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Higher derivatives estimate for the 3D Navier–Stokes equation

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Abstract

In this article, a nonlinear family of spaces, based on the energy dissipation, is introduced. This family bridges an energy space (containing weak solutions to Navier–Stokes equation) to a critical space (invariant through the canonical scaling of the Navier–Stokes equation). This family is used to get uniform estimates on higher derivatives to solutions to the 3D Navier–Stokes equations. Those estimates are uniform, up to the possible blowing-up time. The proof uses blow-up techniques. Estimates can be obtained by this means thanks to the galilean invariance of the transport part of the equation.

MSC: 76D05; 35Q30

Keywords: Navier–Stokes equation; Fluid mechanics; Blow-up techniques

1. Introduction

In this paper, we investigate estimates of higher derivatives of solutions to the incompressible Navier–Stokes equations in dimension 3, namely:

$$\begin{aligned}&\partial_t u + \operatorname{div}(u \otimes u) + \nabla P - \Delta u = 0, \quad t \in (0, \infty),\ x \in \mathbb{R}^3,\\ &\operatorname{div} u = 0.\end{aligned} \tag{1}$$

The initial value problem is endowed with the conditions:

$$u(0, \cdot) = u^0 \in L^2\big(\mathbb{R}^3\big).$$

The existence of weak solutions for this problem was proved long ago by Leray [11] and Hopf [8]. For this, Leray introduces a notion of weak solution. He shows that for any initial value with finite energy $u^0 \in L^2(\mathbb{R}^3)$ there exists a function $u \in L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; \dot{H}^1(\mathbb{R}^3))$ verifying (1) in the sense of distribution. From that time on, much effort has been made to establish results on the uniqueness and regularity of weak solutions. However those two questions remain yet mostly open. Especially it is not known until now if such a weak solution can develop singularities in finite time, even considering smooth initial data. We present our main result on a laps of time $(0, T)$ where the solution is indeed smooth (with possible blow-ups both at $t = 0$ and $t = T$). We will carefully show,

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however, that the estimates do not depend on the blow-up time T, but only on $\|u^0\|_{L^2}$ and $\inf(t, 1)$. The aim of this paper is to show the following theorem.

Theorem 1. *For any $t_0 > 0$, any Ω bounded subset of $(t_0, \infty) \times \mathbb{R}^3$, any integer $n \geqslant 1$, any $\gamma > 0$, and any $p \geqslant 0$ such that*

$$\frac{4}{p} > n + 1, \tag{2}$$

there exists a constant C, such that the following property holds.

For any smooth solution u of (1) on $(0, T)$ (with possible blow-up at 0 and T), we have

$$\|\nabla^n u\|_{L^p(\Omega \cap [(0,T)\times\mathbb{R}^3])} \leqslant C\big(\|u^0\|_{L^2(\mathbb{R}^3)}^{2(1+\gamma)/p} + 1\big).$$

Note that the constant C does not depend on the solution u nor on the blowing-up time T.

Note that for $n \geqslant 3$ we consider L^p spaces with $p < 1$. Those spaces are not complete for the weak topology. For this reason the result cannot be easily extend to general weak solutions after the possible blow-up time. However, up to $n = 2$, the result can be proved in this context. For this reason, along the proof, we will always consider suitable weak solutions, following [2]. That is, solutions verifying in addition to (1) the generalized energy inequality in the sense of distribution:

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}\left(u\frac{|u|^2}{2}\right) + \operatorname{div}(uP) + |\nabla u|^2 - \Delta\frac{|u|^2}{2} \leqslant 0, \quad t \in (0, \infty),\ x \in \mathbb{R}^3. \tag{3}$$

Moreover, by interpolation, the result of Theorem 1 can be extended to the whole real derivative coefficients, $1 < d \leqslant 2$, for $\|\Delta^{d/2}u\|_{L^p}$ with

$$\frac{4}{p} > d + 1.$$

Our result can be seen as a kind of anti-Sobolev result. Indeed, as we will see later, $\|\nabla u\|_{L^2}^2$ is used as a pivot quantity to control higher derivatives on the solution. The result for $d = 2$ was already obtained with completely different techniques by Constantine [4]. It has been extended in a slightly better space by Lions [13]. He shows that $\nabla^2 u$ can be bounded in the Lorentz space $L^{4/3,\infty}$.

In a standard way, using the energy inequality and interpolation, we get estimates on $\Delta^{d/2}u \in L^p((0,\infty)\times\mathbb{R}^3)$ for

$$\frac{5}{p} = d + \frac{3}{2}, \quad 0 \leqslant d \leqslant 1. \tag{4}$$

The Serrin–Prodi conditions (see [18,5,20]) ensure the regularity for solutions such that $\Delta^{d/2}u \in L^p((0,\infty)\times\mathbb{R}^3)$ for

$$\frac{5}{p} = d + 1, \quad 0 \leqslant d < \infty. \tag{5}$$

Those two families of spaces are given by an affine relation on d with respect to $1/p$ with slope 5. Notice that the family of spaces present in Theorem 1 has a different slope. Imagine, that we were able to extend this result along the same line with $d < 1$. For $d = 0$, we would obtain almost $u \in L^4((0,\infty)\times\mathbb{R}^3)$, which would imply that the energy inequality (3) is an equality (see [21]). Notice also that the line of this new family of spaces crosses the line of the critical spaces (5) at $d = -1$, $1/p = 0$. This point corresponds (at least formally) to the Tataru and Koch result on regularity of solutions small in $L^\infty(0,\infty; BMO^{-1}(\mathbb{R}^3))$ (see [9]). However, at this time, due to the "anti-Sobolev" feature of the proof, obtaining results for $d < 1$ seems out of reach. Note that different higher derivatives estimates have been obtained by Foiaş, Guillopé, and Temam [6]. In a different direction, Giga and Sawada studied higher derivatives of mild solutions to Navier–Stokes equations to obtain the space analyticity of those solutions (see [7]).

To see where lie the difficulties, let us focus on the result on the third derivatives. Consider the gradient of the Navier–Stokes equations (1),

$$\partial_t \nabla u - \Delta \nabla u = -\nabla u \cdot \nabla u - \nabla^2 P - (u \cdot \nabla)\nabla u.$$

Note that the two first right-hand side terms lie in $L^1((0,\infty) \times \mathbb{R}^3)$ (for the pressure term, see [13]). Parabolic regularity are not complete in L^1. This justify the fact that we miss the limit case L^1. But, surprisingly, the worst term is the transport one $(u \cdot \nabla)\nabla u$. To control it in L^1 using the control on $D^2 u$ in $L^{4/3,\infty}$ of Lions [13], we would need $u \in L^{4,1}$, which is not known. To overcome this difficulty, we will consider the solution in another frame, locally, by the following flow.

The idea of the proof comes from the result of partial regularity obtained by Caffarelli, Kohn and Nirenberg [2]. This paper extended the analysis about the possible singular points set, initialized by Scheffer in a series of paper [14–17]. The main remark in [2] is that the dissipation of entropy

$$\mathcal{D}(u) = \int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt \tag{6}$$

has a scaling, through the standard invariance of the equation, which is far more powerful that any other quantities from the energy scale (4). Let us be more specific. The standard invariance of the equation gives that for any $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^3$ and $\varepsilon > 0$, if u is a suitable solution of the Navier–Stokes equations (1), (3), then

$$u_\varepsilon(t,x) = \varepsilon u\big(t_0 + \varepsilon^2 t, x_0 + \varepsilon x\big) \tag{7}$$

is also solution to (1), (3). The dissipation of energy of this quantity is then given by

$$\mathcal{D}(u_\varepsilon) = \varepsilon^{-1} \mathcal{D}(u).$$

This power of ε made possible in [2] to show that the Hausdorff dimension of the set of blow-up points is at most 1. This was a great improvement of the result obtained by Scheffer who gives 5/3 as an upper bound for the Hausdorff dimension of this set. We can notice that it is what we get considering the quantity of the energy scale (4) with $d = 0$, $p = 10/3$:

$$\mathcal{F}(u) = \int_0^\infty \int_{\mathbb{R}^3} |u|^{10/3} \, dx \, dt.$$

Indeed:

$$\mathcal{F}(u_\varepsilon) = \varepsilon^{-5/3} \mathcal{F}(u).$$

The idea of this paper is to give a quantitative version of the result of [2], in the sense, of getting control of norms of the solution which have the same nonlinear scaling that $\mathcal{D}$. Indeed, for any norm of the nonlinear scaling (2), we have (in the limit case)

$$\|\nabla^n u_\varepsilon\|_{L^p}^p = \varepsilon^{-1} \|\nabla^n u\|_{L^p}^p.$$

The paper is organized as follows. In the next section, we give some preliminaries and fix some notations. We introduce the local frame the following flow in the third section. The fourth section is dedicated to a local result providing a universal control of the higher derivatives of u from a local control of the dissipation of the energy $\|\nabla u\|_{L^2}^2$ and a corresponding quantity on the pressure (see Proposition 10). Ideally, we would like to consider a quantity on the pressure which has the same nonlinear scaling as $\mathcal{D}(u)$. The corresponding quantity is $\|\nabla^2 P\|_{L^1}$. Unfortunately, we need a slightly better integrability in time for the local study. This is the reason why we miss the limit case $L^{p,\infty}$ with

$$\frac{4}{p} = n + 1.$$

This is also the reason why we need to work with fractional Laplacian for the pressure: $\|\Delta^{-s} \nabla^2 P\|_{L^p}$ with $0 < s < 1/2$. In the last section, we show how this local study leads to our main theorem.

2. Preliminaries and notations

We gather in this section some elementary lemmas which will be useful later. They are either well-known or follow standard ideas. We do not claim any originality in this section.

Let us denote $Q_r = (-r^2, 0) \times B_r$ where $B_r = B(0, r)$, the ball in $\mathbb{R}^3$ of radius r and centered at 0.

For $F \in L^p(\mathbb{R}^+ \times \mathbb{R}^3)$, we define the Maximal function in x only by

$$MF(t,x) = \sup_{r>0} \frac{1}{r^3} \int_{B_r} \left|F(t, x+y)\right| dy.$$

We recall that for any $1 < p < \infty$, there exists C_p such that for any $F \in L^p(\mathbb{R}^+ \times \mathbb{R}^3)$,

$$\|MF\|_{L^p(\mathbb{R}^+\times\mathbb{R}^3)} \leqslant C_p \|F\|_{L^p(\mathbb{R}^+\times\mathbb{R}^3)}.$$

We begin with an interpolation lemma. It is a straightforward consequence of a result in [1]. We state it here for further reference.

Lemma 2. *For any function F such that $(-\Delta)^{d_1/2} F$ lies in $L^{p_1}(0,\infty; L^{q_1}(\mathbb{R}^3))$ and*

$$(-\Delta)^{d_2/2} F \in L^{p_2}\big(0,\infty; L^{q_2}\big(\mathbb{R}^3\big)\big)$$

with

$$d_1, d_2 \in \mathbb{R}, \qquad 1 \leqslant p_1, p_2 \leqslant \infty, \qquad 1 < q_1, q_2 < \infty,$$

we have $(-\Delta)^{d/2} F \in L^p(0,\infty; L^q(\mathbb{R}^3))$ with

$$\big\|(-\Delta)^{d/2} F\big\|_{L^p(0,\infty;L^q(\mathbb{R}^3))} \leqslant \big\|(-\Delta)^{d_1/2} F\big\|^{\theta}_{L^{p_1}(0,\infty;L^{q_1}(\mathbb{R}^3))} \big\|(-\Delta)^{d_2/2} F\big\|^{1-\theta}_{L^{p_2}(0,\infty;L^{q_2}(\mathbb{R}^3))},$$

for any d, p, q such that

$$\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2},$$
$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2},$$
$$d = \theta d_1 + (1-\theta) d_2,$$

where $0 < \theta < 1$.

Proof. Exercise 31, p. 168 in [1] shows that for any $0 < t < \infty$, we have

$$\big\|(-\Delta)^{d/2} F(t)\big\|_{L^p(\mathbb{R}^3)} \leqslant \big\|(-\Delta)^{d_1/2} F(t)\big\|^{\theta}_{L^{p_1}(\mathbb{R}^3)} \big\|(-\Delta)^{d_2/2} F(t)\big\|^{1-\theta}_{L^{p_2}(\mathbb{R}^3)}.$$

Interpolation in the time variable gives the result. □

In the second lemma we show that we can control a local L^1 norm on a function f by its mean value and some local control on the maximal function of $(-\Delta)^{-s}\nabla f$, $0 < s < 1/2$. This extends the fact that we can control the local L^1 norm by the mean value and a local L^p norm of the gradient. But due to the nonlocal feature of the fractional Laplacian, we need to consider the maximal function to recapture all the information needed.

Lemma 3. *Let $0 < s < 1/2$, $q \geqslant 1$, $p \geqslant 1$. For any $\phi \in C^\infty(\mathbb{R}^3)$, $\phi \geqslant 0$, compactly supported in B_1 with $\int_{\mathbb{R}^3} \phi(x)\,dx = 1$, there exists $C > 0$ such that, for any function $f \in L^q(\mathbb{R}^3)$ with $(-\Delta)^{-s}\nabla f \in L^p(\mathbb{R}^3)$, we have $f \in L^1(B_1)$ and*

$$\|f\|_{L^1(B_1)} \leqslant C\left(\left|\int_{\mathbb{R}^3} f(x)\phi(x)\,dx\right| + \big\|M\big((-\Delta)^{-s}\nabla f\big)\big\|_{L^p(B_1)}\right).$$

Proof. Let us denote $g=(-\Delta)^{-s}\nabla f$. Since $f\in L^q(\mathbb{R}^3)$, we have

$$f=-(-\Delta)^{s-1}\operatorname{div} g.$$

So, for any $x\in B_1$,

$$f(x)=C_s\int_{\mathbb{R}^3}\frac{g(y)}{|x-y|^{2(1+s)}}\cdot\frac{(x-y)}{|x-y|}\,dy,$$

and

$$f(x)-\int_{\mathbb{R}^3}\phi(z)f(z)\,dz=C_s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\phi(z)g(y)\left(\frac{(x-y)/|x-y|}{|x-y|^{2(1+s)}}-\frac{(z-y)/|z-y|}{|y-z|^{2(1+s)}}\right)dy\,dz.$$

Note that, for $k\geqslant 2$, $y\in B_{2^k}\setminus B_{2^{k-1}}$, $x\in B_1$, $z\in B_1$, we have

$$\left|\frac{(x-y)/|x-y|}{|x-y|^{2(1+s)}}-\frac{(z-y)/|z-y|}{|y-z|^{2(1+s)}}\right|\leqslant\frac{C}{2^{k(3+2s)}}.$$

Moreover

$$\begin{aligned}&\int_{B_1}\int_{B_1}\int_{B_2}\phi(z)\big|g(y)\big|\left|\frac{(x-y)/|x-y|}{|x-y|^{2(1+s)}}-\frac{(z-y)/|z-y|}{|y-z|^{2(1+s)}}\right|dy\,dz\,dx\\&\quad\leqslant\int_{B_3}\int_{B_1}\int_{B_2}\frac{\phi(z)|g(y)|}{|x|^{2(1+s)}}\,dy\,dz\,dx+\int_{B_1}\int_{B_3}\int_{B_2}\frac{\sup|\phi||g(y)|}{|z|^{2(1+s)}}\,dy\,dz\,dx\\&\quad\leqslant 2C_s\|g\|_{L^1(B_1)}\leqslant 2C_s\|Mg\|_{L^1(B_1)},\end{aligned}$$

since $2(1+s)<3$. Hence

$$\begin{aligned}\left\|f-\int\phi(z)f(z)\,dz\right\|_{L^1(B_1)}&\leqslant\int_{B_1}\int_{B_1}\int_{B_2}\phi(z)\big|g(y)\big|\left|\frac{(x-y)/|x-y|}{|x-y|^{2(1+s)}}-\frac{(z-y)/|z-y|}{|y-z|^{2(1+s)}}\right|dy\,dz\,dx\\&\quad+\sum_{k=2}^{\infty}\int_{B_1}\int_{B_1}\int_{(B_{2^k}\setminus B_{2^{k-1}})}\phi(z)\big|g(y)\big|\left|\frac{(x-y)/|x-y|}{|x-y|^{2(1+s)}}-\frac{(z-y)/|z-y|}{|y-z|^{2(1+s)}}\right|\\&\leqslant 2C_s\|Mg\|_{L^1(B_1)}+C\sum_{k=2}^{\infty}\int_{B_{2^k}}\frac{|g(y)|}{2^{k(3+2s)}}\,dy\\&\leqslant 2C_s\|Mg\|_{L^1(B_1)}+8C\sum_{k=2}^{\infty}2^{-2sk}\frac{1}{|B_{2^{k+1}}|}\int_{B_1}\int_{B_{2^{k+1}}}\big|g(y+u)\big|\,dy\,du\\&\leqslant 2C_s\|Mg\|_{L^1(B_1)}+C\|Mg\|_{L^1(B_1)}\sum_{k=2}^{\infty}\big[2^{-2s}\big]^k\\&\leqslant C_s\|Mg\|_{L^1(B_1)},\end{aligned}$$

whenever $0<s<1/2$. □

We give now very standard results of parabolic regularity. There are not even optimal, but enough for our study.

Lemma 4. *For any $1<p<\infty$, $t_0>0$, there exists a constant C such that the following is true. Let $f,g\in L^p((-t_0,0)\times\mathbb{R}^3)$ be compactly supported in B_1 in x. Then there exists a unique $u\in L^p(-t_0,0;W^{1,p}(\mathbb{R}^3))$ solution to*

$$\partial_t u - \Delta u = g + \operatorname{div} f, \quad -t_0 \leqslant t \leqslant 0,\ x \in \mathbb{R}^3,$$
$$u(-t_0, x) = 0, \quad x \in \mathbb{R}^3.$$

Moreover,

$$\|u\|_{L^p(-t_0,0;W^{1,p}(B_1))} \leqslant C\big(\|f\|_{L^p((-t_0,0)\times\mathbb{R}^3)} + \|g\|_{L^p((-t_0,0)\times\mathbb{R}^3)}\big). \tag{8}$$

If $g \in L^1(-t_0, 0; L^\infty(\mathbb{R}^3))$ and $f \in L^1(-t_0, 0; W^{1,\infty}(\mathbb{R}^3))$, then

$$\|u\|_{L^\infty((-t_0,0)\times\mathbb{R}^3)} \leqslant C\big(\|g\|_{L^1(-t_0,0;L^\infty(\mathbb{R}^3))} + \|f\|_{L^1(-t_0,0;W^{1,\infty}(\mathbb{R}^3))}\big).$$

Proof. We get the solution using the Green function*:*

$$u(t,x) = \int_{-t_0}^{t} \frac{1}{4\pi(t-s)^{3/2}} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4(t-s)}} \big(g(s,y) + \operatorname{div} f(s,y)\big)\,dy\,ds.$$

From this formulation, using that $z^n e^{-z^2}$ are bounded functions, we find that

$$\big|u(t,x)\big| \leqslant C\frac{\|f\|_{L^1((-t_0,0)\times B_1)} + \|g\|_{L^1((-t_0,0)\times B_1)}}{|x|^3}, \quad \text{for } |x| > 2,\ -t_0 \leqslant t < 0. \tag{9}$$

Standard Solonnikov's parabolic regularization result gives (8) (see for instance [19]). Finally, if $g \in L^1(-t_0, 0; L^\infty(\mathbb{R}^3))$ and $f \in L^1(-t_0, 0; W^{1,\infty}(\mathbb{R}^3))$, then the function

$$v(t,x) = \int_0^t \big(\big\|g(s)\big\|_{L^\infty} + \big\|\operatorname{div} f(s)\big\|_{L^\infty}\big)\,ds$$

is a supersolution thanks to (9). The global bound follows. □

The last lemma of this section is a standard decomposition of the pressure term as a close range part and a long range part.

Lemma 5. *Let $\overline{B}$ and $\underline{B}$ be two balls such that*

$$\overline{B} \Subset \underline{B}.$$

Then for any $1 < p < \infty$, there exists a constant $C > 0$ and a family of constants $\{C_{d,q} \setminus d, q$ integers$\}$ (depending only on p, $\underline{B}$ and $\overline{B}$) such that for any $R \in L^1(\underline{B})$ and $A \in [L^p(\underline{B})]^{N\times N}$ symmetric matrix, verifying

$$-\Delta R = \operatorname{div}\operatorname{div} A, \quad \text{in } \underline{B},$$

we have a decomposition

$$R = R_1 + R_2,$$

with, for any integer $q \geqslant 0$, $d \geqslant 0$:

$$\|R_1\|_{L^p(\overline{B})} \leqslant C\|A\|_{L^p(\underline{B})},$$
$$\big\|\nabla^d R_2\big\|_{L^\infty(\overline{B})} \leqslant C_{d,q}\big(\|A\|_{L^1(\underline{B})} + \|R\|_{W^{-q,1}(\underline{B})}\big).$$

Moreover, if A is Lipschitzian, then we can choose R_1 such that

$$\|R_1\|_{L^\infty(\overline{B})} \leqslant C\big(\|\nabla A\|_{L^\infty(\underline{B})} + \|A\|_{L^\infty(\underline{B})}\big).$$

Proof. Let B^* be a ball such that

$$\overline{B} \Subset B^* \Subset \underline{B},$$

with a distance between $\overline{B}$ and B^{*c} bigger that $D/2$, where D is the distance between $\overline{B}$ and $\underline{B}^c$. Consider a smooth nonnegative cut-off function ψ, $0 \leqslant \psi \leqslant 1$ such that

$$\begin{aligned} \psi(x) &= 1 \quad \text{in } B^*, \\ &= 0 \quad \text{in } \underline{B}^c. \end{aligned}$$

Then the function ψR (defined in $\mathbb{R}^3$) is solution in $\mathbb{R}^3$ to

$$-\Delta(\psi R) = \operatorname{div}\operatorname{div}(\psi A) + R\Delta\psi + A : \nabla^2\psi - 2\operatorname{div}\{\nabla\psi \cdot A + R\nabla\psi\}.$$

We denote

$$\begin{aligned} R_1 &= (-\Delta)^{-1}\operatorname{div}\operatorname{div}(\psi A), \\ R_2 &= (-\Delta)^{-1}\big(R\Delta\psi + A : \nabla^2\psi - 2\operatorname{div}\{\nabla\psi \cdot A + R\nabla\psi\}\big). \end{aligned}$$

We have, on $\overline{B}$, $R = R_1 + R_2$. The operator $(-\Delta)^{-1}\operatorname{div}\operatorname{div}$ is a Riesz operator, so there exists a constant (depending only on p and ψ) such that

$$\begin{aligned} \|R_1\|_{L^p(\mathbb{R}^3)} &\leqslant C\|\psi A\|_{L^p(\mathbb{R}^3)} \leqslant C\|A\|_{L^p(\underline{B})}, \\ \|R_1\|_{C^\alpha(\mathbb{R}^3)} &\leqslant C\|\psi A\|_{C^\alpha(\mathbb{R}^3)} \leqslant C\big(\|\nabla A\|_{L^\infty(\underline{B})} + \|A\|_{L^\infty(\underline{B})}\big). \end{aligned}$$

Using the fact that $\nabla\psi$ and $\nabla^2\psi$ vanishes on $B^* \cup \underline{B}^c$, we have for any $x \in \overline{B}$:

$$\begin{aligned} \big|\nabla^d R_2(x)\big| &= \left| \int_{\mathbb{R}^3} \nabla^d\left(\frac{1}{|x-y|}\right)\big(R\Delta\psi + A : \nabla^2\psi\big)(y)\,dy + 2\int_{\mathbb{R}^3} \nabla^{d+1}\left(\frac{1}{|x-y|}\right)\{\nabla\psi \cdot A + R\nabla\psi\}(y)\,dy \right| \\ &\leqslant \|\nabla^2\psi\|_{L^\infty}\|A\|_{L^1(\underline{B})} \sup_{|x-y|\geqslant D/2}\left|\nabla^d\left(\frac{1}{|x-y|}\right)\right| \\ &\quad + 2\|\nabla\psi\|_{L^\infty}\|A\|_{L^1(\underline{B})} \sup_{|x-y|\geqslant D/2}\left|\nabla^{d+1}\left(\frac{1}{|x-y|}\right)\right| \\ &\quad + \|R\|_{W^{-q,1}(\underline{B})} \sup_{|x-y|\geqslant D/2}\left|\nabla^q\left[\nabla^d\left(\frac{1}{|x-y|}\right)\Delta\psi\right]\right| \\ &\quad + 2\|R\|_{W^{-q,1}(\underline{B})} \sup_{|x-y|\geqslant D/2}\left|\nabla^q\left[\nabla^{d+1}\left(\frac{1}{|x-y|}\right)\nabla\psi\right]\right| \\ &\leqslant C_d\left[\left(\frac{2}{D}\right)^{d+2} + \left(\frac{2}{D}\right)^{d+1}\right]\|A\|_{L^1(\underline{B})} + C_{d,q}\left[\left(\frac{2}{D}\right)^{d+1} + \left(\frac{2}{D}\right)^{q+d+2}\right]\|R\|_{W^{-q,1}(\underline{B})}. \qquad \square \end{aligned}$$

3. Blow-up method along the trajectories

Our result relies on a local study, which was the keystone of the partial regularity result of [2] (see [12] for an other proof). We use, here, the version of [22]. This version is better for our purpose because it requires a bound on the pressure only in L^p in time for any $p > 1$.

Proposition 6. *(See [22].) For any $p > 1$, there exists $\eta > 0$, such that the following property holds. For any u, suitable weak solution to the Navier–Stokes equations* (1), (3)*, in Q_1, such that*

$$\sup_{-1<t<0}\left(\int_{B_1} \big|u(t,x)\big|^2\,dx\right) + \int_{Q_1} |\nabla u|^2\,dx\,dt + \int_{-1}^{0}\left(\int_{B_1} |P|\,dx\right)^p dt \leqslant \eta, \tag{10}$$

we have

$$\sup_{(t,x)\in Q_{1/2}} \left|u(t,x)\right| \leqslant 1.$$

As explained in the introduction, the proof of Theorem 1 relies on this local control. From there we can get control on higher derivatives of u. We first show the following lemma. It introduces the pivot quantity. Note that the ideal pivot quantity would be $\|\nabla u\|^2_{L^2(L^2)} + \|\nabla^2 P\|_{L^1(L^1)}$. This is because this quantity scales as $1/\varepsilon$ through the canonical scaling. However, to use Proposition 6 locally, we need a better integrability in time on the pressure. For this reason, we add the quantity on the pressure involving the fractional Laplacian. We get a better integrability in time on the pressure, at the cost of a slightly worst rate of change in ε through the canonical scaling. Finally, due to the nonlocal character of the fractional Laplacian, the maximal function is used in order to recapture all the local information needed (see Lemma 3).

Lemma 7. *For any* $0 < \delta < 1$*, there exists* $\gamma > 0$ *and a constant* $C > 0$ *such that for any* u *solution to* (1), (3)*, with* $u^0 \in L^2(\mathbb{R}^3)$*, we have*

$$\int_0^\infty \int_{\mathbb{R}^3} \left(\left|M\left((-\Delta)^{-\delta/2}\nabla^2 P\right)\right|^{1+\gamma} + \left|\nabla^2 P\right| + |\nabla u|^2\right) dx\,dt \leqslant C\left(\left\|u^0\right\|^2_{L^2(\mathbb{R}^3)} + \left\|u^0\right\|^{2(1+\gamma)}_{L^2(\mathbb{R}^3)}\right).$$

Moreover, γ *converges to 0 when* δ *converges to 0.*

Proof. Integrating in x the energy equation (3) gives that

$$\int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2\,dx\,dt \leqslant \left\|u^0\right\|^2_{L^2(\mathbb{R}^3)}, \tag{11}$$

together with

$$\|u\|^2_{L^\infty(0,\infty;L^2(\mathbb{R}^3))} \leqslant \left\|u^0\right\|^2_{L^2(\mathbb{R}^3)}.$$

By Sobolev embedding and interpolation, this gives in particular that

$$\|u\|^2_{L^4(0,\infty;L^3(\mathbb{R}^3))} \leqslant C\left\|u^0\right\|^2_{L^2(\mathbb{R}^3)}. \tag{12}$$

For the pressure, we have $\nabla^2 P \in L^1(\mathcal{H})$ (see Lions [13]). Indeed,

$$\nabla^2 P = \left(\nabla^2\Delta^{-1}\right)\sum_{ij} \partial_i u_j \partial_j u_i = \left(\nabla^2\Delta^{-1}\right)\sum_i (\partial_i u)\cdot\nabla u_i.$$

For any i, we have $\operatorname{rot}(\nabla u_i) = 0$ and $\operatorname{div}\ \partial_i u = 0$. Hence, from the div-rot lemma (see Coifman, Lions, Meyer and Semmes [3]), we have

$$\left\|\sum_i \partial_i u\cdot\nabla u_i\right\|_{L^1(\mathcal{H})} \leqslant \|\nabla u\|^2_{L^2}.$$

But $\nabla^2\Delta^{-1}$ is a Riesz operator (in x only) which is bounded from $\mathcal{H}$ to $\mathcal{H}$. Hence:

$$\left\|\nabla^2 P\right\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)} \leqslant C\left\|\nabla^2 P\right\|_{L^1(\mathbb{R}^+;\mathcal{H}(\mathbb{R}^3))} \leqslant C\|\nabla u\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^3)}. \tag{13}$$

Using the Sobolev imbedding with Hardy space (see Lemarié-Rieusset [10, Thm. 6.9]), we get from the second estimate of (13) that for any $0 < s < 1$,

$$\left\|(-\Delta)^{-s/2}\nabla^2 P\right\|_{L^1(0,\infty;L^p(\mathbb{R}^3))} \leqslant C\left\|u^0\right\|^2_{L^2} \tag{14}$$

for

$$\frac{1}{p} = 1 - \frac{s}{3},$$

we have also

$$(-\Delta)^{-1/2}\nabla^2 P = \sum_{ij}\big[(-\Delta)^{-3/2}\nabla^2\partial_i\big](\partial_j u_i u_j).$$

The operators $(-\Delta)^{-3/2}\nabla^2\partial_i$ are Riesz operators so, together with (11), (12), we have

$$\big\|(-\Delta)^{-1/2}\nabla^2 P\big\|_{L^{4/3}(0,\infty;L^{6/5}(\mathbb{R}^3))} \leqslant C\big\|u^0\big\|^2_{L^2(\mathbb{R}^3)}. \tag{15}$$

By interpolation with (14), using Lemma 2 with $\theta = 1/(1+4s)$, we find

$$\big\|M\big[(-\Delta)^{-\delta/2}\nabla^2 P\big]\big\|_{L^{1+\gamma}((0,\infty)\times\mathbb{R}^3)} \leqslant C\big\|u^0\big\|^2_{L^2(\mathbb{R}^3)}$$

with

$$\delta = \frac{5s}{1+4s}, \qquad \gamma = \frac{s}{1+3s}.$$

Note that γ converges to 0 when δ goes to 0. This, together with (13) and (11), gives the result. □

Let us fix from now on a smooth cut-off function $0 \leqslant \phi \leqslant 1$ compactly supported in B_1 and such that

$$\int_{\mathbb{R}^3} \phi(x)\,dx = 1. \tag{16}$$

For any $\varepsilon > 0$, we define

$$u_\varepsilon(t,x) = \int_{\mathbb{R}^3} \phi(y)u(t, x+\varepsilon y)\,dy. \tag{17}$$

Note that $u_\varepsilon \in L^\infty(0,\infty; C^\infty(\mathbb{R}^3))$ and $\operatorname{div} u_\varepsilon = 0$. We define the flow:

$$\begin{aligned} &\frac{\partial X}{\partial s} = u_\varepsilon\big(s, X(s,t,x)\big), \\ &X(t,t,x) = x. \end{aligned} \tag{18}$$

Note that the flow X depends on ε. Consider, for any $0 < \delta < 1$ and $\eta^* > 0$:

$$\Omega_\varepsilon^\delta = \left\{(t,x) \in \big(4\varepsilon^2, \infty\big)\times\mathbb{R}^3 \;\Big|\; \frac{1}{\varepsilon}\int_{t-4\varepsilon^2}^{t}\int_{B_{2\varepsilon}} F^\delta\big(s, X(s,t,x)+y\big)\,ds\,dy \leqslant \eta^*\varepsilon^\delta\right\},$$

where

$$F^\delta(t,x) = \big|M\big((-\Delta)^{-\delta/2}\nabla^2 P\big)\big|^{1+\gamma} + |\nabla u|^2 + \big|\nabla^2 P\big|,$$

and γ is defined in Lemma 7. We then have the following lemma.

Lemma 8. *There exists a constant C such that for any $0 < \varepsilon < 1$, $0 < \delta < 1$, and $\eta^* > 0$ we have*

$$\big|\big[\Omega_\varepsilon^\delta\big]^c\big| \leqslant C\left(\frac{\|u^0\|^2_{L^2(\mathbb{R}^3)} + \|u^0\|^{2(1+\gamma)}_{L^2(\mathbb{R}^3)}}{\eta^*}\right)\varepsilon^{4-\delta}.$$

Proof. Define for $t > 4\varepsilon^2$,

$$F_\varepsilon^\delta(t,x) = \frac{1}{(2\varepsilon)^5} \int_{t-4\varepsilon^2}^{t} \int_{B_{2\varepsilon}} F^\delta\big(s, X(s,t,x)+y\big)\,ds\,dy. \tag{19}$$

We have

$$\begin{aligned}
\int_{4\varepsilon^2}^{\infty}\int_{\mathbb{R}^3} F_\varepsilon^\delta(t,x)\,dx\,dt &= \int_{4\varepsilon^2}^{\infty}\int_{\mathbb{R}^3} \frac{1}{(2\varepsilon)^5}\int_{-4\varepsilon^2}^{0}\int_{B_{2\varepsilon}} F^\delta\big(t+s, X(t+s,t,x)+y\big)\,ds\,dy\,dx\,dt \\
&= \frac{1}{(2\varepsilon)^5}\int_{B_{2\varepsilon}}\int_{-4\varepsilon^2}^{0}\int_{4\varepsilon^2}^{\infty}\int_{\mathbb{R}^3} F^\delta\big(t+s, X(t+s,t,x)+y\big)\,dx\,dt\,ds\,dy \\
&= \frac{1}{(2\varepsilon)^5}\int_{B_{2\varepsilon}}\int_{-4\varepsilon^2}^{0}\int_{4\varepsilon^2}^{\infty}\int_{\mathbb{R}^3} F^\delta(t+s, z+y)\,dz\,dt\,ds\,dy \\
&\leqslant \left(\frac{1}{(2\varepsilon)^5}\int_{B_{2\varepsilon}}\int_{-4\varepsilon^2}^{0} ds\,dy\right)\int_0^\infty\int_{\mathbb{R}^3} F^\delta(\underline{t},\underline{z})\,d\underline{z}\,d\underline{t} \\
&= \int_0^\infty\int_{\mathbb{R}^3}\left(\left|M\big((-\Delta)^{-\delta/2}\nabla^2 P\big)\right|^{1+\gamma} + |\nabla u|^2 + \left|\nabla^2 P\right|\right)dx\,dt.
\end{aligned}$$

In the second equality, we have used Fubini, in the third we have used the fact that X is an incompressible flow. In the fourth equality we did the change of variable in (t,z)

$$\underline{t} = t+s, \qquad \underline{z} = y+z.$$

We then find, thanks to Tchebychev inequality,

$$\left|\left\{F_\varepsilon^\delta(t,x) \geqslant \frac{\eta^*\varepsilon^\delta}{2(2\varepsilon)^4}\right\}\right| \leqslant 2^5\frac{\int_0^\infty\int_{\mathbb{R}^3} F_\varepsilon^\delta(t,x)\,dx\,dt}{\eta^*}\varepsilon^{4-\delta}.$$

We conclude thanks to Lemma 7. □

We fix $\delta > 0$. For any fixed $(t,x) \in \Omega_\varepsilon^\delta$ with $t \geqslant 4\varepsilon^2$, we define v_ε, P_ε, (depending on this fixed point (t,x)) as functions of two local new variables $(s,y) \in Q_2$:

$$v_\varepsilon(s,y) = \varepsilon u\big(t+\varepsilon^2 s, X\big(t+\varepsilon^2 s,t,x\big)+\varepsilon y\big) - \varepsilon u_\varepsilon\big(t+\varepsilon^2 s, X\big(t+\varepsilon^2 s,t,x\big)\big), \tag{20}$$

$$P_\varepsilon(s,y) = \varepsilon^2 P\big(t+\varepsilon^2 s, X\big(t+\varepsilon^2 s,t,x\big)+\varepsilon y\big) + \varepsilon y\partial_s\big[u_\varepsilon\big(t+\varepsilon^2 s, X\big(t+\varepsilon^2 s,t,x\big)\big)\big]. \tag{21}$$

We have the following proposition.

Proposition 9. *The function* $(v_\varepsilon, P_\varepsilon)$ *is solution to* (1), (3) *for* $(s,y) \in (-4,0)\times\mathbb{R}^3$. *It verifies*:

$$\int_{\mathbb{R}^3}\phi(y)v_\varepsilon(s,y)\,dy = 0, \quad s \geqslant -4, \tag{22}$$

$$\int_{-4}^{0}\int_{B_2}|\nabla v_\varepsilon|^2\,dy\,ds \leqslant \eta^*, \tag{23}$$

$$\int_{-4}^{0}\int_{B_2}\left|\nabla^2 P_\varepsilon\right| dy\,ds \leqslant \eta^*, \tag{24}$$

$$\int_{-4}^{0}\int_{B_2}\left|M\left[(-\Delta)^{-\delta/2}\nabla^2 P_\varepsilon\right]\right|^{1+\gamma} dy\,ds \leqslant \eta^*. \tag{25}$$

Proof. The fact that $(v_\varepsilon, P_\varepsilon)$ is solution to (1), (3) and verifies (22) comes from its definition (20), (21), (16) and (17). We have

$$\begin{aligned}
&\int_{Q_2}\left(|\nabla v_\varepsilon|^2+\left|\nabla^2 P_\varepsilon\right|\right) dy\,ds+\int_{Q_2}\left|M\left[(-\Delta)^{-\delta/2}\nabla^2 P_\varepsilon\right]\right|^{1+\gamma} dy\,ds\\
&\quad=\int_{Q_2}\left(\varepsilon^4\left(|\nabla u|^2+\left|\nabla^2 P\right|\right)+\varepsilon^{(4-\delta)(1+\gamma)}\left|M\left[(-\Delta)^{-\delta/2}\nabla^2 P\right]\right|^{1+\gamma}\right)\left(t+\varepsilon^2 s, X\left(t+\varepsilon^2 s,t,x\right)+\varepsilon y\right) dy\,ds\\
&\quad\leqslant \frac{1}{\varepsilon^{1+\delta}}\int_{t-4\varepsilon^2}^{t}\int_{B_{2\varepsilon}}\left(|\nabla u|^2+\left|\nabla^2 P\right|+M\left[(-\Delta)^{-\delta/2}\nabla^2 P\right]^{1+\gamma}\right)\left(s, X(s,t,x)+y\right) ds\,dy\\
&\quad\leqslant \eta^*.
\end{aligned} \tag{26}$$

In the first equality, we used the definition of v_ε and P_ε, in the second, we used the change of variable $(t+\varepsilon^2 s, \varepsilon y)\to(s,y)$ (together with the fact that $\delta<4$ and $\gamma\geqslant 0$), and the last inequality comes from the fact that (t,x) lies in $\Omega_\varepsilon^\delta$. □

Our aim is to apply proposition 6 to v_ε. It will be a consequence of the following section.

4. Local study

This section is dedicated to the following proposition.

Proposition 10. *For any $\gamma>0$ and any $0<\delta<1$, there exists a constant $\overline{\eta}<1$, and a sequence of constants $\{C_n\}$ such that for any solution (u,P) of* (1), (3) *in Q_2 verifying*

$$\int_{\mathbb{R}^3}\phi(y)u(t,x)\,dx=0,\quad t\geqslant -4, \tag{27}$$

$$\int_{-4}^{0}\int_{B_2}|\nabla u|^2\,dx\,dt\leqslant\overline{\eta}, \tag{28}$$

$$\int_{-4}^{0}\int_{B_2}\left|\nabla^2 P\right| dx\,dt\leqslant\overline{\eta}, \tag{29}$$

$$\int_{-4}^{0}\int_{B_2}\left|M\left[(-\Delta)^{-\delta/2}\nabla^2 P\right]\right|^{1+\gamma} dx\,dt\leqslant\overline{\eta}, \tag{30}$$

the velocity u is infinitely differentiable in x at $(0,0)$ *and*

$$\left|\nabla^n u(0,0)\right|\leqslant C_n.$$

Proof. We want to apply Proposition 6. Then, by a bootstrapping argument we will get uniform controls on higher derivatives. For this, we first need a control of u in $L^\infty(L^2)$ and a control on P in $L^{\gamma+1}(L^1)$. The equation is on ∇P (not the pressure itself). Therefore, changing P by $P - \int_{B_2} \phi P\, dx$ we can assume without loss of generality that

$$\int_{\mathbb{R}^3} \phi(x) P(t,x)\, dx = 0, \quad -4 < t < 0.$$

To get a control in $L^{1+\gamma}(L^1)$ on the pressure it is then enough to control ∇P.

Step 1: Control on u in $L^\infty(L^{3/2})$ in $Q_{3/2}$. Thanks to hypothesis (27), there exists a constant C, depending only on ϕ, such that for any $-4 < t < 0$,

$$\|u(t)\|_{L^6(B_2)} \leqslant C \|\nabla u(t)\|_{L^2(B_2)}. \tag{31}$$

So

$$\|(u\cdot\nabla)u\|_{L^1(-4,0;L^{3/2}(B_2))} \leqslant C\|\nabla u\|^2_{L^2(Q_2)} \leqslant C\bar{\eta}.$$

We need the same control on ∇P. First, multiplying (1) by $\phi(x)$, integrating in x, and using hypothesis (27), we find for any $-4 < t < 0$,

$$\int \phi(x)(u\cdot\nabla)u\, dx + \int \phi(x)\nabla P\, dx - \int \Delta\phi u\, dx = 0. \tag{32}$$

So

$$\left\| \int \phi(x)\nabla P\, dx \right\|_{L^1(-4,0)} \leqslant C\big(\|\nabla u\|^2_{L^2(Q_2)} + \|u\|_{L^2(-4,0;L^6(B_2))}\big) \leqslant C\sqrt{\bar{\eta}}.$$

But, as for u,

$$\left\| \nabla P - \int \phi\nabla P\, dx \right\|_{L^1(-4,0;L^{3/2}(B_2))} \leqslant C\|\nabla^2 P\|_{L^1(Q_2)}.$$

So, finally

$$\big\| |(u\cdot\nabla)u| + |\nabla P| \big\|_{L^1(-4,0;L^{3/2}(B_2))} \leqslant C\sqrt{\bar{\eta}}. \tag{33}$$

Note that

$$\frac{3}{2}\frac{u}{|u|^{1/2}}\partial_t u = \frac{3}{2}\frac{1}{|u|^{1/2}}\partial_t \frac{|u|^2}{2} = \frac{3}{2}|u|^{1/2}\partial_t|u| = \partial_t |u|^{3/2},$$
$$\frac{3}{2}\frac{u}{|u|^{1/2}}\Delta u = \frac{3}{2}\operatorname{div}\left(\frac{u}{|u|^{1/2}}\nabla u\right) - \frac{3}{2}\frac{|\nabla u|^2}{|u|^{1/2}} + \frac{3}{4}\frac{|\nabla|u||^2}{|u|^{1/2}} \leqslant \Delta|u|^{3/2},$$

since $|\nabla u| \geqslant |\nabla|u||$.

We consider $\psi_1 \in C^\infty(\mathbb{R}^4)$ a nonnegative function compactly supported in Q_2 with $\psi_1 = 1$ in $Q_{3/2}$ and

$$|\nabla_{t,x}\psi_1| + |\nabla^2_{t,x}\psi_1| \leqslant C.$$

Multiplying (1) by $(3/2)\psi_1(t,x)u/|u|^{1/2}$ and integrating in x gives

$$\begin{aligned}
&\frac{d}{dt}\int \psi_1(t,x)|u|^{3/2}\, dx \\
&\quad\leqslant \int \big(|\partial_t\psi_1| + |\Delta\psi_1|\big)|u|^{3/2}\, dx + \frac{3}{2}\big\|\psi_1^{1/3}|u|^{1/2}\big\|_{L^3(\mathbb{R}^3)}\big\|\psi_1^{2/3}\big((u\cdot\nabla)u + \nabla P\big)\big\|_{L^{3/2}(B_2)} \\
&\quad\leqslant \int \big(|\partial_t\psi_1| + |\Delta\psi_1|\big)|u|^{3/2}\, dx + \frac{3}{2}\left(\int \psi_1(t,x)|u|^{3/2}\, dx\right)^{1/3}\big\|\big((u\cdot\nabla)u + \nabla P\big)\big\|_{L^{3/2}(B_2)} \\
&\quad\leqslant \alpha(t)\left(1 + \int \psi_1(t,x)|u|^{3/2}\, dx\right),
\end{aligned}$$

with

$$\alpha(t) = \int \big(|\partial_t \psi_1| + |\Delta\psi_1|\big)|u|^{3/2}\,dx + \frac{3}{2}\big\|\big((u\cdot\nabla)u + \nabla P\big)\big\|_{L^{3/2}(B_2)}.$$

Thanks to (31) and (33),

$$\|\alpha\|_{L^1(-4,0)} \leqslant C\sqrt{\overline{\eta}}.$$

Denoting $Y(t) = 1 + \int \psi_1(t,x)|u|^{3/2}\,dx$, we have

$$\dot{Y} \leqslant \alpha Y, \quad Y(-4) = 1.$$

Gronwall's lemma gives that for any $-4 < t < 0$ we have

$$Y(t) \leqslant \exp\left(\int_{-4}^{t} \alpha(s)\,ds\right).$$

Hence, for $\overline{\eta}$ small enough:

$$\|u\|_{L^\infty(-(3/2)^2,0;L^{3/2}(B_{3/2}))} \leqslant C\overline{\eta}^{1/3}. \tag{34}$$

Step 2: Control on u in $L^\infty(L^2)$ in Q_1. We consider $\psi_2 \in C^\infty(\mathbb{R}^4)$ a nonnegative function compactly supported in $Q_{3/2}$ with $\psi_2 = 1$ in Q_1 and

$$|\nabla_{t,x}\psi_2| + \big|\nabla^2_{t,x}\psi_2\big| \leqslant C.$$

Multiplying inequality (3) by ψ_2 and integrating in x gives

$$\frac{d}{dt}\left(\int \psi_2 \frac{|u|^2}{2}\,dx\right) \leqslant \int u\cdot\nabla\psi_2\left(\frac{|u|^2}{2} + P\right)dx + \int (\partial_t\psi_2 + \Delta\psi_2)\frac{|u|^2}{2}\,dx.$$

Equalities (31) together with (33) and Sobolev imbedding gives

$$\big\||u|^2 + P\big\|_{L^1(-(3/2)^2,0;L^3(B_{3/2}))} \leqslant C\overline{\eta}^{1/2}.$$

Together with (34), this gives that

$$\|u\|_{L^\infty(-1,0;L^2(B_1))} \leqslant C\overline{\eta}^{1/4}. \tag{35}$$

Step 3. L^∞ bound in $Q_{1/2}$. We need now to get better integrability in time on the pressure.

From (32) and (35), we get

$$\left\|\int \phi(x)\nabla P\,dx\right\|_{L^2(-1,0)} \leqslant C\sqrt{\overline{\eta}}.$$

With Lemma 3 and (30), this gives for $\gamma < 1$

$$\|\nabla P\|_{L^{1+\gamma}(-1,0;L^1(B_1))} \leqslant C\sqrt{\overline{\eta}}.$$

Together with (35), (28), and Proposition 6, this shows that for $\overline{\eta}$ small enough, we have

$$|u| \leqslant 1 \quad \text{in } Q_{1/2}.$$

Step 4: Obtaining more regularity. We now obtain higher derivative estimates by a standard bootstrapping method. We give the details carefully to ensure that the bounds obtained are universal, that is, do not depend on the actual solution u.

For $n \geqslant 1$ we define $r_n = 2^{-n-3}$, $\overline{B}_n = B_{r_n}$ and $\overline{Q}_n = Q_{r_n}$. We denote also $\overline{\psi}_n$ such that $0 \leqslant \overline{\psi}_n \leqslant 1$, $\overline{\psi}_n \in C^\infty(\mathbb{R}^4)$,

$$\overline{\psi}_n(t,x) = 1, \quad (t,x) \in \overline{Q}_n,$$
$$\qquad = 0, \quad (t,x) \in \overline{Q}_{n-1}^c.$$

For every n we have

$$\partial_t \nabla^n u + \operatorname{div} A_n + \nabla R_n - \Delta \nabla^n u = 0, \tag{36}$$

with

$$A_n = \nabla^n (u \otimes u), \quad R_n = \nabla^n P.$$

So we have

$$\|A_n\|_{L^p(\overline{Q}_{n-1})} \leqslant C_n \|u\|^2_{L^{2p}(-r_{n-1}^2,0;W^{n,2p}(\overline{B}_{n-1}))} \tag{37}$$

and thanks to Lemma 5, we can split R_n as

$$R_n = R_{1,n} + R_{2,n},$$

with

$$\|R_{1,n}\|_{L^p(\overline{Q}_{n-1})} \leqslant C_n \|A_n\|_{L^p(\overline{Q}_{n-2})}, \tag{38}$$

$$\begin{aligned}\|R_{2,n}\|_{L^1(-r_{n-1}^2,0;W^{2,\infty}(\overline{B}_{n-1}))} &\leqslant C_n\big(\|A_n\|_{L^p(\overline{Q}_{n-2})} + \|\nabla P\|_{L^1(\overline{Q}_{n-2})}\big)\\ &\leqslant C_n\big(\|A_n\|_{L^p(\overline{Q}_{n-2})} + 1\big).\end{aligned} \tag{39}$$

Moreover we have:

$$\begin{aligned}\partial_t\big(\overline{\psi}_n \nabla^n u\big) - \Delta\big(\overline{\psi}_n \nabla^n u\big) = {} & -\operatorname{div}(A_n \overline{\psi}_n) + \nabla\overline{\psi}_n A_n - \nabla(\overline{\psi}_n R_n) + (\nabla\overline{\psi}_n) R_n\\ & + \Delta\overline{\psi}_n \nabla^n u - 2\operatorname{div}\big(\nabla\overline{\psi}_n \nabla^n u\big) + (\partial_t \overline{\psi}_n)\nabla^n u.\end{aligned}$$

Note that $\overline{\psi}_n \nabla^n u = 0$ on $\partial\overline{Q}_{n-1}$. So

$$\overline{\psi}_n \nabla^n u = V_{1,n} + V_{2,n} \tag{40}$$

with

$$\begin{aligned}\partial_t V_{1,n} - \Delta V_{1,n} = {} & -\operatorname{div}(A_n\overline{\psi}_n) + \nabla\overline{\psi}_n A_n - \nabla(\overline{\psi}_n R_{1,n}) + (\nabla\overline{\psi}_n) R_{1,n} + \Delta\overline{\psi}_n \nabla^n u\\ & - 2\operatorname{div}\big(\nabla\overline{\psi}_n \nabla^n u\big) + (\partial_t\overline{\psi}_n)\nabla^n u\\ = {} & F_n,\end{aligned}$$

$$V_{1,n} = 0 \quad \text{for } t = -r_{n-1}^2,$$

and

$$\partial_t V_{2,n} - \Delta V_{2,n} = -\nabla(\overline{\psi}_n R_{2,n}) + R_{2,n}(\nabla\overline{\psi}_n),$$
$$V_{2,n} = 0 \quad \text{for } t = -r_{n-1}^2.$$

Thanks to (37) and (38), we have

$$\|F_n\|_{L^p(-r_{n-1}^2,0;W^{-1,p}(\overline{B}_{n-1}))} \leqslant C_n\big(1 + \|u\|^2_{L^{2p}(-r_{n-2}^2,0;W^{n,2p}(\overline{B}_{n-2}))}\big).$$

So, from Lemma 4,

$$\|V_{1,n}\|_{L^p(-r_{n-1}^2,0;W^{1,p}(\overline{B}_{n-1}))} \leqslant C\|F_n\|_{L^p(-r_{n-1}^2,0;W^{-1,p}(\mathbb{R}^3))},$$

$$\begin{aligned}\|V_{2,n}\|_{L^\infty(-r_{n-1}^2,0;W^{1,\infty}(\overline{B}_{n-1}))} &\leqslant C\|\overline{\psi}_n \nabla R_{2,n}\|_{L^1(-r_{n-1}^2;W^{1,\infty}(\mathbb{R}^3))} + C\big\|R_{2,n}(\nabla\overline{\psi}_n)\big\|_{L^1(-r_{n-1}^2 W^{1,\infty}(\mathbb{R}^3))}\\ &\leqslant C_n\big(1 + \|u\|^2_{L^{2p}(-r_{n-2}^2,0;W^{n,2p}(\overline{B}_{n-2}))}\big),\end{aligned}$$

where we have used (37) and (39) in the last line.

Hence, from (40) and using that $\overline{\psi}_n = 1$ on $\overline{Q}_n$, we have for any $1 < p < \infty$,

$$\left\|\nabla^n u\right\|_{L^p(-r_n^2,0;W^{1,p}(\overline{B}_n))} \leqslant C_n\left(1 + \|u\|^2_{L^{2p}(-r_{n-2}^2,0;W^{n,2p}(\overline{B}_{n-2}))}\right).$$

By induction we find that for any $n \geqslant 1$, and any $1 \leqslant p < \infty$, there exists a constant $C_{n,p}$ such that

$$\|u\|_{L^{2^{-n}p}(-r_n^2,0;W^{n,2^{-n}p}(\overline{B}_n))} \leqslant C_{n,p}.$$

This is true for any p, so for n fixed, taking p big enough and using Sobolev imbedding, we show that for any $1 \leqslant q < \infty$, there exists a constant $C_{n,q}$ such that

$$\|u\|_{L^q(-r_{n+1}^2,0;W^{n,\infty}(\overline{B}_{n+1}))} \leqslant C_{n,q}.$$

As (37), we get that

$$\|A_n\|_{L^1(-r_{n+3}^2,0;W^{2,\infty}(\overline{B}_{n+3}))} \leqslant C_n.$$

Thanks to Lemma 5, we get

$$\|R_{1,n}\|_{L^1(-r_{n+4}^2,0;W^{1,\infty}(\overline{B}_{n+4}))} \leqslant C_n,$$
$$\|R_{2,n}\|_{L^1(-r_{n+4}^2,0;W^{1,\infty}(\overline{B}_{n+4}))} \leqslant C_n.$$

Hence

$$\left\|\partial_t \nabla^n u\right\|_{L^1(-r_{n+4}^2,0;L^\infty(\overline{B}_{n+4}))} \leqslant C_n,$$

and finally

$$\left\|\nabla^n u\right\|_{L^\infty(\overline{Q}_{n+4})} \leqslant C_n. \qquad \square$$

5. From local to global

Let us fix $\delta > 0$. We take $\eta^* \leqslant \overline{\eta}$ and consider any $\varepsilon > 0$ such that $4\varepsilon^2 \leqslant t_0$. Then from Proposition 10 and Proposition 9, for any $(t,x) \in \Omega_\varepsilon^\delta \cap \{t \geqslant t_0\}$, we have

$$\left|\nabla_y^n v_\varepsilon(0,0)\right| \leqslant C_n,$$

where v_ε is defined by (20). But for any $n \geqslant 1$, we have

$$\nabla_y^n v_\varepsilon(0,0) = \varepsilon^{n+1} \nabla^n u(t,x).$$

Hence

$$\left|\left\{(t,x) \in \Omega \setminus \left|\nabla^n u(t,x)\right| \geqslant \frac{C_n}{\varepsilon^{n+1}}\right\}\right| \leqslant \left|\left[\Omega_\varepsilon^\delta\right]^c\right|.$$

And thanks to Lemma 8, this measure is smaller than

$$\frac{C}{\eta^*}\left(\left\|u^0\right\|^2_{L^2(\mathbb{R}^3)} + \left\|u^0\right\|^{2(\gamma+1)}_{L^2(\mathbb{R}^3)}\right)\varepsilon^{4-\delta}.$$

We denote

$$R = \left(1 + \frac{4}{t_0}\right)^{\frac{n+1}{2}}.$$

For $k \geqslant 1$, we use our estimate with $\varepsilon^{n+1} = R^{-k}$ to get

$$\left|\left\{(t,x) \in \Omega \setminus \frac{|\nabla^n u(t,x)|}{C_n} \geqslant R^k\right\}\right| \leqslant \frac{C(1 + \|u^0\|^{2(\gamma+1)}_{L^2(\mathbb{R}^3)})}{R^{k\frac{4-\delta}{n+1}}}.$$

So, for $p < \frac{4-\delta}{n+1}$,

$$\begin{aligned}\left\|\frac{\nabla^n u}{C_n}\right\|^p_{L^p(\Omega)} &\leqslant \left|\left\{(t,x)\in\Omega\setminus\frac{|\nabla^n u(t,x)|}{C_n}\leqslant R\right\}\right|R^p+\sum_{k=1}^{\infty}R^{(k+1)p}\left|\left\{(t,x)\in\Omega\setminus\frac{|\nabla^n u(t,x)|}{C_n}\geqslant R^k\right\}\right|\\ &\leqslant|\Omega|R^p+CR^p\left(1+\left\|u^0\right\|^{2(\gamma+1)}_{L^2(\mathbb{R}^3)}\right)\sum_{k=1}^{\infty}R^{k(p-\frac{4-\delta}{n+1})}\\ &\leqslant|\Omega|R^p+\frac{CR^p}{1-R^{p-\frac{4-\delta}{n+1}}}\left(1+\left\|u^0\right\|^{2(\gamma+1)}_{L^2(\mathbb{R}^3)}\right).\end{aligned}$$

The results holds for any $\delta > 0$ which ends the proof of Theorem 1.

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