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Lane–Emden problems: Asymptotic behavior of low energy nodal solutions ☆

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Abstract

We study the nodal solutions of the Lane–Emden–Dirichlet problem

$$\begin{cases} -\Delta u = |u|^{p-1}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ and $p > 1$. We consider solutions u_p satisfying

$$p \int_\Omega |\nabla u_p|^2 \to 16\pi e \quad \text{as } p \to +\infty \tag{$*$}$$

and we are interested in the shape and the asymptotic behavior as $p \to +\infty$.

First we prove that $(*)$ holds for least energy nodal solutions. Then we obtain some estimates and the asymptotic profile of this kind of solutions. Finally, in some cases, we prove that pu_p can be characterized as the difference of two Green's functions and the nodal line intersects the boundary of Ω, for large p.

MSC: primary 35J91; secondary 35B32

Keywords: Superlinear elliptic boundary value problem; Least energy nodal solution; Asymptotic behavior; Variational methods

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1. Introduction

We consider the superlinear elliptic boundary value problem

$$\begin{cases} -\Delta u = |u|^{p-1}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \qquad (\mathscr{P}_p)$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ and $p > 1$.

By standard variational methods we know that problem ($\mathscr{P}_p$) has a positive ground state solution. Moreover many other results about the multiplicity and the qualitative properties of positive solutions in various types of domains have been obtained in the last decades.

In this paper we are interested in studying sign changing solutions of ($\mathscr{P}_p$). In contrast with the case of positive solutions not much is known on nodal solutions of ($\mathscr{P}_p$), in particular about their qualitative behavior. Let us therefore recall some recent results. In the paper [10] A. Castro, J. Cossio and J.M. Neuberger proved the existence of a nodal solution with least energy among nodal solutions, which is therefore referred to as the *least energy nodal solution* of problem ($\mathscr{P}_p$). T. Bartsch and T. Weth showed that these solutions possess exactly two nodal regions and have Morse index two (see [3]). Since positive ground state solutions have the symmetries of the domain Ω, if Ω is convex, by the classical result of [14], a natural question is whether least energy nodal solutions also inherit the symmetries of the domain Ω. In [2] A. Aftalion and F. Pacella proved that, in a ball or in an annulus, a least energy nodal solution cannot be radial. In fact, in dimension N, they cannot be even with respect to more than $N-1$ orthogonal directions. They also proved that the nodal set touches the boundary. On the other hand, T. Bartsch, T. Weth and M. Willem in [4] and F. Pacella and T. Weth in [19], with different methods, obtained partial symmetry results: they showed that on a radial domain, a least energy nodal solution u has the so-called foliated Schwarz symmetry, i.e. u can be written as $u(x) = \tilde{u}(|x|, \xi \cdot x)$, where $\xi \in \mathbb{R}^N$ and $\tilde{u}(r, \cdot)$ is nondecreasing for every $r > 0$. In fact, as they are not radial, $\tilde{u}(r, \cdot)$ is increasing. In dimension N, it implies that the least energy nodal solutions are even with respect to $N-1$ orthogonal directions. Concerning the "last direction", in [8,15], D. Bonheure, V. Bouchez, C. Grumiau, C. Troestler and J. Van Schaftingen proved that for p close to 1 the least energy nodal solution must be odd with respect to this direction. Moreover, it is unique up to a rotation. For general open bounded domains, they prove that least energy nodal solutions must respect the symmetries of their orthogonal projection on the second eigenspace of $-\Delta$ when p is close to 1.

In this paper we study the profile and other qualitative properties of low energy nodal solutions of problem ($\mathscr{P}_p$) as $p \to +\infty$ and $\Omega \subseteq \mathbb{R}^2$ is any bounded smooth domain. For ground state positive solutions the same analysis has been done by X. Ren and J. Wei in [21] and [20], obtaining, in particular, L^∞ estimates. This result has been improved by Adimurthi and M. Grossi in [1] (see also [11]) who computed the exact value of the L^∞-norm at the limit, by a different approach.

Here by low energy we mean that we are interested in the families of nodal solutions $(u_p)_{p>1}$ satisfying

$$p \int_\Omega |\nabla u_p|^2 \to 16\pi e \quad \text{as } p \to +\infty. \qquad \text{(A)}$$

Note that as a consequence of [21] and as it will be clear later, this kind of solutions cannot have more than 2 nodal regions for p large.

Let us observe that there are nodal solutions of ($\mathscr{P}_p$) satisfying (A). In fact least energy nodal solutions are among those and we have:

Theorem 1. *The condition* (A) *holds for any family of least energy nodal solutions.*

To describe our results we need some notations. In $H_0^1(\Omega)$, we use the scalar product $(u, v) = \int_\Omega \nabla u \cdot \nabla v$ and denote by $\|\cdot\|_q$ the usual norm in $L^q(\Omega)$ and by $d(x, D)$ the distance between a point $x \in \mathbb{R}^2$ and the set $D \subseteq \mathbb{R}^2$. Let us consider a family of nodal solutions $(u_p)_{p>1}$. Throughout the paper, we assume that u_p are low energy solutions, i.e. (A) holds. The positive part u_p^+ (resp. negative part u_p^-) is defined as $u_p^+ := \max(u_p, 0)$ (resp. $u_p^- := \min(u_p, 0)$).

Let us define the families $(x_p^+)_{p>1}$ (resp. $(x_p^-)_{p>1}$) of maximum (resp. minimum) points in Ω of u_p, i.e. $u_p(x_p^+) = \|u_p^+\|_\infty$ and $u_p(x_p^-) = -\|u_p^-\|_\infty$ and assume w.l.o.g. that $u_p(x_p^+) = \|u_p\|_\infty$, i.e. $u_p(x_p^+) \geqslant -u_p(x_p^-)$. To start with, we prove that x_p^+ cannot go "too fast" to the boundary of Ω which is the key point to make some rescaling around x_p^+ and obtain a limit profile on $\mathbb{R}^2$. More precisely we prove that $\frac{d(x_p^+,\partial\Omega)}{\varepsilon_p} \to +\infty$ (see Proposition 3.1), where

$$\varepsilon_p^{-2} := p u_p\big(x_p^+\big)^{p-1}.$$

Then we get the following result.

Theorem 2. *The scaling of u_p around x_p^+:*

$$z_p(x) := \frac{p}{u_p(x_p^+)}\big(u_p\big(\varepsilon_p x + x_p^+\big) - u_p\big(x_p^+\big)\big)$$

defined on $\Omega^+(\varepsilon_p) := \frac{\Omega - x_p^+}{\varepsilon_p}$ converges, as $p \to \infty$ to a function z in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$. Moreover z must solve the equation $-\Delta z = e^z$ on $\mathbb{R}^2$, $z \leqslant 0$, $z(0) = 0$, $\int_{\mathbb{R}^2} e^z = 8\pi$ and $z(x) = \log\big(\frac{1}{(1+\frac{1}{8}|x|^2)^2}\big)$.

As a consequence of the previous theorem, we deduce that $\varepsilon_p^{-1} d(x_p^+, NL_p) \to +\infty$ as $p \to \infty$, where NL_p denotes the nodal line of u_p. So, in some sense, the rescaled solution about x_p^+ ignores the other nodal domain of u_p. This implies that we can repeat the same kind of rescaling argument in the positive nodal domain $\tilde{\Omega}_p^+ := \{x \in \Omega\colon u_p(x) > 0\}$ of u_p. Hence, defining $\tilde{\Omega}^+(\varepsilon_p) := \frac{\tilde{\Omega}_p^+ - x_p^+}{\varepsilon_p}$, we get the analogous of Theorem 2:

Theorem 3. *The function $z_p : \tilde{\Omega}^+(\varepsilon_p) \to \mathbb{R}$ converges, as $p \to +\infty$, to a function z in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ as $p \to \infty$. Moreover z must solve the equation $-\Delta z = e^z$ on $\mathbb{R}^2$, $z \leqslant 0$, $z(0) = 0$, $\int_{\mathbb{R}^2} e^z = 8\pi$ and $z(x) = \log\big(\frac{1}{(1+\frac{1}{8}|x|^2)^2}\big)$.*

At this point, to the aim of studying the negative part u_p^-, let us observe that we can have two types of families of solutions satisfying the assumption (A), the ones which satisfy

(B) there exists $K \geqslant 0$ such that $p(u_p(x_p^+) + u_p(x_p^-)) \to K$;

and the ones which satisfy

(B') $p(u_p(x_p^+) + u_p(x_p^-)) \to \infty$.

The meaning of (B) is that the speeds of convergence of the maximum and the minimum of u_p (multiplied by p) are comparable. Instead the condition (B') implies that one of the two values converges faster than the other one.

Remark 4. It is easy to see that nodal solutions of type (B) exist. Indeed, if Ω is a ball, it is enough to consider the antisymmetric, with respect to a diameter, solution with two nodal regions. We believe that also solution of type (B') should exist and we conjecture that the radial solution in the ball, with two nodal regions, should be of type (B'). However, the complete characterization of low energy solutions in the ball will be analyzed in a subsequent paper.

In this paper we investigate the alternative (B) that we conjecture holding for the least energy nodal solutions.

First, we prove that, as for x_p^+, the condition (B) implies that $\varepsilon_p^{-1} d(x_p^-, \partial\Omega) \to +\infty$ as $p \to \infty$. Then we get the following result.

Theorem 5. *If (B) holds then the scaling of u_p around x_p^-*

$$z_p^-(x) := \frac{p}{u_p(x_p^+)}\big(-u_p\big(\varepsilon_p x + x_p^-\big) - u_p\big(x_p^+\big)\big)$$

defined on $\Omega^-(\varepsilon_p) := \frac{\Omega - x_p^-}{\varepsilon_p}$ *converges, as* $p \to +\infty$*, to a function* z *in* $C^2_{\mathrm{loc}}(\mathbb{R}^2)$*. Moreover* z *must solve the equation* $-\Delta z = e^z$ *on* $\mathbb{R}^2$*,* $z \leqslant 0$*,* $\int_{\mathbb{R}^2} e^z = 8\pi$ *and* $z(x) = \log(\frac{\mu}{(1+\frac{\mu}{8}|x|^2)^2})$ *for some* $0 < \mu \leqslant 1$*. When* $K = 0$ *in condition* (B)*, we get* $\mu = 1$*.*

As for the case of x_p^+, as a consequence of Theorem 5, we get that $\varepsilon_p^{-1} d(x_p^-, NL_p) \to +\infty$, which allows to do the same rescaling in the negative nodal domain $\tilde{\Omega}_p^- := \{x \in \Omega\colon u_p(x) < 0\}$, obtaining the analogous of Theorem 5.

Theorem 6. *If* (B) *holds, the function*

$$z_p^-(x) := \frac{p}{\|u_p\|_\infty}\big(-u_p^-\big(\varepsilon_p x + x_p^-\big) - \|u_p\|_\infty\big)$$

defined on $\tilde{\Omega}^-(\varepsilon_p) := \frac{\tilde{\Omega}_p^- - x_p^-}{\varepsilon_p}$ *converges, as* $p \to +\infty$*, to a function* z *in* $C^2_{\mathrm{loc}}(\mathbb{R}^2)$*. Moreover* z *must solve the equation* $-\Delta z = e^z$ *on* $\mathbb{R}^2$*,* $z \leqslant 0$*,* $\int_{\mathbb{R}^2} e^z = 8\pi$ *and* $z(x) = \log(\frac{\mu}{(1+\frac{\mu}{8}|x|^2)^2})$ *for some* $0 < \mu \leqslant 1$*. When* $K = 0$ *in condition* (B)*, we get* $\mu = 1$*.*

Remark 7. Another natural condition to make the rescaling in the negative nodal domain without assuming condition (B) could be to consider the parameter

$$\tilde{\varepsilon}_p^{-2} = p\big|u_p^-\big(x_p^-\big)\big|^{p-1}$$

which is now just related to the negative part of u (we are not using the L^∞-norm of u_p but the L^∞-norm of u_p^-) and assume that $\tilde{\varepsilon}_p^{-1} d(x_p^-, NL_p) \to +\infty$ (as before NL_p is the nodal line of u_p). This assumption is essentially equivalent to condition (B) and allows to prove that $\tilde{\varepsilon}_p^{-1} d(x_p^-, \partial\Omega) \to +\infty$ (see Proposition 3.3). Then one could repeat the proof of Theorem 6 obtaining for $z_p(x) := \frac{p}{u_p(x_p^-)}(u_p^-(\tilde{\varepsilon}_p x + x_p^-) - u_p(x_p^-))$ the same assertion as for z_p^-.

If the positive part of u, i.e. u_p^+, as a solution of $(\mathscr{P}_p)$ in $\tilde{\Omega}^+(\varepsilon_p)$, has Morse index one then the previous results allow to obtain the exact value of the limits of $\|u_p^\pm\|_\infty$, as $p \to +\infty$.

Theorem 8. *Let us assume that the Morse index of* u_p^+ *as a solution of* $(\mathscr{P}_p)$ *in* $\tilde{\Omega}_p^+$ *is one. Then we have:* $\|u_p^+\|_\infty \to e^{1/2}$*. If also* (B) *holds then* $\|u_p^-\|_\infty \to e^{1/2}$*.*

The result of the previous statement is similar to the one obtained in [1] for the least energy positive solution of $(\mathscr{P}_p)$.

Let us remark that the additional assumption on the Morse index of u_p^+ holds for any nodal solutions with Morse index 2, hence, in particular, for least energy nodal solutions.

Our last result gives the asymptotic behavior of the nodal solutions in the whole domain Ω.

Let us denote by $G(x,y) = -\frac{1}{2\pi}\log|x-y| + H(x,y)$ the Green's function of Ω and by H its regular part. Finally, let $x^\pm$ be the limit point of $x_p^\pm$ as $p \to +\infty$.

Theorem 9. *Under the same hypothesis of Theorem 8,* pu_p *converges, as* $p \to +\infty$*, to the function* $8\pi e^{1/2}(G(\cdot, x^+) - G(\cdot, x^-))$ *in* $\mathscr{C}^2_{\mathrm{loc}}(\bar{\Omega} \setminus \{x^-, x^+\})$ *and* $x^+ \neq x^- \in \Omega$*. Moreover the limit points* x^+ *and* x^- *satisfy the system*

$$\begin{cases} \dfrac{\partial G}{\partial x_i}\big(x^+, x^-\big) - \dfrac{\partial H}{\partial x_i}\big(x^+, x^+\big) = 0, \\ \dfrac{\partial G}{\partial x_i}\big(x^-, x^+\big) - \dfrac{\partial H}{\partial x_i}\big(x^-, x^-\big) = 0, \end{cases}$$

for $i = 1, 2$*. Finally, the nodal line of* u_p *intersects the boundary of* Ω *for* p *large.*

The result of Theorem 9 gives a very accurate description of the profile of the low energy solutions of type (B) in terms of the Green function of Ω and of its regular part. It is also remarkable that the property that the nodal line intersects $\partial\Omega$ holds for this kind of solutions in any bounded domain Ω, extending so the result proved in [2] for least energy solutions in balls or annulus. It is also reminiscent of the property of the second eigenfunction of the laplacian in planar convex domains (see [18]), though we are not analyzing the case of p close to 1 as in [8,15].

Let us remark that nodal solutions with this property have been constructed in [13,12].

Finally we would like to point out that our analysis is similar to the one carried out in [5–7] for low energy nodal solutions of an almost critical problem or of the Brezis–Nirenberg problem in dimension $N \geqslant 3$. However, the techniques and the proofs are completely different since in [5–7] the nodal solutions whose energy is close to $2S_N$ (S_N is the best Sobolev constant in $\mathbb{R}^N$) can be written almost explicitly.

The outline of the paper is as follows. In Section 2, we recall the variational characterization of the problem and we prove Theorem 1 and some useful asymptotic estimates. In Section 3, we show that x_p^+ cannot go too fast to the boundary and then prove Theorem 2 and Theorem 5 using a rescaling argument on the whole domain Ω. Then, using a rescaling argument on the nodal domains, we prove Theorem 3 and Theorem 6. In Section 4, we improve the bounds given in Section 2 to obtain Theorem 8. Finally, in Section 5, we prove Theorem 9.

2. Variational setting and estimates

We recall that solutions of problem $(\mathscr{P}_p)$ are the critical points of the energy functional $\mathscr{E}_p$ defined on $H_0^1(\Omega)$ by

$$\mathscr{E}_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}.$$

The Nehari manifold $\mathscr{N}_p$ and the nodal Nehari set $\mathscr{M}_p$ are defined by

$$\mathscr{N}_p := \{u \in H_0^1(\Omega) \setminus \{0\}: \langle \mathrm{d}\mathscr{E}_p(u), u\rangle = 0\}, \qquad \mathscr{M}_p := \{u \in H_0^1(\Omega): u^\pm \in \mathscr{N}_p\},$$

where $u^+(x) := \max(u(x), 0)$ and $u^-(x) := \min(u(x), 0)$. If $u \in H_0^1(\Omega)$, $u^+ \neq 0$ and $u^- \neq 0$ then $u \in \mathscr{M}_p$ if and only if

$$\int_\Omega |\nabla u^+|^2 = \int_\Omega |u^+|^{p+1} \quad \text{and} \quad \int_\Omega |\nabla u^-|^2 = \int_\Omega |u^-|^{p+1}. \tag{1}$$

For any $u \neq 0$ fixed, there exists a unique multiplicative factor α such that $\alpha u \in \mathscr{N}_p$. If u changes sign then there exists a unique couple (α_+, α_-) such that $\alpha_+ u^+ + \alpha_- u^- \in \mathscr{M}_p$.

The interest of $\mathscr{N}_p$ (resp. $\mathscr{M}_p$) comes from the fact that it contains all the non-zero (resp. sign-changing) critical points of $\mathscr{E}_p$. If u minimizes $\mathscr{E}_p$ on $\mathscr{N}_p$ (resp. $\mathscr{M}_p$) then u is a (resp. nodal) solution of problem $(\mathscr{P}_p)$ usually referred to as the *ground state solutions* (resp. *least energy nodal solutions*). So, we need to solve

$$\inf\left\{\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |\nabla u|^2\right\} \quad \text{on} \int_\Omega |\nabla u^\pm|^2 = \int_\Omega (u^\pm)^{p+1}$$

to characterize the least energy nodal solutions.

Theorem 2.1. *(See T. Bartsch, T. Weth [3].) There exists a least energy nodal solution of problem $(\mathscr{P}_p)$ which has exactly two nodal domains and Morse index* 2.

To start with, we show that each family of least energy nodal solutions for problem $(\mathscr{P}_p)$ is a family of low energy nodal solutions, i.e. satisfies condition (A) of the Introduction. To this aim let us prove an upper bound and a control on the energy.

Lemma 2.2. *Let $(u_p)_{p>1}$ be a family of least energy nodal solutions of problem $(\mathscr{P}_p)$. For any $\varepsilon > 0$, there exists p_ε such that, for any $p \geqslant p_\varepsilon$,*

$$p\mathscr{E}_p(u_p) = p\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |\nabla u_p|^2 \leqslant 8\pi e + \varepsilon.$$

Proof. Let $a, b \in \Omega$. Let us consider $0 < r < 1$ such that $B(a,r), B(b,r) \subseteq \Omega$ and $B(a,r) \cap B(b,r) = \emptyset$. Then, we define a cut-off function $\varphi : \Omega \to [0,1]$ in $\mathscr{C}_0^\infty(\Omega)$ such that

$$\varphi(x) := \begin{cases} 1 & \text{if } |x-a| < r/2, \\ 0 & \text{if } |x-a| \geqslant r. \end{cases}$$

First we introduce the family of functions $\bar{W}_p : \Omega \to \mathbb{R}$ which are defined on $B(a,r)$ as

$$\bar{W}_p(x) := \varphi(x)\sqrt{e}\left(1 + \frac{z(\frac{x-a}{\varepsilon_p})}{p}\right)$$

where $z(x) = -2\log(1 + \frac{|x|^2}{8})$ and $\varepsilon_p^2 := \frac{1}{p\sqrt{e^{p-1}}}$. The functions $\bar{W}_p$ vanish outside the ball $B(a,r)$. We claim that

$$\int_\Omega |\bar{W}_p|^{p+1} = \frac{8\pi e}{p} + o(1/p),$$
$$\int_\Omega |\nabla \bar{W}_p|^2 = \frac{8\pi e}{p} + o(1/p).$$

Indeed, setting $\frac{x-a}{\varepsilon_p} = \psi$ and using the fact that $\int_{\mathbb{R}^2} e^z = 8\pi$,

$$\begin{aligned}\int_\Omega |\bar{W}_p|^{p+1} &= (\sqrt{e})^{p+1}\varepsilon_p^2 \int_{\frac{\Omega-a}{\varepsilon_p}} \varphi(\varepsilon\psi + a)^{p+1}\left(1 + \frac{z(\psi)}{p}\right)^{p+1} \mathrm{d}\psi \\ &= \frac{e}{p}\left(\int_{\mathbb{R}^2} e^z + o(1)\right) \\ &= \frac{8\pi e}{p} + o(1/p).\end{aligned}$$

Concerning $\int_\Omega |\nabla \bar{W}_p|^2$, we get that

$$\begin{aligned}\int_\Omega |\nabla \bar{W}_p|^2 = {} & \int_\Omega \varphi^2(x)\left|\nabla\left(\sqrt{e}\left(1 + \frac{z((x-a)/\varepsilon_p)}{p}\right)\right)\right|^2 + \int_\Omega \left|\nabla\varphi(x)\right|^2\left(\sqrt{e}\left(1 + \frac{z((x-a)/\varepsilon_p)}{p}\right)\right)^2 \\ & + 2\int_\Omega \varphi(x)\sqrt{e}\left(1 + \frac{z((x-a)/\varepsilon_p)}{p}\right)\nabla\varphi(x) \cdot \nabla\left(\sqrt{e}\left(1 + \frac{z((x-a)/\varepsilon_p)}{p}\right)\right).\end{aligned}$$

The first term gives

$$\begin{aligned}&\int_\Omega \varphi^2(x)\left|\nabla\left(\sqrt{e}\left(1 + \frac{z((x-a)/\varepsilon_p)}{p}\right)\right)\right|^2 \\ &\quad = \frac{e}{p^2}\int_\Omega 16\varphi^2(x)\frac{|x-a|^2}{(8\varepsilon_p^2 + |x-a|^2)^2} \\ &\quad = \frac{16e}{p^2}\left\{\int_{B(a,r/2)} \frac{|x-a|^2}{(8\varepsilon_p^2 + |x-a|^2)^2} + \int_{\Omega\setminus B(a,r/2)} \varphi^2(x)\frac{|x-a|^2}{(8\varepsilon_p^2 + |x-a|^2)^2}\right\} \\ &\quad = \frac{16e}{p^2}\left(2\pi\int_0^{r/2} \frac{\psi^3}{(8\varepsilon_p^2 + \psi^2)^2} + O(1)\right).\end{aligned}$$

Setting $\psi^2 = t$ and integrating, we get

$$\int_0^{r/2} \frac{\psi^3}{(8\varepsilon_p^2 + \psi^2)^2} = \frac{1}{2}\log\left|\frac{\frac{r}{2} + 8\varepsilon_p^2}{8\varepsilon_p^2}\right| + \frac{1}{2}\left(\frac{8\varepsilon_p^2}{8\varepsilon_p^2 + \frac{r}{2}} - 1\right) = -\log|\varepsilon_p| + O(1).$$

So, we get

$$\begin{aligned}\int_\Omega \varphi^2(x)\left|\nabla\left(\sqrt{e}\left(1 + \frac{z((x-a)/\varepsilon_p)}{p}\right)\right)\right|^2 &= -\frac{32\pi e}{p^2}\big(\log\varepsilon_p + O(1)\big)\\ &= \frac{-32e\pi}{p^2}\left(-\frac{p-1}{4} + o(p) + O(1)\right)\\ &= \frac{8\pi e}{p} + o(1/p).\end{aligned}$$

The second term gives the existence of a constant $K > 0$ such that

$$\begin{aligned}\int_\Omega \left|\nabla\varphi(x)\right|^2\left(\sqrt{e}\left(1 + \frac{z((x-a)/\varepsilon_p)}{p}\right)\right)^2 &= \int_{B(a,r)\setminus B(a,r/2)} \left|\nabla\varphi(x)\right|^2\left(\sqrt{e}\left(1 + \frac{z((x-a)/\varepsilon_p)}{p}\right)\right)^2\\ &\leqslant K\frac{(1 + 2\max_{x\in\Omega\setminus B(a,1/p)}|\log(\frac{|x-a|^2 p}{8})| + K)^2}{p^2}\\ &= o(1/p).\end{aligned}$$

The third term can be treated with similar techniques. So, finally, we get

$$\int_\Omega |\nabla\bar{W}_p|^2 = \frac{8\pi e}{p} + o(1/p)$$

which proves the claim.

Then, we define the family of test functions $W_p : \Omega \to \mathbb{R}$ which are defined on $B(a,r)$ as $\bar{W}_p$ and on $B(b,r)$ as the odd reflection of $\bar{W}_p$. The functions W_p vanish outside the two balls $B(a,r)$ and $B(b,r)$. So, $\|\nabla W_p^\pm\|_2^2 = \frac{8\pi e}{p} + o(1/p)$ and $\|W_p^\pm\|_{p+1}^{p+1} = \frac{8\pi e}{p} + o(1/p)$. Clearly, the unique multiplicative factor $\alpha_p := \alpha_p^+$ such that $\alpha_p^+ W_p^+ \in \mathcal{N}_p$ equals the unique multiplicative factor α_p^- such that $\alpha_p^- W_p^- \in \mathcal{N}_p$. To characterize it, we need to solve

$$\alpha_p^2\|\nabla W_p^\pm\|_2^2 = \alpha_p^{p+1}\|W_p^\pm\|_{p+1}^{p+1}.$$

It implies that

$$\alpha_p = \left(\frac{\int_\Omega |\nabla W_p^\pm|^2}{\int_\Omega |W_p^\pm|^{p+1}}\right)^{\frac{1}{p-1}} \to 1. \tag{2}$$

So, as u_p is a minimum for the H_0^1-norm on $\mathcal{M}_p$ and $\int_\Omega |\nabla u_p|^2 = \int_\Omega |\nabla u_p^+|^2 + \int_\Omega |\nabla u_p^-|^2$, we conclude that

$$p\left(\frac{1}{2} - \frac{1}{p+1}\right)\|\nabla u_p\|_2^2 \leqslant p\left(\frac{1}{2} - \frac{1}{p+1}\right)2(\alpha_p)^2\int_\Omega \left|\nabla W_p^+\right|^2.$$

As the right-hand side converges to $8\pi e$, we get the assertion. □

Lemma 2.3. *Let $(u_p)_{p>1}$ be a family of least energy nodal solutions of problem ($\mathcal{P}_p$). For any $\varepsilon > 0$, there exists p_ε such that, for any $p \geqslant p_\varepsilon$,*

$$p\mathcal{E}_p(u_p) = p\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |\nabla u_p|^2 \geqslant 8\pi e - \varepsilon.$$

Proof. To do this, we prove that for any sequence $p_n \to +\infty$ $\liminf_{n\to+\infty} p_n(\frac{1}{2} - \frac{1}{p_n+1})\int_\Omega |\nabla u_{p_n}^\pm|^2 \geqslant 4\pi e$. On one hand, $1 = \frac{\int_\Omega (u_{p_n}^\pm)^{p_n+1}}{\int_\Omega |\nabla u_{p_n}^\pm|^2}$. On the other hand, in [21, p. 752], it is proved that, for any $t > 1$, $\|u\|_t \leqslant D_t t^{1/2} \|\nabla u\|_2$ where $D_t \to (8\pi e)^{-1/2}$ is independent of u in $H_0^1(\Omega)$.

So, we obtain

$$1 \leqslant D_{p_n+1}^{p_n+1} (p_n+1)^{\frac{p_n+1}{2}} \left(\int_\Omega |\nabla u_{p_n}^\pm|^2 \right)^{\frac{p_n-1}{2}},$$

i.e. $\int_\Omega |\nabla u_{p_n}^\pm|^2 \geqslant D_{p_n+1}^{-2\frac{p_n+1}{p_n-1}} (p_n+1)^{-\frac{p_n+1}{p_n-1}}$. Thus,

$$\left(\frac{1}{2} - \frac{1}{p_n+1}\right)(p_n+1)^{\frac{p_n+1}{p_n-1}} \int_\Omega |\nabla u_{p_n}^\pm|^2 \geqslant \left(\frac{1}{2} - \frac{1}{p_n+1}\right) D_{p_n+1}^{-2\frac{p_n+1}{p_n-1}}.$$

As $\frac{p_n}{(p_n+1)^{\frac{p_n+1}{p_n-1}}}$ converges to 1 and the right-hand side converges to $4\pi e$, we get the assertion. □

Proof of Theorem 1. It follows from Lemma 2.2 and Lemma 2.3. □

Remark 2.4. The proof of Lemma 2.3 does not depend on the fact that u_{p_n} is a least energy nodal solution. Indeed, for any $(u_p)_{p>1}$ verifying (A), as $p \to +\infty$, we get

- $p(\frac{1}{2} - \frac{1}{p})\int_\Omega |\nabla u_p^\pm|^2 \to 4\pi e$, $p\int_\Omega |\nabla u_p^\pm|^2 \to 8\pi e$ and $p\int_\Omega |\nabla u_p|^2 \to 16\pi e$.
- $\mathscr{E}_p(u_p) \to 0$, $\int_\Omega |\nabla u_p|^2 \to 0$, $\int_\Omega |\nabla u_p^-|^2 \to 0$ and $\int_\Omega |\nabla u_p^+|^2 \to 0$.

Moreover the proof of Lemma 2.3 implies, as corollary, that u_p has 2 nodal domains for p large.

From now on, throughout the paper, we consider a family $(u_p)_{p>1}$ of nodal solutions for which (A) holds. The following result shows an asymptotic lower bound for the L^∞-norms of u_p^+ and u_p^-. We denote by $\lambda_1(D)$ the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in a domain D and by $x_p^\pm$ both the maximum or the minimum point of u_p, as defined in the Introduction.

Proposition 2.5. *For any $p > 1$ we have that $|u_p(x_p^\pm)| \geqslant \lambda_1^{\frac{1}{p-1}}$ where $\lambda_1 := \lambda_1(\Omega)$.*

Proof. Using Poincaré's inequality, we get

$$1 = \frac{\int_\Omega |u_p^\pm|^{p+1}}{\int_\Omega |\nabla u_p^\pm|^2} \leqslant \frac{|u_p(x_p^\pm)|^{p-1} \int_\Omega (u_p^\pm)^2}{\int_\Omega |\nabla u_p^\pm|^2} \leqslant |u_p(x_p^\pm)|^{p-1} \lambda_1^{-1}(\tilde{\Omega}_p^\pm),$$

where $\tilde{\Omega}_p^\pm$ are the nodal domains of u_p. As $\tilde{\Omega}_p^\pm \subseteq \Omega$, we have $\lambda_1(\tilde{\Omega}_p^\pm) \geqslant \lambda_1$ which ends the proof. □

Remark 2.6. We have:

- For any $\varepsilon > 0$, $|u_p(x_p^\pm)| \geqslant 1 - \varepsilon$ for p large. In particular this holds for $\|u_p\|_{L^{+\infty}}$.
- By Remark 2.4, as $|u_p(x_p^\pm)|^{p-1}$ is bounded from below, $\frac{|u_p(x_p^\pm)|^{p-1}}{\int_\Omega |\nabla u_p^\pm|^2}$ and $\frac{|u_p(x_p^\pm)|^{p-1}}{\int_\Omega |\nabla u_p|^2}$ converge to $+\infty$ when $p \to +\infty$.

The next result gives a direct argument to prove that the L^∞-norms of u_p^+ and u_p^- are bounded. It will be improved in the next sections.

Proposition 2.7. *We have that $u_p(x_p^\pm)$ is bounded as $p \to +\infty$.*

Proof. Let us make the proof for the positive case. By Proposition 2.5, we only have to prove that $u_p(x_p^+)$ is bounded from above. Let us denote by G the Green's function on Ω. As $|G(x,y)| \leqslant C|\log|x-y||$ for any $x, y \in \Omega$ and some independent constant $C > 0$, using the Hölder inequality we have

$$\begin{aligned} u_p(x_p^+) &= \int_\Omega G(x_p^+, y)|u_p(y)^{p-1}|u_p(y)\,\mathrm{d}y \\ &\leqslant C\int_\Omega \big|\log|x_p^+ - y|\big|\big|u_p(y)^p\big|\,\mathrm{d}y \\ &\leqslant C\Big(\int_\Omega \big|\log|x_p^+ - y|\big|^{p+1}\,\mathrm{d}y\Big)^{\frac{1}{p+1}}\Big(\int_\Omega |u_p|^{p+1}\Big)^{\frac{p}{p+1}}. \end{aligned}$$

Since $p\int_\Omega |u_p|^{p+1} \to 16\pi e$ as $p \to +\infty$ (see Remark 2.4), it is enough to show the existence of a constant $C > 0$ such that

$$\int_\Omega \big|\log|x_p^+ - y|\big|^{p+1}\,\mathrm{d}y \leqslant C(p+1)^{p+2}.$$

Let us consider $R > 0$ such that $\Omega \subseteq B(x_p, R)$ for all n. Then there exists a constant $K > 0$ such that

$$\int_\Omega \big|\log|x_p^+ - y|\big|^{p+1}\,\mathrm{d}y \leqslant \int_{B(x_p,R)} \big|\log|x_p^+ - y|\big|^{p+1}\,\mathrm{d}y = K\int_0^R |\log r|^{p+1} r\,\mathrm{d}r.$$

Integrating $([p]+1)$-times by parts, we get

$$\begin{aligned} \int_\Omega \big|\log|x_p^+ - y|\big|^{p+1}\,\mathrm{d}y &\leqslant K\big\{\big|\log(R)\big|^{p+1} + (p+1)\big|\log(R)\big|^p + \cdots + (p+1)\cdots\big(p-[p]+2\big) \\ &\quad \times \big|\log(R)\big|^{p-[p]+1}\big\} + K(p+1)p\cdots\big(p-[p]+1\big)\int_0^R |\log r|^{p-[p]} r\,\mathrm{d}r. \end{aligned}$$

Thus, there exists C such that for large n

$$\int_\Omega \big|\log|x_p^+ - y|\big|^{p+1}\,\mathrm{d}y \leqslant C(p+1)^{p+2},$$

which ends the proof. □

3. Asymptotic behavior

For the rest of the paper, w.l.o.g., let us assume that $\|u_p\|_\infty = u_p(x_p^+)$ for any $p > 1$.

In this section we use several rescaling arguments to characterize the asymptotic behavior of $u_p^\pm$.

Let us define $\varepsilon_p^2 := \frac{1}{pu_p(x_p^+)^{p-1}} \to 0$ by Remark 2.6.

3.1. Control close to the boundary

We prove that x_p^+ cannot go to the boundary of Ω too fast.

Proposition 3.1. *We have*

$$\frac{d(x_p^+,\partial\Omega)}{\varepsilon_p}\to+\infty \tag{3}$$

as $p\to+\infty$.

Proof. Let us argue by contradiction and assume that, for a sequence $p_n\to+\infty$, $\frac{d(x_{p_n}^+,\partial\Omega)}{\varepsilon_{p_n}}\to l\geqslant 0$ and that $x_{p_n}^+\to x_*\in\partial\Omega$ (i.e. $\frac{d(x_{p_n}^+,x_*)}{\varepsilon_{p_n}}\to l$).

First, we treat the case when $\partial\Omega$ is flat around x_*. We consider a semi-ball D centered in x_* with radius R such that $D\subseteq\Omega$ and the diameter of D belongs to $\partial\Omega$. For large n, let us remark that $x_{p_n}^+$ belongs to D. Then, on $A:=B(x_*,R)$, we consider the function $u_{p_n}^*$ which is defined as u_{p_n} on D and as the odd reflection of u_{p_n} on $A\setminus D$. It is a solution of $-\Delta u=|u|^{p_n-1}u$ on A. For large n, we consider

$$z_{p_n}^*(x):=\frac{p_n}{u_{p_n}^*(x_{p_n}^+)}\bigl(u_{p_n}^*\bigl(\varepsilon_{p_n}x+x_{p_n}^+\bigr)-u_{p_n}^*\bigl(x_{p_n}^+\bigr)\bigr) \tag{4}$$

on $\Omega_{p_n}^*:=\frac{A-x_{p_n}^+}{\varepsilon_{p_n}}\to\mathbb{R}^2$. On $\Omega_{p_n}^*$, we get from (4)

$$\begin{cases}-\Delta z_{p_n}^*=\left|1+\dfrac{z_{p_n}^*}{p_n}\right|^{p_n-1}\left(1+\dfrac{z_{p_n}^*}{p_n}\right),\\[2mm] \left|1+\dfrac{z_{p_n}^*}{p_n}\right|\leqslant 1.\end{cases}$$

Let us fix $R>0$. For large n, $B(0,R)\subseteq\Omega_{p_n}^*$ and we consider the problem

$$\begin{cases}-\Delta w_{p_n}=\left|1+\dfrac{z_{p_n}^*}{p_n}\right|^{p_n-1}\left(1+\dfrac{z_{p_n}^*}{p_n}\right), & \text{in } B(0,R),\\ w_{p_n}=0, & \text{on } \partial B(0,R).\end{cases}$$

Since, by (4), $|1+\frac{z_{p_n}^*}{p_n}|\leqslant 1$, we have that $|w_{p_n}|$ is uniformly bounded by a constant C independent of n by the maximum principle and the regularity theory. Moreover, because $z_{p_n}\leqslant 0$, we have that $\psi_{p_n}=z_{p_n}-w_{p_n}$ is a harmonic function which is uniformly bounded above. By Harnack's inequality, ψ_{p_n} is bounded in $L^\infty(B(0,R))$ or tends to $-\infty$ on each compact set of $B(0,R)$. As $\psi_{p_n}(0)=z_{p_n}(0)-w_{p_n}(0)\geqslant -C$, we get that ψ_{p_n} and z_{p_n} are uniformly bounded on each compact set of $B(0,R)$.

Since we are assuming that $\frac{d(x_{p_n},x_*)}{\varepsilon_{p_n}}\to l$ we get that $y_n:=\frac{x_*-x_{p_n}^+}{\varepsilon_{p_n}}\in B[0,l+1]$ for large n and $z_{p_n}(y_n)=-p_n\to-\infty$ which is a contradiction.

Next, we treat the case when $\partial\Omega$ is not locally flat around x_* but is a $\mathscr{C}^1$-curve. We consider a $\mathscr{C}^1$-domain D which is the intersection of a fixed neighborhood of x_* and Ω. Let us define the square $Q:=(-1,1)^2$, $Q^+:=(-1,1)\times(0,1)\subseteq Q$ and $S:=(-1,1)\times\{0\}$.

We consider the change of variables $\varphi:D\to Q^+$ and $\varphi(D\cap\partial\Omega)=S$ (see [9] to get that φ is well-defined and can be assumed to be $\mathscr{C}^1(\bar D)$). Moreover $\varphi^-1\in\mathscr{C}^1(\bar Q^+)$.

We fix a positive function $\theta\in C^2$ such that $\theta\circ\varphi^{-1}:\bar Q^+\to\mathbb{R}$ equals 0 on $\partial Q^+\setminus S$ and $\partial_\nu\theta\circ\varphi^{-1}=0$ on S where ∂_ν denotes the normal derivative. We extend $\theta\circ\varphi^{-1}$ on Q by even symmetry with respect to S.

On Q, we define $\tilde u_{p_n}$ as $\theta(\varphi^{-1}(\cdot))u_{p_n}(\varphi^{-1}(\cdot))$ on Q^+ and the odd symmetric function on $Q\setminus Q^+$. Since θu_{p_n} solves

$$-\Delta u=\theta|u_{p_n}|^{p_n-1}u_{p_n}-2\nabla\theta\nabla u_{p_n}-(\Delta\theta)u_{p_n}=:g_{p_n} \tag{5}$$

with Dirichlet boundary conditions on D, by the change of variables $y=\varphi(x)$, we get that $\tilde u_{p_n}$ solves for some matrix A_{p_n}

$$-div(A_{p_n}\nabla u)=h_{p_n}$$

with Dirichlet boundary conditions on Q and where h_{p_n} is $g_{p_n} \circ \varphi^{-1}$ on Q^+ and the antisymmetric on $Q \setminus Q^+$. Coming back to Ω by the change of variables $x = \varphi^{-1}(y)$ we get that $\theta u^*_{p_n} = \tilde{u}_{p_n}(\varphi(\cdot))$ solves $-\Delta u = h_{p_n} \circ \varphi$ on $A := \varphi^{-1}(Q)$.

As θ is positive, it implies that $u^*_{p_n}$ solves $-\Delta u = |u|^{p_n-1}u$ on A.

We conclude by working in the same way as in the first case. □

3.2. *Rescaling argument in Ω around x_p^+: limit equation in $\mathbb{R}^2$*

The idea is inspired by [1]. Let us consider $\Omega^+(\varepsilon_p) := \frac{\Omega - x_p^+}{\varepsilon_p}$ and $z_p : \Omega^+(\varepsilon_p) \to \mathbb{R}$ the scaling of u_p around x_p^+:

$$z_p(x) := \frac{p}{u_p(x_p^+)}\big(u_p\big(\varepsilon_p x + x_p^+\big) - u_p\big(x_p^+\big)\big). \tag{6}$$

Proof of Theorem 2. Let p_n be a sequence, $p_n \to +\infty$. As in the previous proof, we have that z_{p_n} solves the equation

$$\begin{cases} -\Delta z_{p_n} = \left|1 + \dfrac{z_{p_n}}{p_n}\right|^{p_n-1}\left(1 + \dfrac{z_{p_n}}{p_n}\right), & \text{in } \Omega^+(\varepsilon_{p_n}),\\ \left|1 + \dfrac{z_{p_n}}{p_n}\right| \leqslant 1, & \\ z_{p_n} = -p_n, & \text{on } \partial\Omega^+(\varepsilon_{p_n}). \end{cases}$$

Let us fix $R > 0$. By Proposition 3.1, we know that $\frac{d(x_{p_n}^+, \partial\Omega)}{\varepsilon_{p_n}} \to +\infty$. So, $\Omega^+(\varepsilon_{p_n})$ "converges" to $\mathbb{R}^2$ as $p_n \to +\infty$, i.e. $B(0,R) \subseteq \Omega^+(\varepsilon_{p_n})$ for large n. Let us consider the problem

$$\begin{cases} -\Delta w_{p_n} = \left|1 + \dfrac{z_{p_n}}{p_n}\right|^{p_n-1}\left(1 + \dfrac{z_{p_n}}{p_n}\right), & \text{in } B(0,R),\\ w_{p_n} = 0, & \text{on } \partial B(0,R). \end{cases}$$

Since, by (6), $|1 + \frac{z_{p_n}}{p_n}| \leqslant 1$, we get that $|w_{p_n}| \leqslant C$ independent of n. By arguing as before, we get that ψ_{p_n} and z_{p_n} are bounded up to a subsequence in $L^\infty(B(0,R))$ for any R.

Thus, by the standard regularity theory, z_{p_n} is bounded in $C^2_{loc}(\mathbb{R}^2)$ and, on each ball, $1 + \frac{z_{p_n}}{p_n} > 0$ for large n. We have that $z_{p_n} \to z$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $-\Delta z = e^z$.

To finish, we prove that $\int_{\mathbb{R}^2} e^z < +\infty$. We have that $z_{p_n} + p_n(\log|1 + \frac{z_{p_n}}{p_n}| - \frac{z_{p_n}}{p_n})$ converges pointwisely to z in $\mathbb{R}^2$. By Fatou's lemma, we deduce

$$\begin{aligned} \int_{\mathbb{R}^2} e^z &\leqslant \lim_n \int_{\Omega^+(\varepsilon_{p_n})} e^{z_{p_n} + p_n(\log|1+\frac{z_{p_n}}{p_n}| - \frac{z_{p_n}}{p_n})} \\ &= \lim_n \int_{\Omega^+(\varepsilon_{p_n})} \left|1 + \frac{z_{p_n}}{p_n}\right|^{p_n} \\ &\leqslant \lim_n \int_\Omega \frac{|u_{p_n}|^{p_n}}{\varepsilon_{p_n}^2 |u_{p_n}(x_{p_n}^+)|^{p_n}} \\ &= \lim_n \int_\Omega \frac{p_n}{|u_{p_n}(x_{p_n}^+)|}|u_{p_n}|^{p_n} \\ &\leqslant \lim_n \frac{p_n}{|u_{p_n}(x_{p_n}^+)|}|\Omega|^{\frac{1}{p_n+1}} \left(\int_\Omega |u_{p_n}|^{p_n+1}\right)^{p_n/(p_n+1)}. \end{aligned}$$

By Proposition 2.5 and Remark 2.4, we deduce that $\int_{\mathbb{R}^2} e^z \leqslant 16\pi e$. The solutions of $-\Delta z = e^z$ with $\int_{\mathbb{R}^2} e^z < +\infty$ are given by $z(x) = \log\left(\frac{\mu}{(1+\frac{\mu}{8}|x-x_0|^2)^2}\right)$ for some $\mu > 0$.

As $z(x) \leqslant z(0) = 0$ for any x, we have that $\mu = 1$ and $x_0 = 0$. Finally, $\int_{\mathbb{R}^2} e^z = 8\pi$. □

3.3. Rescaling argument in the positive nodal domain

Theorem 2 implies directly a control on $d(x_p^+, NL_p)$ where NL_p denotes the nodal line of u_p.

Proposition 3.2. *We have*

$$\frac{d(x_p^+, NL_{p_n})}{\varepsilon_p} \to +\infty \tag{7}$$

as $p \to +\infty$.

Proof. If the assertion is not true then, for a sequence $p_n \to +\infty$ the level curve $C_{p_n}(z_{p_n}) = \{x \in \Omega^+(\varepsilon_{p_n}), z_{p_n}(x) = -p_n\}$ intersects $B(0, R)$ for some large $R > 0$. This is a contradiction since z_{p_n} is uniformly bounded in all balls. □

Proof of Theorem 3. By Proposition 3.2, we can repeat the proof of Theorem 2 for the rescaled function $z_p(x)$ in $\tilde{\Omega}_p^+$. □

3.4. Rescaling argument on Ω around x_p^-

Let us consider $\Omega^-(\varepsilon_p) := \frac{\Omega - x_p^-}{\varepsilon_p}$ and $z_p^- : \Omega^-(\varepsilon_p) \to \mathbb{R}$ the scaling of u_p around x_p^-:

$$z_p^-(x) := \frac{p}{u_p(x_p^+)}\left(-u_p\left(\varepsilon_p x + x_p^-\right) - u_p\left(x_p^+\right)\right). \tag{8}$$

To obtain the same kind of result as that of Theorem 2, we need

$$\frac{d(x_p^-, \partial\Omega)}{\varepsilon_p} \to +\infty \tag{9}$$

as $p \to +\infty$. To get (9) we can repeat step by step the proof of Proposition 3.1. The only delicate point is the use of Harnack's inequality when we need that $\psi_p(0)$ is bounded from below. Nevertheless, requiring that $p(u_p(x_p^+) + u_p(x_p^-))$ is bounded (alternative (B) in the Introduction) we get the boundness of $\psi_p(0)$ and so (9) holds. This explains the role of condition (B) in getting Theorem 5.

Proof of Theorem 5. It is obtained following step by step the proof of Theorem 2. The constant μ in the limit function z can be different from 1 because

$$z(0) = \lim_{p\to+\infty} z_p^-(0) = \lim_{p\to+\infty} \frac{p}{u_p(x_p^+)}\left(-u_p\left(x_p^-\right) - u_p\left(x_p^+\right)\right) \neq 0$$

whenever K (in condition (B)) is not zero. □

3.5. Rescaling argument in the negative nodal domain

We would like to obtain a result similar to that of Theorem 3 for the function u_p^- defined in the negative nodal domain $\tilde{\Omega}_p^-$. We consider solutions satisfying condition (B). By Theorem 5, working in the same way as in the proof of Proposition 3.2, we get that $\varepsilon_p^{-1} d(x_p^-, NL_p) \to +\infty$.

Proof of Theorem 6. As (9) is satisfied when (B) holds, we can repeat the proof of Theorem 3, taking into account the remark in the proof of Theorem 5. □

We conclude this section by explaining Remark 7 of the Introduction. Let us now consider the "natural" rescaling coefficient

$$\tilde{\varepsilon}_p^2 := \frac{1}{p|u_p^-(x_p^-)|^{p-1}} \to 0$$

since $\liminf_{p\to+\infty} |u_p(x_p^-)|^{p-1} \geqslant 1$ (see Remark 2.6). We would like to control the rescaling of u_p^- around x_p^-

$$z_p(x) := \frac{p}{u_p(x_p^-)}\big(u_p^-(\tilde{\varepsilon}_p x + x_p^-) - u_p(x_p^-)\big).$$

The same argument as in the proof of Proposition 3.1 and Theorem 2 does not work as we might loose the essential estimate $|1+\frac{z_p}{p}| \leqslant 1$ in the proof. So, we do not get Proposition 3.2 for x_p^- and we need to assume that $\tilde{\varepsilon}_p^{-1} d(x_p^-, NL_p) \to +\infty$.

Proposition 3.3. *Assume that* $\tilde{\varepsilon}_p^{-1} d(x_p^-, NL_p) \to +\infty$ *as* $p \to +\infty$, *with* $\tilde{\varepsilon}_p$ *defined as* $\tilde{\varepsilon}_p^2 := \frac{1}{p|u_p^-(x_p^-)|^{p-1}}$. *Then* $\tilde{\varepsilon}_p^{-1} d(x_p^-, \partial\Omega) \to +\infty$ *as* $p \to +\infty$.

Proof. Let us work by contradiction and assume that, for a sequence $p_n \to +\infty$, $\frac{d(x_{p_n}^-, \partial\Omega)}{\tilde{\varepsilon}_{p_n}} \to l \geqslant 0$. Let us also assume w.l.o.g. that $x_{p_n}^- \to x_* \in \partial\Omega$ (i.e. $\frac{d(x_{p_n}^-, x_*)}{\tilde{\varepsilon}_{p_n}} \to l$).

As $\tilde{\varepsilon}_{p_n}^{-1} d(x_{p_n}^-, NL_{p_n}) \to +\infty$, we can construct a sequence of $\mathscr{C}^1$-domains D_{p_n} which are the intersection between a neighborhood V_{p_n} of x_* and $\tilde{\Omega}_{p_n}^-$ such that

$$\tilde{\varepsilon}_{p_n}^{-1} d(x_*, \partial D_{p_n} \setminus \partial\Omega) \to +\infty.$$

For large n, we have that $x_{p_n}^-$ belongs to D_{p_n}. So, as u_{p_n} stays negative in D_{p_n}, we can argue in the same way as in Proposition 3.1 to conclude the proof. □

By working in the same way as in Theorem 2 or Theorem 3, Proposition 3.3 allows to make the rescaling in the negative nodal domain $\tilde{\Omega}_p^-$, so to obtain for $\frac{p}{u_p(x_p^-)}(u_p^-(\tilde{\varepsilon}_p x + x_p^-) - u_p(x_p^-))$ the same assertion as for z_p in Theorem 3.

4. L^∞-estimates

In the last two sections, we will work in the positive and negative nodal domains. While dealing with the positive nodal domain, z_p will always denote the rescaled function used in Theorem 2. For the negative one, the expression of z_p can be defined as in Theorem 6 when (B) holds (and so with $\varepsilon_p^{-2} = p|u_p(x_p^+)|^{p-1}$).

Let us point out that some proofs will be given just for the positive case, the negative one being similar.

Proposition 4.1. *For any sequence* $p_n \to +\infty$ *we have* $\limsup_{n\to+\infty} |u_{p_n}^{\pm}(x_{p_n}^{\pm})| \leqslant e^{1/2}$.

Proof. Let us prove the assertion for the positive case. By Fatou's lemma, we have

$$\begin{aligned}
1 &= \frac{\int_\Omega |u_{p_n}^+|^{p_n+1}}{\|u_{p_n}^+\|_{p_n+1}^{p_n+1}} = \left(\frac{|u_{p_n}^+(x_{p_n}^+)|}{\|u_{p_n}^+\|_{p_n+1}}\right)^{p_n+1} \varepsilon_{p_n}^2 \int_{\tilde{\Omega}^+(\varepsilon_{p_n})} \left|1+\frac{z_{p_n}}{p_n}\right|^{p_n+1} \\
&= \frac{|u_{p_n}^+(x_{p_n}^+)|^2}{p_n\|u_{p_n}^+\|_{p_n+1}^{p_n+1}} \int_{\tilde{\Omega}^+(\varepsilon_{p_n})} \left|1+\frac{z_{p_n}}{p_n}\right|^{p_n+1} \\
&\geqslant \frac{\limsup_{n\to+\infty} |u_{p_n}^+(x_{p_n}^+)^2|}{8\pi e} \int_{\mathbb{R}^2} e^z.
\end{aligned}$$

As $\int_{\mathbb{R}^2} e^z = 8\pi$, the proof is complete. □

Now, we study the equality in the last statement. We will show that $\int_{\tilde{\Omega}^{\pm}(\varepsilon_{p_n})}|1+\frac{z_{p_n}}{p_n}|^{p_n+1}$ converges to $\int_{\mathbb{R}^2} e^z$ with no mass lost at infinity.

Let us consider the linearized operators

$$L_p^+(v) = -\Delta v - p\left|u_p^+\right|^{p-1} v$$

for $v : \tilde{\Omega}_p^+ \to \mathbb{R}$ and let us denote by $\lambda_i(L_p^+)$ the eigenvalues of L_p^+ with homogeneous Dirichlet boundary conditions.

Our aim is to prove Theorem 9, therefore we assume that the Morse index of u_p^+ in Ω^+ is 1. Hence we have

$$\lambda_1\left(L_p^+\right) < 0 \quad \text{and} \quad \lambda_2\left(L_p^+\right) \geqslant 0 \quad \text{in } \tilde{\Omega}_p^+.$$

Then, for $D \subseteq \tilde{\Omega}^+(\varepsilon_p)$, let us consider $L_{p,D}^+(v) = -\Delta v - \frac{|u_p^+(\varepsilon_p x + x_p^+)|^{p-1}}{|u_p^+(x_p^+)|^{p-1}} v$ and denote by $\lambda_i(L_{p,D}^+)$ the corresponding Dirichlet eigenvalues. By scaling, we get

Lemma 4.2. $\lambda_1(L^+_{p,\tilde{\Omega}^+(\varepsilon_p)}) < 0$ *and* $\lambda_2(L^+_{p,\tilde{\Omega}^+(\varepsilon_p)}) \geqslant 0$.

Lemma 4.3. *Let $p \to +\infty$, then there exists $r > 0$ such that $\lambda_1(L^+_{p,B(0,r)}) < 0$ for large p.*

Proof. Let us consider $w_p = x.\nabla z_p + \frac{2}{p-1} z_p + \frac{2p}{p-1}$. We have that w_p satisfies $-\Delta w = \frac{|u_p^+(\varepsilon_p x + x_p^+)|^{p-1}}{|u_p^+(x_p^+)|^{p-1}} w$.

We also have $w_p(0) \to 2$. As $z_p \to z = \log\left(\frac{1}{(1+\frac{|x|^2}{8})^2}\right)$, for $|x| = r$, we get

$$w_p(x) \to -\frac{4r^2}{8+r^2} + 2$$

as $p \to +\infty$. So, for large r, $w_p \to \alpha < 0$ on $\partial B[0,r]$.

Let us fix such an r. By considering $A_p := \{x \in B(0,r)\colon\ w_p > 0\}$ and the function $\bar{w}_p$ equals to w_p on A_p (0 otherwise), we get

$$\int_{B(0,r)} |\nabla \bar{w}_p|^2 - \int_{B(0,r)} \frac{|u_p^+(\varepsilon_p x + x_p^+)|^{p-1}}{|u_p^+(x_p^+)|^{p-1}} \bar{w}_p^2 = 0,$$

which implies our statement. □

Lemma 4.4. *For p large, $\lambda_1(L^+_{p,\tilde{\Omega}^+(\varepsilon_p)\setminus B(0,r)}) > 0$, where r is given by Lemma 4.3.*

Proof. If $\lambda_1(L^+_{p,\tilde{\Omega}^+(\varepsilon_p)\setminus B(0,r)})$ was negative then, by Lemma 4.3 we would have $\lambda_2(L^+_{p,\tilde{\Omega}^+(\varepsilon_p)}) < 0$ which contradicts Lemma 4.2. □

Proof of Theorem 8. Since we are analyzing nodal solutions which satisfy condition (B), it is enough to prove that $\|u_p^+\|_\infty \to \sqrt{e}$ as $p \to +\infty$. Let us argue by contradiction and assume that for a sequence $p_n \to +\infty$, by Proposition 4.1, $\lim_{n\to\infty} |u_{p_n}^+(x_{p_n}^+)| = \lim_{n\to\infty} \|u_{p_n}^+\|_\infty < e^{1/2}$. We claim that this implies $z_{p_n}(x) - z(x) \leqslant C$ on $\tilde{\Omega}^+(\varepsilon_{p_n})$ uniformly.

Indeed, z_{p_n} converges to z on each compact set. In particular, on $B(0,r)$, where r is given by Lemma 4.3. So, it is enough to check what happens in $\tilde{\Omega}^+(\varepsilon_{p_n}) \setminus B(0,r)$.

On one hand, $-\Delta z = e^z \geqslant |1+\frac{z}{p}|^p$ for any $p > 1$. On the other hand, by computing $z_{p_n} - z$ on $\partial\tilde{\Omega}^+(\varepsilon_{p_n}) \setminus B[0,r]$, we get for some uniform constant C

$$z_{p_n}(x) - z(x) = -p_n - \log\left(\frac{1}{(1+\frac{|x|^2}{8})^2}\right)$$

$$\begin{aligned}
&\leqslant -p_n - \log\left(C\frac{\varepsilon_{p_n}^4}{\mathrm{d}(x_{p_n}^+,\partial\Omega)^4}\right)\\
&\leqslant -p_n + 2\log\bigl(p_n\bigl|u_{p_n}^+\bigl(x_{p_n}^+\bigr)\bigr|^{p_n-1}\bigr) + 4\log\bigl(\mathrm{d}\bigl(x_{p_n}^+,\partial\Omega\bigr)\bigr) + C\\
&\leqslant -p_n + 2\log\bigl(p_n\bigl|u_{p_n}^+\bigl(x_{p_n}^+\bigr)\bigr|^{p_n-1}\bigr) + C\\
&\leqslant -p_n + 2\log\bigl(\bigl|u_{p_n}^+\bigl(x_{p_n}^+\bigr)\bigr|^{p_n-1}\bigr) + C \leqslant C
\end{aligned}$$

where we used that $|u_{p_n}^+(x_{p_n}^+)| < e^{1/2}$ and $\mathrm{d}(x_{p_n}^+,\partial\Omega) \leqslant C$ for large n (by contradiction).

We also have the estimate on $\partial B(0,r)$ because we have the convergence on each compact set.

Finally, by convexity, we have

$$-\Delta z_{p_n} + \Delta z \leqslant \left|1+\frac{z_{p_n}}{p_n}\right|^{p_n-1}(z_{p_n}-z) = \frac{|u_{p_n}^+(\varepsilon_{p_n}x + x_{p_n}^+)|^{p_n-1}}{|u_{p_n}(x_{p_n}^+)|^{p_n-1}}(z_{p_n}-z).$$

Since the maximum principle holds in $\tilde\Omega^+(\varepsilon_{p_n}) \setminus B(0,r)$ for $L^+_{p_n,\tilde\Omega^+(\varepsilon_{p_n})\setminus B(0,r)}$ (see Lemma 4.4), we deduce our claim.

From this claim, we obtain that $\int_{\tilde\Omega^+(\varepsilon_{p_n})}|1+\frac{z_{p_n}}{p_n}|^{p_n+1}$ converges to $\int_{\mathbb{R}^2} e^z$. So,

$$\begin{aligned}
1 &= \frac{\int_\Omega |u_{p_n}^+|^{p_n+1}}{\|u_{p_n}^+\|_{p_n+1}^{p_n+1}} = \frac{|u_{p_n}^+(x_{p_n}^+)|^{p_n+1}}{\|u_{p_n}^+\|_{p_n+1}^{p_n+1}}\varepsilon_{p_n}^2 \int\limits_{\tilde\Omega^+(\varepsilon_{p_n})}\left|1+\frac{z_{p_n}}{p_n}\right|^{p_n+1}\\
&= \frac{\|u_{p_n}^+\|_\infty^2}{8\pi e + o(1)}\bigl(8\pi + o(1)\bigr),
\end{aligned}$$

which proves that $\lim_{n\to\infty}\|u_{p_n}^+\|_\infty = e^{1/2}$, which is a contradiction. □

5. Green's characterizations

To start with, we observe that Theorem 8 gives a direct way to prove the convergence of $\int_{\tilde\Omega^\pm(\varepsilon_p)}|1+\frac{z_p}{p}|^{p+1}$, as $p\to+\infty$.

Proposition 5.1. *As* $p\to+\infty$ $\int_{\tilde\Omega^\pm(\varepsilon_p)}|1+\frac{z_p}{p}|^{p+1} \to \int_{\mathbb{R}^2} e^z = 8\pi$.

Proof. Let us give the proof for the positive case. For any $n\in\mathbb{N}$, we have

$$\int\limits_{\tilde\Omega^+(\varepsilon_p)}\left|1+\frac{z_p}{p}\right|^{p+1} = \frac{p\int_\Omega |u_p^+|^{p+1}}{|u_p^+(x_p^+)|^2}.$$

As the right-hand side converges to 8π, we obtain our statement. □

The previous result implies a similar statement where the exponent $p+1$ is replaced by p.

Proposition 5.2. *We have*

$$\int\limits_{\tilde\Omega^\pm(\varepsilon_{p_n})}\left|1+\frac{z_p}{p}\right|^p \to 8\pi$$

as $p\to+\infty$.

Proof. Let us give the proof for the positive case. On one hand, as $|1+\frac{z_p}{p}| \leqslant 1$, we have $\int_{\tilde\Omega^+(\varepsilon_p)}|1+\frac{z_p}{p}|^{p+1} \leqslant \int_{\tilde\Omega^+(\varepsilon_p)}|1+\frac{z_p}{p}|^p$. By Proposition 5.1, we get $8\pi \leqslant \liminf_{p\to+\infty}\int_{\tilde\Omega^+(\varepsilon_p)}|1+\frac{z_p}{p}|^p$.

On the other hand, as $\int_{\tilde\Omega^+(\varepsilon_p)}|1+\frac{z_p}{p}|^{p+1}\to 8\pi$ and $|1+\frac{z_p}{p}|^{p+1}\to e^z$ with $\int_{\mathbb{R}^2}e^z=8\pi$ (see Theorem 3 and Theorem 6), we have

$$\forall\varepsilon>0,\ \exists R_\varepsilon>0 \text{ and } p_\varepsilon\colon \forall p>p_\varepsilon \text{ and } R>R_\varepsilon,\qquad \int\limits_{\tilde\Omega^+(\varepsilon_p)\cap\{|x|>R\}}\left|1+\frac{z_p}{p}\right|^{p+1}\leqslant\varepsilon. \tag{10}$$

By interpolation, we get for any $\varepsilon>0$ that

$$\begin{aligned}\int\limits_{\tilde\Omega^+(\varepsilon_p)}\left|1+\frac{z_p}{p}\right|^p&=\int\limits_{\tilde\Omega^+(\varepsilon_p)\cap\{|x|\leqslant R_\varepsilon\}}\left|1+\frac{z_p}{p}\right|^p+\int\limits_{\tilde\Omega^+(\varepsilon_p)\cap\{|x|>R_\varepsilon\}}\left|1+\frac{z_p}{p}\right|^p\\&\leqslant\int\limits_{\tilde\Omega^+(\varepsilon_p)\cap\{|x|\leqslant R_\varepsilon\}}\left|1+\frac{z_p}{p}\right|^p+\Bigg(\int\limits_{\tilde\Omega^+(\varepsilon_p)\cap\{|x|>R_\varepsilon\}}\left|1+\frac{z_p}{p}\right|^{p+1}\Bigg)^{\frac{p}{p+1}}\big|\tilde\Omega^+(\varepsilon_p)\big|^{\frac{1}{p+1}}\\&\leqslant\int\limits_{\tilde\Omega^+(\varepsilon_p)\cap\{|x|\leqslant R_\varepsilon\}}\left|1+\frac{z_p}{p}\right|^p+\Bigg(\int\limits_{\tilde\Omega^+(\varepsilon_p)\cap\{|x|>R_\varepsilon\}}\left|1+\frac{z_p}{p}\right|^{p+1}\Bigg)^{\frac{p}{p+1}}|\Omega|^{\frac{1}{p+1}}\varepsilon_p^{\frac{-1}{p+1}}.\end{aligned}$$

As $\int_{\tilde\Omega^+(\varepsilon_p)\cap\{|x|\leqslant R_\varepsilon\}}|1+\frac{z_p}{p}|^p\to C\leqslant 8\pi$ and $\varepsilon_p^{\frac{-1}{p+1}}\to e^{1/4}$ as $p\to+\infty$, we get by (10) that, for any $\varepsilon>0$, there exists $\bar p>0$ such that if $p>\bar p$ then

$$\int\limits_{\tilde\Omega^+(\varepsilon_p)}\left|1+\frac{z_p}{p}\right|^p\leqslant(8\pi+\varepsilon)+\varepsilon^{\frac{p}{p+1}}\big(e^{\frac14}+\varepsilon\big),$$

which implies our statement. □

Let us denote by G the Green's function of Ω and by $x^\pm\in\bar\Omega$ the limit points of $x_p^\pm$ as $p\to+\infty$.

Lemma 5.3. *Let $x\neq x^\pm$. We have*

$$\int\limits_{\tilde\Omega^\pm(\varepsilon_p)}G\big(x,\varepsilon_p\psi+x_p^\pm\big)\left|1+\frac{z_p}{p}\right|^p\mathrm{d}\psi\to 8\pi G\big(x,x^\pm\big).$$

Proof. Let us make the proof for the positive case. Let us fix $x\neq x^+$ and consider $\alpha>0$ such that $B(x,\alpha)\subseteq\Omega$ and $d(x^+,B(x,\alpha))=\beta>0$. We have

$$\begin{aligned}\int\limits_{\tilde\Omega^+(\varepsilon_p)}G\big(x,\varepsilon_p\psi+x_p^+\big)\left|1+\frac{z_p}{p}\right|^p\mathrm{d}\psi&=\int\limits_{\tilde\Omega^+(\varepsilon_p)\setminus\frac{B(x,\alpha)-x_p^+}{\varepsilon_p}}G\big(x,\varepsilon_p\psi+x_p^+\big)\left|1+\frac{z_p}{p}\right|^p\mathrm{d}\psi\\&\quad+\int\limits_{\frac{B(x,\alpha)-x_p^+}{\varepsilon_p}}G\big(x,\varepsilon_p\psi+x_p^+\big)\left|1+\frac{z_p}{p}\right|^p\mathrm{d}\psi.\end{aligned}$$

Arguing as in Proposition 5.2, since $G(x,\varepsilon_p\psi+x_p^+)$ converges uniformly to $G(x,x^+)$ on each compact set of $\mathbb{R}^2$, $G(x,\cdot)$ is bounded on $\Omega\setminus B(x,\alpha)$ and $d(\frac{B(x,\alpha)-x_p^+}{\varepsilon_p},0)\to+\infty$, we get that the first integral converges to $8\pi G(x,x^+)$. Concerning the second integral, since $x^+\notin B(x,\alpha)$, we derive that $\int_{\frac{B(x,\alpha)-x_p^+}{\varepsilon_p}}|1+\frac{z_p}{p}|^p=p\int_{B(x,\alpha)}|u_p^+|^p\to 0$. From the last statement we deduce that we can apply Lemma 3.5 in [20] and obtain that $\frac{u_p}{p\int_{B(x,\alpha)}|u_p^+|^p}$ is bounded in $B(x,\alpha)$ and hence $u_p(x)<\frac12$ in $B(x,\alpha)$. Then

$$\int_{\frac{B(x,\alpha)-x_p^+}{\varepsilon_p}} G\left(x, \varepsilon_p\psi + x_p^+\right)\left|1 + \frac{z_p}{p}\right|^p = p \int_{B(x,\alpha)} G(x,y)\left|u_p^+(y)\right|^p \mathrm{d}y$$

$$\leqslant p\left(\frac{1}{2}\right)^p \int_{B(x,\alpha)} G(x,y) = o(1),$$

which gives our claim. □

Let us remark that the convergence in Lemma 5.3 is uniform in x in $\mathscr{C}^0_{\mathrm{loc}}(\Omega \setminus \{x^+\})$.

Proposition 5.4. *Under the same assumptions as in Theorem 8, the following alternatives hold*:

(1) $d(x_p^+, \partial\Omega) \to 0$ *and* $d(x_p^-, \partial\Omega) \not\to 0$. *Then the function* pu_p *converges, up to a subsequence, to the negative function* $-8\pi e^{1/2} G(\cdot, x^-)$ *in* $\mathscr{C}^1_{\mathrm{loc}}(\bar\Omega \setminus \{x^-\})$;
(2) $d(x_p^-, \partial\Omega) \to 0$ *and* $d(x_p^+, \partial\Omega) \not\to 0$. *Then the function* pu_p *converges, up to a subsequence, to the positive function* $8\pi e^{1/2} G(\cdot, x^+)$ *in* $\mathscr{C}^1_{\mathrm{loc}}(\bar\Omega \setminus \{x^+\})$;
(3) $d(x_p^+, \partial\Omega)$ *and* $d(x_p^-, \partial\Omega) \not\to 0$. *Then* pu_p *converges, up to a subsequence, to* $8\pi e^{1/2}(G(\cdot, x^+) - G(\cdot, x^-))$ *in* $\mathscr{C}^1_{\mathrm{loc}}(\bar\Omega \setminus \{x^-, x^+\})$ *with* $x^+ \neq x^-$, $x^+, x^- \in \Omega$;
(4) $d(x_p^+, \partial\Omega) \to 0$ *and* $d(x_p^-, \partial\Omega) \to 0$. *Then* $pu_p \to 0$ *in* $\mathscr{C}^1_{\mathrm{loc}}(\bar\Omega \setminus \{x^-, x^+\})$.

In the case (3), *the limit points* x^+ *and* x^- *satisfy the system*

$$\begin{cases} \dfrac{\partial G}{\partial x_i}\left(x^+, x^-\right) - \dfrac{\partial H}{\partial x_i}\left(x^+, x^+\right) = 0, \\ \dfrac{\partial G}{\partial x_i}\left(x^-, x^+\right) - \dfrac{\partial H}{\partial x_i}\left(x^-, x^-\right) = 0, \end{cases} \tag{11}$$

for $i = 1, 2$, *where, as in the Introduction,* $H(x, y)$ *is the regular part of the Green function. Moreover the nodal line of* u_p *intersects the boundary* $\partial\Omega$ *for* p *large.*

Proof. We have

$$u_p(x) = \int_\Omega G(x,y)\left|u_p(y)\right|^{p-1} u_p(y)\,\mathrm{d}y$$

$$= \int_{\tilde\Omega_p^+} G(x,y)\left|u_p(y)\right|^p \mathrm{d}y - \int_{\tilde\Omega_p^-} G(x,y)\left|u_p(y)\right|^p \mathrm{d}y.$$

Let us just treat the first member of the sum. The second one can be treated in the same way. With the change of variables $y = \varepsilon_p\psi + x_p^+$, we get

$$\int_{\tilde\Omega_p^+} G(x,y)\left|u_p(y)\right|^p \mathrm{d}y = \int_{\tilde\Omega^+(\varepsilon_p)} \frac{1}{p|u_p^+(x_p^+)|^{p-1}} G\left(x, \varepsilon_p\psi + x_p^+\right)\left|u_p^+\left(\varepsilon_p\psi + x_p^+\right)\right|^p \mathrm{d}\psi$$

$$= \int_{\tilde\Omega^+(\varepsilon_p)} G\left(x, \varepsilon_p\psi + x_p^+\right) \frac{||u_p^+(\varepsilon_p\psi + x_p^+)| - \|u_p^+\|_\infty + \|u_p^+\|_\infty|^p}{p\|u_p^+\|_\infty^{p-1}}\,\mathrm{d}\psi$$

$$= \int_{\tilde\Omega^+(\varepsilon_p)} G\left(x, \varepsilon_p\psi + x_p^+\right) \frac{|\frac{\|u_p^+\|_\infty z_p}{p} + \|u_p^+\|_\infty|^p}{p\|u_p^+\|_\infty^{p-1}}\,\mathrm{d}\psi$$

$$= \frac{\|u_p^+\|_\infty}{p} \int\limits_{\tilde{\Omega}^+(\varepsilon_p)} G\big(x, \varepsilon_p \psi + x_p^+\big)\left|1 + \frac{z_p}{p}\right|^p \mathrm{d}\psi.$$

As $\|u_p^+\|_\infty \to e^{1/2}$ and $\int_{\tilde{\Omega}^+(\varepsilon_p)} G(x, \varepsilon_p\psi + x_p^+)|1 + \frac{z_p}{p}|^p \,\mathrm{d}\psi$ converges to $8\pi G(x, x^+)$ (see Lemma 5.3), by working in the same way with the second part of the sum, we have

$$pu_p \to 8\pi e^{1/2}\big(G\big(., x^+\big) - G\big(., x^-\big)\big) \tag{12}$$

in $\mathscr{C}^0_{\mathrm{loc}}(\Omega \setminus \{x^+, x^-\})$, up to a subsequence. By regularity, it implies the convergence in $\mathscr{C}^1_{\mathrm{loc}}(\Omega \setminus \{x^+\})$ (see [16]).

Observing that $G(., x^+) = 0$ when $x^+ \in \partial\Omega$, we get the alternatives. In the third case, we prove that $x^+ \neq x^-$ as follows. Indeed, arguing by contradiction, we have that $x^+ = x^-$. Then, $pu_p \to 0$ in $\mathscr{C}^1(\bar{\omega})$ where ω is a neighborhood of the boundary $\partial\Omega$. By the Pohozaev identity, multiplying by p^2, we get

$$\frac{p^2}{p+1}\int\limits_{\Omega} |u_p|^{p+1} = \frac{1}{4}\int\limits_{\partial\Omega} (x \cdot \nu)\big(\partial_\nu(pu_p)\big)^2.$$

As the left-hand side converges to $16\pi e$ (see Remark 2.4) and the right-hand side converges to 0 (since $pu_p \to 0$ in $\mathscr{C}^1(\bar{\omega})$), we get a contradiction.

Now, we prove that x^+ and x^- solve the system (11). Concerning the location of x^+ and x^-, we use a Pohozaev-type identity. For $i = 1, 2$ let us multiply Eq. ($\mathscr{P}_p$) by $\frac{\partial u_p}{\partial x_i}$ and integrate on $B_R(x^+) \subseteq \Omega$, the ball centered at x^+ and radius R. We have that,

$$0 = \frac{2}{p+1}\int\limits_{\partial B_R(x^+)} |u_p|^{p+1}\nu_i + \int\limits_{\partial B_R(x^+)} \frac{\partial u_p}{\partial x_i}\frac{\partial u_p}{\partial \nu} - \frac{1}{2}\int\limits_{\partial B_R(x^+)} |\nabla u_p|^2 \nu_i = I_1 + I_2 + I_3 \tag{13}$$

where ν_i are the components of the normal direction.

From (12) we get that

$$p^2 I_1 = O\left(\frac{1}{2}\right)^p \quad \text{as } p \to +\infty. \tag{14}$$

Multiplying (13) by p^2 and using (12) and (14) we deduce

$$\int\limits_{\partial B_R(x^+)} \frac{\partial(G(., x^+) - G(., x^-))}{\partial x_i}\frac{\partial(G(., x^+) - G(., x^-))}{\partial \nu} - \frac{1}{2}\int\limits_{\partial B_R(x^+)} \big|\nabla\big(G\big(., x^+\big) - G\big(., x^-\big)\big)\big|^2 \nu_i = 0. \tag{15}$$

The last integral was computed in [17, pp. 511–512] which gives

$$\nabla\big(G\big(x^+, x^-\big) - H\big(x^+, x^+\big)\big) = 0. \tag{16}$$

Repeating the same procedure in $B_R(x^-)$ we derive that

$$\nabla\big(G\big(x^-, x^+\big) - H\big(x^-, x^-\big)\big) = 0 \tag{17}$$

which gives the claim.

To conclude the proof, we show that the nodal line of u_p intersects the boundary $\partial\Omega$ for p large. If not, u_p is a one-signed function in a neighborhood of $\partial\Omega$, which, by Höpf's lemma, implies that $\partial_\nu pu_p$ is one-signed on $\partial\Omega$ for large p. On the other hand, as $x^+ \neq x^-$ and $\int_{\partial\Omega} \partial_\nu(G(\cdot, x^+) - G(\cdot, x^-)) = 0$, the normal derivative of the limit function changes its sign along $\partial\Omega$. It contradicts the $\mathscr{C}^1$-convergence of pu_p to $8\pi\sqrt{e}(G(\cdot, x^+) - G(\cdot, x^-))$ in a compact neighborhood of $\partial\Omega$. □

Proof of Theorem 9. We need to prove that the cases (1), (2) and (4) in Proposition 5.4 cannot happen. To start with, we focus on the case (4). Arguing by contradiction, let us assume that x^+ and x^- belong to $\partial\Omega$. Let $D \subseteq \Omega$ be an open domain which is the intersection between a neighborhood of x^+ and Ω. We assume w.l.o.g. that $x^- \notin \bar{D}$ when $x^+ \neq x^-$ and $x^- \notin \partial\bar{D}$ when $x^+ = x^-$. We have that $pu_p \to 0$ in $\mathscr{C}^1_{\mathrm{loc}}(\bar{D} \setminus \{x^+\})$. Using the same notations as in the proof of Proposition 3.1 (for $Q, Q^+, S, \ldots$), we consider the change of variables $\varphi : D \to Q^+$ and $\varphi(D \cap \partial\Omega) = S$. Moreover $\varphi^{-1} \in \mathscr{C}^1$. Then, we define $D^* := \varphi^{-1}(Q)$ and u_p^* which is u_p on D and the odd tubular reflection on $D^* \setminus D$ (as in the proof of Proposition 3.1). We get that u_p^* solves $-\Delta u = |u|^{p-1}u$ on D^* and $pu_p^* \to 0$ in $\mathscr{C}^1(\bar{\omega}^*)$ where $\omega^* \subseteq D^*$ is a neighborhood of the boundary ∂D^* avoiding x^+. Using the Pohozaev identity and multiplying by p^2, we get the existence of constants K, K^* and K^{**} such that

$$\frac{p^2}{p+1}\int_{D^*}\left|u_p^*\right|^{p+1} = K\int_{\partial D^*}(x\cdot\nu)\left(\partial_\nu\left(pu_p^*\right)\right)^2 \mathrm{d}\tau + K^*\int_{\partial D^*}(x\cdot\nu)\left(\partial_\tau\left(pu_p^*\right)\right)^2 \mathrm{d}\tau + K^{**}\frac{p^2}{p+1}\int_{\partial D^*}\left|u_p^*\right|^{p+1}. \quad (18)$$

As $pu_p^* \to 0$ in $\mathscr{C}^1(\bar{D}^* \setminus \{x^+\})$, the right-hand side is converging to zero. To get a contradiction, we prove that the left-hand side is not converging to zero. For this, we claim that $p\int_{D^*}|u_p^*|^{p+1}$ converges to $C \geqslant 8\pi e$. If not, as $p\int_\Omega |u_p^-|^{p+1} \to 8\pi e$ and $p\int_\Omega |u_p|^{p+1} \to 16\pi e$, we get the existence of a positive constant ψ such that

$$\int_{\Omega\setminus(D^*\cup B(x^-,\delta))} p|u_p||u_p|^p > \psi$$

for any $\delta > 0$ and large p. It contradicts $pu_p \to 0$ in $\mathscr{C}^1(\bar{\Omega} \setminus \{x^+\})$.

To finish, let us prove that the case (1) cannot happen (the case (2) is similar). Working in the same way, we construct an open domain $x^+ \in D^*$ such that u_p^* solves $-\Delta u = |u|^{p-1}u$ on D^* and $pu_p^* \to G(\cdot, x^-)$ in $\mathscr{C}^1(\bar{\omega}^*)$ where ω^* is any compact set in $\bar{D}^* \setminus \{x^+\}$. Using again the Pohozaev identity and multiplying by p^2, we get Eq. (18). Working as previously, as $pu_p \to G(., x^-)$ and $u_p \to 0$ in $\mathscr{C}^1_{\mathrm{loc}}(\Omega \setminus \{x^+\})$, the left-hand side converges to $C \geqslant 8\pi e$. Concerning the right-hand side, as $G(., x^-) \in \mathscr{C}^1(\bar{\Omega})$ and $G(\cdot, x^-) = 0$ on $\partial\Omega$, we can consider D^* small enough such that the two last terms converge to constants less than $8\pi e/3$. For the first one, as there exists a constant $K > 0$ such that $|\nabla G(x, y)| \leqslant \frac{K}{|x-y|}$, we get that $(\partial_\nu G(\cdot, x^-))^2$ is bounded in a neighborhood of x^+. Taking D^* small enough, we also get that the first term converges to a constant less than $8\pi e/3$ which is a contradiction. □

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