circ u)\right| \leqslant \sigma \quad \text{for every } h, \tag{5.2}$$

where the set $(\varphi_h)$ satisfies (WS). We denote by $MBV(\mathbf{R}^k, M)$ the space of such functions and by $|Du|$ the least possible measure $\sigma$ in (5.2).

For our purposes, it is also necessary to work with the classical definition of function of bounded variation defined on intervals of the real line (see [16, 4.5.10]): if $u : (a, b) \to M$ is a Borel function we let

$$\text{ess-Var}_a^b\, u = \sup\left\{\sum_{k=1}^N d_M\big(\tilde{u}(x_{k-1}), \tilde{u}(x_k)\big),\ a < x_0 < \cdots < x_N < b,\ x_k \in D\right\} \tag{5.3}$$

where $D$ is any countable dense set in $(a,b)\setminus S_u$. As it is proved in [16, 4.5.10] and in [5, Remark 2.2], $u\in BV((a,b))$ if and only if ess-$\mathrm{Var}_a^b u$ is finite and ess-$\mathrm{Var}_a^b u=|Du|((a,b))$. Hypothesis (WS) comes into play when dealing with many measure theoretic properties of the space $MBV$. For instance:

**Lemma 5.7.** *If $u\in MBV((a,b),M)$ then the approximate upper limit of incremental quotient $\delta_x u$ is finite $\mathscr{L}^1$-almost everywhere in $(a,b)$.*

**Proof.** We can assume with no loss of generality that $b-a<\infty$. Then, the composition $u_h:=\varphi_h\circ u$ belongs to $BV((a,b))$: hence there exists an $\mathscr{L}^1$-negligible set $N_h\subset(a,b)$ such that

$$u_h(x)-u_h(y)=Du_h\big((x,y)\big)\quad \forall x,y\notin N_h.$$

Moreover by Vitali's covering theorem the set

$$N'=\left\{x\in(a,b)\colon \limsup_{r\downarrow 0}\frac{|Du|(B_r(x))}{2r}=\infty\right\}$$

where the upper density $\Theta_1^*(|Du|,\cdot)$ is infinite is Lebesgue negligible; since

$$\big|u(x)-u(y)\big|\leqslant|Du|\big((x,y)\big)\quad \forall x,y\in(a,b)\setminus\bigcup_h N_h,$$

therefore $\delta_x u\leqslant\Theta_1^*(|Du|,x)<\infty$ for $x\notin N'\cup\bigcup_h N_h$. □

In particular, by Theorem 5.3(ii), for all $u\in MBV(\mathbf{R}^k,(\mathbf{B}_0(E),\mathcal{G}))$ there exist Borel sets $B_n$ and constants $L_n$ such that

$$\mathcal{G}\big(u(x_1),u(x_2)\big)\leqslant L_n|x_1-x_2|\quad\forall x_1,x_2\in B_n\quad\text{and}\quad \mathscr{L}^k\bigg(\mathbf{R}^k\setminus\bigcup_n B_n\bigg)=0. \tag{5.4}$$

## 6. Rectifiability of flat currents with finite size

This section contains the main rectifiability result of this paper.

**Theorem 6.1** *(Rectifiability of currents of finite size). For every flat current $T\in\mathbf{F}_k(E)$ with finite size the measure $\mu_T$ is concentrated on a countably $\mathscr{H}^k$-rectifiable set. The least one, up to $\mathscr{H}^k$-null sets, is given by*

$$set(T):=\left\{x\in E\colon \limsup_{r\downarrow 0}\frac{\mu_T(B_r(x))}{r^k}>0\right\}. \tag{6.1}$$

Notice that, even for flat chains for finite mass, the theorem provides no information on the rectifiability of the measure $\|T\|$, which fails to be true in general. So, our goal is to find a countably $\mathscr{H}^k$-rectifiable set $\Sigma$ such that $\mu_T(E\setminus\Sigma)=0$. We start by proving the existence of a countably $\mathscr{H}^k$-rectifiable set $\Sigma=\Sigma_\pi$ satisfying

$$\mu_{T,\pi}(E\setminus\Sigma_\pi)=0 \tag{6.2}$$

for a fixed $\pi\in\mathrm{Lip}_1(E,\mathbf{R}^k)$: since $\mathbf{S}(T)$ is finite, for $\mathscr{L}^k$-almost every $x\in\mathbf{R}^k$ the slice $T_x=\langle T,\pi,x\rangle$ has finite size, hence by Theorem 3.3 it is a finite sum of Dirac's masses. Therefore $\mathrm{spt}(T_x)=\{y\in E\colon \|T_x\|(\{y\})>0\}$, moreover $T_x=X_x+(\partial Y)_x$ which entails $\|T_x\|\leqslant\|X_x\|+\|(\partial Y)_x\|$ again almost everywhere. This implies that

$$\mathrm{spt}(T_x)\subset\big\{y\in E\colon \|X_x\|\big(\{y\}\big)>0\big\}\cup\big\{y\in E\colon \big\|(\partial Y)_x\big\|\big(\{y\}\big)>0\big\}. \tag{6.3}$$

So, in order to investigate the rectifiability of the measure $\mu_{T,\pi}=\int_{\mathbf{R}^k}\mathscr{H}^0\llcorner\mathrm{spt}(T_x)\,dx$ we will prove that there are countably many Lipschitz graphs that contain the right-hand side of (6.3), for $\mathscr{L}^k$-almost every $x$. Since $X\in\mathbf{F}_k(E)$ is a flat current with finite mass, the statement regarding its atoms has already been obtained in the proof of [12, Theorem 3.2]. The result reads:

**Theorem 6.2.** *Let $X \in \mathbf{F}_k(E)$ be a flat current of finite mass. Then, for all $\pi \in \mathrm{Lip}(E, \mathbf{R}^k)$ there exists a countably $\mathscr{H}^k$-rectifiable set $\Sigma_{X,\pi}$ such that, for $\mathscr{L}^k$-a.e. $x \in \mathbf{R}^k$, the atoms of the measure $\langle X, \pi, x\rangle$ are contained in $\Sigma_{X,\pi}$.*

The strategy of the proof is to use the fact that flat currents $T$ with finite mass can be approximated in the stronger mass norm by normal currents $T_h$; the approximation in mass norm is inherited by the slices and implies that, up to a Lebesgue negligible set of points $x$, the atoms of $T_x$ are atoms of one of the measures $(T_h)_x$. The validity of the result for normal currents goes back to [7]. Actually one can even prove, arguing as in Section 6.3, that a countably $\mathscr{H}^k$-rectifiable set can be chosen independently of $\pi$, but we shall not need this fact.

Our new contribution is the analogous statement for $\partial Y$, which need not have finite mass. Recall that the mail idea of the classical proof for normal currents in [7] is that the slicing map of a normal current has bounded variation, if we measure the distance between slices using the flat norm. This property uses in a crucial way the duality property of the flat norm $\mathbf{F}$:

$$\mathbf{F}(T) = \sup_{\|\phi\|_\infty \leqslant 1,\ \|d\phi\|_\infty \leqslant 1} T(\phi).$$

Unfortunately our hybrid distance $\mathcal{G}$ does not seem to have a similar duality property. Instead, we consider the classical definition of function of bounded variation recalled in Section 5 to prove the theorem in dimension $k = 1$. Then, the total differential Theorem 5.3 and Proposition 5.4 will allow us to pass to the general dimension.

### *6.1. The 1-dimensional case*

First of all we fix a map $\pi \in \mathrm{Lip}_1(E)$.

**Proposition 6.3.** *Let $T \in \mathbf{F}_1(E)$ be a flat 1-current with finite size, let us write $T = X + \partial Y$ with $\mathbf{M}(X) + \mathbf{M}(Y) < \infty$ and denote by $Q_x$ the slicing map*

$$Q_x : \mathbf{R} \to \mathbf{B}_0(E), \qquad Q_x = \langle T - X, \pi, x\rangle = \langle \partial Y, \pi, x\rangle.$$

*Then $Q_x \in MBV(\mathbf{R}, (\mathbf{B}_0, \mathcal{G}))$ and $|DQ_x|(\mathbf{R}) \leqslant \mathbf{M}(X) + \mathbf{S}(T)$.*

**Proof.** Since $\mu_\pi$ is a finite measure, for almost every $x$ the support of $\langle T, \pi, x\rangle$ is finite. By Theorem 3.3 we know that $\langle T, \pi, x\rangle$ must have finite mass. Therefore $Q_x = \langle T, \pi, x\rangle - \langle X, \pi, x\rangle$ belongs to $\mathbf{M}_0(E)$. Moreover $Q_x$ is a boundary

$$Q_x = \langle T - X, \pi, x\rangle = \partial\big(\partial Y \llcorner \{\pi < x\}\big) - \partial^2 Y \llcorner \{\pi < x\} = \partial\big((T - X) \llcorner \{\pi < x\}\big),$$

which proves that the map $Q$ takes values in $\mathbf{B}_0(E)$. These properties hold whenever the slices exist and restrictions can be made: as explained in Section 2.2 these operations are meaningful in a set of full measure. Therefore for every $x_1 < x_2$ both outside a set of measure zero we can perform the following computation:

$$Q_{x_2} - Q_{x_1} = \langle T - X, \pi, x_2\rangle - \langle T - X, \pi, x_1\rangle = \partial\big((T - X) \llcorner \{x_1 \leqslant \pi < x_2\}\big),$$

hence

$$\mathcal{G}(Q_{x_2}, Q_{x_1}) \leqslant \mathbf{M}\big(X \llcorner \{x_1 \leqslant \pi < x_2\}\big) + \mathbf{S}\big(T \llcorner \{x_1 \leqslant \pi < x_2\}\big).$$

Therefore choosing $x_0 < x_1 < \cdots < x_N$, from (5.3), we obtain that $|DQ_x|(\mathbf{R}) = \text{ess-Var}_{-\infty}^{+\infty} Q_x \leqslant \mathbf{M}(X) + \mathbf{S}(T)$, which is the thesis. □

**Theorem 6.4.** *Let $Q \in MBV(\mathbf{R}, (\mathbf{B}_0, \mathcal{G}))$. There exists an $\mathscr{L}^1$-negligible set $\Lambda \subset \mathbf{R}$ such that the set of atoms*

$$\Sigma_Q = \big\{y \in E:\ \text{there exists } x \in \mathbf{R} \setminus \Lambda \text{ such that } \|Q_x\|\big(\{y\}\big) > 0\big\}$$

*is countably $\mathscr{H}^1$-rectifiable. In particular, for all $T \in \mathbf{F}_1(E)$ with finite size and all $\pi \in \mathrm{Lip}(E)$ this property holds for the map $Q_x = \langle T, \pi, x\rangle$.*

**Proof.** Fix $\varepsilon, \delta > 0$ and let $\Lambda = \mathbf{R} \setminus \bigcup_n B_n$ be the Lebesgue negligible set, where $B_n$ are the Borel sets given by Theorem 5.3(ii) in which the estimate (5.4) holds:

$$\mathcal{G}(Q_{x_1}, Q_{x_2}) \leqslant L_n|x_1 - x_2| \quad \forall x_1, x_2 \in B_n, \tag{6.4}$$

for suitable constants $L_n$. We then take the set $\Sigma_{\varepsilon,\delta,n}$ of points $y \in E$ such that for some $x \in B_n$:

(a) $\|Q_x\|(\{y\}) \geqslant \varepsilon$,
(b) $\|Q_x\|(B_{2\delta}(y) \setminus \{y\}) \leqslant \frac{\varepsilon}{8}$.

It is easy to notice that with this choice of $\Lambda$ the set $\Sigma_{Q_x}$ is the union of $\Sigma_{\varepsilon,\delta,n}$ for a countable set of parameters $\varepsilon$ and $\delta$, therefore it is sufficient to our purpose to prove the rectifiability of the latter sets. In addition the hypothesis of separability allows us to write $E$ as a countable union of disjoint Borel sets $E_k^\delta$ of diameter at most $\delta$, and again it is sufficient to prove the rectifiability of $\Sigma_{\varepsilon,\delta,n,k} := \Sigma_{\varepsilon,\delta,n} \cap E_k^\delta$. Let us take two points $y_1$ and $y_2$ in $\Sigma_{\varepsilon,\delta,n,k}$ and let $x_1 \leqslant x_2$ be their basepoints in $B_n$: we know that $d = d(y_1, y_2) \leqslant \delta$. Take $T, X \in \mathbf{F}_1(E)$ such that

$$Q_{x_1} - Q_{x_2} = \partial(X + T) \quad \text{and} \quad \mathbf{M}(X) + \mathbf{S}(T) \leqslant 2\mathcal{G}(Q_{x_1}, Q_{x_2}). \tag{6.5}$$

Then either $\mathbf{S}(T) \geqslant \frac{d}{9}$ or not. In the first case

$$d(y_1, y_2) \leqslant 9\mathbf{S}(T) \leqslant 18\mathcal{G}(Q_{x_1}, Q_{x_2}),$$

and since $x_1, x_2 \in B_n$, we obtain by (6.4)

$$d(y_1, y_2) \leqslant 18L_n|x_1 - x_2|. \tag{6.6}$$

In the latter case $\mathbf{S}(T) < \frac{d}{9}$, hence by definition of size, slicing $T$ with the distance function from $y_1$, we infer that

$$T_r = \langle T, d(y_1, \cdot), r\rangle = 0$$

for radii $r$ in at least $\frac{8}{9}$ of the segment $[0, d]$. Therefore we can find radii $\rho_1 \in (0, d/3)$, $\rho_2 \in (2d/3, d)$ satisfying

$$T_{\rho_1} = T_{\rho_2} = 0 \quad \text{and} \quad \mathbf{M}(X_{\rho_1}) + \mathbf{M}(X_{\rho_2}) \leqslant \frac{9}{d}\mathbf{M}(X). \tag{6.7}$$

In order to remove the ring $\mathcal{R} = \{\rho_1 \leqslant d(y_1, \cdot) < \rho_2\}$ from $T$ and $X$ we set $T' = T \llcorner (E \setminus \mathcal{R})$ and $X' = X \llcorner (E \setminus \mathcal{R})$. We obtain, as in (4.2),

$$\partial\big(T' + X'\big) = \big[\partial(T + X)\big] \llcorner (E \setminus \mathcal{R}) + X_{\rho_1} - X_{\rho_2} = (Q_{x_1} - Q_{x_2}) \llcorner (E \setminus \mathcal{R}) + X_{\rho_1} - X_{\rho_2}. \tag{6.8}$$

Take now a Lipschitz function $\phi$ such that $0 \leqslant \phi \leqslant 1$, $\phi = 1$ in $B_{d/3}(y_1)$, $\phi = 0$ in $E \setminus B_{2d/3}(y_1)$, and $\mathrm{Lip}(\phi) \leqslant 3/d$. By hypothesis (b) above

$$\begin{aligned}\big|(Q_{x_1} - Q_{x_2}) \llcorner \mathcal{R}(\phi)\big| &\leqslant \|Q_{x_1}\|(\chi_{\mathcal{R}}\phi) + \|Q_{x_2}\|(\chi_{\mathcal{R}}\phi)\\ &\leqslant \|Q_{x_1}\|\big(B_{2\delta}(y_1) \setminus \{y_1\}\big) + \|Q_{x_2}\|\big(B_{2\delta}(y_2) \setminus \{y_2\}\big) \leqslant \frac{\varepsilon}{4},\end{aligned} \tag{6.9}$$

since $\mathcal{R} \subset B_{2\delta}(y_i) \setminus \{y_i\}$. The first two assumptions on $\phi$ imply that

$$\big|(Q_{x_1} - Q_{x_2})(\phi) - Q_{x_1}\big(\{y_1\}\big)\big| \leqslant \|Q_{x_1}\|\big(B_{\frac{2d}{3}}(y_1) \setminus \{y_1\}\big) + \|Q_{x_2}\|\big(B_{\frac{2d}{3}}(y_1)\big) \leqslant \frac{\varepsilon}{4},$$

which gives

$$\big|(Q_{x_1} - Q_{x_2})(\phi)\big| \geqslant \big|Q_{x_1}\big(\{y_1\}\big)\big| - \big|(Q_{x_1} - Q_{x_2})(\phi) - Q_{x_1}\big(\{y_1\}\big)\big| \geqslant \frac{3}{4}\varepsilon. \tag{6.10}$$

Putting together (6.9) and (6.10) we obtain

$$\big|(Q_{x_1} - Q_{x_2}) \llcorner (E \setminus \mathcal{R})(\phi)\big| \geqslant \big|(Q_{x_1} - Q_{x_2})(\phi)\big| - \big|(Q_{x_1} - Q_{x_2}) \llcorner \mathcal{R}(\phi)\big| \geqslant \frac{\varepsilon}{2}. \tag{6.11}$$

We can now test Eq. (6.8) with $\phi$:

$$\begin{aligned}\frac{\varepsilon}{2} &\overset{(6.11)}{\leqslant} \big|(Q_{x_1} - Q_{x_2}) \llcorner (E \setminus \mathcal{R})(\phi)\big| = \big|\big(T' + X'\big)(d\phi) + (X_{\rho_2} - X_{\rho_1})(\phi)\big|\\ &\leqslant \big|\big(T' + X'\big)(d\phi)\big| + \mathbf{M}(X_{\rho_1}) + \mathbf{M}(X_{\rho_2}) \overset{(6.7)}{\leqslant} \big|\big(T' + X'\big)(d\phi)\big| + \frac{9}{d}\mathbf{M}(X).\end{aligned} \tag{6.12}$$

Since $\phi$ is constant in a neighborhood of $B_{\rho_1}(y_1)$ and in a neighborhood of $E \setminus B_{\rho_2}(y_1)$, we deduce from Lemma 6.5 (splitting $T' + X'$ in two parts) that $(T' + X')(d\phi) = 0$. Hence, estimates (6.12) and (6.5) yield

$$\frac{\varepsilon}{2} \leqslant \frac{18}{d}\mathcal{G}(Q_{x_1}, Q_{x_2}) \leqslant \frac{18L_n}{d}|x_1 - x_2|, \tag{6.13}$$

since we took $x_i \in B_n$. Hence putting together the two cases (6.6) and (6.13) we obtain

$$d(y_1, y_2) \leqslant \max\left\{18L_n, \frac{36L_n}{\varepsilon}\right\}|x_1 - x_2|. \tag{6.14}$$

In particular for every $x \in \mathbf{R} \setminus \Lambda$ there exists at most one atom $y$ of $Q_x$ in the set $\Sigma_{\varepsilon,\delta,n,k}$, denoted by $f(x)$: let $D_{\varepsilon,\delta,n,k} \subset \mathbf{R} \setminus \Lambda$ denote the set of points $x$ where this atom exists. The estimate (6.14) implies that the map $f : D_{\varepsilon,\delta,n,k} \to E$ has a global Lipschitz extension.

Finally, the last part of the statement follows by Proposition 6.3. □

**Lemma 6.5.** *Let $T \in \mathbf{F}_k(E)$ and $u \in \mathrm{Lip}(E)$. For $\mathscr{L}^1$-almost every $t \in \mathbf{R}$ the following property holds*:

$$\partial\big(T \llcorner \{u < t\}\big)(\phi) = 0$$

*for every $\phi \in \mathrm{Lip}_b(E)$ constant in a neighborhood of $\{u < t\}$.*

**Proof.** By definition there exists a sequence of normal currents $(T_h)$ satisfying $\sum_h \mathbf{F}(T_h - T) < \infty$, so that for almost every $t$ it holds $\mathbf{F}(\partial(T_h \llcorner \{u < t\}) - \partial(T \llcorner \{u < t\})) \to 0$. Since $T_h$ has finite mass, we can write $\partial(T_h \llcorner \{u < t\})(\phi) = T_h(\chi_{\{u<t\}} d\phi)$ and we can use the locality property of finite mass metric currents [7, Theorem 3.5] to get $T_h(\chi_{\{u<t\}} d\phi) = 0$. Passing to the limit in $h$ the statement follows. □

*6.2. The general k-dimensional case*

In this section we analyze the general case $k \geqslant 1$.

We shall need two technical lemmas. The first one provides a useful commutativity property of the iterated slice operator, the second one provides a measurable selection, see for instance [10, III.6, III.11].

**Lemma 6.6** *(Commutativity of slices). Let $T \in \mathbf{F}_k(E)$ and let $\pi = (p, q)$ satisfy $p \in \mathrm{Lip}(E, \mathbf{R}^{m_1})$, $q \in \mathrm{Lip}(E, \mathbf{R}^{m_2})$, $m_i \geqslant 1$ and $m_1 + m_2 \leqslant k$. Then*

$$\big\langle \langle T, p, z\rangle, q, y\big\rangle = (-1)^{m_1 m_2}\big\langle \langle T, q, y\rangle, p, z\big\rangle \quad \text{for } \mathscr{L}^{m_1+m_2}\text{-a.e. } (z, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}. \tag{6.15}$$

**Proof.** If $T \in \mathbf{N}_k(E)$ it is immediate to check that $\langle\langle T, q, y\rangle, p, z\rangle$ satisfy property (2.4) of Section 2.2 and

$$\int \psi(y, z)\big\langle \langle T, q, y\rangle, p, z\big\rangle dy\, dz = T \llcorner \psi(p, q)\, dq \wedge dp = (-1)^{m_1+m_2} T \llcorner \psi(p, q)\, dp \wedge dq,$$

hence (6.15) holds. The general case can be achieved choosing a sequence $(T_h) \subset \mathbf{N}_k(E)$ with $\sum_h \mathbf{F}(T_h - T) < \infty$. □

**Lemma 6.7.** *Let us assign for all $x \in \mathbf{R}^k$ a finite set $A(x) \subset E$, and let us assume that $\{x\colon A(x) \cap C \neq \emptyset\}$ is Lebesgue measurable for all closed sets $C \subset E$. Then the sets*

$$B_n := \big\{x \in \mathbf{R}^k\colon \operatorname{card} A(x) = n\big\}, \quad n \geqslant 1,$$

*are Lebesgue measurable and there exist Lebesgue measurable maps $f_1, \ldots, f_n : B_n \to E$ such that*

$$A(x) = \big\{f_1(x), \ldots, f_n(x)\big\} \quad \text{for } \mathscr{L}^k\text{-a.e. } x \in B_n. \tag{6.16}$$

We are ready prove the rectifiability of the atoms of $\langle \partial Y, \pi, x\rangle$ for general $k \geqslant 1$ and, as a consequence, the rectifiability of $\mu_{T,\pi}$.

**Theorem 6.8.** *Let $\pi \in \mathrm{Lip}(E, \mathbf{R}^k)$ and suppose $T = X + \partial Y \in \mathbf{F}_k(E)$ has finite size, with $\mathbf{M}(X) + \mathbf{M}(Y) < \infty$. Then there exists a Lebesgue negligible set $\Lambda \subset \mathbf{R}^k$ such that the set of atoms*

$$\Sigma_{\partial Y,\pi} = \bigl\{y \in E\colon \text{ there exists } x \in \mathbf{R}^k \setminus \Lambda \text{ such that } \bigl\|(\partial Y)_x\bigr\|\bigl(\{y\}\bigr) > 0\bigr\}$$

*is a countably $\mathscr{H}^k$-rectifiable set. In particular*

$$\mu^*_{\partial Y,\pi} = \int_{\mathbf{R}^k} \mathscr{H}^0 \llcorner \mathrm{Atoms}\bigl(\langle \partial Y, \pi, x\rangle\bigr)\, dx \tag{6.17}$$

*is concentrated on a countably $\mathscr{H}^k$-rectifiable set.*

**Proof.** First of all notice that the statement of the theorem allows us to ignore sets of atoms whose projection under $\pi$ is Lebesgue negligible. We will split the family of atoms in countably many subfamilies (indexed by $m$ and $n$), according to their weight and the cardinality in each fiber.

Since $T$ has finite size and $X$ has finite mass, by Theorem 3.3 for almost every $x \in \mathbf{R}^k$ the equality

$$Q_x = \langle \partial Y, \pi, x\rangle = \langle T - X, \pi, x\rangle$$

implies that $Q_x$ has finite mass, so for every $m \geqslant 1$ the set of points $y \in E$ such that $\|Q_x\|(\{y\}) \geqslant 1/m$ is finite almost everywhere. We fix a representative $Q_x$ of the slicing map and denote by $\mathcal{N}$ the Lebesgue negligible set of points where $Q_x$ has infinite mass.

**Step 1.** In this step we view the set of atoms with weight larger than $1/m$ as images of suitable maps defined on subsets of $\mathbf{R}^k$. To this aim, consider the set-valued function

$$A_m(x) := \begin{cases} \{y \in \pi^{-1}(x)\colon \|Q_x\|(\{y\}) \geqslant \frac{1}{m}\} & \text{if } x \in \mathbf{R}^k \setminus \mathcal{N}, \\ \emptyset & \text{if } x \in \mathcal{N} \end{cases}$$

and notice that it fulfills the measurability assumption of Lemma 6.7. Indeed, let $C \subset E$ be compact and let $\{y_q\}$ be dense in $E$. We claim that for all $x \notin \mathcal{N}$ it holds

$$\exists y \in C\colon \|Q_x\|\bigl(\{y\}\bigr) \geqslant \frac{1}{m} \quad \Longleftrightarrow \quad \forall \ell\ \exists q\colon \|Q_x\|\bigl(B_{\frac{1}{\ell}}(y_q) \cap C\bigr) \geqslant \frac{1}{m}.$$

The implication $\Rightarrow$ is trivial by density; if on the other hand there is a sequence $(y_{q(\ell)})$ such that $\|Q_x\|(B_{\frac{1}{\ell}}(y_{q(\ell)}) \cap C) \geqslant \frac{1}{m}$, any limit point $\bar{y}$ must belong to $C$ and satisfies $\|Q_x\|(B_{\frac{1}{n}}(\bar{y}) \cap C) \geqslant \frac{1}{m}$ for any given $n$, so that $y \in A_m(x)$. Hence $\{x\colon A_m(x) \cap C \neq \emptyset\}$ can be written as

$$\bigcap_{\ell} \bigcup_{q} \biggl\{x \in \mathbf{R}^k \setminus \mathcal{N}\colon \|Q_x\|\bigl(B_{\frac{1}{\ell}}(y_q) \cap C\bigr) \geqslant \frac{1}{m}\biggr\}. \tag{6.18}$$

The map $x \mapsto \|Q_x\|(B)$ is measurable for every Borel set $B$ and for every $T \in \mathbf{F}_k(E)$ (see [9, Section 3] for the proof of this result), hence the set in (6.18) is Lebesgue measurable. Since any closed set $C$ is a countable union of compact sets we obtain that $A_m$ satisfies the measurability assumption of Lemma 6.7. As a consequence, for all $n \geqslant 1$ we obtain disjoint measurable sets $B_n = \{x\colon \mathscr{H}^0(A_m(x)) = n\}$ and measurable maps $f_1, \ldots, f_n$ satisfying (6.16).

**Step 2.** In order to show that the collection of atoms is countably $\mathscr{H}^k$-rectifiable, modulo sets with Lebesgue negligible projection on $\mathbf{R}^k$, we can use Lusin's theorem and the inner regularity of the Lebesgue measure to restrict the domain of the functions $f_1, \ldots, f_n$ to a compact set $C \subset B_n$ and assume that these restrictions are continuous. Notice that since $f_i(x) \neq f_j(x)$ whenever $x \in B_n$ and $i \neq j$ we can also assume that the sets $K_i := f_i(C)$, $i = 1, \ldots, n$, are pairwise disjoint, by a further decomposition of $C$ in countably many pieces. Observe also that $\pi : K_i \to C$ is injective and its inverse is $f_i$. In order to prove the theorem it suffices to show that the sets $K_i \setminus \pi^{-1}(V_i)$ for suitable Lebesgue negligible sets $V_i \subset \mathbf{R}^k$, are countably $\mathscr{H}^k$-rectifiable: we fix an index $i$ once and for all.

Writing $x = (z, t)$ with $z \in \mathbf{R}^{k-1}$ and $t \in \mathbf{R}$, let us consider the sets

$$C_z := \bigl\{t \in \mathbf{R}\colon (z, t) \in C\bigr\}, \qquad K_{iz} := \bigl\{x \in K_i\colon (\pi_1, \ldots, \pi_{k-1})(x) = z\bigr\}$$

and the maps $g_{iz}(t) := f_i(z,t) : C_z \to K_{iz}$. We claim that, for $\mathscr{L}^{k-1}$-a.e. $z$, $\delta_t g_{iz} < \infty$ $\mathscr{L}^1$-a.e. in $C_z$. Indeed,

$$Q_x = \langle S_z, \pi_k, t\rangle \quad \text{with } S_z := \big\langle T - X, (\pi_1, \dots, \pi_{k-1}), z\big\rangle$$

we know that for $\mathscr{L}^{k-1}$-a.e. $z$ the flat chain $S_z \in \mathbf{F}_1(E)$ is the sum of a flat current with finite size and of a flat current with finite mass and (thanks to Lemma 6.6) $\langle S_z, \pi_k, t\rangle = Q_x$ for $\mathscr{L}^1$-a.e. $t \in \mathbf{R}$. It follows that $\langle S_z, \pi_k, t\rangle \llcorner K_i = Q_x \llcorner K_i$ is a Dirac mass concentrated on $g_{iz}(t)$ for $\mathscr{L}^1$-a.e. $t \in C_z$.

We fix now a point $z$ with these properties: combining Theorem 6.4 (applied to the part with fine size of $S_z$) and Theorem 6.2 (applied to the part with finite mass of $S_z$) we get a countably $\mathscr{H}^1$-rectifiable set $G_z \subset E$ and an $\mathscr{L}^1$-negligible set $N_z \subset \mathbf{R}$ such that the atoms of $\langle S_z, \pi_k, t\rangle$ lying in $K_i$ are contained in $G_z$ for all $t \in \mathbf{R} \setminus N_z$. We denote by $\tilde{K}_{iz} \subset G_z$ the set

$$\tilde{K}_{iz} := \big\{g_{iz}(t):\ t \in C_z \setminus N_z\big\}$$

which is countably $\mathscr{H}^1$-rectifiable as well and contained in $K_{iz}$. Also $\mathscr{L}^1(\pi_k(K_{iz} \setminus \tilde{K}_{iz})) = 0$ because this set is contained in $N_z$. Since $\pi_k|_{K_{iz}}$ is injective, we can now apply Proposition 5.4 with $K = \tilde{K}_{iz}$ and $\Gamma = K_{iz}$ to obtain that $\delta_t((\pi_k)|_{K_{iz}})^{-1} < \infty$ $\mathscr{L}^1$-a.e. on $\pi_k(\tilde{K}_{iz})$ and therefore $\mathscr{L}^1$-a.e. on $\pi_k(K_{iz})$. But, since the inverse of $\pi|_{K_i}$ is $f_{j_i}$, the inverse of $(\pi_k)|_{K_{iz}}$ is $g_{iz}$. It follows that $\delta_t g_{iz} < \infty$ $\mathscr{L}^1$-a.e. on $C_z$. This proves the claim.

Using the commutativity of the slice operator, we see that a similar property is fulfilled by $f_i$ with respect to the other $(k-1)$ variables, hence Theorem 5.3(i) ensures that $\delta_x f_i < \infty$ $\mathscr{L}^k$-a.e. on $C$. This ensures that Theorem 5.3(ii) is applicable to $f_i$, so that we can cover $\mathscr{L}^k$-almost all of $C$ with Borel sets $C_k$ such that the restriction of $f$ to $C_k$ is Lipschitz. Since $f(\bigcup_k C_k)$ is countably $\mathscr{H}^k$-rectifiable, we can choose $V_i = C \setminus \bigcup_k C_k$ to conclude the proof. □

### 6.3. *Proof of the main result*

In this section we prove Theorem 6.1. Let $T = X + \partial Y$. For a given $\pi \in \mathrm{Lip}_1(E, \mathbf{R}^k)$, Theorems 6.2 and 6.8 provide us two countably $\mathscr{H}^k$-rectifiable sets $\Sigma_{X,\pi}$ and $\Sigma_{\partial Y,\pi}$ such that $\mu_{X,\pi}$ is concentrated on $\Sigma_{X,\pi}$ and $\mu_{\partial Y,\pi}$ is concentrated on $\Sigma_{\partial Y,\pi}$. In particular $\mu_{T,\pi}$ is concentrated on the countably $\mathscr{H}^k$-rectifiable set $\Sigma_{T,\pi} = \Sigma_{X,\pi} \cup \Sigma_{\partial Y,\pi}$. Consider for any $n \in \mathbf{N}$ a finite set $J_n \subset \mathrm{Lip}_1(E;\mathbf{R}^k)$ of projections such that

$$\mu_T(E) \leqslant \Big(\bigvee_{\pi \in J_n} \mu_{T,\pi}\Big)(E) + 2^{-n}$$

(its existence is a direct consequence of (2.16)). Then, denoting by $J$ the union of the sets $J_n$, the measure

$$\sigma := \bigvee_{\pi \in J} \mu_{T,\pi}$$

is smaller than $\mu_T$ and with the same total mass, hence it coincides with $\mu_T$. Since $J$ is countable, a countably $\mathscr{H}^k$-rectifiable concentration set $\Sigma$ for $\mu_T$ can be obtained by taking the union $\bigcup_{\pi \in J} \Sigma_{T,\pi}$.

Finally, defining $\mathit{set}(T)$ as in (6.1), since $\mu_T$ is concentrated on $\Sigma$ the spherical differentiation theory gives $\Theta^*_k(\mu_T, x) = 0$ for $\mathscr{H}^k$-a.e. $x \in E \setminus \Sigma$, hence $\mathit{set}(T) \subset \Sigma$ up to $\mathscr{H}^k$-negligible sets.

## 7. Characterization of the size measure

In this section we improve the result of Theorem 6.1, showing a formula for the density of $\mu_T$ with respect to $\mathscr{H}^k \llcorner \mathit{set}(T)$ that involves only the local geometry of $\mathit{set}(T)$. We start by stating some differentiability properties of Lipschitz maps and rectifiable sets contained in [6].

### 7.1. *Dual of separable Banach spaces*

It is helpful for our purposes to consider the dual of a separable Banach space $Y$, like $\ell^\infty$, as an ambient space. There are mainly two reasons for this. First, we can gain some linear structure by embedding $E$ into the vector space $\ell^\infty$. In fact, since $E$ is separable, we let $\{x_k\}_{k\in\mathbf{N}}$ be a dense subset. The map $j : E \to \ell^\infty$ defined by

$$j(x) = \big(d(x,x_0) - d(x_0,x_0), d(x,x_1) - d(x_1,x_0), \dots\big)$$

provides an isometric embedding of $E$ into $\ell^\infty$. We can therefore assume $E \subset Y$.

The second reason is related to the Rademacher-type Theorem 3.5 of [6]: given $f \in \mathrm{Lip}(\mathbf{R}^k, Y)$, for $\mathscr{L}^k$-a.e. $x \in \mathbf{R}^k$ there exists a linear map $wd_x f : \mathbf{R}^k \to Y$ such that

$$w^* - \lim_{y \to x} \frac{f(y) - f(x) - wd_x f(y-x)}{|y-x|} = 0 \quad \text{and}$$
$$\lim_{y \to x} \frac{\|f(y) - f(x)\| - \|wd_x f(y-x)\|}{|y-x|} = 0.$$

The map $wd_x f$ is called the $w^*$-differential of $f$ at $x$.

### 7.2. Approximate tangent space and tangential differentiability

Let $S \subset Y$ be a countably $\mathscr{H}^k$-rectifiable set. For $\mathscr{H}^k$-almost every $x \in S$ there exists a vector space $\mathrm{Tan}^{(k)}(S, x) \subset Y$ of dimension $k$, called the approximate tangent space to $S$ at $x$. This space is defined by setting

$$\mathrm{Tan}^{(k)}\big(S, f(x)\big) = wd_x f_i\big(\mathbf{R}^k\big) \quad \text{for } \mathscr{L}^k\text{-a.e. } x \in A_i,$$

whenever $f_i$ satisfy (5.1). It can be proved that this definition does not depend on the particular choice of parametrization $f_i$; moreover this space is actually independent of the chosen embedding $j$, since its norm (inherited by the inclusion in $Z$) depends only on the distance $d$ of space $E$.

If now $\pi \in \mathrm{Lip}(S, Z)$ and $Z$ is the dual of a separable Banach space, for $\mathscr{H}^k$-almost every $x \in S$ there exists a linear map

$$d_x^S \pi : \mathrm{Tan}^{(k)}(S, x) \to Z$$

called the tangential differential of $\pi$ at $x$. As before such map can be characterized by requiring that

$$wd_x(\pi \circ f) = d^S_{f(x)} \pi \circ wd_x f$$

for any parametrization $f$ as in (5.1).

### 7.3. Jacobians and the area formula

Given a linear map $L : V \to W$ between two Banach spaces $V$ and $W$, with $\dim(V) = k$, we let

$$\mathbf{J}_k(L) = \frac{\omega_k}{\mathscr{H}^k_V(\{x:\ \|L(x)\| \leqslant 1\})} = \frac{\mathscr{H}^k_W(\{L(x):\ \|x\| \leqslant 1\})}{\omega_k}, \tag{7.1}$$

where $\omega_k$ is the $k$-dimensional Hausdorff measure of the unit ball in $\mathbf{R}^k$. We also recall that $\omega_k$ is actually the Hausdorff measure of the unit ball in any Banach space, see [21, Lemma 6]. The importance of $\mathbf{J}_k$ relies on the following general area formula:

$$\int_S \mathbf{J}_k\big(d_x^S \pi\big)\, d\mathscr{H}^k(x) = \int_Z \mathscr{H}^0\big(S \cap \pi^{-1}(y)\big)\, d\mathscr{H}^k(y). \tag{7.2}$$

We restrict our attention to the Euclidean case $Z = \mathbf{R}^k$. In order to relate $\mu_T$ to $\mathscr{H}^k \llcorner set(T)$ we need to calculate the supremum of the $k$-Jacobians $\mathbf{J}_k(d^S \pi)$ among all possible functions $\pi$. As explained in the next two lemmas, it turns out that this quantity depends only on the norm of tangent space $\mathrm{Tan}^{(k)}(S, x)$.

Let $V$ be a $k$-dimensional Banach space and denote by $B_1^V$ its unit ball. We call ellipsoid any set $R = L(B)$, where $L : \mathbf{R}^k \to V$ is a linear map and $B$ is a ball in the Euclidean space $\mathbf{R}^k$. The supremum

$$\lambda_V := \sup\left\{ \frac{\mathscr{H}^k(B_1^V)}{\mathscr{H}^k(R)}:\ B_1^V \subset R,\ R \text{ ellipsoid} \right\} \tag{7.3}$$

is called the area factor of $V$, and is clearly related to the problem of finding the best ellipsoid enclosing a convex set in $\mathbf{R}^k$. For instance if $V$ is a Hilbert space, then the spectral theorem implies $\lambda_V = 1$. The following lemma relates $\lambda_V$ to the $k$-Jacobian of linear maps [7, Lemma 9.2]:

**Lemma 7.1.** *Let $V$ be a $k$-dimensional Banach space. Then*

$$\lambda_V = \sup\{\mathbf{J}_k(\zeta):\ \zeta : V \to \mathbf{R}^k \text{ linear},\ \mathrm{Lip}(\zeta) \leqslant 1\}.$$

**Proof.** Without loss of generality we can assume that the map $\zeta$ is nonsingular. Then, the ellipsoid $\{v \in V:\ |\zeta(v)| \leqslant 1\} = \zeta^{-1}(B_1)$ contains $B_1^V$ if and only if $\mathrm{Lip}(\zeta) \leqslant 1$. Hence for such maps the area formula implies that

$$\mathbf{J}_k(\zeta) = \frac{\omega_k}{\mathscr{H}^k(\{v \in V:\ |\zeta(v)| \leqslant 1\})} = \frac{\mathscr{H}^k(B_1^V)}{\mathscr{H}^k(\{v \in V:\ |\zeta(v)| \leqslant 1\})} \leqslant \lambda_V.$$

On the other hand by definition any nontrivial ellipsoid $R = L(B)$ can be written as $\zeta^{-1}(B)$, for some linear map $\zeta$ and some Euclidean ball $B$, just setting $\zeta = L^{-1}$. By possibly rescaling one can assume that $B$ has radius 1. At this point $R = \zeta^{-1}(B_1) = \{v \in V:\ |\zeta(v)| \leqslant 1\}$ and the same inequality as above completes the proof. □

Also, we shall need the following density result [7, Lemma 9.4]:

**Lemma 7.2.** *Let $\Pi_k(Y)$ be the collection of all $w^*$-continuous linear maps $\pi : Y \to \mathbf{R}^k$, with $\dim(\pi(Y)) = k$. There exists a countable set $\{\pi^j\} \subset \Pi_k(Y)$ such that $\mathrm{Lip}(\pi^j) = 1$ for every $j \in \mathbf{N}$ and*

$$\sup_j \mathbf{J}_k\big(\pi^j\big|_V\big) = \sup\{\mathbf{J}_k(\pi|_V):\ \pi \in \Pi_k(Y),\ \mathrm{Lip}(\pi) \leqslant 1\}$$

*for any $k$-dimensional subspace $V \subset Y$.*

The proof of this lemma relies on the fact that the pseudodistance

$$\gamma\big(\pi, \pi'\big) := \sup\{\big|\big|\pi(x)\big| - \big|\pi'(x)\big|\big|:\ \|x\| \leqslant 1\}$$

makes the space $\Pi_k(Y)$ separable and that if $\gamma(\pi_h, \pi) \to 0$ then

$$\mathscr{H}^k\big(\{v \in V:\ \big|\pi(v)\big| \leqslant 1\}\big) = \lim_h \mathscr{H}^k\big(\{v \in V:\ \big|\pi_h(v)\big| \leqslant 1\}\big),$$

which, according to (7.1), makes the map $\pi \mapsto \mathbf{J}_k(\pi|_V)$ $\gamma$-continuous.

**Theorem 7.3** *(Characterization of $\mu_T$). For any $T \in \mathbf{F}_k(E)$ with finite size it holds*

$$\mu_T = \lambda \mathscr{H}^k \llcorner set(T),$$

*where $\lambda(x) = \lambda_{\mathrm{Tan}^{(k)}(set(T),x)}$ is the function defined in (7.3).*

**Proof.** The area formula (7.2) implies that if $\pi \in \mathrm{Lip}_1(E, \mathbf{R}^k)$ and $A \subset E$ is a Borel set, then

$$\mu_{T,\pi}(A) = \int_{\mathbf{R}^k} \mathscr{H}^0\big(A \cap set(T) \cap \pi^{-1}(y)\big)\, dy = \int_{A \cap set(T)} \mathbf{J}_k\big(d_x^{set(T)}\pi\big)\, d\mathscr{H}^k(x),$$

so that

$$\mu_T = \bigvee_{\pi \in \mathrm{Lip}_1(E,\mathbf{R}^k)} \mathbf{J}_k\big(d^{set(T)}\pi\big)\mathscr{H}^k \llcorner set(T). \tag{7.4}$$

It is now immediately clear by Lemma 7.1 that $\mu_T \geqslant \lambda \mathscr{H}^k \llcorner set(T)$. On the other hand, choosing $\pi$ to be one element of the countable family of maps $\pi^j$ provided by Lemma 7.2, $\mu_T$ can be bounded below by $\sup_j \mathbf{J}_k(d^{set(T)}\pi^j)\mathscr{H}^k \llcorner set(T)$. Lemmas 7.1 and 7.2 give

$$\sup_j \mathbf{J}_k\big(\pi^j\big|_{\mathrm{Tan}^{(k)}(set(T),\cdot)}\big) = \lambda_{\mathrm{Tan}^{(k)}(set(T),\cdot)}$$

and so $\mu_T = \lambda \mathscr{H}^k \llcorner set(T)$. □

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