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Bilinear Estimates Associated to the Schrödinger Equation with a Nonelliptic Principal Part

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Abstract. We discuss bilinear estimates of tempered distributions in the Fourier restriction spaces for the two-dimensional Schödinger equation whose principal part is the d'Alembertian. We prove that the bilinear estimates hold if and only if the tempered distributions are functions.

Keywords. Schrödinger equation, bilinear estimates, local smoothing effect, the Strichartz estimate

Mathematics Subject Classification (2000). 42B35, 35B65, 35Q55

1. Introduction

This paper is devoted to studying bilinear estimates of tempered distributions in the Fourier restriction spaces related with the two-dimensional Schrödinger equation whose principal part is the d'Alembertian. The Fourier restriction spaces were originated by Bourgain in his celebrated papers [1] and [2] to establish time-local or time-global well-posedness of the initial value problem for one-dimensional nonlinear Schrödinger equations and the Korteweg-de Vries equation in $L^2(\mathbb{R})$ respectively. Generally speaking, to solve the initial value problem for nonlinear dispersive partial differential equations which can be treated by the classical energy method, one usually analyzes the interactions of propagation of singularities in nonlinearity in detail, and applies the regularity properties of free propagators to the resolution of singularities. It is well-known that propagators of some classes of linear dispersive equations with constant coefficients have local smoothing effects (see, e.g., [3]), and dispersion properties (see, e.g., [7] and [14]). Surprisingly, the Fourier restriction spaces automatically work for both of the analysis of the interactions of propagation of singularities in the

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frequency space and the application of the regularity properties of free propagators. For this reason, many applications and refinements of the method of the Fourier restriction spaces have been investigated in the last decade; see, e.g., [4], [10]–[12], [15, 16] and references therein.

Here we state the definition of the Fourier restriction spaces. The Fourier transform of a function $f(x,t)$ of $(x,t)=(x_1,\dots,x_n,t)\in\mathbb{R}^{n+1}$ is defined by

$$\tilde{f}(\xi,\tau)=(2\pi)^{-\frac{n+1}{2}}\iint_{\mathbb{R}^{n+1}} e^{-it\tau-ix\cdot\xi}f(x,t)dxdt,$$

where $i=\sqrt{-1}$, $(\xi,\tau)=(\xi_1,\dots,\xi_n,\tau)\in\mathbb{R}^{n+1}$ and $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$. Let $a(\xi)$ be a real polynomial of $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$. Set $\partial_t=\frac{\partial}{\partial t}$, $\partial_j=\frac{\partial}{\partial x_j}$, $D_t=-i\partial_t$, $D_j=-i\partial_j$, $D=(D_1,\dots,D_n)$, $|\xi|=\sqrt{\xi\cdot\xi}$, $\langle\tau\rangle=\sqrt{1+\tau^2}$, and $\langle\xi\rangle=\sqrt{1+|\xi|^2}$. For $s,b\in\mathbb{R}$, the Fourier restriction space $X^{s,b}=X^{s,b}(\mathbb{R}^{n+1})$ associated to the differential operator $D_t-a(D)$ is the set of all tempered distributions f on $\mathbb{R}^{n+1}$ satisfying

$$\|f\|_{s,b}=\left(\iint_{\mathbb{R}^{n+1}}\left|\langle\tau-a(\xi)\rangle^b\langle\xi\rangle^s\tilde{f}(\xi,\tau)\right|^2 d\xi\,d\tau\right)^{\frac{1}{2}}<+\infty.$$

The free propagator $e^{ita(D)}$ of a differential equation $(D_t-a(D))u=0$ is defined by

$$e^{ita(D)}\phi(x)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{ix\cdot\xi+ita(\xi)}\hat{\phi}(\xi)\,d\xi,$$

where $\hat{\phi}$ is the Fourier transform of ϕ in $x\in\mathbb{R}^n$, that is,

$$\hat{\phi}(\xi)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\phi(x)\,dx.$$

In one-dimensional case, bilinear estimates in the Fourier restriction spaces associated to D_t-D^2 and D_t-D^3 were completed. More precisely, in [10] and [11], Kenig, Ponce and Vega refined the bilinear estimates in the Fourier restriction spaces with some negative indices $s<0$. Nakanishi, Takaoka and Tsutsumi in [12] constructed sequences of tempered distributions breaking the bilinear estimates to show the optimality of the indices $s<0$ used in [10] and [11].

In [15] Tao investigated the bilinear estimates associated to $a(\xi)=|\xi|^2$ with $n\geqslant 2$. He dealt with some equivalent estimates of the integral of trilinear form, and pointed out that the worst singularity occurs when an orthogonal relationship of three phases in that integral holds. Particularly in case $n=2$, Colliander, Delort, Kenig and Staffilani succeeded in overcoming this difficulty by the dyadic decomposition in not only the sizes of phases but also the angles among them. See [4] for the detail. Combining the above results for $a(\xi)=|\xi|^2$ with $n=1,2$, we have the following.

Theorem 1.1 ([4, 10]). *Let $n = 1, 2$, and let $a(\xi) = |\xi|^2$.*

(i) *For any $s \in (-\frac{3}{4}, 0]$, there exist $b \in (\frac{1}{2}, 1)$ and $C > 0$ such that*

$$\|uv\|_{s,b-1} \leqslant C\|u\|_{s,b}\,\|v\|_{s,b}, \tag{1}$$

$$\|\bar{u}\bar{v}\|_{s,b-1} \leqslant C\|u\|_{s,b}\,\|v\|_{s,b}. \tag{2}$$

(ii) *For any $s \in (-\frac{1}{4}, 0]$, there exist $b \in (\frac{1}{2}, 1)$ and $C > 0$ such that*

$$\|\bar{u}v\|_{s,b-1} \leqslant C\|u\|_{s,b}\,\|v\|_{s,b}. \tag{3}$$

(iii) *For any $s < -\frac{3}{4}$ and for any $b \in \mathbb{R}$, the estimates* (1) *and* (2) *fail to hold, and for any $s < -\frac{1}{4}$ and for any $b \in \mathbb{R}$,* (3) *fails to hold.*

Here we mention a few remarks. First, the difference between (i) and (ii) are basically due to the structure of the products. In view of Hörmander's theorem concerned with the microlocal condition on the multiplication of distributions (see [13, Theorem 0.4.5] for instance), $u\bar{u}$ needs more smoothness of u than u^2 and $\bar{u}^2$ to make sense. Secondly, the local smoothing effect and the dispersion property of the fundamental solution $e^{it|D|^2}$ are strongly reflected in these bilinear estimates. These are applied to solving the initial value problem for some nonlinear Schrödinger equations in a class of tempered distributions which are not necessarily functions. Indeed, by using the technique developed in [10] together with the estimates (1), (2) and (3), one can prove time-local well-posedness of the initial value problem for quadratic nonlinear Schrödinger equations of the form

$$D_t u - |D|^2 u = N_j(u, u) \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \tag{4}$$

$$u(x, 0) = u_0(x) \qquad \text{in } \mathbb{R}^n, \tag{5}$$

in Sobolev space $H^s(\mathbb{R}^n)$ with $s \in (-\frac{3}{4}, 0]$ for $j = 1, 2$ and $s \in (-\frac{1}{4}, 0]$ for $j = 3$, respectively. Here $n = 1, 2$, $u(x, t)$ is a complex-valued unknown function of (x, t), u_0 is a given initial data, $N_1(u, v) = uv$, $N_2(u, v) = \bar{u}\bar{v}$, $N_3(u, v) = \bar{u}v$, $H^s(\mathbb{R}^n) = \langle D\rangle^{-s}L^2(\mathbb{R}^n)$, and $L^2(\mathbb{R}^n)$ is the set of all square-integrable functions on $\mathbb{R}^n$.

Some two-dimensional nonlinear dispersive equations with a nonelliptic principal part arise in classical mechanics. For example, the Ishimori equation ([8])

$$D_t u - (D_1^2 - D_2^2)u = \frac{-2\bar{u}}{1 + |u|^2}\Big((D_1 u)^2 - (D_2 u)^2\Big) + i(D_2\phi D_1 u + D_1\phi D_2 u),$$

$$\phi = -4i|D|^{-2}\left(\frac{D_1\bar{u}D_2 u - D_1 u D_2\bar{u}}{1 + |u|^2}\right),$$

and the hyperbolic–elliptic Davey-Stewartson equation ([5])

$$D_t u - (D_1^2 - D_2^2)u = -|u|^2 u - u D_1^2 |D|^{-2}(|u|^2)$$

are well-known two-dimensional nonlinear dispersive equations. It is easy to see that $e^{it(D_1^2-D_2^2)}$ has exactly the same local smoothing and dispersion properties of $e^{it(D_1^2+D_2^2)}$ since $a(\xi)=\xi_1^2 \pm \xi_2^2$ are two-dimensional nondegenerate quadratic forms. If the gradient $a'(\xi)$ of a quadratic form $a(\xi)$ does not vanish for $\xi \neq 0$, then $e^{ita(D)}$ gains $\frac{1}{2}$-spatial differentiation globally in time and locally in space. If the Hessian $a''(\xi)$ of an n-dimensional quadratic form $a(\xi)$ is a nonsingular matrix, then the distribution kernel of $e^{ita(D)}$ in $\mathbb{R}^n \times \mathbb{R}^n$ is estimated by $O(|t|^{-\frac{n}{2}})$ for all $t \in \mathbb{R}$ (see, e.q., [9]). Then, we expect that the bilinear estimates for $a(\xi)=\xi_1^2-\xi_2^2$ are the same as those for $a(\xi)=\xi_1^2+\xi_2^2$. The purpose of this paper is to examine this expectation. However, our answer is negative. More precisely, our results are the following.

Theorem 1.2. *Let $n=2$, and let $a(\xi)=\xi_1^2-\xi_2^2$.*

(i) *For $s \geq 0$, there exists $b \in (\frac{1}{2},1)$ and $C>0$ such that the estimates* (1), (2) *and* (3) *hold.*

(ii) *For any $s<0$ and for any $b \in \mathbb{R}$, the estimates* (1), (2) *and* (3) *fail to hold.*

Note that our results are independent of the structure of products. In other words, our results depend only on the properties of $a(\xi)$, in particular, on the noncompactness of the zeros of $a(\xi)$.

We shall prove Theorem 1.2 in the next section. On one hand, we directly compute trilinear forms in the phase space to show (i) of Theorem 1.2. We see that the Strichartz estimates work for making use of the regularity property of the free propagator $e^{it(D_1^2-D_2^2)}$ to prove (i).

On the other hand, to prove (ii) of Theorem 1.2, we construct two sequences of real-analytic functions for which the bilinear estimates break down. We observe that one cannot make full use of the regularity properties of $e^{it(D_1^2-D_2^2)}$ for the negative index s. More precisely, if $s<0$, then these properties cannot work effectively near the set of zeros of $a(\xi)$, that is, the hyperbola in $\mathbb{R}^2$.

Finally, we remark that our results seem to be strongly related with the recent results on bilinear estimates of the two dimensional Fourier restriction problems by Tao and Vargas in [17] and [18]. They obtained bilinear estimates of two functions restricted on the unit paraboloid in the phase space. Their method of proof does not work for the restriction on the hyperbolic paraboloid.

2. Proof of Theorem 1.2

Fix $a(\xi)=\xi_1^2-\xi_2^2$. Note that $a(\xi)=a(-\xi)$ for any $\xi \in \mathbb{R}^2$. First, we prove (i) of Theorem 1.2. Secondly, we prove a lemma needed in the proof of (i). Lastly, we conclude this paper by proving (ii) of Theorem 1.2.

Proof of (i) *of Theorem* 1.2. Let $s \geq 0$ and $\frac{1}{2} < b < 1$. We employ the idea of trilinear estimates developed in [15]. In view of the duality argument, we have only to show that there exists a positive constant C depending only on s and b such that

$$|I| \leqslant C\|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)},$$

where

$$I = \int_{A_1}\int_{A_2} \frac{\langle \mu_0\rangle^s \langle \mu_1\rangle^{-s} \langle \mu_2\rangle^{-s} f(\mu_0,\tau_0) g(\mu_1,\tau_1) h(\mu_2,\tau_2)}{\langle \tau_0 + a(\mu_0)\rangle^{1-b} \langle \tau_1 \pm a(\mu_1)\rangle^b \langle \tau_2 \pm a(\mu_2)\rangle^b} d\tau_0 d\tau_1 d\tau_2 d\mu_0 d\mu_1 d\mu_2$$

and, A_1 and A_2 are defined by

$$A_1 = \{(\mu_0,\mu_1,\mu_2) \in \mathbb{R}^6 \mid \mu_0 + \mu_1 + \mu_2 = 0\}$$
$$A_2 = \{(\tau_0,\tau_1,\tau_2) \in \mathbb{R}^3 \mid \tau_0 + \tau_1 + \tau_2 = 0\}.$$

By using the pairs of signatures $\pm a(\mu_1)$ and $\pm a(\mu_2)$ in I, we can prove (1), (2) and (3) together. More precisely, the pairs $(-,-)$, $(+,+)$ and $(+,-)$ correspond to (1), (2) and (3) respectively. Since $\langle \mu_1 + \mu_2\rangle^s \leqslant 2^s \langle \mu_1\rangle^s \langle \mu_2\rangle^s$ for $s \geq 0$, a simple computation gives

$$\begin{aligned}
|I| &= \Bigg| \int_{\mathbb{R}^4}\int_{\mathbb{R}^2} \frac{f(-\mu_1-\mu_2, -\tau_1-\tau_2) g(\mu_1,\tau_1) h(\mu_2,\tau_2)}{\langle \tau_1 \pm a(\mu_1)\rangle^b \langle \tau_2 \pm a(\mu_2)\rangle^b} \\
&\qquad \times \frac{\langle \mu_1+\mu_2\rangle^s \langle \mu_1\rangle^{-s} \langle \mu_2\rangle^{-s}}{\langle -\tau_1 - \tau_2 + a(\mu_1+\mu_2)\rangle^{1-b}} \, d\tau_1\, d\tau_2\, d\mu_1\, d\mu_2 \Bigg| \\
&\leqslant 2^s \int_{\mathbb{R}^4}\int_{\mathbb{R}^2} \frac{|f(-\mu_1-\mu_2,-\tau_1-\tau_2)||g(\mu_1,\tau_1)||h(\mu_2,\tau_2)|}{\langle \tau_1 \pm a(\mu_1)\rangle^b \langle \tau_2 \pm a(\mu_2)\rangle^b} \, d\tau_1\, d\tau_2\, d\mu_1\, d\mu_2 \\
&= (2\pi)^{-\frac{3}{2}} 2^s \int_{\mathbb{R}^2}\int_{\mathbb{R}} \mathscr{F}^{-1}_{\xi,\tau}[|f|](x,t) G(x,t) H(x,t)\, dt\, dx,
\end{aligned}$$

where

$$G(x,t) = \int_{\mathbb{R}^2}\int_{\mathbb{R}} e^{i(x\cdot\mu + t\tau)} \frac{|g(\mu,\tau)|}{\langle \tau \pm a(\mu)\rangle^b} \, d\tau\, d\mu$$
$$H(x,t) = \int_{\mathbb{R}^2}\int_{\mathbb{R}} e^{i(x\cdot\mu + t\tau)} \frac{|h(\mu,\tau)|}{\langle \tau \pm a(\mu)\rangle^b} \, d\tau\, d\mu,$$

and $\mathscr{F}^{-1}_{\xi,\tau}$ denotes the inverse Fourier transform on ξ and τ, that is,

$$\mathscr{F}^{-1}_{\xi,\tau}[\tilde{f}](x,t) = (2\pi)^{-\frac{3}{2}} \iint_{\mathbb{R}^3} e^{it\tau + ix\cdot\xi} \tilde{f}(\xi,\tau) d\xi d\tau.$$

The estimates of G and H are the following:

Lemma 2.1. *For $b > \frac{1}{2}$, there exists $C_1 = C_1(b) > 0$ such that for any g, $h \in L^2(\mathbb{R}^3)$*

$$\|G\|_{L^4(\mathbb{R}^3)} \leqslant C_1 \|g\|_{L^2(\mathbb{R}^3)}, \quad \|H\|_{L^4(\mathbb{R}^3)} \leqslant C_1 \|h\|_{L^2(\mathbb{R}^3)},$$

where $L^4(\mathbb{R}^3)$ is the set of all Lebesgue measurable functions of $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$ satisfying

$$\|F\|_{L^4(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}} |F(x,t)|^4 \, dt dx \right)^{\frac{1}{4}} < +\infty.$$

By using Lemma 2.1, the Hölder inequality and the Plancherel formula, we deduce

$$|I| \leqslant \|f\|_{L^2(\mathbb{R}^3)} \|G\|_{L^4(\mathbb{R}^3)} \|H\|_{L^4(\mathbb{R}^3)} \leqslant C \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)},$$

which was to be established.

Proof of Lemma 2.1. We show the estimate of G. Changing a variable by $\tau = \lambda \mp a(\xi)$, we deduce

$$\begin{aligned} G(x,t) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{i(x\cdot\xi + t\tau)} \frac{|g(\xi,\tau)|}{\langle \tau \pm a(\xi) \rangle^b} \, d\tau \, d\xi \\ &= \int_{\mathbb{R}} e^{it\lambda} \langle \lambda \rangle^{-b} \left(\int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{\mp ita(\xi)} |g(\xi, \lambda \mp a(\xi))| \, d\xi \right) d\lambda \\ &= \int_{\mathbb{R}} e^{it\lambda} \langle \lambda \rangle^{-b} e^{\mp ita(D)} \psi_\lambda(x) \, d\lambda, \end{aligned}$$

where $(\psi_\lambda)^\wedge(\xi) = 2\pi |g(\xi, \lambda \mp a(\xi))|$. Applying the Minkowski inequality, we get

$$\begin{aligned} \|G\|_{L^4(\mathbb{R}^3)} &= \left(\iint_{\mathbb{R}^3} \left| \int_{\mathbb{R}} e^{it\lambda} \langle \lambda \rangle^{-b} e^{\mp ita(D)} \psi_\lambda(x) d\lambda \right|^4 dt \, dx \right)^{\frac{1}{4}} \\ &\leqslant \int_{\mathbb{R}} \left(\iint_{\mathbb{R}^3} \left| e^{it\lambda} \langle \lambda \rangle^{-b} e^{\mp ita(D)} \psi_\lambda(x) \right|^4 dt \, dx \right)^{\frac{1}{4}} d\lambda \\ &= \int_{\mathbb{R}} \langle \lambda \rangle^{-b} \left(\iint_{\mathbb{R}^3} \left| e^{\mp ita(D)} \psi_\lambda(x) \right|^4 dt dx \right)^{\frac{1}{4}} d\lambda. \end{aligned} \tag{6}$$

Since $a(\xi)$ is a two-dimensional nondegenerate quadratic form of ξ, the so-called Strichartz estimate

$$\|e^{\pm ita(D)} u\|_{L^4(\mathbb{R}^3)} \leqslant C \|u\|_{L^2(\mathbb{R}^2)}$$

holds (see, e.g., [6, Appendix]). Using this, the Schwarz inequality with $b > \frac{1}{2}$ and the Plancherel formula, we obtain

$$\begin{aligned}
\|G\|_{L^4(\mathbb{R}^3)} &\leqslant C \int_{\mathbb{R}} \langle\lambda\rangle^{-b} \|\psi_\lambda\|_{L^2(\mathbb{R}^2)}\, d\lambda \\
&\leqslant C(b) \left(\int_{\mathbb{R}} \|\psi_\lambda\|^2_{L^2(\mathbb{R}^2)}\, d\lambda \right)^{\frac{1}{2}} \\
&= 2\pi C(b) \left(\int_{\mathbb{R}} \int_{\mathbb{R}^2} |g(\xi, \lambda \mp a(\xi))|^2\, d\xi\, d\lambda \right)^{\frac{1}{2}} \\
&= 2\pi C(b) \left(\int_{\mathbb{R}} \int_{\mathbb{R}^2} |g(\xi, \lambda)|^2\, d\xi\, d\lambda \right)^{\frac{1}{2}} \\
&= 2\pi C(b) \|g\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

This completes the proof of Lemma 2.1. □

Proof of (ii) *of Theorem* 1.2. Basically we show the optimality in the bilinear estimates by constructing suitable Knapp-type counterexamples as in [10].

First, we prove the case $j = 1$. Fix $s < 0$ and $b \in \mathbb{R}$. Set $B = \max\{1, |b|\}$ for short. Suppose that there exists a positive constant $C > 0$ such that the bilinear estimate (1) holds for any u, $v \in L^2(\mathbb{R}^3)$. For $N = 1, 2, 3, \ldots$, set

$$\widetilde{u_N}(\xi_1, \xi_2, \tau) = \chi_{Q_N}(\xi_1, \xi_2, \tau), \quad \widetilde{v_N}(\xi_1, \xi_2, \tau) = \chi_{Q_N}(-\xi_1, -\xi_2, -\tau),$$

where χ_A is the characteristic function of a set A, and

$$Q_N = \left\{ (\xi_1, \xi_2, \tau) \in \mathbb{R}^3 \;\middle|\; N \leqslant \xi_1 + \xi_2 \leqslant 2N, |\xi_1 - \xi_2| \leqslant \frac{1}{4N}, |\tau| \leqslant \frac{1}{2} \right\}.$$

Note that

$$Q_N \subset \left\{ (\xi_1, \xi_2, \tau) \in \mathbb{R}^3 \;\middle|\; |\tau \pm a(\xi)| \leqslant 1, \frac{N}{2} \leqslant |\xi| \leqslant 2N \right\}, \tag{7}$$

since

$$\begin{gathered}
-\frac{1}{2} \leqslant a(\xi) = (\xi_1 + \xi_2)(\xi_1 - \xi_2) \leqslant \frac{1}{2} \\
\frac{N^2}{2} \leqslant |\xi|^2 = \frac{(\xi_1 + \xi_2)^2}{2} + \frac{(\xi_1 - \xi_2)^2}{2} \leqslant 2N^2 + \frac{1}{32N^2}.
\end{gathered}$$

By using (7), we deduce

$$\begin{aligned}
\|u_N\|_{s,b} &= \left(\iint_{Q_N} \langle \tau - a(\xi) \rangle^{2b} \langle \xi \rangle^{2s} d\tau d\xi \right)^{\frac{1}{2}} \\
&\leqslant 2^{B-s} N^s \left(\iint_{Q_N} d\tau d\xi \right)^{\frac{1}{2}} \\
&= 2^{B-s-1} N^s,
\end{aligned} \tag{8}$$

and

$$
\begin{aligned}
\|v_N\|_{s,b} &= \left(\iint_{Q_N} \langle \tau + a(\xi) \rangle^{2b} \langle \xi \rangle^{2s} d\tau d\xi \right)^{\frac{1}{2}} \\
&\leqslant 2^{B-s} N^s \left(\iint_{Q_N} d\tau d\xi \right)^{\frac{1}{2}} \\
&= 2^{B-s-1} N^s. \qquad (9)
\end{aligned}
$$

A simple computation shows that for large $N \in \mathbb{N}$,

$$
\begin{aligned}
\widetilde{u_N v_N}(\xi, \tau) &= (2\pi)^{-\frac{3}{2}} \iint_{\mathbb{R}^3} \chi_{Q_N}(\xi - \eta, \tau - \lambda) \chi_{Q_N}(-\eta, -\lambda) d\eta d\lambda \\
&= (2\pi)^{-\frac{3}{2}} \iint_{Q_N} \chi_{Q_N}(\xi + \eta, \tau + \lambda) d\eta d\lambda \\
&\geqslant \frac{1}{2^{6+\frac{1}{2}} \pi^{\frac{3}{2}}} \chi_{R_N}(\xi, \tau),
\end{aligned}
$$

where

$$
R_N = \left\{ (\xi_1, \xi_2, \tau) \in \mathbb{R}^3 \,\middle|\, |\xi_1 + \xi_2| \leqslant \frac{N}{2}, |\xi_1 - \xi_2| \leqslant \frac{1}{8N}, |\tau| \leqslant \frac{1}{4} \right\}.
$$

Since $R_N \subset \left\{ (\xi_1, \xi_2, \tau) \in \mathbb{R}^3 \,\middle|\, |\tau \pm a(\xi)| \leqslant 1, |\xi| \leqslant \frac{N}{2} \right\}$, we get

$$
\begin{aligned}
\|u_N v_N\|_{s,b-1} &= \left(\iint_{\mathbb{R}^3} |\widetilde{u_N v_N}(\xi, \tau)|^2 \langle \tau - a(\xi) \rangle^{2(b-1)} \langle \xi \rangle^{2s} d\tau d\xi \right)^{\frac{1}{2}} \\
&\geqslant \frac{1}{2^{6+\frac{1}{2}} \pi^{\frac{3}{2}}} \left(\iint_{R_N} \langle \tau - a(\xi) \rangle^{2(b-1)} \langle \xi \rangle^{2s} d\tau d\xi \right)^{\frac{1}{2}} \\
&\geqslant 2^{B-6-\frac{1}{2}} \pi^{-\frac{3}{2}} N^s \left(\iint_{R_N} d\tau d\xi \right)^{\frac{1}{2}} \\
&= 2^{B-8-\frac{1}{2}} \pi^{-\frac{3}{2}} N^s. \qquad (10)
\end{aligned}
$$

Substitute (8), (9) and (10) into (1). Then we have $2^{B-8-\frac{1}{2}} \pi^{-\frac{3}{2}} N^s \leqslant 2^{2B-2s-2} N^{2s}$, which becomes $2^{-B+2s-6-\frac{1}{2}} \pi^{-\frac{3}{2}} \leqslant N^s$. Since $s < 0$, the right hand side of the above goes to zero as $N \to \infty$ while the left hand side is a strictly positive constant depending only on $s < 0$ and $b \in \mathbb{R}$. This is contradiction. Then, this completes the proof of the case $j = 1$.

The cases $j = 2, 3$ are proved in the same way. Let Q_N be the same as above. For $j = 2$, set

$$
\widetilde{u_N}(\xi, \tau) = \chi_{Q_N}(-\xi, -\tau), \quad \widetilde{v_N}(\xi, \tau) = \chi_{Q_N}(\xi, \tau),
$$

and for $j = 3$, set

$$\widetilde{u_N}(\xi,\tau) = \chi_{Q_N}(-\xi,-\tau), \quad \widetilde{v_N}(\xi,\tau) = \chi_{Q_N}(-\xi,-\tau).$$

We omit the detail about the cases $j = 2, 3$. □

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