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On the energy critical Schrödinger equation in $3D$ non-trapping domains

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Abstract

We prove that the quintic Schrödinger equation with Dirichlet boundary conditions is locally well posed for $H_0^1(\Omega)$ data on any smooth, non-trapping domain $\Omega \subset \mathbb{R}^3$. The key ingredient is a smoothing effect in $L_x^5(L_t^2)$ for the linear equation. We also derive scattering results for the whole range of defocusing sub quintic Schrödinger equations outside a star-shaped domain.
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1. Introduction

The Cauchy problem for the semilinear Schrödinger equation in $\mathbb{R}^3$ is by now relatively well understood: after seminal results by Ginibre and Velo [10] in the energy class for energy subcritical equations, the issue of local well-posedness in the critical Sobolev spaces ($\dot{H}^{\frac{3}{2}-\frac{2}{p-1}}$) was settled in [7]. Scattering for large time was proved in [10] for energy subcritical defocusing equations, while the energy critical (quintic) defocusing equation was only recently successfully tackled in [9]. The local well-posedness relies on Strichartz estimates, while scattering results combine these local results with suitable non-concentration arguments based on Morawetz type estimates. On domains, the same set of problems remains an elusive target, due to the difficulty in obtaining Strichartz estimates in such a setting. In [2], the authors proved Strichartz estimates with a half-derivative loss on non-trapping domains: the non-trapping assumption is crucial in order to rely on the local smoothing estimates. However, the loss resulted in well-posedness results for strictly less than cubic non-linearities; this was later improved to cubic non-linearities in [1] (combining local smoothing and semiclassical Strichartz near the boundary) and in [11] (on the exterior of a ball, through precised smoothing effects near the boundary). Recently there were two significant improvements, following different strategies:

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- in [16], Luis Vega and the second author obtain an $L^4_{t,x}$ Strichartz estimate which is scale invariant. However, one barely misses $L^4_t(L^\infty(\Omega))$ control for H^1_0 data, and therefore local well-posedness in the energy space was improved to all subcritical (less than quintic) non-linearities, but combining this Strichartz estimate with local smoothing close to the boundary and the full set of Strichartz estimates in $\mathbb{R}^3$ away from it. Scattering was also obtained for the cubic defocusing equation, but the lack of a good local well-posedness theory at the scale invariant level ($\dot H^{\frac{1}{2}}$) led to a rather intricate incremental argument, from scattering in $\dot H^{\frac{1}{4}}$ to scattering in H^1_0;
- in [13], the first author proved the full set of Strichartz estimates (except for the endpoint) outside strictly convex obstacles, by following the strategy pioneered in [17] for the wave equation, and relying on the Melrose–Taylor parametrix. In the case of the Schrödinger equation, one obtains Strichartz estimates on a semiclassical time scale (taking advantage of a "finite speed of propagation" principle at this scale), and then upgrades them to large time results by combining them with the smoothing effect (see [3] for a nice presentation of such an argument, already implicit in [19]). Therefore, one obtains the exact same local well-posedness theory as in the $\mathbb{R}^3$ case, including the quintic non-linearity, and scattering holds for all sub quintic defocusing non-linearities, taking advantage of the a priori estimates from [16].

In the present work, we aim at providing a local well-posedness theory for the quintic non-linearity outside non-trapping obstacles, a case which is not covered by [13]. From explicit computations with gallery modes [12], one knows that the full set of optimal Strichartz estimates does not hold for the Schrödinger equation on a domain whose boundary has at least one geodesically convex point; while this does not preclude a scale invariant Strichartz estimate with a loss (like the $L^4_t(L^\infty_x)$ estimate in $\mathbb{R}^3$ which is enough to solve the quintic NLS), it suggests to bypass the issue and use a different set of estimates, which we call smoothing estimates: in $\mathbb{R}^3$, these estimates may be stated as follows,

$$\left\|\exp(it\Delta)f\right\|_{L^4_x(L^2_t)} \lesssim \|f\|_{\dot H^{-\frac{1}{4}}}, \tag{1.1}$$

from which one can infer various estimates by using Sobolev in time and/or in space. Formally, (1.1) is an immediate consequence of the Stein–Tomas restriction theorem in $\mathbb{R}^3$ (or, more accurately, its dual version, on the extension): let $\tau > 0$ be a fixed radius, one sees $\hat f(\xi)$ as a function on $|\xi| = \sqrt{\tau}$, and applies the extension estimate, with δ the Dirac function and $\mathcal{F}$ the space Fourier transform

$$\left\|\mathcal{F}^{-1}\big(\delta\big(\tau - |\xi|^2\big)\hat f(\xi)\big)\right\|_{L^4_x} \lesssim \left\|\hat f(\xi)\right\|_{L^2(|\xi|=\sqrt{\tau})}.$$

Summing over τ yields the L^2 norm of f on the right-hand side, while on the left we use Plancherel in time and Minkowski to get (1.1). A similar estimate holds for the wave equation, replacing $\sqrt{\tau} = |\xi|$ by $\tau = \pm|\xi|$, and usually goes under the denomination of square function (in time) estimates. In a compact setting (e.g. compact manifolds) a substitute for the Stein–Tomas theorem is provided by L^p eigenfunction estimates, or better yet, spectral cluster estimates. In the context of a compact manifold with boundaries, such spectral cluster estimates were recently obtained by Smith and Sogge in [18], and provided a key tool for solving the critical wave equation on domains, see [4,6]. In this paper, we apply the same strategy to the Schrödinger equation:

- we derive an $L^5(\Omega; L^2_I)$ "smoothing" estimate for spectrally localized data on compact manifolds with boundaries, from the spectral cluster $L^5(\Omega)$ estimate; here I is a time interval whose size is such that $|I||\sqrt{-\Delta_D}| \sim 1$;
- we decompose the solution to the linear Schrödinger equation on a non-trapping domain into two main regions: close to the boundary, where we can view the region as embedded into a $3D$ punctured torus, to which the previous semiclassical estimate may be applied, and then summed up using the local smoothing effect; and far away from the boundary where the $\mathbb{R}^3$ estimates hold.
- Finally, we patch together all estimates to obtain an estimate which is valid on the whole exterior domain. Local well-posedness in the critical Sobolev space $\dot H^{\frac{3}{2}-\frac{2}{p-1}}$ immediately follows for $3 + 2/5 < p \leqslant 5$, and together with the a priori estimates from [16], this implies scattering for the defocusing equation for $3 + 2/5 < p < 5$. The remaining range $3 \leqslant p \leqslant 3 + 2/5$ is sufficiently close to 3 that, as alluded to in [16], a suitable modification of the arguments from [16] yields scattering as well.

Remark 1.1. Clearly, such smoothing estimates are better suited to "large" values of p: the restriction $3 + 2/5 < p$ for the critical well-posedness is directly linked to the exponent 5 in the spectral cluster estimates; in $\mathbb{R}^3$, where the correct (and optimal!) exponent is 4, one may solve down to $p = 3$ by this method, while the Strichartz estimates allow to solve at scaling level all the way to the L^2 critical value $p = 1 + 4/3$.

2. Statement of results

Let Θ be a compact, non-trapping obstacle in $\mathbb{R}^3$ and set $\Omega = \mathbb{R}^3 \setminus \Theta$. By Δ_D we denote the Laplace operator with constants coefficients on Ω. We write $L^p_x = L^p(\Omega)$ and $\dot{H}^\sigma$ for the Lebesgue and Sobolev spaces on Ω. For $s \in \mathbb{R}$, $p, q \in [1, \infty]$ we denote by $\dot{B}^{s,q}_p(\Omega) = \dot{B}^{s,q}_p$ Besov spaces on Ω, where the spectral localization in their definition is meant to be with respect to Δ_D; it will be useful to introduce Banach-valued Besov spaces $\dot{B}^{s,q}_p(L^r_t)$, and we refer to Section 4 (Definition 4.1) for relevant definitions. Whenever L^p_t is replaced by L^p_T, it means that the time integration is restricted to the interval $(-T, T)$. Notice that $\dot{H}^s = \dot{B}^{s,2}_2$ (using the spectral decomposition).

Remark 2.1. A reader who is well acquainted with Besov spaces in $\mathbb{R}^n$ may think our spaces are indeed equivalent to the usual ones, which are defined on domains via extensions or with finite differences when the regularity s is strictly positive. While most likely true, these equivalences are non-trivial to prove and we elected to just define the spaces we need in a way which is convenient for our purposes. We however have a (small) price to pay in Appendix A where several non-linear mappings are proved within our framework.

We aim at studying well-posedness for the energy critical equation on $\Omega \times \mathbb{R}$, with Dirichlet boundary condition,

$$i\partial_t u + \Delta_D u = \pm|u|^4 u, \qquad u_{|\partial\Omega} = 0, \qquad u_{|t=0} = u_0 \tag{2.1}$$

and more generally

$$i\partial_t u + \Delta_D u = \pm|u|^{p-1} u, \qquad u_{|\partial\Omega} = 0, \qquad u_{|t=0} = u_0 \tag{2.2}$$

with $p < 5$.

Theorem 2.1 *(Well-posedness for the quintic Schrödinger equation). Let $u_0 \in H^1_0(\Omega)$. There exists $T(u_0)$ such that the quintic non-linear equation* (2.1) *admits a solution u in $C([-T, T], H^1_0(\Omega))$; moreover, u is unique in $\dot{B}^{1,2}_5(L^{\frac{20}{11}}_T) \cap L^{\frac{20}{3}}_x L^{40}_T$. If the data is small, then the solution is global in time and scatters in H^1_0.*

The previous theorem extends to the following subcritical range:

Theorem 2.2. *Let $3 + \frac{2}{5} < p < 5$, $s_p = \frac{3}{2} - \frac{2}{p-1}$ and $u_0 \in \dot{H}^{s_p}$. There exists $T(u_0)$ such that the non-linear equation* (2.2) *admits a solution u in $C([-T, T], \dot{H}^{s_p})$, and u is unique in $\dot{B}^{s_p,2}_5(L^{\frac{20}{11}}_T) \cap L_x^{\frac{5(p-1)}{3}} L_T^{10(p-1)}$. Moreover the solution is global in time and scatters in $\dot{H}^{s_p}$ if the data is small.*

Remark 2.2. We elected to state both theorems for Dirichlet boundary conditions mostly for sake of simplicity. Indeed, both results hold with Neumann boundary conditions, at least for $p \geqslant 4$: the key ingredients for the required linear estimates are known to hold for Neumann, see [18,2], while the non-linear mappings from our appendix rely on [14] (where all relevant estimates can be proved to hold in the Neumann case).

Finally, we consider the long time asymptotics for (2.2) in the defocusing case, namely the + sign on the left; in this situation, we are indeed restricted to the Dirichlet boundary conditions, as we rely on a priori estimates from [16].

Theorem 2.3. *Assume the domain Ω to be the exterior of a star-shaped compact obstacle* (*which implies Ω is non-trapping*). *Let $3 \leqslant p < 5$, and $u_0 \in H^1_0(\Omega)$. There exists a unique global in time solution u, which is in the energy* *class,* $C(\mathbb{R}, H_0^1(\Omega))$*, to the non-linear equation* (2.2) *in the defocusing case* (+ *sign in* (2.2))*. Moreover, this solution scatters for large times*: *there exist two scattering states* $u^\pm \in H_0^1(\Omega)$ *such that*

$$\lim_{t\leftarrow\pm\infty}\|u(x,t)-e^{it\Delta_D}u^\pm\|_{H_0^1(\Omega)}=0.$$

As mentioned in the introduction, the (global) existence part was dealt with in [16]; for the scattering part, the $p=3$ case was also dealt with in [16]. In the setting of Theorem 2.2, one may adapt the usual argument from the $\mathbb{R}^n$ case, combining a priori estimates and a good Cauchy theory at the critical regularity; this provides a very short argument in the range $3+2/5<p<5$. In the remaining range, namely $3<p\leqslant 3+2/5$, one unfortunately needs to adapt the intricate proof from [16], and this leads to a much lengthier proof; we provide it mostly for the sake of completeness. This type of argument may however be of relevance in other contexts.

3. Smoothing type estimates

We start with definitions and notations. Let $\psi(\xi^2)\in C_0^\infty(\mathbb{R}\setminus\{0\})$ and $\psi_j(\xi^2)=\psi(2^{-2j}\xi^2)$. On the domain Ω, one has the spectral resolution of the Dirichlet Laplacian, and we may define a smooth spectral projection $\Delta_j=\psi_j(-\Delta_D)$ as a bounded operator on L^2. Moreover, this operator is bounded on L^p for all p, and if f is Hilbert-valued and such that $\|\|f\|_H\|_{L^p(\Omega)}<+\infty$, then Δ_j is bounded as well on $L^p(H)$. We refer to [14] for an extensive discussion and references. We simply point out that if $H=L_t^2$, then Δ_j is continuous on all $L_x^pL_t^q$ by interpolation with the obvious $L_t^p(L_x^p)$ bound and duality.

In this section we focus on estimates for the linear Schrödinger equation on $\Omega\times\mathbb{R}$ with Dirichlet boundary conditions,

$$i\partial_t u_L+\Delta_D u_L=0,\qquad u_{L|\partial\Omega}=0,\qquad u_{L|t=0}=u_0. \tag{3.1}$$

Theorem 3.1. *The following local smoothing estimate holds for the homogeneous linear equation* (3.1)*,*

$$\|\Delta_j u_L\|_{L_x^5L_t^2}+2^{-2j}\|\partial_t\Delta_j u_L\|_{L_x^5L_t^2}\lesssim 2^{-\frac{j}{10}}\|\Delta_j u_0\|_{L_x^2}. \tag{3.2}$$

Moreover, let $2\leqslant q\leqslant\infty$*, then*

$$\|\Delta_j u_L\|_{L_x^5L_t^q}+2^{-2j}\|\partial_t\Delta_j u_L\|_{L_x^5L_t^q}\lesssim 2^{-j(\frac{2}{q}-\frac{9}{10})}\|\Delta_j u_0\|_{L_x^2}. \tag{3.3}$$

Consider now the inhomogeneous equation,

$$i\partial_t v+\Delta_D v=F,\qquad v_{|\partial\Omega}=0,\qquad v_{|t=0}=0. \tag{3.4}$$

From Theorem 3.1, we will obtain the following set of estimates:

Theorem 3.2. *Let* $2\leqslant q\leqslant+\infty$*,* $2<r\leqslant+\infty$*, then*

$$\|\Delta_j v\|_{C_t(L_x^2)}+2^{j(\frac{2}{q}-\frac{9}{10})}\|\Delta_j v\|_{L_x^5L_t^q}+2^{j(\frac{2}{q}-\frac{29}{10})}\|\partial_t\Delta_j v\|_{L_x^5L_t^q}\lesssim 2^{-j(\frac{4}{r}-\frac{9}{5})}\|\Delta_j F\|_{L_x^{\frac{5}{4}}L_t^{r'}}, \tag{3.5}$$

with $1/r+1/r'=1$ *and* v *solution to* (3.4).

Combining the previous theorems with the results from [16], we finally state the set of estimates which will be used later for

$$i\partial_t u+\Delta_D u=F_1+F_2,\qquad u_{|\partial\Omega}=0,\qquad v_{|t=0}=u_0. \tag{3.6}$$

Theorem 3.3. *Let* $2\leqslant q\leqslant+\infty$*,* $2<r\leqslant+\infty$*, then*

$$\begin{aligned}&\|\Delta_j u\|_{C_t(L_x^2)}+2^{j(\frac{2}{q}-\frac{9}{10})}\|\Delta_j u\|_{L_x^5L_t^q}+2^{j(\frac{2}{q}-\frac{29}{10})}\|\partial_t\Delta_j u\|_{L_x^5L_t^q}+2^{-\frac{3}{4}j}\|\Delta_j u\|_{L_{t,x}^4}+2^{-\frac{11}{4}j}\|\partial_t\Delta_j u\|_{L_{t,x}^4}\\&\quad\lesssim\|\Delta_j u_0\|_{L_x^2}+2^{-j(\frac{4}{r}-\frac{9}{5})}\|\Delta_j F_1\|_{L_x^{\frac{5}{4}}L_t^{r'}}+2^{-\frac{1}{4}j}\|\Delta_j F_2\|_{L_{t,x}^{\frac{4}{3}}},\end{aligned} \tag{3.7}$$

with $1/r+1/r'=1$ *and* u *solution to* (3.6).

3.1. Proof of Theorem 3.1

Let $P_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$, then $P_j \Delta_j = \Delta_j$ because of support properties. We now split the solution of the linear equation $\Delta_j u_L = P_j \Delta_j u_L$ as a sum of two terms $P_j \chi \Delta_j u_L + P_j(1-\chi)\Delta_j u_L$, where $\chi \in C_0^\infty(\mathbb{R}^3)$ is compactly supported and $\chi = 1$ near the boundary $\partial\Omega$.

3.1.1. "Far" from the boundary: $P_j(1-\chi)\Delta_j u_L$

In this case the spectral localization provided by P_j is useless until the very last step and we therefore drop it. Set $w_j(t,x) = (1-\chi)\Delta_j e^{it\Delta_D} u_0(x)$. Then w_j satisfies

$$\begin{cases} i\partial_t w_j + \Delta_D w_j = -[\Delta_D, \chi]\Delta_j u_L, \\ w_j|_{t=0} = (1-\chi)\Delta_j u_0. \end{cases} \tag{3.8}$$

Since $\chi = 1$ near the boundary $\partial\Omega$, we can view the solution to (3.8) as a solution to the Schrödinger equation in $\mathbb{R}^3$. Consequently, the Duhamel formula writes

$$w_j(t,x) = e^{it\Delta_0}(1-\chi)\Delta_j u_0 - \int_0^t e^{i(t-s)\Delta_0}[\Delta_D, \chi]\Delta_j u_L(s)\,ds, \tag{3.9}$$

where Δ_0 is the free Laplacian on $\mathbb{R}^3$. Hence, for $e^{it\Delta_0}(1-\chi)\Delta_j u_0$, usual Strichartz estimates hold. We now have to deal with the second term in the right-hand side of (3.9). Ideally, one would like to remove the time restriction $s < t$ and use a variant of the Christ–Kiselev lemma. However, this would miss the endpoint case $q = 2$. Instead, we recall the following lemma:

Lemma 3.1. *(See Staffilani and Tataru [19].) Let $x \in \mathbb{R}^n$, $n \geqslant 3$ and let $f(x,t)$ be compactly supported in space, such that $f \in L_t^2(H^{-\frac{1}{2}})$. Then the solution w to $(i\partial_t + \Delta_0)w = f$ with $w_{|t=0} = 0$, is such that*

$$\|w\|_{L_t^2(L_x^{\frac{2n}{n-2}})} \lesssim \|f\|_{L_t^2(H^{-\frac{1}{2}})}. \tag{3.10}$$

In fact, one may shift regularity in (3.10) without difficulty. Now, the proof in [19] relies on a decomposition into traveling waves, to which homogeneous estimates are then applied. We can therefore use the $L_x^4(L_t^2)$ smoothing estimate, Sobolev in space, and extend the conclusion of Lemma 3.1 to

$$\|w\|_{L_x^5(L_t^2)} \lesssim \|f\|_{L_t^2(H^{-\frac{1}{2}-\frac{1}{10}})}, \tag{3.11}$$

where we chose to conveniently shift the regularity to the right hand-side.

We now take $f = -[\Delta_D, \chi]\Delta_j u_L \in L_t^2 H_{\mathrm{comp}}^{-1/2-1/10}(\Omega)$ and

$$\big\|[\Delta_D, \chi]\Delta_j u_L\big\|_{L^2 H_{\mathrm{comp}}^{-1/2-1/10}} \lesssim \|\Delta_j u_L\|_{L^2 \dot H^{1/2-1/10}(\Omega)} \lesssim \|\Delta_j u_0\|_{\dot H^{-1/10}(\Omega)},$$

from which the smoothing estimate for solutions to (3.8) follows

$$\begin{aligned} \big\|(1-\chi)\Delta_j u_L\big\|_{L^5(\mathbb{R}^3)L_t^2} &\lesssim \big\|(1-\chi)\Delta_j u_0\big\|_{\dot H^{-\frac{1}{10}}(\mathbb{R}^3)} + \big\|[\Delta_D, \chi]\Delta_j u_L\big\|_{L^2 H_{\mathrm{comp}}^{-1/2-1/10}} \\ &\lesssim \|\Delta_j u_0\|_{\dot H^{-\frac{1}{10}}(\Omega)}. \end{aligned} \tag{3.12}$$

We conclude using the continuity properties of P_j which were recalled at the beginning of Section 3 (e.g. see [14, Corollary 2.5]). In fact, using (3.12), we get

$$\begin{aligned} \big\|P_j(1-\chi)\Delta_j u_L\big\|_{L_x^5 L_t^2} &\lesssim \big\|(1-\chi)\Delta_j u_L\big\|_{L_x^5 L_t^2} \\ &\lesssim 2^{-\frac{j}{10}}\|\Delta_j u_0\|_{L^2(\Omega)}, \end{aligned}$$

where we have used the spectral localization Δ_j to estimate

$$\|\Delta_j u_0\|_{\dot H^\sigma(\Omega)} \simeq 2^{\sigma j}\|\Delta_j u_0\|_{L^2(\Omega)}.$$

3.1.2. "Close" to the boundary: $P_j \chi \Delta_j u_L$

For $l \in \mathbb{Z}$ let $\varphi_l \in C_0^\infty(((l-1/2)\pi, (l+1)\pi))$ equal to 1 on $[l\pi, (l+1/2)\pi]$. We set $v_j = P_j \chi \Delta_j u_L$ and for $l \in \mathbb{Z}$ we set $v_{j,l} = \varphi_l(2^j t) v_j$. We have

$$\begin{aligned}\|v_j\|^2_{L^5_x L^2_t} &= \Big\| \sum_{l\in\mathbb{Z}} v_{j,l} \Big\|^2_{L^5_x L^2_t} \simeq \Big\| \Big\| \sum_{l\in\mathbb{Z}} v_{j,l} \Big\|^2_{L^2_t} \Big\|_{L^{5/2}_x} \\ &\lesssim \Big\| \sum_{l\in\mathbb{Z}} \|v_{j,l}\|^2_{L^2_t} \Big\|_{L^{5/2}_x} \leqslant \sum_{l\in\mathbb{Z}} \|v_{j,l}\|^2_{L^5_x L^2_t}, \end{aligned} \tag{3.13}$$

where for the first inequality rests upon almost orthogonality in time of the $(\varphi_l)_l$. In order to estimate $\|v_j\|^2_{L^5_x L^2_t}$ it will be thus sufficient to estimate each $\|v_{j,l}\|^2_{L^5_x L^2_t}$. We derive an equation for $\tilde{v}_{j,l} \overset{\text{def}}{=} \varphi_l(2^j t)\chi \Delta_j u_L$:

$$i\partial_t \tilde{v}_{j,l} + \Delta_D \tilde{v}_{j,l} = -\big(\varphi_l\big(2^j t\big)[\Delta_D, \chi]\Delta_j u_L - i 2^j \varphi_l'\big(2^j t\big)\chi \Delta_j u_L\big), \tag{3.14}$$

and we stress that $\tilde{v}_{j,l}$ vanishes outside the time interval $(2^{-j}(l-1/2)\pi, 2^{-j}(l+1)\pi)$. Let $V_{j,l}$ be the right-hand side in (3.14), namely

$$V_{j,l} \overset{\text{def}}{=} -\varphi_l\big(2^j t\big)[\Delta_D, \chi]\Delta_j u_L + i 2^j \varphi_l'\big(2^j t\big)\chi \Delta_j u_L. \tag{3.15}$$

Let $Q \subset \mathbb{R}^3$ be an open cube which is sufficiently large so that $\partial\Omega$ is contained in the interior of Q. We now view Q as a compact, boundary less, manifold with periodic boundary conditions on ∂Q, and we denote by S the punctured torus $Q \setminus \Theta$ (recall that $\Omega = \mathbb{R}^3 \setminus \Theta$). Let also $\Delta_S \overset{\text{def}}{=} \sum_{j=1}^3 \partial_j^2$ denote the Laplace operator on the compact domain S.

On S, we may define a spectral localization operator using eigenvalues λ_k and eigenvectors e_k of Δ_S: if $f = \sum_k c_k e_k$, then

$$\Delta_j^S f = \psi\big(2^{-2j}\Delta_S\big) f = \sum_k \psi\big(2^{-2j}\lambda_k^2\big) c_k e_k. \tag{3.16}$$

We also define $P_j^S = \Delta_{j-1}^S + \Delta_j^S + \Delta_{j+1}^S$.

Remark 3.1. Notice that in a neighborhood of the boundary, the domains of Δ_S and Δ_D coincide, thus if $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$ is supported near $\partial\Omega$ then

$$\Delta_S \tilde{\chi} = \Delta_D \tilde{\chi}.$$

In order to apply estimates on the manifold S, we will need to re-localize close to the obstacle. Consider $\chi_1 \in C_0^\infty(\mathbb{R}^3)$ supported near the boundary and equal to 1 on the support of $\tilde{\chi}$, we will write

$$\chi_1 P_j \tilde{\chi} = \chi_1 P_j^S \tilde{\chi} + \chi_1\big(P_j - P_j^S\big)\tilde{\chi}, \tag{3.17}$$

and our expectation will be that the difference term is smoothing.

In what follows let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$ be equal to 1 on the support of χ and be supported in a neighborhood of $\partial\Omega$ such that on its support the operator $-\Delta_D$ coincide with $-\Delta_S$. From their support properties in space, we have $\tilde{v}_{j,l} = \tilde{\chi}\tilde{v}_{j,l}$ and $V_{j,l} = \tilde{\chi} V_{j,l}$. Consequently $\tilde{v}_{j,l}$ will also solve the following equation on our compact manifold S

$$\begin{cases} i\partial_t \tilde{v}_{j,l} + \Delta_S \tilde{v}_{j,l} = V_{j,l}, \\ \tilde{v}_{j,l}|_{t<h(l-1/2)\pi} = 0, \quad \tilde{v}_{j,l}|_{t>h(l+1)\pi} = 0. \end{cases} \tag{3.18}$$

Therefore we can write the Duhamel formula either for the last equation (3.18) on S, or for Eq. (3.14) on Ω. We now apply P_j and use that $v_{j,l} = P_j \tilde{v}_{j,l}$, $\tilde{\chi}\tilde{v}_{j,l} = \tilde{v}_{j,l}$ and $P_j \tilde{\chi} = \chi_1 P_j^S \tilde{\chi} + (1-\chi_1) P_j \tilde{\chi} + \chi_1(P_j - P_j^S)\tilde{\chi}$, which yields

$$v_{j,l}(t,x)=\chi_1\int_{h(l-1/2)\pi}^{t} e^{i(t-s)\Delta_S}P_j^S V_{j,l}(s,x)\,ds$$
$$+(1-\chi_1)\int_{h(l-1/2)\pi}^{t} e^{i(t-s)\Delta_D}P_j V_{j,l}(s,x)\,ds$$
$$+\chi_1\big(P_j-P_j^S\big)\tilde{v}_{j,l},\tag{3.19}$$

where we conveniently chose to write Duhamel on S for the first term and Duhamel on Ω for the second one, which allows to commute the flow under the time integral. Denote by $v_{j,l,m}$ the first term in the right-hand side of (3.19) by $v_{j,l,f}$ the second one and $v_{j,l,s}$ the last one. We deal with them separately. To estimate the $L_x^5L_t^2$ norm of the $v_{j,l,f}$ we notice that its support is far from the boundary: as such, estimates on the $L_x^5L_t^2$ norm will follow from Section 3.1.1. Indeed, we get

$$\big\|(1-\chi_1)P_je^{i(t-s)\Delta_D}V_{j,l}\big\|_{L_x^5L_t^2}\lesssim\|P_jV_{j,l}\|_{\dot{H}^{-1/10}(\Omega)}\simeq 2^{-\frac{j}{10}}\|P_jV_{j,l}\|_{L^2(\Omega)}.\tag{3.20}$$

We then apply the Minkowski inequality to deduce

$$\left\|(1-\chi_1)\int_{h(l-1/2)\pi}^{t} P_je^{i(t-s)\Delta_D}V_{j,l}(s,x)\,ds\right\|_{L_x^5L_t^2}$$
$$\leqslant 2^{-j/2}\bigg(\int_{I_{j,l}}\big\|(1-\chi_1)P_je^{i(t-s)\Delta_D}V_{j,l}(s,.)\big\|^2_{L^5(\Omega)L^2(I_{j,l})}\,ds\bigg)^{1/2},\tag{3.21}$$

where we denoted $I_{j,l}=[2^{-j}(l-1/2)\pi,2^{-j}(l+1)\pi]$ and we used the Cauchy–Schwartz inequality. Using (3.20) we finally get

$$\|v_{j,l,f}\|_{L^5(\Omega)L^2(I_{j,l})}\leqslant 2^{-j(1/2+1/10)}\|P_jV_{j,l}\|_{L^2(I_{j,l})L^2(\Omega)}.\tag{3.22}$$

To estimate the $L_x^5L_t^2$ norm of the main contribution $v_{j,l,m}$ we need the following:

Proposition 3.1. *Let $j\geqslant 0$, $I_j=(-\pi 2^{-j},\pi 2^{-j})$, $\tilde{\chi}\in C_0^\infty(\mathbb{R}^3)$ be supported near $\partial\Omega$ and $V_0\in L^2(\Omega)$. Then there exists $C>0$ independent of j such that for the solution $e^{it\Delta_S}P_j^S\tilde{\chi}V_0$ of the linear Schrödinger equation on S with initial data $P_j^S\tilde{\chi}V_0$ we have*

$$\big\|e^{it\Delta_S}P_j^S\tilde{\chi}V_0\big\|_{L^5(S)L_t^2(I_j)}\leqslant C2^{-\frac{j}{10}}\big\|P_j^S\tilde{\chi}V_0\big\|_{L^2(S)}.\tag{3.23}$$

We postpone the proof of Proposition 3.1 to Section 3.4.

Using the fact that $v_{j,l}$ is supported in time in $I_{j,l}=[2^{-j}(l-1/2)\pi,2^{-j}(l+1)\pi]$, the Minkowski inequality, Proposition 3.1 with $\tilde{\chi}=1$ on the support of χ and with $V_0=V_{j,l}$, and since $\tilde{\chi}_1v_{j,l,m}=v_{j,l,m}$ for any $\tilde{\chi}_1\in C^\infty(\mathbb{R}^3)$ with $\tilde{\chi}_1=1$ on the support of χ_1, we obtain

$$\|v_{j,l,m}\|_{L_x^5L^2(I_{j,l})}=\|\tilde{\chi}_1v_{j,l,m}\|_{L_x^5L^2(I_{j,l})}=\|v_{j,l,m}\|_{L^5(S)L^2(I_{j,l})}$$
$$\leqslant\int_{2^{-j}(l-1)\pi}^{2^{-j}(l+1)\pi}\big\|e^{i(t-s)\Delta_S}P_j^SV_{j,l}(s,.)\big\|_{L^5(S)L^2(I_{j,l})}\,ds$$
$$\leqslant 2^{-\frac{j}{10}}\int_{I_{j,l}}\big\|P_j^SV_{j,l}(s)\big\|_{L^2(S)}\,ds$$

$$\leqslant 2^{-\frac{j}{10}} \int_{I_{j,l}} \|\tilde{\chi} V_{j,l}(s)\|_{L^2(S)}\, ds$$

$$\leqslant 2^{-\frac{j}{10}} \int_{I_{j,l}} \|\tilde{\chi} V_{j,l}(s)\|_{L^2(\Omega)}\, ds \tag{3.24}$$

where we used again $V_{j,l} = \tilde{\chi} V_{j,l}$ to switch S and Ω as well as continuity of Δ_j^S on $L^2(S)$. Using the Cauchy–Schwartz inequality in (3.24) yields

$$\|v_{j,l,m}\|_{L^5_x L^2(I_{j,l})} \lesssim 2^{-j(1/2+1/10)} \|V_{j,l}\|_{L^2(I_{j,l})L^2(\Omega)}. \tag{3.25}$$

We postpone for a moment further treatment of the right-hand side of this inequality, and now turn to the difference term $v_{j,l,s}$. We rely on the following smoothing lemma, whose proof is postponed to Section 3.2.

Lemma 3.2. *Let $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on a fixed neighborhood of the support of $\tilde{\chi}$. Then we have for all $N \in \mathbb{N}$,*

$$\|v_{j,l,s}\|_{L^5(\Omega)L^2(I_{j,l})} \leqslant C_N 2^{-Nj} \|V_{j,l}(x,s)\|_{L^2(I_{j,l},L^2(\Omega))}. \tag{3.26}$$

Using this lemma, we get for $v_{j,l,s}$ an estimate which matches (3.25): picking $N = 1$ is enough. From there, using (3.13), (3.22), (3.25), we write

$$\|P_j \chi \Delta_j u_L\|^2_{L^5_x L^2_t} \lesssim 2^{-2j(\frac{1}{2}+\frac{1}{10})} \sum_{l\in\mathbb{Z}} \|P_j V_{j,l}(s)\|^2_{L^2(I_{j,l})L^2(\Omega)} \tag{3.27}$$

and we are left with the right-hand side in (3.27). Using the explicit expression of $V_{j,l}$ given in (3.15), we have

$$\begin{aligned}\|V_{j,l}(s)\|_{L^2(I_{j,l})L^2(\Omega)} \lesssim \big(&\|\varphi_l(2^j t)[\Delta_D, \chi]\Delta_j u_L\|_{L^2(I_{j,l})L^2(\Omega)} \\ &+ 2^j \|\varphi_l'(2^j t)\chi \Delta_j u_L\|_{L^2(I_{j,l})L^2(\Omega)}\big).\end{aligned} \tag{3.28}$$

As $[\Delta_D, \chi]$ is bounded from H_0^1 to L^2, we get

$$\|P_j V_{j,l}\|_{L^2(I_{j,l})L^2(\Omega)} \lesssim \|\chi_1 \Delta_j u_L\|_{L^2(I_{j,l})H_0^1(\Omega)} + 2^j \|\chi \Delta_j u_L\|_{L^2(I_{j,l})L^2(\Omega)}. \tag{3.29}$$

Now, let us recall the following local smoothing result on a non-trapping domain:

Lemma 3.3. *(See Burq, Gérard, Tzvetkov [2, Proposition 2.7].) Assume that $\Omega = \mathbb{R}^3 \setminus \Theta$, where $\Theta \neq \emptyset$ is a non-trapping obstacle. Then, for every $\tilde{\chi} \in C_0^\infty(\mathbb{R}^3)$, and $\sigma \in [-1/2, 1]$,*

$$\|\tilde{\chi} \Delta_j u_L\|_{L^2(\mathbb{R},\dot{H}^{\sigma+1/2}(\Omega))} \leqslant C \|\Delta_j u_0\|_{H^\sigma(\Omega)}, \tag{3.30}$$

where, as usual, $u_L(t,x) = e^{-it\Delta_D} u_0(x)$.

Applying Lemma 3.3 to the right-hand side of (3.29),

$$\begin{aligned}\|P_j \chi \Delta_j u_L\|^2_{L^5_x L^2_t} &\lesssim 2^{-2j(\frac{1}{2}+\frac{1}{10})} \sum_{l\in\mathbb{Z}} \big(\|\chi_1 \Delta_j u_L\|^2_{L^2(I_{j,l})H_0^1(\Omega)} + 2^{2j} \|\chi \Delta_j u_L\|^2_{L^2(I_{j,l})L^2(\Omega)}\big) \\ &\lesssim 2^{-\frac{2j}{10}} \big(2^{-j} \|\Delta_j u_0\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} + 2^j \|\Delta_j u_0\|^2_{\dot{H}^{-\frac{1}{2}}(\Omega)}\big) \\ &\lesssim 2^{-\frac{2j}{10}} \|\Delta_j u_0\|^2_{L^2(\Omega)},\end{aligned}$$

which is the first part of inequality (3.2). Therefore we have proved Theorem 3.1 $q = 2$, but without the time derivative term. We now use the Gagliardo–Nirenberg inequality in order to deduce (3.3) for every $q \geqslant 2$. We have

$$\|\Delta_j u_L\|_{L^\infty_t} \lesssim \|\Delta_j u_L\|^{1/2}_{L^2_t} \|\Delta_j \partial_t u_L\|^{1/2}_{L^2_t}.$$

Taking the L^5_x norms and using Cauchy–Schwartz yields

$$\|\Delta_j u_L\|^5_{L^5_x L^\infty_t} \lesssim \|\Delta_j u_L\|^{5/2}_{L^5 L^2_t} \|\Delta_j \partial_t u_L\|^{5/2}_{L^5_x L^2_t}. \tag{3.31}$$

It remains to estimate $\|\Delta_j \partial_t u_L\|_{L^5_x L^2_t}$: notice that since $u_L = e^{-it\Delta_D} u_0$

$$\Delta_j \partial_t u_L = -i\Delta_D \Delta_j u_L = i2^{2j} Q_j u_L,$$

where $Q_j = \psi_1(2^{-2j}\Delta_D)$ and $\psi_1(x) = x\psi(x) \in C^\infty_0(\mathbb{R} \setminus \{0\})$. Therefore

$$\|\Delta_j \partial_t u_L\|_{L^5_x L^2_t} \leqslant C2^{j(2-1/10)} \|\tilde{\Delta}_j u_0\|_{L^2(\Omega)}, \tag{3.32}$$

consequently

$$\|\Delta_j \partial_t u_L\|_{L^5_x L^q_t} \leqslant C2^{-j(2/q-9/10)} \|\Delta_j u_0\|_{L^2(\Omega)}$$

and Theorem 3.1 is proved.

3.2. *Proof of Lemma 3.2*

In order to prove the lemma, one would like to rewrite $P_j = \psi(2^{-2j-2}\Delta_D) + \psi(2^{-2j}\Delta_D) + \psi(2^{-2j+2}\Delta_D)$ as a solution of the wave equation, using $h = 2^{-j}$ as a time. Then by finite speed of propagation we could switch Δ_D and Δ_S. However the inverse Fourier transform (in $|\xi|$) of $\Psi(|\xi|) = \psi(|\xi|^2/4) + \psi(|\xi|^2) + \psi(4|\xi|^2)$ is only Schwartz class, rather than compactly supported. The tails will eventually account for the right-hand side of (3.26). We now turn to the details: let $\varphi_0(y), \varphi(y)$ be even functions which are compactly supported (and away from zero for $\varphi(y)$) and such that

$$\varphi_0(y) + \sum_{k\geqslant 1} \varphi(2^{-k} y) = 1.$$

We decompose $\hat{\Psi}(y)$ using this resolution of the identity, and set with obvious notations

$$\Psi(|\xi|) = \sum_{k\in\mathbb{N}} \phi_k(|\xi|),$$

where, being Schwartz class, the ϕ_k have good bounds, $\hat{\phi}_0 \in L^\infty$ and for $k \geqslant 1$

$$\forall N \in \mathbb{N}, \quad \|\hat{\phi}_k\|_\infty = \|\hat{\Psi}(y)\varphi(2^{-k}y)\|_\infty \leqslant C_N 2^{-kN}. \tag{3.33}$$

At fixed k, we write (abusing notation and letting Δ be either Δ_D or Δ_S)

$$\phi_k(h\sqrt{-\Delta})\tilde{\chi}\tilde{v}_{j,l} = \frac{1}{2\pi}\int e^{iyh\sqrt{-\Delta}} \tilde{\chi}(x)\tilde{v}_{j,l}(x)\hat{\phi}_k(y)\,dy.$$

Notice that $\phi_k(y)$ is compactly supported, in fact its support is roughly $|y| \in [2^{k-1}, 2^{k+1}]$. As such the y integral is a time average of half-wave operators, which have finite speed of propagation. Therefore if the "time" $|yh| \leqslant 1$, we can add another cut-off function χ_1 which is equal to one on the domain of dependency of $\tilde{\chi}$ on this time scale, and such that χ_1 is indifferently defined on S or Ω: namely, for $k \lesssim j$,

$$\begin{aligned}\phi_k(h\sqrt{-\Delta_S})\tilde{\chi}\tilde{v}_{j,l} &= \chi_1(x)\phi_k(h\sqrt{-\Delta_S})\tilde{\chi}\tilde{v}_{j,l}\\ &= \chi_1(x)\frac{1}{2\pi}\int e^{iyh\sqrt{-\Delta}}\tilde{\chi}(x)\tilde{v}_{j,l}(x)\hat{\phi}_k(y)\,dy,\\ \phi_k(2^{-j}\sqrt{-\Delta_S})\tilde{\chi}\tilde{v}_{j,l} &= \chi_1(x)\phi_k(2^{-j}\sqrt{-\Delta_D})\tilde{\chi}\tilde{v}_{j,l}.\end{aligned} \tag{3.34}$$

From this identity, we obtain

$$v_{j,l,s} = \chi_1(x)\sum_{j\lesssim k}\left(\phi_k(2^{-j}\sqrt{-\Delta_D}) - \phi_k(2^{-j}\sqrt{-\Delta_S})\right)\tilde{\chi}(x)\tilde{v}_{j,l}. \tag{3.35}$$

At this point the difference in (3.35) is irrelevant and we estimate both terms using Sobolev embedding and energy estimates. Abusing notations, with $\Delta \in \{\Delta_D, \Delta_S\}$, we have

$$\begin{aligned}\left\|\chi_1\phi_k\left(2^{-j}\sqrt{-\Delta}\right)\tilde{\chi}\tilde{v}_{j,l}\right\|_{L_x^5L_t^2(I_{j,l})} &\leqslant \left\|\chi_1\phi_k\left(2^{-j}\sqrt{-\Delta}\right)\tilde{\chi}\tilde{v}_{j,l}\right\|_{L_t^2(I_{j,l})L_x^5} \\ &\leqslant 2^{-\frac{j}{2}}\left\|\chi_1\phi_k\left(2^{-j}\sqrt{-\Delta}\right)\tilde{\chi}\tilde{v}_{j,l}\right\|_{L_t^\infty(I_{j,l})L_x^5} \\ &\lesssim 2^{-\frac{j}{2}}\left\|\phi_k\left(2^{-j}\sqrt{-\Delta}\right)\tilde{\chi}\tilde{v}_{j,l}\right\|_{L_t^\infty(I_{j,l})H^{\frac{1}{2}}(\Omega)} \\ &\lesssim C_N 2^{-\frac{j}{2}-kN}\|\tilde{\chi}\tilde{v}_{j,l}\|_{L_t^\infty(I_{j,l})H^{\frac{1}{2}}(\Omega)} \end{aligned} \tag{3.36}$$

where we used Minkowski, Hölder, (non-sharp!) Sobolev and (3.33). We now use Lemma 3.3 (but for the inhomogeneous equation): by the dual estimate of (3.30), we estimate the right-hand side of (3.36)

$$\|\tilde{v}_{j,l}\|_{L_t^\infty(I_{j,l})H^{\frac{1}{2}}(\Omega)} \lesssim \|V_{j,l}\|_{L_t^2(I_{j,l},L^2(\Omega))}.$$

Summing in k and relabeling N, we have

$$\|v_{j,l,s}\|_{L_x^5L_t^2(I_{j,l})} \leqslant C_N 2^{-jN}\|V_{j,l}\|_{L_t^2(I_{j,l},L^2(\Omega))}, \tag{3.37}$$

which concludes the proof of Lemma 3.2.

3.3. Proof of Theorems 3.2 and 3.3

We recall a lemma due to Christ and Kiselev [8]. We state the corollary we will use, with only the time variable: we refer to [5] for a simple direct proof of all the different cases we use, with Banach-valued $L_t^p(B)$ spaces or $B(L_t^p)$. Its use in the context of reversed norms $L_x^q(L_t^p)$ goes back to [15] and it greatly simplifies obtaining inhomogeneous estimates from homogeneous ones.

Lemma 3.4. *(See Christ and Kiselev [8].) Consider a bounded operator*

$$T : L^r(\mathbb{R}) \to L^q(\mathbb{R})$$

given by a locally integrable kernel $K(t,s)$. Suppose that $r<q$. Then the restricted operator

$$T_R f(t) = \int_{s<t} K(t,s) f(s)\, ds$$

is bounded from $L^r(\mathbb{R})$ to $L^q(\mathbb{R})$ and

$$\|T_R\|_{L^r(\mathbb{R})\to L^q(\mathbb{R})} \leqslant C\left(1-2^{-(1/q-1/r)}\right)^{-1}\|T\|_{L^r(\mathbb{R})\to L^q(\mathbb{R})}.$$

From the lemma, the proof of the inhomogeneous set of estimates in Theorem 3.2 is routine from the homogeneous estimates in Theorem 3.1 and the Duhamel formula. Combining both homogeneous and inhomogeneous estimates yields Theorem 3.3.

3.4. Proof of Proposition 3.1

Let S denote the compact domain defined above. Define $(e_n)_n$ to be the eigenfunctions of $-\Delta_S$ with eigenvalues λ_n^2. Then this collection of functions is an eigenbasis of $L^2(S)$. Following [4], we define an abstract self-adjoint operator on $L^2(S)$ as follows:

$$A_h(e_n) \stackrel{\text{def}}{=} -\left[h\lambda_n^2\right]e_n.$$

Here $[\lambda]$ denotes the integer part of λ. Notice that, on band-limited functions of $L^2(S)$, $A_h = [h\Delta_S]$, where this equality makes sense through the spectral theorem.

We first prove estimates for the linear Schrödinger equation on the compact domain S, with spectrally localized initial data. Set $h = 2^{-j}$, and let us provide estimates on a new evolution equation, where $h\Delta_S$ is replaced by A_h.

Lemma 3.5. *Let $0 < h \leqslant 1$, $q \geqslant 2$, $I_h = (-\pi h, \pi h)$, $\tilde\chi \in C_0^\infty(\mathbb{R}^3)$ be supported near $\partial\Omega$ and $V_0 \in L^2(\Omega)$. There exists $C > 0$ independent of h such that*

$$\left\| e^{i\frac{t}{h}A_h} P_j^S \tilde\chi V_0 \right\|_{L^5(S)L^q(I_h)} \leqslant C h^{2/q - 9/10} \left\| P_j^S \tilde\chi V_0 \right\|_{L^2(S)}. \tag{3.38}$$

We postpone the proof of Lemma 3.5 and proceed with the proof of Proposition 3.1. Denote by $V_h(t,x) \stackrel{\text{def}}{=} e^{it\Delta_S} P_j^S \tilde\chi V_0(x)$, then

$$(ih\partial_t + A_h)V_h = (ih\partial_t + h\Delta_S)V_h + (A_h - h\Delta_S)V_h = (A_h - h\Delta_S)e^{it\Delta_S} P_j^S \tilde\chi V_0.$$

Writing the Duhamel formula for V_h yields

$$V_h(t,x) = e^{i\frac{t}{h}A_h} P_j^S \tilde\chi V_0(x) - \frac{i}{h}\int_0^t e^{i\frac{(t-s)}{h}A_h}(A_h - h\Delta_S)e^{is\Delta_S} P_j^S \tilde\chi V_0(x)\,ds. \tag{3.39}$$

Using (3.38) with $q = 2$, (3.39), the Minkowski inequality and boundedness of the operator

$$\left\| e^{i\frac{t}{h}A_h} P_j^S \right\|_{L^2(S)\to L^5(S)L^2(I_h)} \lesssim 2^{-\frac{j}{10}} \sim h^{1/10}$$

(which follows from the postponed proof of Lemma 3.5), we obtain

$$\begin{aligned}\left\| e^{it\Delta_S} P_j^S \tilde\chi V_0 \right\|_{L^5(S)L^2(I_h)} \lesssim h^{\frac{1}{10}} \Bigg(& \left\| P_j^S \tilde\chi V_0 \right\|_{L^2(S)} \\ & + \frac{1}{h}\left\| (A_h - h\Delta_S)e^{is\Delta_S} P_j^S \tilde\chi V_0 \right\|_{L^1(-h\pi,h\pi)L^2(S)} \Bigg),\end{aligned} \tag{3.40}$$

where, to handle the second term in the right-hand side of (3.39), we used that A_h commutes with the spectral localization P_j^S. Changing variables $s = h\tau$ in the second term in the right-hand side of (3.40) yields

$$\begin{aligned}\frac{1}{h}\left\| (A_h - h\Delta_S)e^{is\Delta_S} P_j^S \tilde\chi V_0 \right\|_{L^1(-h\pi,h\pi)L^2(S)} &= \int_{-\pi}^{\pi} \left\| (A_h - h\Delta_S)e^{i\tau h\Delta_S} P_j^S \tilde\chi V_0 \right\|_{L^2(S)} d\tau \\ &\lesssim 2\pi \left\| P_j^S \tilde\chi V_0 \right\|_{L^2(S)},\end{aligned} \tag{3.41}$$

where we now used boundedness of $A_h - h\Delta_S$ on $L^2(S)$ and mass conservation for the linear Schrödinger flow on S. Combining (3.40) and (3.41),

$$\left\| e^{it\Delta_S} P_j^S \tilde\chi V_0 \right\|_{L^5(S)L^2(I_h)} \lesssim h^{1/10} \left\| P_j^S \tilde\chi V_0 \right\|_{L^2(S)},$$

which ends the proof of Proposition 3.1.

We now return to Lemma 3.5 for the remaining part of this section. Writing $P_j^S V_0 = \sum_n \Psi(h^2\lambda_n^2) V_{\lambda_n} e_n$, we decompose (for $0 < h \leqslant 1/4$)

$$e^{i\frac{t}{h}A_h} P_j^S V_0(t,x) = \sum_{k\in\mathbb{N}} e^{i\frac{t}{h}k} v_k(x)$$

with

$$v_k(x) = \sum_{\lambda=(k2^j)^{1/2}}^{((k+1)2^j)^{1/2}-1} \sum_{\lambda_n\in[\lambda,\lambda+1)} \Psi\big(h^2\lambda_n^2\big) V_{\lambda_n} e_n = \sum_{\lambda=(k2^j)^{1/2}}^{((k+1)2^j)^{1/2}-1} \Pi_\lambda\big(P_j^S V_0\big),$$

where Π_λ denotes the spectral projector $\Pi_\lambda = 1_{\sqrt{-\Delta_S}\in[\lambda,\lambda+1)}$. Let us estimate the $L^5(S)L^q(I_h)$ norm of $e^{i\frac{t}{h}A_h} P_j^S V_0$:

$$\begin{aligned}
\left\| e^{i\frac{t}{h}A_h} P_j^S V_0 \right\|^2_{L^5(S)L^q(I_h)} &\lesssim h^{2/q} \left\| \left\| e^{isA_h} P_j^S V_0 \right\|^2_{L^q_s(-\pi,\pi)} \right\|_{L^{5/2}(S)} \\
&\lesssim h^{2/q} \left\| \left\| e^{isA_h} P_j^S V_0 \right\|^2_{H^{1/2-1/q}(s\in(-\pi,\pi))} \right\|_{L^{5/2}(S)} \\
&\lesssim h^{2/q} \left\| \sum_{k\in\mathbb{N}} (1+k)^{2(\frac{1}{2}-\frac{1}{q})} \left\| e^{isk} v_k(x) \right\|^2_{L^2_s(-\pi,\pi)} \right\|_{L^{5/2}(S)} \\
&\lesssim h^{2/q} \sum_{k\in\mathbb{N}} (1+k)^{1-2/q} \left\| e^{isk} v_k(x) \right\|^2_{L^5(S)L^2(-\pi,\pi)} \\
&\lesssim h^{2/q} \sum_{k\in\mathbb{N}} (1+k)^{1-2/q} \left\| e^{isk} v_k(x) \right\|^2_{L^2(-\pi,\pi)L^5(S)},
\end{aligned}$$

where we used the Sobolev (in time) embedding $H^{1/2-1/q} \subset L^q$ and Plancherel in time. We now recall a result of [18] of Smith and Sogge on the spectral projector Π_λ:

Theorem 3.4. *(See Smith and Sogge [18].) Let S be a compact manifold of dimension* 3*, then*

$$\|\Pi_\lambda\|_{L^2(S)\to L^5(S)} \leqslant \lambda^{2/5}.$$

Using Theorem 3.4 we have

$$\begin{aligned}
\left\| e^{i\frac{t}{h}A_h} P_j^S V_0 \right\|^2_{L^5(S)L^q(I_h)} &\lesssim h^{2/q} \sum_{1/4h-1\leqslant k\leqslant 4/h} (1+k)^{1-2/q+4/5} \left\| P_j^S V_0 \right\|^2_{L^2(S)} \\
&\lesssim \sum_{hk\in[1/4,4]} k^{1-4/q+4/5} \left\| P_j^S V_0 \right\|^2_{L^2(S)} \\
&\lesssim \left\| P_j^S V_0 \right\|^2_{\dot H^{2/q-9/10}(S)},
\end{aligned}$$

as for $hk > 4$ or $h(k+1) < 1/4$ and $\lambda_n \in [(k2^j)^{1/2}, ((k+1)2^j)^{1/2})$ we have $\tilde\Psi(h^2\lambda_n^2) = 0$ and on the other hand for these values of k we have

$$k/\sqrt{2} \leqslant \left(k2^j\right)^{1/2} \leqslant \lambda_n \leqslant \left((k+1)2^j\right)^{1/2} \leqslant \sqrt{2}(k+1), \qquad h \leqslant 5(k+1)^{-1}.$$

This completes the proof of Lemma 3.5.

4. Local existence

In this section we prove Theorem 2.1.

Definition 4.1. Let $u \in \mathcal{S}'(\mathbb{R}\times\Omega)$ and let $\Delta_j = \psi(-2^{-2j}\Delta_D)$ be a spectral localization with respect to the Dirichlet Laplacian Δ_D in the x variable, such that $\sum_j \Delta_j = Id$ and let $S_j = \sum_{k<j} \Delta_j$. We introduce two families of Besov spaces, $\dot B^{s,q}_p$ and its "Banach-valued" counterpart, $\dot B^{s,q}_p(L^r_t)$ as follows: we say that $u \in \dot B^{s,q}_p$ (resp. $u \in \dot B^{s,q}_p(L^r_t)$) if

$$\left(2^{js}\|\Delta_j u\|_{L^p_x}\right)_{j\in\mathbb{Z}} \in l^q \quad \left(\text{resp. } \left(2^{js}\|\Delta_j u\|_{L^p_x L^r_t}\right)_{j\in\mathbb{Z}} \in l^q\right),$$

and $\sum_j \Delta_j f$ converges to f in $\mathcal{S}'$. If L^r_t is replaced by L^r_T (resp. L^r_I), the time integration is meant to be over a time interval $(-T,T)$ (resp. I). Moreover, when $s<0$, Δ_j may be replaced by S_j in the norm and both norms are equivalent.

Consider $u_0 \in \dot H^1_0$ and u_L the solution to the linear equation (3.1). Applying Theorem 3.1 with $q = 2, 5$ and taking $s = 1$ in the definition above we obtain

$$u_L \in \dot B_5^{1+\frac{1}{10},2}\left(L^2_t\right) \cap \dot B_5^{\frac{1}{2},2}\left(L^5_t\right) \quad \text{and} \quad \partial_t u_L \in \dot B_5^{-\frac{3}{2},2}\left(L^5_t\right).$$

We now apply Lemma A.5 from Appendix A, which is a variant on Gagliardo–Nirenberg, and

$$u_L \in L^{20/3}_x L^{40}_t,$$

and consequently

$$u_L^4 \in L_x^{5/3} L_t^{10} \quad \text{as well as} \quad |u_L|^4 u_L \in \dot{B}^{1,2}_{\frac{5}{4}}\big(L_t^{\frac{20}{11}}\big)$$

which should be enough to iterate.

Remark 4.1. One may make several unrelated remarks. First, one could dispense with the use of Lemma 3.1, miss the endpoint $q = 2$ and still get the exact same non-linear results, as there is room (due to the use of Sobolev embedding) in all mapping estimates. Moreover, as soon as we use an estimate with a (however small) gain in regularity, we do not need Lemma A.5, as we could use a simpler embedding in a Besov space of negative regularity and play regularities against each other. In fact, in the same spirit as [15] one could replace the critical Sobolev norm by a Besov norm $\dot{B}_2^{s_p,\infty}$. We elected to select the simplest choice for the fixed point.

For $T > 0$ let $X_T \stackrel{\text{def}}{=} \dot{B}^{1,2}_5(L_T^{\frac{20}{9}}) \cap L^{\frac{20}{3}} L_T^{40}$ and for $u \in X_T$ set $F(u) \stackrel{\text{def}}{=} |u|^4 u$.

Proposition 4.1. *Define a non-linear map ϕ as follows,*

$$\phi(u)(t) \stackrel{\text{def}}{=} \int\limits_{s<t} e^{i(t-s)\Delta_D} F\big(u(s)\big)\, ds.$$

Then

$$\big\|\phi(u)\big\|_{C_T(\dot{H}_0^1)} + \big\|\phi(u)\big\|_{X_T} \lesssim \big\|F(u)\big\|_{\dot{B}^{1,2}_{5/4}(L_T^{20/11})} \lesssim \|u\|^5_{X_T}, \tag{4.1}$$

and

$$\big\|\phi(u) - \phi(v)\big\|_{X_T} \lesssim \big\|F(u) - F(v)\big\|_{\dot{B}^{1,2}_{5/4}(L_T^{20/11})} \lesssim \|u - v\|_{X_T}\big(\|u\|_{X_T} + \|v\|_{X_T}\big)^4. \tag{4.2}$$

Theorem 3.2 (shifting the regularity to $s = 1$) provides the Besov component of the X_T and Lemma A.5 provides the space–time Lebesgue component of X_T in the left inequality of both estimates (4.1) and (4.2). Now, Lemma A.4 in Appendix A provides the non-linear part (right inequality) of both estimates (note however that, as $p = 5$ is an integer, one can prove directly the non-linear mappings by chain rule and Hölder, see Appendix A).

One may now set up the usual fixed point argument in X_T if T is sufficiently small or if the data is small. This concludes the proof of Theorem 2.1 (scattering for small data follows the usual way from the global in time space–time estimates).

We now consider local well-posedness for $p < 5$, e.g. Theorem 2.2. The critical Sobolev exponent w.r.t. scaling is $s_p = 3/2 - 2/(p-1)$. Exactly as before, we have by Theorem 3.1

$$u_L \in \dot{B}_5^{s_p+\frac{1}{10},2}\big(L_T^2\big) \cap \dot{B}_4^{s_p-\frac{1}{4},2}\big(L_T^4\big) \quad \text{and} \quad \partial_t u_L \in \dot{B}_4^{s_p-\frac{1}{4}-2,2}\big(L_T^4\big). \tag{4.3}$$

Again, by interpolation and Lemma A.5, we have $u_L \subset \dot{B}_5^{s_p,2}(L_T^{20/9}) \cap L_x^{5(p-1)/3} L_T^{10(p-1)}$, which proves to be the convenient space to set up the fixed point.

Remark 4.2. Some numerology is in order: if one were only to have the $L_x^5 L_t^2$ smoothing estimate and use Sobolev (in time and in space), it would require $5(p-1)/3 \geqslant 5$, namely $p \geqslant 4$. However, we have the Strichartz estimate from [16], which allows $5(p-1)/3 \geqslant 4$, or $p \geqslant 3 + 2/5$.

Let $X_T \stackrel{\text{def}}{=} \dot{B}_5^{s_p,2}(L_T^{\frac{20}{9}}) \cap L^{5(p-1)/3} L_T^{10(p-1)}$. Again from Lemma A.4 in Appendix A, the non-linear mapping verifies

$$\big\|F(u) - F(v)\big\|_{\dot{B}^{s_p,2}_{5/4}(L_T^{20/11})} \lesssim \|u - v\|_{X_T}\big(\|u\|^{p-1}_{X_T} + \|v\|^{p-1}_{X_T}\big)$$

and existence and uniqueness follow by fixed point, once we remark that from Theorem 3.3, we actually obtain

$$u \in \dot{B}_5^{s_p+\frac{1}{10},2}\left(L_T^2\right) \cap \dot{B}_4^{s_p-\frac{1}{4},2}\left(L_T^4\right), \tag{4.4}$$

as well as

$$\partial_t u \in \dot{B}_5^{s_p-\frac{19}{10},2}\left(L_T^2\right) \cap \dot{B}_4^{s_p-\frac{9}{4},2}\left(L_T^4\right), \tag{4.5}$$

which provides the Lebesgue component of the X_T through Lemma A.5. This additional estimates will also be useful for the scattering part in the next section.

4.1. Scattering for $3+2/5<p<5$

We now deal with scattering in the same range of $p \in (3+2/5, 5)$: from [16], we have an a priori bound

$$\|S_j u\|_{L_t^4 L_x^4}^4 \lesssim \|u\|_{L_t^4 L_x^4}^4 \lesssim \|u_0\|_{L_x^2}^3 \sup_t \|u\|_{H_0^1} \leqslant M^{\frac{3}{2}} E^{\frac{1}{2}},$$

where M and E are the conserved charge and Hamiltonian,

$$M = \int_\Omega |u|^2\,dx \quad \text{and} \quad E = \int_\Omega |\nabla u|^2 + \frac{2}{p+1}|u|^{p+1}\,dx. \tag{4.6}$$

Notice how this estimate is below the critical scaling s_p, as the right-hand side regularity is $s=1/4$. From the energy a priori bound and Sobolev embedding, one has on the other hand

$$\|S_j u\|_{L_{t,x}^\infty} \lesssim 2^{\frac{j}{2}} \sup_t \|u\|_{H_0^1} \lesssim 2^{\frac{j}{2}} E^{\frac{1}{2}},$$

where the right-hand side regularity is $s=1$. Interpolating between the two bounds to get the critical regularity s_p on the right-hand side yields the following a priori bound

$$\|S_j u\|_{L_{t,x}^{\frac{6(p-1)}{5-p}}} \lesssim C(M,E) 2^{j(\frac{1}{2}-\frac{5-p}{3(p-1)})}. \tag{4.7}$$

In order to proceed with the usual scattering argument, we need to revisit the fixed point, or more precisely the non-linear estimate on $F(u)$. We may split the bound (4.7) into N bounds on time intervals I_k, $1 \leqslant k \leqslant N$, such that

$$\|S_j u\|_{L_{x,I_k}^{\frac{6(p-1)}{5-p}}} \lesssim \varepsilon 2^{j(\frac{1}{2}-\frac{5-p}{3(p-1)})}, \tag{4.8}$$

which, provided we can use it in evaluating $F(u)$, will provide smallness on the I_k intervals.

Let I be any time interval, and define Y_I to be

$$Y_I = \left\{ u \text{ s.t. } u \in \dot{B}_5^{s_p+\frac{1}{10},2}\left(L_I^2\right) \cap \dot{B}_4^{s_p-\frac{1}{4},2}\left(L_I^4\right) \text{ and } \partial_t u \in \dot{B}_5^{s_p-\frac{19}{10},2}\left(L_I^2\right) \cap \dot{B}_4^{s_p-\frac{9}{4},2}\left(L_I^4\right) \right\}.$$

Notice how, for $I=[-T,T]$, $Y_T \subset X_T$, and from the previous section, the local in time solution u is in fact in Y_T.

As $6(p-1)/(5-p) > 5(p-1)/3 > 4$, one may interpolate the bound (4.8) on any time interval I_k with

$$\|\Delta_j u\|_{L_x^4 L_{I_k}^4} \lesssim 2^{-j(s_p-\frac{1}{4})} \|u\|_{Y_{I_k}} \mu_j, \quad \text{with } \left\|(\mu_j)_j\right\|_{l^2} = 1.$$

We pick θ such that

$$\frac{3}{5(p-1)} = \frac{\theta}{4} + (1-\theta)\frac{5-p}{6(p-1)}, \quad \text{e.g. } \theta = \frac{2}{5}\frac{(5p-7)}{(5p-13)}.$$

Therefore,

$$\|\Delta_j u\|_{L_{x,I_k}^{\frac{5(p-1)}{3}}} \lesssim 2^{-\frac{1}{p-1}j} \|u\|_{Y_{I_k}}^\theta \varepsilon^{1-\theta} \mu_j^\theta, \quad \text{with } \left(\mu_j^\theta\right)_j \in l^{5\frac{5p-13}{5p-7}} \subset l^{\frac{5(p-1)}{3}}.$$

Combining this bound with

$$\|\partial_t \Delta_j u\|_{L^4_x L^4_{I_k}} \lesssim 2^{-j(s_p-\frac{9}{4})}\|u\|_{Y_{I_k}}\lambda_j, \quad \text{with } \big\|(\lambda_j)_j\big\|_{l^2}=1,$$

and the Gagliardo–Nirenberg Lemma A.5 in Appendix A, we finally get for some $0<\eta<\theta$,

$$\|u\|_{L_x^{\frac{5(p-1)}{3}} L_{I_k}^{10(p-1)}} \lesssim \|u\|^{\eta}_{Y_{I_k}} \varepsilon^{1-\eta}.$$

From there, we can now estimate, through Lemma A.4 in Appendix A,

$$\big\|F(u)\big\|_{\dot B^{s_p,2}_{5/4}(L^{20/11}_{I_k})} \lesssim \|u\|_{Y_{I_k}} \|u\|^{p-1}_{L_x^{\frac{5(p-1)}{3}} L_{I_k}^{10(p-1)}},$$

and on any interval I_k, the Duhamel formula yields

$$\|u\|_{Y_{I_k}} \lesssim E^{s_p} M^{1-s_p} + \|u\|^{1+(p-1)\eta}_{Y_{I_k}} \varepsilon^{(p-1)(1-\eta)},$$

from which we get $\|u\|_{Y_{\mathbb{R}}}<+\infty$ provided we chose ε small enough to contract the Y_{I_k} norms on each I_k. Scattering follows the usual way from the global space–time bound in $Y_{\mathbb{R}}$.

4.2. *Scattering for* $3 \leqslant p \leqslant 3+2/5$

In this part we consider the remaining case, e.g. non-linearities which are close to 3 and for which our main results do not provide a scale-invariant local Cauchy theory. As mentioned before, this case will be dealt with using the approach from [16]. As such, this entire Subsection is somewhat disconnected from the rest of the paper; the combination of several technical difficulties makes it lengthy and cumbersome, but we hope the underlying strategy is clear. We have two a priori bounds on the non-linear equation at our disposal: local smoothing, which is at the scale of $\dot H^{\frac12}$ regularity for the data, and an $L^4_{t,x}$ space–time bound, which is at the scale of $\dot H^{\frac14}$ regularity for the data. Both are below the scale of critical H^s regularity, which is $s_p=\frac32-\frac{2}{(p-1)}$. Interpolation with the energy bound provides bounds at the critical level, but the lack of flexible scale-invariant estimates on the inhomogeneous problem make them seemingly useless. As such, one has to improve both the local smoothing bound and the $L^4_{t,x}$ space–time bounds obtained in [16], to reach critical scaling and beyond. This is accomplished through several steps, which we informally summarize as follows:

- improve the space–time bounds by splitting the solution (and therefore the equation) into two pieces: far and close to the boundary. As the resulting commutator source term can only be handle at $H^{\frac12}$ regularity, this will improve estimates from $\dot H^{\frac14}$ regularity to $\dot H^{\frac12-\varepsilon}$ regularity, which is still below scale invariance;
- combine this improved estimates with the energy bound to obtain yet again better space–time bounds through the equation (but splitting now the source term in close and far away terms). As an added bonus we also improve our local smoothing estimate; moreover we now go beyond scale-invariance;
- turn the crank a few more times, going back and forth between estimates on the split equations and estimates on the equation with split source terms, until we reach the correct set of estimates to prove scattering at the H^1_0 regularity. It is worth noticing that the numerology gets worse with $p>3+2/5$, and that the forthcoming argument would probably break down before even reaching $p=4$.

We start by stating a few linear estimates which will be needed in the proof and are simple consequences of our Theorem 3.3 by summing over dyadic frequencies.

Lemma 4.1. *(See [16, Lemma 5.4].) Let Ω be a non-trapping domain and denote by $u_L=e^{it\Delta_D}u_0$ the linear flow for the Schrödinger equation on Ω with Dirichlet boundary conditions. Then*

$$\big\|e^{it\Delta_D}u_0\big\|_{L^4_t \dot W^{s,4}(\Omega)} \lesssim \|u_0\|_{\dot H_0^{s+\frac14}(\Omega)}. \tag{4.9}$$

Denote by w the solution of the inhomogeneous equation, e.g. $w=\int_0^t e^{i(t-s)\Delta_D} f(s)\,ds$, then

$$\|w\|_{C_t \dot H_0^{s+\frac14}(\Omega)} + \|w\|_{L^4_t \dot W^{s,4}} \lesssim \|f\|_{L_t^{\frac43} \dot W^{s+\frac12,\frac43}}. \tag{4.10}$$

The next lemma is just the Christ–Kiselev lemma again, stated in a form which is convenient for later use.

Lemma 4.2. *(See [16, Lemma 5.6].) Let $U(t)$ be a one parameter group of operators, $1 \leqslant r < q \leqslant \infty$, H a Hilbert space and B_r and B_q two Banach spaces. Suppose that*

$$\|U(t)\varphi\|_{L_t^q(B_q)} \lesssim \|\varphi\|_H \quad \text{and} \quad \left\| \int_s U(-s)g(s)\,ds \right\|_H \lesssim \|g\|_{L_t^r(B_r)},$$

then

$$\left\| \int_{s<t} U(t-s)g(s)\,ds \right\|_{L_t^q(B_q)} \lesssim \|g\|_{L_t^r(B_r)}.$$

Finally, we recall that we have Lemma 3.1 at our disposal, should we need the endpoint Strichartz on the left-hand side in Lemma 4.2, provided that we used a (dual) local smoothing norm on the right-hand side.

In what follows we shall write $p = 3 + 2\eta$, with $\eta \in [0, 1/5]$. We recall all a priori bounds at our disposal: the first two are uniform in time bounds for the $L^2(\Omega)$ and $H_0^1(\Omega)$ norms of the solution to the defocusing NLS, irrespective of the power p, and were already stated in the previous section, see (4.6). The next two were obtained in [16], again in the defocusing case and irrespective of p: a space–time norm estimate

$$\|u\|_{L_t^4(L^4(\Omega))} \leqslant E^{\frac{1}{8}} M^{\frac{3}{8}}, \tag{4.11}$$

which has the same scaling as $\dot{H}^{\frac{1}{4}}$ for the data; and a local smoothing norm estimate

$$\|\nabla u\|_{L_t^2(L^2(K))} \leqslant C(K) E^{\frac{1}{4}} M^{\frac{1}{4}}, \tag{4.12}$$

which has the same scaling as $\dot{H}^{\frac{1}{2}}$ for the data; here K is meant to be a compact set which includes the obstacle, and (4.12) holds only under the star-shaped condition on the obstacle, while proving (4.11) makes an essential use of (4.12).

As we will split our solution u in χu (near the boundary) and $(1-\chi)u$ (far from the boundary), the proof involves two different families of Besov spaces, depending on context:

- $\dot{B}_p^{s,q}(\Omega)$, which we used up to now, which is defined on the domain, and which will be used for χu (as a solution of an equation on the domain);
- $\dot{B}_p^{s,q}(\mathbb{R}^3)$, which is defined in the whole space, and which will be used for $(1-\chi)u$ (as a solution of an equation in $\mathbb{R}^3$).

We point out that all the non-linear mappings from Appendix A hold equally true for both families (the proofs apply verbatim for $\Omega = \mathbb{R}^3$, where the spectral localization reduces to the Fourier transform one). Moreover, for functions which are supported away from the boundary, the Sobolev space $\dot{H}^s(\mathbb{R}^3)$ and $\dot{H}^s(\Omega) = \dot{B}_2^{s,2}(\Omega)$ coincides for $0 \leqslant s \leqslant 1$ and we therefore drop the domain, and retain $\dot{H}^s$ as a convenient notation. We proceed similarly for $L^p(\Omega)$ and $L^p(\mathbb{R}^3)$, using L^p (functions defined on Ω may suitably be extended by zero outside Ω).

We start with proving

Proposition 4.2. *Let u be a solution to the non-linear problem* (2.2). *Let $\chi \in C_0^2(\mathbb{R}^3)$ be a smooth function equal to* 1 *near $\partial\Omega$. Then*

$$\chi u \in L_t^4 \dot{B}_4^{1/4-\eta,2}(\Omega) \quad \text{and} \quad (1-\chi)u \in L_t^2 \dot{B}_6^{1/2-\eta,2}\big(\mathbb{R}^3\big). \tag{4.13}$$

Remark 4.3. Notice that our cut χ is only C^2 rather than C^∞, and this will remain so for the rest of the section. This is in no way a difficulty, and it allows to conveniently take $\chi = \chi_1^p$ or $\chi = \chi_1^{p-1}$, where $\chi_1 \in C_0^2$ as an admissible cut if we need, as $p - 1 > 2$. This is particularly convenient for non-linear mappings where all factors can be considered "equal". Alternatively, one may retain C_0^∞ cuts and play with at least 3 overlapping ones, as was done in [16], at the expense of desymmetrizing various non-linear estimates. These are (mildly annoying) considerations that the reader should ignore at first read.

Proof. In order to prove the proposition, we split Eq. (2.2), treating differently the neighborhood of the boundary (using local smoothing type arguments) and spatial infinity (where the full range of sharp Strichartz estimates holds).

Consider the equation satisfied by χu,

$$(i\partial_t + \Delta_D)(\chi u) = \chi |u|^{2+2\eta} u - [\chi, \Delta_D]u. \tag{4.14}$$

We need to show that the non-linear term belongs to $L_t^2 H_{\mathrm{comp}}^{-\eta}$. The commutator term is controlled by $\|\tilde\chi u\|_{L_t^2 H_{\mathrm{comp}}^1}$ for some $\tilde\chi \in C_0^2(\mathbb{R}^3)$ equal to 1 on the support of χ and it belongs to $L_t^2 L_{\mathrm{comp}}^2 \subset L_t^2 H_{\mathrm{comp}}^{-\eta}$. We now deal with the non-linear term: let q be such that $\dot B_q^{1,2}(\Omega) \subset H^{-\eta}$, hence $1 - \frac{3}{q} = -\eta - \frac{3}{2}$. Then $\frac{1}{q} = \frac{1}{2} + \frac{2(1+\eta)}{6}$ and

$$\big\|\chi |u|^{2(1+\eta)} u\big\|_{L_t^2 H_{\mathrm{comp}}^{-\eta}} \lesssim \big\|\chi |u|^{2(1+\eta)} u\big\|_{L_t^2 \dot B_q^{1,2}(\Omega)} \lesssim \|\chi_1 u\|_{L_t^2 H^1} \|\chi_1 u\|_{L_t^\infty L_x^6}^{2(1+\eta)},$$

where $\chi_1^p = \chi$ and we used Lemma A.4 in Appendix A with $u \in L_t^\infty H^1 \subset L_t^\infty L_x^6$ as well as $\chi_1 u \in L_t^2 H_{\mathrm{comp}}^1$. Hence the right-hand side in (4.14) is in $L_t^2 H_{\mathrm{comp}}^{-\eta}$ and we can apply Lemma 4.2 with $L^q(B_q) = L_t^4 \dot W^{1/4-\eta,4}(\Omega)$, $H = H^{1/2-\eta}$ and $L^r(B_r) = L_t^2 H_{\mathrm{comp}}^{-\eta}$. This gives the first assertion in (4.13). Let us deal now with $(1-\chi)u$ which is solution to

$$(i\partial_t + \Delta_0)\big((1-\chi)u\big) = (1-\chi)|u|^{2+2\eta} u + [\chi, \Delta]u, \tag{4.15}$$

where Δ_0 denotes the free Laplacian (notice that we can consider (4.15) in the whole space $\mathbb{R}^3$ since both source terms vanish near the boundary $\partial\Omega$). The commutator term is dealt with exactly as in the previous part and is therefore in $L_t^2 L_{\mathrm{comp}}^2$.

Let $v \stackrel{\mathrm{def}}{=} (1-\chi_1)u$ for some $\chi_1 \in C_0^2(\mathbb{R}^3)$ such that $(1-\chi_1)^p = 1-\chi$. In order to prove (4.13) we only need to prove $|v|^{2+2\eta} v \in L_t^2 \dot B_{6/5}^{1/2-\eta,2}(\mathbb{R}^3)$, since then we may apply the dual endpoint Strichartz estimates (from the $\mathbb{R}^3$ case) on the non-linear term. Using the embedding $\dot B_1^{1-\eta,2}(\mathbb{R}^3) \subset \dot B_{6/5}^{1/2-\eta,2}(\mathbb{R}^3)$, it suffices to get $|v|^{2+2\eta} v \in L_t^2 \dot B_1^{1-\eta,2}(\mathbb{R}^3)$. When evaluating $|v|^{2+2\eta} v$ we take advantage of the energy bound and Sobolev embedding on v, $L_t^\infty H^1 \subset L_t^\infty \dot B_q^{1-\eta,2}(\mathbb{R}^3)$ with $\frac{1}{q} = \frac{1}{2} - \frac{\eta}{3}$. On the other hand, from our a priori bound from [16], we have $v \in L_t^4 L_x^4$, while $v \in L_t^\infty H^1 \subset L_t^\infty L_x^6$ and hence by interpolation with weights $1/(1+\eta)$ and $\eta/(1+\eta)$, we get $v \in L_t^{4(1+\eta)} L_x^{\frac{12(1+\eta)}{3+2\eta}}$. Consequently, using Lemma A.4 in Appendix A, we get

$$\big\||v|^{2+2\eta} v\big\|_{L_t^2 \dot B_{6/5}^{1/2-\eta,2}(\mathbb{R}^3)} \lesssim \big\||v|^{2+2\eta} v\big\|_{L_t^2 \dot B_1^{1-\eta,2}(\mathbb{R}^3)} \lesssim \|v\|_{L_t^\infty \dot B_q^{1-\eta,2}(\mathbb{R}^3)} \|v\|_{L_t^{4(1+\eta)} L_x^{12\frac{(1+\eta)}{(3+2\eta)}}}^{2(1+\eta)}.$$

This achieves the proof of Proposition 4.2. □

The next iterative step will be the following lemma:

Proposition 4.3. *Let u be a solution to the non-linear problem* (2.2). *Then*

$$u \in L_t^4 \dot W^{1/4+\eta,4}(\Omega) \cap L_t^2 H_{\mathrm{comp}}^{1+\eta}(\Omega). \tag{4.16}$$

Proof. The split of the equation into equations for χu and $(1-\chi)u$ is no longer of any use: the resulting commutator source term is no better than $[\chi, \Delta]u \in L_t^2 L_{\mathrm{comp}}^2$. However we now have estimates from Proposition 4.2 which turn out to be good enough that splitting the non-linear term in (2.2) in two parts, using the partition $\chi + (1-\chi) = 1$ will allow us to use the somewhat restricted set of inhomogeneous estimates we have for the equation on a domain. Setting $g_1 \stackrel{\mathrm{def}}{=} \chi |u|^{2+2\eta} u$, $g_2 \stackrel{\mathrm{def}}{=} (1-\chi)|u|^{2+2\eta} u$ and using Duhamel formula, we have

$$u(t,x) = e^{it\Delta_D} u_0 + \int_0^t e^{i(t-s)\Delta_D} g_1(s)\, ds + \int_0^t e^{i(t-s)\Delta_D} g_2(s)\, ds; \tag{4.17}$$

the idea is then that one may use (4.10) on the g_1 Duhamel term, while the g_2 term may be handled in $L^1_t\dot H^s$ for a suitable s.

Lemma 4.3. *Recall* $v \stackrel{def}{=} (1-\chi_1)u$, *where* $\chi_1 \in C^2_0(\mathbb{R}^3)$ *is such that* $(1-\chi_1)^p = 1-\chi$. *We have*

$$g_2 \in L^2_t \dot B^{1/2,2}_{6/5}(\mathbb{R}^3) \quad \text{and} \quad v \in L^2_t \dot B^{1/2,2}_{6}(\mathbb{R}^3). \tag{4.18}$$

Moreover, $g_2 \in L^1_t \dot H^{\frac12+\eta}$ *and*

$$\left\| \int_0^t e^{i(t-s)\Delta_D} g_2(s)\,ds \right\|_{L^4_t \dot B^{1/4+\eta,2}_4(\Omega)\cap L^2_t H^{1+\eta}_{\mathrm{comp}}} \lesssim \|g_2\|_{L^1_t \dot H^{\frac12+\eta}}. \tag{4.19}$$

Proof. From Proposition 4.2, the energy and mass bound, and interpolation, we have

$$v \in L^2_t \dot B^{1/2-\eta,2}_6(\mathbb{R}^3) \cap L^\infty_t \dot H^{\frac12-\eta} \subset L^4_t L^q \quad \text{for } \frac1q = \frac16 + \frac\eta3,$$

hence $v \in L^4_t L^q \cap L^\infty_t L^6$. We now interpolate again to obtain $v \in L^{4(1+\eta)}_t L^{r(1+\eta)}$, where $\frac2r = \frac13 + \eta$. Therefore, the non-linear term $g_2 = |v|^{2+2\eta} v$ belongs to $L^2_t \dot B^{1-3\eta,2}_{6/5}(\mathbb{R}^3)$. Indeed, let $\frac1m = \frac12 + \frac2r = \frac56 + \eta$, then by Lemma A.4

$$\|g_2\|_{L^2_t \dot B^{1-3\eta,2}_{6/5}(\mathbb{R}^3)} \lesssim \|g_2\|_{L^2_t \dot B^{1,2}_m(\mathbb{R}^3)} \lesssim \|v\|_{L^\infty_t \dot H^1} \|v\|^{2(1+\eta)}_{L^{4(1+\eta)}_t L^{r(1+\eta)}}. \tag{4.20}$$

If $1-3\eta \geqslant 1/2$, (4.18) follows, but unfortunately this covers only $\eta \leqslant 1/6$. It remains to deal with the situation $\eta \in (1/6, 1/5]$. In this case we use the equation on v (obtained by replacing χ by χ_1 in (4.15)) to get

$$v \in L^2_t \dot B^{1-3\eta,2}_6(\mathbb{R}^3). \tag{4.21}$$

In fact, the commutator term $[\chi_1, \Delta]u$ is in $L^2_t L^2_{\mathrm{comp}}$ and, consequently, it also belongs to $L^2_t H^{1/2-3\eta}(\Omega)$ since in this case $1/2 - 3\eta < 0$, while $(1-\chi_1)|v|^{2+2\eta} v \in L^2_t \dot B^{1-3\eta,2}_{6/5}(\mathbb{R}^3)$ as shown before. In order to estimate g_2 we interpolate (4.21) with $v \in L^\infty_t H^1$, which yields

$$v \in L^{2(3+2\eta)}_t \dot B^{\frac{3-\eta}{3+2\eta},2}_\lambda(\mathbb{R}^3) \quad \text{for } \frac1\lambda = \frac16 \frac{1}{(3+2\eta)} + \frac12 \frac{(2+2\eta)}{(3+2\eta)}. \tag{4.22}$$

From (4.22) and Lemma A.4, we get $g_2 \in L^1_t H^{1/2}$ (notice that the regularity of v is $(3-\eta)(3+2\eta) > 1/2$).

Using the equation satisfied by v and Duhamel formula we can write

$$v(t,x) = e^{it\Delta_0}(1-\chi_1)u_0 + \int_0^t e^{i(t-s)\Delta_0}\big(g_2 + [\chi_1, \Delta]u\big)(s)\,ds. \tag{4.23}$$

Using Lemma 4.2 with $L^q(B_q) = L^2_t \dot B^{1/2,2}_6(\mathbb{R}^3)$, $L^r(B_r) = L^1_t \dot H^{1/2}$, the first term in the integral in the right-hand side of (4.23) belongs to $L^2_t \dot B^{1/2,2}_6(\mathbb{R}^3)$. Using Lemma 3.1, we also obtain

$$\left\| \int_0^t e^{i(t-s)\Delta}[\chi_1, \Delta]u(s)\,ds \right\|_{L^2_t \dot B^{1/2,2}_6(\mathbb{R}^3)} \lesssim \big\|[\chi_1, \Delta]u\big\|_{L^2_t L^2_{\mathrm{comp}}}.$$

Finally, the linear evolution $e^{it\Delta_0}(1-\chi_1)u_0$ is evidently in $L^2_t \dot B^{1/2,2}_6(\mathbb{R}^3)$ and we obtain (4.18).

For the last part of the proof of Lemma 4.3 we shall use less information than that, precisely we only need the fact that for $\epsilon > 0$ small enough we have

$$v \in L^2_t \dot B^{1/2-\epsilon,2}_6(\mathbb{R}^3) \subset L^2_t L^{\frac3\epsilon}_x. \tag{4.24}$$

We now refine our knowledge on $g_2 = v|v|v^{1+2\eta}$: using (4.24), by interpolation with the energy bound we have $v \in L_t^{2(1+\eta)} L^{\frac{3(1+\eta)}{2\epsilon+\eta}}$. From Lemma A.4 and the energy bound $v \in L_t^\infty H^1$, the source term g_2 can be estimated as follows

$$\|g_2\|_{L_t^1 H^{1-\eta-2\epsilon}} \lesssim \|v\|^{2(1+\eta)}_{L_t^{2(1+\eta)} L^{\frac{3(1+\eta)}{2\epsilon+\eta}}} \|v\|_{L_t^\infty H^1}. \tag{4.25}$$

Using again Lemma 4.2, this time with $L^q(B_q) = L_t^4 \dot B_4^{3/4-\eta-2\epsilon,2}(\Omega)$, $H = \dot H^{1-\eta-2\epsilon}$ and $L^r(B_r) = L_t^1 H^{1-\eta-2\epsilon}$, we get by interpolation

$$\begin{aligned}\left\| \int_0^t e^{i(t-s)\Delta_D} g_2(s)\,ds \right\|_{L_t^4 \dot B_4^{1/4+\eta,2}(\Omega)} &\lesssim \left\| \int_0^t e^{i(t-s)\Delta_D} g_2(s)\,ds \right\|^\theta_{L_t^4 B_4^{3/4-\eta-2\epsilon,2}(\Omega)} \|u\|^{1-\theta}_{L^4_{t,x}} \\ &\lesssim \|g_2\|_{L_t^1 \dot H^{1-\eta-2\epsilon}} + \|u\|_{L^4_{t,x}};\end{aligned} \tag{4.26}$$

where for the first (interpolation) inequality in (4.26) we used that $3/4 - \eta - 2\epsilon > 1/4 + \eta$ if ϵ is sufficiently small (take $0 < \epsilon \leqslant 1/20$ for example).

On the other hand, by Lemma 4.2 again,

$$\left\| \int_0^t e^{i(t-s)\Delta_D} g_2(s)\,ds \right\|_{L_t^2 H^{1+\eta}_{\mathrm{comp}}} \lesssim \|g_2\|_{L_t^1 H^{1/2+\eta}} \lesssim \|g_2\|_{L_t^1 H^{1-\eta-2\epsilon}}, \tag{4.27}$$

which finally achieves the proof of Lemma 4.3. □

It remains now to deal with the Duhamel term coming from g_1 in (4.17).

Lemma 4.4. *Suppose that we know moreover that*

$$u \in L_t^4 \dot B_4^{\sigma,2}(\Omega), \quad \text{where } \sigma = \frac{1}{4} + \frac{\eta}{1+\eta}, \tag{4.28}$$

then

$$g_1 \in L_t^{4/3} \dot B_{4/3}^{3/4+\eta}(\Omega) \quad \text{and} \quad \int_0^t e^{i(t-s)\Delta_D} g_1(s)\,ds \in L_t^4 \dot B_4^{1/4+\eta,2}(\Omega) \cap L_t^2 H^{1+\eta}_{\mathrm{comp}}. \tag{4.29}$$

Taking the lemma for granted, we can complete the proof of Proposition 4.3: using Lemmas 4.3, 4.4, the fact that the linear flow is in $L_t^\infty H^1 \cap L_t^2 H^{3/2}_{\mathrm{comp}}$ and Duhamel formula (4.17), estimate (4.16) follows immediately.

Proof of Lemma 4.4. The a priori bound (4.28) gives

$$u \in L_t^4 \dot B_4^{\sigma,2}(\Omega) \subset L_t^4 L^q \quad \text{for } \frac{1}{q} = \frac{1}{4} - \frac{\sigma}{3},$$

and consequently $u \in L_t^4 L_x^{6(1+\eta)/(1-\eta)}$. On the other hand, interpolating between $L_t^2 H^1_{\mathrm{comp}}$ and $L_t^\infty H_0^1$ gives $\chi u \in L_t^r H^1_{\mathrm{comp}}$ for every $r \in [2,\infty]$. Therefore, with $\chi_1^p = \chi$, we can estimate by Lemma A.4

$$\left\| \chi |u|^{2+2\eta} u \right\|_{L_t^{4/3} \dot B_M^{1,2}(\Omega)} \lesssim \|\chi_1 u\|_{L_t^{4/(1-2\eta)} H^1_{\mathrm{comp}}} \|\chi_1 u\|^{2(1+\eta)}_{L_t^4 L_x^{6(1+\eta)/(1-\eta)}}, \tag{4.30}$$

where $\frac{1}{M} = \frac{1}{2} + \frac{1-\eta}{3} = \frac{5}{6} - \frac{\eta}{3}$. It remains to notice that for M defined above, the embedding $\dot B_M^{1,2}(\Omega) \subset \dot B_{4/3}^{3/4+\eta,2}(\Omega)$ holds (indeed, $1 > 3/4 + \eta$ and $1 - 3/M = 3/4 + \eta - 9/4$) and to use again Lemmas 4.2, 3.1. Another application of Lemma 4.2 with $L^q(B_q) = L_t^2 H^{1+\eta}_{\mathrm{comp}}$, $H = H^{1/2+\eta}_{\mathrm{comp}}$ and $L^r(B_r) = L_t^{4/3} \dot B_{4/3}^{3/4+\eta,2}(\Omega)$ achieves the proof of (4.29) and Lemma 4.4. □

End of the proof of Proposition 4.3. In order to complete the proof of Proposition 4.3 it remains to prove that (4.28) holds indeed, since we have used it to deduce (4.16). Let $0 < T < \infty$ be small enough, so that by the local existence theory (see [16]) the $L^4_T \dot{B}^{\sigma,2}_4(\Omega)$ norm of u is finite; in fact, the same can be said with σ replaced by $\eta + \frac{1}{4}$. We shall prove that $T = \infty$ is allowed. For this, we interpolate between $L^4_t \dot{B}^{1/4-\eta,2}_4(\Omega)$ and $L^4_T \dot{B}^{1/4+\eta,2}_4(\Omega)$ with interpolation exponent $\theta = \frac{\eta}{2(1+\eta)}$ to obtain an estimate on the $L^4_T \dot{B}^{\sigma,2}_4(\Omega)$ norm, where $\sigma = 1/4 + \eta/(1+\eta)$:

$$\|u\|_{L^4_T \dot{B}^{\sigma,2}_4(\Omega)} \leqslant \|u\|^{\theta}_{L^4_t \dot{B}^{1/4-\eta,2}_4(\Omega)} \|u\|^{1-\theta}_{L^4_T \dot{B}^{1/4+\eta,2}_4(\Omega)}. \tag{4.31}$$

Recall that from Proposition 4.2 we have now a uniform bound,

$$\|u\|_{L^4_t \dot{B}^{1/4-\eta,2}_4(\Omega)} \lesssim C(E, M), \tag{4.32}$$

and from Lemma 4.3 we consequently also have a uniform bound on the Duhamel part coming from g_2, see (4.19). Finally, using (4.29) for g_1 and the uniform bounds we already have for the linear part and the g_2 part,

$$\|u\|_{L^4_T \dot{B}^{1/4+\eta,2}_4(\Omega)} \lesssim C_1(E, M) + C_2(E, M) \|\chi u\|^{1/2-\eta}_{L^2_t H^1_{\mathrm{comp}}} \|u\|^{2(1+\eta)}_{L^4_T \dot{B}^{\sigma,2}_4(\Omega)}. \tag{4.33}$$

Plugging (4.32), (4.33) in (4.31) yields

$$\|u\|_{L^4_T \dot{B}^{\sigma,2}_4(\Omega)} \leqslant C_3(E, M) + C_4(E, M) \|\chi u\|^{\gamma}_{L^2_t H^1_{\mathrm{comp}}} \|u\|^{\rho}_{L^4_T \dot{B}^{\sigma,2}_4(\Omega)}, \tag{4.34}$$

where $\rho, \gamma > 0$. The coefficients are uniformly bounded, and a splitting time argument performed on the $L^2_t H^1_{\mathrm{comp}}$ norm which is finite provides global in time control of u in $L^4_t \dot{B}^{\sigma,2}_4(\Omega)$. This finally completes the proof of Proposition 4.3. $\square$

Remark 4.4. Remark that $L^4_t(\dot{B}^{\sigma,2}_4(\Omega))$ with $\sigma = \frac{1}{4} + \frac{\eta}{1+\eta}$ is scale invariant with respect to the critical regularity s_p. As such, it makes sense that it plays a pivotal role in the argument. Having reached (and in fact, gone beyond) critical scaling in our a priori estimates, the remaining part of the argument is somewhat less involved.

At this point of the proof, we could establish scattering in the scale-invariant Sobolev space; however we want to reach H^1_0. Recall that we may write

$$\left\| u(t,x) - e^{it\Delta_D}\left(u_0 + \int_0^{+\infty} e^{-is\Delta_D}|u|^{p-1}u(s)\,ds\right)\right\|_{H^1} = \left\| \int_t^{+\infty} e^{i(t-s)\Delta_D}|u|^{p-1}u(s)\,ds\right\|_{H^1},$$

from which we wish to use Duhamel to get

$$\left\| \int_t^{+\infty} e^{i(t-s)\Delta_D}|u|^{p-1}u(s)\,ds\right\|_{H^1} \lesssim \|g_1\|_{L^{4/3}(t,+\infty;\, \dot{B}^{5/4,2}_{4/3}(\Omega))} + \|g_2\|_{L^1(t,+\infty;H^1)}, \tag{4.35}$$

from which scattering easily follows (the same argument applies at $t = -\infty$ as well).

Therefore we focus on the right-hand side and start with the easiest part, which is g_2.

Lemma 4.5. *We have* $g_2 = (1-\chi)|u|^{p-1}u \in L^1_t H^1$.

Proof. We start by proving that

$$v = (1-\chi_1)u \in L^{2(1+\eta)}_t L^\infty_x. \tag{4.36}$$

Remark 4.5. Notice that if we have (4.36) the proof is over since then, by Lemma A.4,

$$\|v|v|^{2+2\eta}\|_{L^1_t H^1} \leqslant \|v\|^{2(1+\eta)}_{L^{2(1+\eta)}_t L^\infty_x} \|v\|_{L^\infty_t H^1}. \tag{4.37}$$

We proceed with (4.36). From Lemma 4.3 we know that $g_2 \in L_t^1 H^{\frac{1}{2}+\eta}$, and from Proposition 4.3 we have $[\chi, \Delta_D]u \in L_t^2 H_{\mathrm{comp}}^{\eta}$; using again the equation for $(1-\chi)u$ and Lemma 4.2,

$$(1-\chi)u \in L_t^2 \dot{B}_6^{\frac{1}{2}+\eta,2}(\mathbb{R}^3) \quad \left(\cap L_t^\infty H^1\right). \tag{4.38}$$

Recall that from Lemma 4.3 we also have $v \in L_t^2 \dot{B}_6^{1/2,2}(\mathbb{R}^3) \cap L_t^\infty H^{1/2}$. The lemma now follows by interpolation and the Gagliardo–Nirenberg inequality (a similar key step exists in [16]). □

Lemma 4.6. *We have* $g_1 = \chi |u|^{p-1} u \in L_t^{4/3} \dot{B}_{4/3}^{5/4,2}(\Omega)$.

Proof. We first prove

$$u \in L_t^{8(1+\eta)} L_x^{8(1+\eta)}. \tag{4.39}$$

Indeed, from Propositions 4.2, 4.3 and interpolation, we get $u \in L_t^4 \dot{B}_4^{1/4+\eta/2,2}(\Omega)$. Interpolating again between this bound and the energy bound $u \in L_t^\infty H^1$, followed by Sobolev embedding yields (4.39). Now we write, by Lemma A.4

$$\|g_1\|_{L_t^{4/3} \dot{B}_{4/3}^{5/4,2}(\Omega)} \lesssim \|\chi_1 u\|_{L_t^2 H_{\mathrm{comp}}^{5/4}} \|\chi_1 u\|^{2(1+\eta)}_{L_t^{8(1+\eta)} L_x^{8(1+\eta)}}, \tag{4.40}$$

and also by the Duhamel formula and the local smoothing estimate on the domain,

$$\|u\|_{L_t^2 H_{\mathrm{comp}}^{5/4}} \leqslant \|u_0\|_{H^{3/4}} + \|g_1\|_{L_t^{4/3} \dot{B}_{4/3}^{1,2}(\Omega)} + \|g_2\|_{L_t^1 H^{3/4}}. \tag{4.41}$$

Certainly, using Lemma 4.5, the g_2 term is bounded. For g_1, we may write

$$\|g_1\|_{L_t^{4/3} \dot{B}_{4/3}^{1,2}(\Omega)} \lesssim \|\chi_1 u\|_{L_t^2 H_{\mathrm{comp}}^{1}} \|\chi_1 u\|^{2(1+\eta)}_{L_t^{8(1+\eta)} L_x^{8(1+\eta)}}; \tag{4.42}$$

and we have reached a point where our right-hand side is uniformly bounded. Consequently the lemma is proved, and this concludes the proof of Theorem 2.3. □

Appendix A

In order to perform the various product estimates, we need a couple of useful lemmas. Observe that with the spectral localization one cannot take advantage of convolution of Fourier supports. In order to avoid cumbersome notations, we only consider functions and Besov spaces which do not depend on time.

We now explain how to re-instate the time dependence in the non-linear estimates: both Δ_j and S_j operators are well defined on $L_t^p L_x^q$ and $L_x^q L_t^p$ for all the pairs (p, q) to be considered: this follows from [14] for the case $L_t^p L_x^q$ where the time norm is harmless. In the case $L_x^q L_t^2$, the arguments from [14] apply as well (heat estimates are proved for data in $L_x^p(H)$ where H is an abstract Hilbert space, and when $H = L_t^2$, the heat kernel is diagonal and therefore Gaussian as well). By interpolation and duality we recover all pairs (p, q).

Remark A.1. In $\mathbb{R}^n$, one may perform product estimates in an easier way because of the convolution of Fourier supports. However, when dealing with non-integer power-like non-linearities, one cannot proceed so easily: the usual route is to use a characterization of Besov spaces via finite differences; here, because of the Banach-valued Besov spaces, we perform a direct argument which is directly inspired by computations in [15], where the same sort of time-valued Besov spaces were unavoidable.

Lemma A.1. *Let f_j be such that $S_j f_j = f_j$, and $\|f_j\|_{L^p} \lesssim 2^{-js}\eta_j$, with $s > 0$ and $(\eta_j)_j \in l^q$. Then $g = \sum_j f_j \in \dot{B}_p^{s,q}$.*

We have, by support conditions,

$$g = \sum_k \Delta_k \sum_{k<j} S_j f_j.$$

Now,

$$\left\|\Delta_k\Big(\sum_{k<j} S_j f_j\Big)\right\|_p \lesssim 2^{-ks}\sum_{k<j} 2^{-s(j-k)}\eta_j,$$

which by an $l^1 - l^q$ convolution provides the result.

Lemma A.2. *Let f_j be such that $(I - S_j)f_j = f_j$, and $\|f_j\|_{L^p} \lesssim 2^{-js}\eta_j$, with $s < 0$ and $(\eta_j)_j \in l^q$. Then $g = \sum_j f_j \in \dot B_p^{s,q}$.*

We have, by support conditions,

$$g = \sum_k \Delta_k \sum_{k>j}(I - S_j)f_j.$$

Now,

$$\left\|\Delta_k\Big(\sum_{k>j}(I - S_j)f_j\Big)\right\|_p \lesssim 2^{-ks}\sum_{k<j} 2^{-s(j-k)}\eta_j,$$

which by an $l^1 - l^q$ convolution provides the result.

Lemma A.3. *Consider $\alpha = 1$ or $\alpha \geqslant 2$, $f \in \dot B_p^{s,q}$ and $g \in L^r$, with $0 < s < 2$, $\frac{1}{m} = \frac{\alpha}{r} + \frac{1}{p}$: let*

$$T_g^\alpha f = \sum_j (S_j g)^\alpha \Delta_j f.$$

Then

$$T_g^\alpha f \in \dot B_m^{s,q}.$$

We split the “paraproduct” $T_g^\alpha f$:

$$T_g^\alpha f = \sum_j S_j\big((S_j g)^\alpha \Delta_j f\big) + \sum_j (I - S_j)\big((S_j g)^\alpha \Delta_j f\big);$$

the first part is easily dealt with by Lemma A.1. For the second one, $K_g f$, taking once again advantage of the spectral supports

$$\Delta_k K_g f = \Delta_k \sum_{j<k}(I - S_j)\big((S_j g)^\alpha \Delta_j f\big).$$

Notice the situation is close to the one in Lemma A.2, but we don’t have a negative regularity for summing. We therefore derive

$$\begin{aligned}
\Delta_D K_g f &= \sum_{j<k}(I - S_j)\Delta_D\big((S_j g)^\alpha \Delta_j f\big)\\
&= \sum_{j<k}(I - S_j)\big(\Delta_D (S_j g)^\alpha \Delta_j f + (\Delta_D \Delta_j f)(S_j g)^\alpha + 2\alpha (S_j g)^{\alpha-1}\nabla S_j g\cdot\nabla\Delta_j f\big)\\
&= \sum_{j<k}(I - S_j)\big(\alpha\Delta_D S_j g (S_j g)^{\alpha-1}\Delta_j f + \alpha(\alpha-1)|\nabla S_j g|^2 (S_j g)^{\alpha-2}\Delta_j f\\
&\quad + (\Delta_D\Delta_j f)(S_j g)^\alpha + 2\alpha (S_j g)^{\alpha-1}\nabla S_j g\cdot\nabla\Delta_j f\big).
\end{aligned}$$

The first two pieces are again easily dealt with Lemma A.2, and the resulting function is in $\dot B_m^{s-2,q}$. The remaining cross term is handled with some help from [14]:

$$\nabla\Delta_j f = \nabla \exp\big(4^{-j}\Delta_D\big)\tilde{\Delta}_j f,$$

where the new dyadic block $\tilde{\Delta}_j$ is built on the function $\tilde{\psi}(\xi) = \exp(|\xi|^2)\psi(\xi)$. From the continuity properties of $\sqrt{s}\nabla\exp(s\Delta_D)$ on L^p, $1 < p < +\infty$, we immediately deduce

$$\|\nabla\Delta_j f\|_p \lesssim 2^j\|\tilde{\Delta}_j f\|_p, \tag{A.1}$$

and we can easily sum and conclude. This will be enough to deal with the critical case, but for differences of non-linear power-like mappings, we need

Lemma A.4. *Consider $\alpha \geqslant 3$, $f, g \in X = \dot{B}^{s,q}_p \cap L^r$, with $0 < s < 2$, $\frac{1}{m} = \frac{\alpha-1}{r} + \frac{1}{p}$: Then, if $F(x) = |x|^{\alpha-1}x$ or $F(x) = |x|^\alpha$,*

$$\big\|F(u) - F(v)\big\|_{\dot{B}^{s,q}_m} \lesssim \|u - v\|_X\big(\|u\|_X^{\alpha-1} + \|v\|_X^{\alpha-1}\big).$$

In order to obtain a factor $u - v$, we write

$$F(u) - F(v) = (u - v)\int_0^1 F'\big(\theta u + (1-\theta)v\big)\,d\theta. \tag{A.2}$$

We need to efficiently split this difference into two paraproducts involving $u - v$ and $F'(w)$ with $w = \theta u + (1-\theta)v$, and this requires an estimate on $F'(w)$: write another telescopic series

$$\begin{aligned}
F'(w) &= \sum_j F'(S_{j+1}w) - F'(S_j w)\\
&= \sum_j S_j\big(F'(S_{j+1}w) - F'(S_j w)\big) + \sum_j (I - S_j)\big(F'(S_{j+1}w) - F'(S_j w)\big)\\
&= S_1 + S_2.
\end{aligned}$$

Exactly as before, the first sum S_1 is easily disposed of with Lemma A.1, as

$$\big|F'(S_{j+1}w) - F'(S_j w)\big| \lesssim |\Delta_j w|\big(|S_{j+1}w|^{\alpha-2} + |S_j w|^{\alpha-2}\big).$$

The second sum S_2 requires again a trick; to avoid unnecessary cluttering, we set $F(x) = x^\alpha$, ignoring the sign issue (recall that $\alpha \geqslant 3$, hence $F'''(x)$ is well defined as a function): we apply Δ_D, let $\beta = \alpha - 1 \geqslant 2$

$$\begin{aligned}
\Delta_D S_2 &= \sum_j (I - S_j)\Delta_D\big((S_{j+1}w)^{\alpha-1} - (S_j w)^{\alpha-1}\big)\\
&= \sum_j (I - S_j)\big(\beta(S_{j+1}w)^{\beta-1}\Delta_D S_{j+1}w - \beta(S_j w)^{\beta-1}\Delta_D S_j w\\
&\quad + \beta(\beta-1)(S_{j+1}w)^{\beta-2}(\nabla S_{j+1}w)^2 - \beta(\beta-1)(S_j w)^{\beta-2}(\nabla S_j w)^2\big).
\end{aligned}$$

We now apply Lemma A.2 after inserting the right factors: we have four types of differences,

$$\begin{aligned}
&\big|\big((S_{j+1}w)^{\beta-1} - (S_j w)^{\beta-1}\big)\Delta_D S_{j+1}w\big| \lesssim C_\beta|\Delta_j w||\Delta_D S_{j+1}|\big(|S_{j+1}w|^{\beta-2} + |S_j w|^{\beta-2}\big),\\
&\big|(S_{j+1}w)^{\beta-1}\Delta_D\Delta_j w\big| \leqslant |\Delta_D\Delta_j w||S_{j+1}w|^{\beta-2},\\
&\big|\big((S_{j+1}w)^{\beta-2} - (S_j w)^{\beta-2}\big)(\nabla S_{j+1}w)^2\big| \lesssim \tilde{C}_\beta|\Delta_j w|^{\beta-2}|\nabla S_{j+1}w|^2,\\
&\big|(S_{j+1}w)^{\beta-2}\big((\nabla S_j w)^2 - (\nabla S_{j+1}w)^2\big)\big| \leqslant |\nabla\Delta_j w|\big(|\nabla S_j w| + |\nabla S_{j+1}w|\big)|S_{j+1}w|^{\beta-2}
\end{aligned}$$

where on the third line we wrote the worst case, namely $2 \leqslant \beta < 3$ (otherwise the power of $\Delta_j w$ in the third bound will be replaced by $|\Delta_j w|(|S_j w|^{\beta-3} + |S_{j+1}w|^{\beta-3})$).

By integrating, applying Hölder and using (A.1) to eliminate the ∇ operator, we obtain as an intermediary result

$$F'(w) \in \dot{B}^{s,q}_\lambda, \quad \text{with } \frac{1}{\lambda} = \frac{\alpha-2}{r} + \frac{1}{p}.$$

We may now go back to the difference $F(u)-F(v)$ as expressed in (A.2) and perform a simple paraproduct decomposition in two terms to which Lemma A.3 may be applied. Observe that there is no difficulty in estimating $F'(w)$ in $L^{m/(\alpha-1)}$, and that the integration in θ is irrelevant. This completes the proof.

We now go back to the first non-linear estimate, namely (4.1), to illustrate how it can be proved directly. We write a telescopic series for the product five factors $u_1, u_2, u_3, u_4, u_5 \in X_T$,

$$u_1u_2u_3u_4u_5 = \sum_j S_{j+1}u_1S_{j+1}u_2S_{j+1}u_3S_{j+1}u_4S_{j+1}u_5 - S_ju_1S_ju_2S_ju_3S_ju_4S_ju_5$$

and we are reduced to studying five sums of the same type, of which the following is generic

$$S_1 = \sum_j \Delta_j u_1 S_ju_2S_ju_3S_ju_4S_ju_5,$$

and we intend to apply Lemma A.3, which is trivially extended to a product of several factors. Then

$$u_k \in \dot{B}_5^{1,2}\big(L_T^{\frac{20}{11}}\big) \cap L_x^{\frac{20}{3}} L_T^{40}$$

is enough, using the first space of the Δ_j factor and the second one for all remaining S_j factors.

We proceed with the low frequencies by proving a suitable Gagliardo–Nirenberg embedding.

Lemma A.5. *We have the following embeddings*:

- *let* $u \in \dot{B}_5^{\frac{1}{2},5}(L_T^5)$ *and* $\partial_t u \in \dot{B}_5^{-\frac{3}{2},5}(L_T^5)$. *Then* $u \in L_x^{\frac{20}{3}} L_T^{40}$;
- *let* $u \in \dot{B}_4^{s_p-\frac{1}{4},4}(L_T^4)$ *and* $\partial_t u \in \dot{B}_4^{s_p-\frac{9}{4},4}(L_T^4)$. *Then* $u \in L_x^{\frac{5(p-1)}{3}} L_T^{10(p-1)}$;
- *let* $u \in \dot{B}_{\frac{5(p-1)}{3}}^{\frac{1}{p-1},\frac{5(p-1)}{3}}(L_T^{\frac{5(p-1)}{3}})$ *and* $\partial_t u \in \dot{B}_4^{s_p-\frac{9}{4},4}(L_T^4)$. *Then* $u \in L_x^{\frac{5(p-1)}{3}} L_T^{10(p-1)}$.

We deal with the first embedding, the other two are similar (but exponents are painful due to the p). Let

$$2^{\frac{1}{2}j}\|\Delta_j u\|_{L_x^5L_T^5} + 2^{-\frac{3}{2}j}\|\partial_t\Delta_j u\|_{L_T^5L_x^5} = \mu_j^{(1)} \in l_j^5,$$

notice we can easily switch time and space Lebesgue norms. Using Gagliardo–Nirenberg in time, we have

$$2^{\frac{1}{6}j}\|\Delta_j u\|_{L_x^5L_T^{30}} \lesssim \mu_j^{(2)} \in l_j^5. \tag{A.3}$$

Using now Gagliardo–Nirenberg in space, we also have (each $\mu_j^{(\cdot)}$ being obtained from the previous one and retaining its summability)

$$2^{-\frac{j}{10}}\|\Delta_j u\|_{L_x^\infty L_T^5} \lesssim 2^{-\frac{j}{10}}\|\Delta_j u\|_{L_T^5L_x^\infty} \lesssim \mu_j^{(3)}$$

and the bound holds for $2^{-2j}\partial_t\Delta_j u$ as well. Yet another Gagliardo–Nirenberg in time provides

$$2^{-\frac{1}{2}j}\|\Delta_j u\|_{L_{T,x}^\infty} \lesssim \mu_j^{(4)}. \tag{A.4}$$

Finally, we take advantage of a discrete embedding between l^1 and weighted l^∞ sequences:

$$\begin{aligned}
|u| &\leqslant \sum_{j<J}|\Delta_j u| + \sum_{j\geqslant J}|\Delta_j u| \\
&\leqslant \sum_{j<J} 2^{\frac{j}{2}}\sup_j 2^{-\frac{j}{2}}|\Delta_j u| + \sum_{j\geqslant J} 2^{-\frac{j}{6}}\sup_j 2^{\frac{j}{6}}|\Delta_j u| \\
&\lesssim 2^{\frac{J}{2}}\sup_j 2^{-\frac{j}{2}}|\Delta_j u| + 2^{-\frac{J}{6}}\sup_j 2^{\frac{j}{6}}|\Delta_j u|, \\
|u|^4 &\lesssim \sup_j 2^{-\frac{j}{2}}|\Delta_j u|\Big(\sup_j 2^{\frac{j}{6}}|\Delta_j u|\Big)^3,
\end{aligned}$$

$$\left\| |u|^4 \right\|_{L_x^{\frac{5}{3}} L_T^{10}} \lesssim \left\| \sup_j 2^{-\frac{j}{2}} |\Delta_j u| \right\|_{L^\infty_{T,x}} \left\| \sup_j 2^{\frac{j}{6}} |\Delta_j u| \right\|^3_{L_x^5 L_T^{30}},$$

$$\|u\|_{L_x^{\frac{20}{3}} L_T^{40}} \lesssim \|u\|^{\frac{1}{4}}_{\dot{B}_\infty^{-\frac{1}{2},\infty}(L_t^\infty)} \|u\|^{\frac{3}{4}}_{\dot{B}_5^{\frac{1}{6},5}(L_t^{30})}.$$

Both other cases, involving $p < 5$, are handled in a similar way, and we leave the details to the reader, sparing him the complete set of exponents (depending on $p!$) that would appear in the proof. For scaling reasons there is actually no need to perform computations (moreover, the second embedding could be seen at as consequence of the third one, but we did not proved Bernstein inequalities): the first embedding, which is related to the critical case, simply illustrates that we can sidestep issues related to the usual Littlewood–Paley theory by using direct arguments.

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