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Multi-peak bound states for Schrödinger equations with compactly supported or unbounded potentials

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Abstract

In this paper, we will study the existence and qualitative property of standing waves $\psi(x,t)=e^{-\frac{iEt}{\varepsilon}}u(x)$ for the nonlinear Schrödinger equation $i\varepsilon\frac{\partial\psi}{\partial t}+\frac{\varepsilon^2}{2m}\Delta_x\psi-(V(x)+E)\psi+K(x)|\psi|^{p-1}\psi=0$, $(t,x)\in\mathbb{R}_+\times\mathbb{R}^N$. Let $G(x)=[V(x)]^{\frac{p+1}{p-1}-\frac{N}{2}}\times[K(x)]^{-\frac{2}{p-1}}$ and suppose that $G(x)$ has k local minimum points. Then, for any $l\in\{1,\dots,k\}$, we prove the existence of the standing waves in $H^1(\mathbb{R}^N)$ having exactly l local maximum points which concentrate near l local minimum points of $G(x)$ respectively as $\varepsilon\to 0$. The potentials $V(x)$ and $K(x)$ are allowed to be either compactly supported or unbounded at infinity. Therefore, we give a positive answer to a problem proposed by Ambrosetti and Malchiodi (2007) [2].

MSC: 35J20; 35J60

Keywords: Bound state; Multi-peak solutions; Nonlinear Schrödinger equation; Potentials compactly supported or unbounded at infinity

1. Introduction

In this paper, we will consider the existence of concentrating solutions for the following nonlinear Schrödinger equations

$$\begin{cases} -\varepsilon^2\Delta u+V(x)u=K(x)u^p, \quad u>0,\ x\in\mathbb{R}^N, \\ u\in H^1\big(\mathbb{R}^N\big), \end{cases} \tag{1.1}$$

where $\varepsilon>0$ is a small number, $1<p<(N+2)/(N-2)$ for $N\geqslant 3$, and $1<p<\infty$ for $N=1,2$, $V(x)\geqslant 0$ has a compact support, and $K(x)\geqslant 0$ may tend to zero or infinity as $|x|\to\infty$, $H^1(\mathbb{R}^N)$ is the usual Sobolev space $\{u\in D^{1,2}(\mathbb{R}^N)\colon \int_{\mathbb{R}^N}|u|^2<\infty\}$.

A basic motivation for the study of (1.1) comes from looking for standing waves of the type

$$\psi(x,t)=e^{-\frac{iEt}{\varepsilon}}u(x)$$

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for the following nonlinear Schrödinger equations

$$i\varepsilon\frac{\partial\psi}{\partial t}=-\frac{\varepsilon^2}{2m}\Delta_x\psi+\big(V(x)+E\big)\psi-K(x)|\psi|^{p-2}\psi,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^N, \tag{1.2}$$

where ε is the Planck constant, i is the imaginary unit. Problem (1.2) arises in many applications. For example, in some problems arising in nonlinear optics, in plasma physics and in condensed matter physics, the presence of many particles leads one to consider nonlinear terms which simulate the interaction effect among them. The function $V(x)$ represents the potential acting on the particle and $K(x)$ represents a particle-interaction term, which avoids spreading of the wave packets in the time-dependent version of the above equation. A solution ψ is referred to as a bound state of (1.2) if $\psi\to 0$ as $|x|\to+\infty$. Furthermore, to describe the transition from quantum to classical mechanics, we let $\varepsilon\to 0$ and thus the existence of solutions $\psi_\varepsilon(x,t)$ of (1.2) for small ε (which is called semiclassic state) has an important physical interest.

There are a lot of works on problem (1.1). In [23], Floer and Weinstein considered the case $N=1$, $p=3$ and $K(x)\equiv 1$. By using a Lyapunov–Schmidt reduction argument, they constructed a positive solution u_ε to problem (1.1) which concentrates around the critical point of potential $V(x)$ as $\varepsilon\to 0$. Their method and results were later generalized by Oh [31,32] to the higher-dimensional case with $1<p<(N+2)/(N-2)$ and multi-bump solutions concentrating near several non-degenerate critical points of $V(x)$ as $\varepsilon\to 0$ were obtained. Existence of solutions concentrating at one or several points to problem (1.1) under different conditions has also been obtained in [4,6,12, 18–21,26,27,30,33,35–37].

We also mention some results on problem (1.1) in the case that $\liminf_{|x|\to\infty}V(x)>0$, $Z=:\{x\in\mathbb{R}^N\colon V(x)=0\}\neq\emptyset$, which was referred to as the critical frequency in [8]. In [8,9], it was shown that problem (1.1) admits a ground state concentrating on Z. Later, if Z consists of several components, Cao and Peng in [15], Cao and Noussair in [13] and Cao, Noussair and Yan in [14] proved that problem (1.1) has multi-peak solutions which concentrate simultaneously on some prescribed components of Z or some points on which $V(x)$ does not vanish. For more results, we can refer to [10,11,34] and the references therein.

In all the works mentioned above, it is worth pointing out that the common assumptions are $\liminf_{|x|\to\infty}V(x)>0$ and $0<K(x)<c$ in $\mathbb{R}^N$ for some $c>0$. The assumption $\liminf_{|x|\to\infty}V(x)>0$ guarantees that the norm $(\int_{\mathbb{R}^N}|\nabla u|^2+V(x)u^2)^{1/2}$ is not weaker than the usual norm of Sobolev space $H^1(\mathbb{R}^N)$, and thus ensures that the solutions of (1.1) are bound states which are the most relevant from the physical point of view. Hence, by the boundedness of $K(x)$, the energy functionals corresponding to the equations are well defined in $H^1(\mathbb{R}^N)$ and the variational theory can be used directly. However, if the potential $V(x)$ decays to zero or is compactly supported, or $K(x)$ approaches infinity at infinity, variational theory in $H^1(\mathbb{R}^N)$ cannot be used, nor can one apply the perturbation methods, as in [4], since the spectrum of the linear operator $-\Delta+V(x)$ is $[0,+\infty)$, see [7]. Therefore, the methods used in the previous papers cannot be employed any more in the present situation.

Recently, the case in which $V(x)$ and $K(x)$ may decay to zero as $|x|\to\infty$ has been investigated. In [1], it is proved that if the potentials $V(x)$ and $K(x)$ are smooth and satisfy

(V_0) $\quad\exists\gamma_0,\gamma_1>0\colon\quad \dfrac{\gamma_0}{1+|x|^\alpha}\leqslant V(x)\leqslant\gamma_1,$

(K_0) $\quad\exists\beta,k\colon\quad 0<K(x)\leqslant\dfrac{k}{1+|x|^\beta},$

then (1.1) has a ground state solution for $0\leqslant\alpha<2$ provided that ε is sufficiently small and $\sigma<p<(N+2)/(N-2)$, where

$$\sigma=\begin{cases}\frac{N+2}{N-2}-\frac{4\beta}{\alpha(N-2)}, & \text{if } 0<\beta<\alpha,\\ 1 & \text{otherwise.}\end{cases}$$

In [3], this result was generalized to the case $1<p<(N+2)/(N-2)$ and $0\leqslant\alpha\leqslant 2$. Moreover, setting

$$G(x)=\big[V(x)\big]^{\frac{p+1}{p-1}-\frac{N}{2}}\big[K(x)\big]^{-\frac{2}{p-1}},$$

which is referred to as a ground energy function introduced in [36], then it is verified that for any isolated stable stationary point x_0 of $G(x)$, Eq. (1.1) has a bound state concentrating at x_0 for ε sufficiently small. When $Z=\{x\in\mathbb{R}^N\colon$

$V(x)=0\} \neq \emptyset$ and $V(x)$, $K(x)$ satisfy (V_0) and (K_0) at infinity respectively, Ambrosetti and Wang in [5] obtained a ground state of (1.1) which concentrates at some point $x^* \in Z$ in the case $0 \leqslant \alpha < 2$ and $\sigma < p < \frac{N+2}{N-2}$. Similar results can also be found in [10,11].

An open problem left, as proposed by Ambrosetti and Malchiodi in [2], is what happens if the potential $V(x)$ decays faster than $|x|^{-2}$ at infinity or has compact support? More precisely, it was illustrated in [2] the following: "Is it possible to handle potentials with faster decay, or compactly supported? ... . Clearly, the approach used so far cannot be repeated. However, any result, positive or negative, would be interesting."

Concerning this open problem, some interesting results appeared recently. In [16], the case $V(x) \sim |x|^\alpha$ at infinity with $\alpha \geqslant -4$ was considered by a constructive argument and multi-peak bound states with prescribed number of maximum points approaching to a local minimum point of the function $G(x)$ as $\varepsilon \to 0$ were obtained. The case that $V(x)$ has compact support, which is the most difficult case, was studied by Fei and Yin in [22], where it was shown that problem (1.1) admits a bound state with exact one local maximum point x_ε which tends to a minimum point of the function $G(x)$ as $\varepsilon \to 0$. The method introduced in [22] is quite new. Precisely, the authors proved a mountain pass solution to a modified problem and then obtained a type of the weak Harnack inequality, by which they proved that the solution decays at infinity at the desired speed and hence is a bound state solving the original problem. However, the solution obtained in [22] is in some sense a least energy solution and indeed one-peaked. Moreover, because of the absence of an exact estimate to the energy related to the solution, their argument cannot be adopted to look for multi-peak solutions with higher energy. In this paper, we intend to focus on this problem.

We assume that $V(x)$, $K(x)$ satisfy the following conditions:

(H_1). $V(x) \in C(\mathbb{R}^N)$ is compactly supported, $V(x) \geqslant 0$ and $V(x) \not\equiv 0$; $K(x) \in C(\mathbb{R}^N)$, $K(x) \geqslant 0$.

(H_2). There exist smooth bounded domains Λ_i of $\mathbb{R}^N$, mutually disjoint such that $V(x) > 0$ and $K(x) > 0$ on $\bar{\Lambda}_i$, $i = 1, \dots, k$, and

$$0 < c_i \equiv \inf_{x \in \Lambda_i} G(x) < \inf_{x \in \partial\Lambda_i} G(x).$$

(H_3). There exist constants $k_1 > 0$, $\beta_1 < 2$ such that

$$0 \leqslant K(x) \leqslant k_1 \big(1 + |x|\big)^{\beta_1} \quad \text{in } \mathbb{R}^N.$$

Our main result for Eq. (1.1) is as follows:

Theorem 1.1. *Assume $N \geqslant 5$ and $\max(1, (N+\beta_1)/(N-2)) < p < (N+2)(N-2)$. Under the assumptions (H_1)–(H_3), there is an $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, problem (1.1) admits a positive solution $u_\varepsilon \in H^1(\mathbb{R}^N)$. Moreover, u_ε possesses exactly k local maxima $x_\varepsilon^i \in \Lambda_i$, $i = 1, \dots, k$, which satisfy that $G(x_\varepsilon^i) \to \inf_{x \in \Lambda_i} G(x)$ as $\varepsilon \to 0$.*

Remark 1.1. We actually prove the existence of solutions concentrating simultaneously at several points of a set $\{\xi_1, \dots, \xi_k\}$, where for each $i = 1, \dots, k$, ξ_i satisfies $G(\xi_i) = c_i$.

Remark 1.2. From the proof of Theorem 1.1, we can see that Theorem 1.1 holds if $V(x)$ is a nonnegative continuous function in $\mathbb{R}^N$. In particular, Theorem 1.1 remains true if $V(x)$ decays faster than $|x|^{-\alpha}$ including the case that $V(x)$ decays faster than $e^{-\beta|x|}$, where $\alpha > 0$, $\beta > 0$. Furthermore, if $V(x) = 1/|x|^2$ near infinity, which is related to the well-known Hardy inequality, our result is also true.

In order to obtain the existence of peaked solutions we use variational methods and penalization techniques. Since $\int_{\mathbb{R}^N} V(x)u^2 < \infty$ cannot guarantee that u is a bound state and $\int_{\mathbb{R}^N} K(x)|u|^{p+1}$ may not make sense for $u \in H^1(\mathbb{R}^N)$ due to the fact that $V(x)$ may vanish or $K(x)$ may be unbounded at infinity, we will employ a penalization argument which needs to modify the nonlinear term $K(x)u^p$ in (1.1). To obtain multi-peak bound states with the desired number of peaks, we must give an exact estimate to the functional energy corresponding to the solution u_ε of the modified problem. For this, we penalize the functional related to the modified problem by adding a penalization $P_\varepsilon(u)$ introduced in [19], which can also be used to prevent the concentration outside of $\bigcup_{i=1}^k \Lambda_i$. To ensure that u_ε solves the original problem (1.1), we use an argument similar to [22] to obtain the weak Harnack inequality and thus find the decay rate of u_ε outside Λ_i. As we can see later, for the case $k = 1$, our method can simplify the proof in [22]. Moreover, our result is true for the case that $V(x)$ decays faster than $|x|^\alpha$ ($\forall \alpha < 0$) at infinity.

This paper is organized as follows: In Section 2 we define the modified functional needed for the proof of Theorem 1.1, and prove some preliminary results. Section 3 is devoted to the proof of Theorem 1.1. In this paper, the notations $C, C_1, \ldots$ denote the generic positive constants depending only on $V(x)$, $K(x)$, p.

2. Preliminaries

Let $\Lambda = \bigcup_{i=1}^{k} \Lambda_i$. For $\xi \in \Lambda$, consider the following equation

$$\begin{cases} -\Delta u(x) + V(\xi)u(x) = K(\xi)u^p(x), \quad u(x) > 0,\ x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{2.1}$$

The functional corresponding to (2.1) is defined as

$$I_\xi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{V(\xi)}{2}\int_{\mathbb{R}^N} |u|^2 - \frac{K(\xi)}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}. \tag{2.2}$$

The following function

$$G(\xi) = \inf_{u\in\mathcal{M}^\xi} I_\xi(u) \tag{2.3}$$

is referred to as ground energy function of (2.1) (see [36]), where $\mathcal{M}^\xi$ is the Nehari manifold defined by

$$\mathcal{M}^\xi = \{u \in H^1(\mathbb{R}^N) \setminus \{0\}\colon \langle I'_\xi(u), u\rangle = 0\}.$$

For further properties of $G(\xi)$, one can see [36]. It is shown in [24,28] that, up to translations, (2.1) has a unique solution $U_\xi(x)$, which is radially symmetric and decays exponentially at infinity. Moreover,

$$G(\xi) = I_\xi(U_\xi). \tag{2.4}$$

Let E_ε be a class of weighted Sobolev space defined as follows

$$\left\{u \in D^{1,2}(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} \left(\varepsilon^2|\nabla u|^2 + V(x)|u|^2\right) < \infty\right\},$$

where $D^{1,2}(\mathbb{R}^N) = \{u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) \mid \nabla u \in L^2(\mathbb{R}^N)\}$. The norm of the space E_ε is denoted by

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} \left(\varepsilon^2|\nabla u|^2 + V(x)|u|^2\right)\right)^{1/2}.$$

Under the assumptions (H_1) and (H_2), we can deduce from the Sobolev inequality the following:

Lemma 2.1. *Assume that (H_1), (H_2) hold true, then for each $\varepsilon \in (0, 1]$, there exists a constant $C_1 > 0$ independent of ε such that*

$$\int_\Lambda K(x)|u|^{p+1} \leqslant C_1\varepsilon^{-\frac{N(p-1)}{2}}\|u\|_\varepsilon^{p+1}, \quad \forall u \in E_\varepsilon. \tag{2.5}$$

Next we define the first modification of our functional. Set

$$f_\varepsilon(x,t) = \min\left\{K(x)\left(t^+\right)^p,\ \frac{\varepsilon^3}{1+|x|^{\theta_0}}t^+,\ \frac{\varepsilon}{1+|x|^N}\right\},$$

where $t^+ = \max\{t, 0\}$, and $\theta_0 > 2$ will be suitably chosen later on.

Define

$$g_\varepsilon(x,\xi) = \chi_\Lambda(x)K(x)\left(\xi^+\right)^p + \left(1-\chi_\Lambda(x)\right)f_\varepsilon(x,\xi),$$

where $\chi_\Lambda(x)$ represents the characteristic function of the set Λ. Denote $G_\varepsilon(x,u) = \int_0^u g_\varepsilon(x,\tau)\,d\tau$.

Now we define the modified functional $L_\varepsilon : E_\varepsilon \to \mathbb{R}$ as

$$L_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(\varepsilon^2|\nabla u|^2 + V(x)u^2\right) - \frac{1}{p+1}\int_{\Lambda} K(x)\left(u^+\right)^{p+1} - \int_{\mathbb{R}^N\setminus\Lambda} F_\varepsilon(x,u),$$

where $F_\varepsilon(x,u) = (1-\chi_\Lambda(x))\int_0^u f_\varepsilon(x,\tau)\,d\tau$.

For $u\in E_\varepsilon$, due to $\theta_0 > 2$, we see that

$$\int_{\mathbb{R}^N\setminus\Lambda} F_\varepsilon(x,u) \leqslant \int_{\mathbb{R}^N\setminus\Lambda} \frac{\varepsilon^3}{1+|x|^{\theta_0}}u^2 \leqslant C\varepsilon^3\Bigg(\int_{\mathbb{R}^N\setminus\Lambda} |u|^{\frac{2N}{N-2}}\Bigg)^{\frac{N-2}{N}}$$
$$\leqslant C\varepsilon^3\int_{\mathbb{R}^N} |\nabla u|^2 \leqslant C\varepsilon\|u\|_\varepsilon^2. \tag{2.6}$$

It follows from (2.5) and (2.6) that $L_\varepsilon(u)$ is well defined in E_ε.

Next, we introduce the second modification of the functional. Assume $G(\xi_i) = c_i$, $\xi_i \in \Lambda_i$. Let $\sigma_i > 0$ be such that

$$\sup_{\Lambda_i} G(x) \leqslant c_i + \sigma_i, \tag{2.7}$$

and assume that

$$\sum_{i=1}^k \sigma_i < \frac{1}{2}\min\{c_i \mid i = 1,\dots,k\}. \tag{2.8}$$

This can be achieved by making Λ_i smaller if necessary. For mutually disjoint open sets $\tilde{\Lambda}_i$ compactly containing Λ_i and satisfying $V(x) > 0$ on the closure of $\tilde{\Lambda}_i$, we define on E_ε the functional

$$L_\varepsilon^i(u) = \frac{1}{2}\int_{\tilde{\Lambda}_i} \left(\varepsilon^2|\nabla u|^2 + V(x)u^2\right) - \frac{1}{p+1}\int_{\Lambda_i} K(x)\left(u^+\right)^{p+1} - \int_{\tilde{\Lambda}_i\setminus\Lambda_i} F_\varepsilon(x,u), \tag{2.9}$$

and the penalization

$$P_\varepsilon(u) = M\sum_{i=1}^k \left\{\left(L_\varepsilon^i(u)_+\right)^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}}(c_i+\sigma_i)^{\frac{1}{2}}\right\}_+^2. \tag{2.10}$$

The constant M will be chosen later. Now set

$$J_\varepsilon(u) = L_\varepsilon(u) + P_\varepsilon(u).$$

Then it follows from Proposition A.1 in Appendix A that functional J_ε is of class C^1. We show next that J_ε satisfies the Palais–Smale condition.

Lemma 2.2. *Let $\{u_n\}$ be a sequence in E_ε such that $J_\varepsilon(u_n)$ is bounded and $J_\varepsilon'(u_n) \to 0$. Then $\{u_n\}$ has a convergent subsequence.*

Proof. We first prove that the sequence $\{u_n\}$ is bounded in E_ε.

Similarly to (2.6), we have

$$\left|\int_{\mathbb{R}^N\setminus\Lambda} f_\varepsilon(x,u)u\right| \leqslant C\varepsilon\|u\|_\varepsilon^2, \quad \forall u \in E_\varepsilon. \tag{2.11}$$

For a fixed $q \in (2, p+1)$, by (2.6) and (2.11), we have

$$\begin{aligned}
L_\varepsilon(u_n)-\frac{1}{q}\big\langle L_\varepsilon'(u_n),u_n\big\rangle &= \left(\frac{1}{2}-\frac{1}{q}\right)\int\limits_{\mathbb{R}^N}\big(\varepsilon^2|\nabla u_n|^2+V(x)u_n^2\big)+\left(\frac{1}{q}-\frac{1}{p+1}\right)\int\limits_{\Lambda}K(x)\big(u_n^+\big)^{p+1}\\
&\quad+\frac{1}{q}\int\limits_{\mathbb{R}^N\setminus\Lambda}f_\varepsilon(x,u_n)u_n-\int\limits_{\mathbb{R}^N\setminus\Lambda}F_\varepsilon(x,u_n)\\
&\geqslant C\int\limits_{\mathbb{R}^N}\big(\varepsilon^2|\nabla u_n|^2+V(x)u_n^2\big).
\end{aligned}\tag{2.12}$$

Similarly, we find

$$L_\varepsilon^i(u_n)-\frac{1}{q}\big\langle L_\varepsilon^{i\,\prime}(u_n),u_n\big\rangle\geqslant C\int\limits_{\tilde\Lambda_i}\big(\varepsilon^2|\nabla u_n|^2+V(x)u_n^2\big).\tag{2.13}$$

On the other hand, noting that

$$\big\langle P_\varepsilon'(u_n),u_n\big\rangle=M\sum_{i=1}^k\big\{\big(L_\varepsilon^i(u_n)_+\big)^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\big\}_+\big(L_\varepsilon^i(u_n)_+\big)^{-\frac12}\big\langle L_\varepsilon^{i\,\prime}(u_n),u_n\big\rangle,$$

we derive from (2.13) that

$$\begin{aligned}
P_\varepsilon(u_n)-\frac{1}{q}\big\langle P_\varepsilon'(u_n),u_n\big\rangle &= M\sum_{i=1}^k\big\{\big(L_\varepsilon^i(u_n)_+\big)^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\big\}_+^2\\
&\quad-\frac1q M\sum_{i=1}^k\big\{\big(L_\varepsilon^i(u_n)_+\big)^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\big\}_+\big(L_\varepsilon^i(u_n)_+\big)^{-\frac12}\big\langle L_\varepsilon^{i\,\prime}(u_n),u_n\big\rangle\\
&= M\sum_{i=1}^k\big\{\big(L_\varepsilon^i(u_n)_+\big)^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\big\}_+\\
&\quad\times\left[\big\{\big(L_\varepsilon^i(u_n)_+\big)^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\big\}_+-\frac1q\big(L_\varepsilon^i(u_n)_+\big)^{-\frac12}\big\langle L_\varepsilon^{i\,\prime}(u_n),u_n\big\rangle\right]\\
&\geqslant M\sum_{i=1}^k\big\{\big(L_\varepsilon^i(u_n)_+\big)^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\big\}_+\\
&\quad\times\big[\big\{\big(L_\varepsilon^i(u_n)_+\big)^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\big\}_+-\big(L_\varepsilon^i(u_n)_+\big)^{\frac12}\big]\\
&\geqslant -M\sum_{i=1}^k\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\big\{\big(L_\varepsilon^i(u_n)_+\big)^{\frac12}-\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\big\}_+\\
&\geqslant -M^{\frac12}\sum_{i=1}^k\varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}P_\varepsilon^{\frac12}(u_n)=-C\varepsilon^{\frac N2}P_\varepsilon^{\frac12}(u_n).
\end{aligned}$$

Hence, by the fact that $P_\varepsilon(u_n)\leqslant M\int_{\mathbb{R}^N}(\varepsilon^2|\nabla u_n|^2+V(x)u_n^2)$, we get

$$\begin{aligned}
C_\varepsilon &\geqslant J_\varepsilon(u_n)-\frac1q\big\langle J_\varepsilon'(u_n),u_n\big\rangle\\
&\geqslant C\int\limits_{\mathbb{R}^N}\big(\varepsilon^2|\nabla u_n|^2+V(x)u_n^2\big)-C\varepsilon^{\frac N2}\left(\int\limits_{\mathbb{R}^N}\big(\varepsilon^2|\nabla u_n|^2+V(x)u_n^2\big)\right)^{\frac12},
\end{aligned}\tag{2.14}$$

which implies that $\{u_n\}$ is bounded in E_ε.

Since $E_\varepsilon \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow H^1_{\mathrm{loc}}(\mathbb{R}^N)$, the boundedness of $\{u_n\}$ in E_ε implies that there exists $u_0 \in E_\varepsilon$ satisfying, after passing to a subsequence if necessary,

$$u_n \to u_0 \quad \text{weakly in } E_\varepsilon, \qquad u_n \to u_0 \quad \text{strongly in } L^q_{\mathrm{loc}}(\mathbb{R}^N),$$

for $2 \leqslant q < 2N/(N-2)$.

Now, to complete the proof, it suffices to prove that $\|u_n\|_\varepsilon \to \|u_0\|_\varepsilon$ as $n \to \infty$.

By $\langle J'_\varepsilon(u_n), u_0\rangle \to 0$, we see

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\varepsilon^2|\nabla u_0|^2 + V(x)u_n u_0\right) - \int_{\Lambda} K(x)\left(u_n^+\right)^p u_0 - \int_{\mathbb{R}^N\setminus\Lambda} f_\varepsilon(x,u_n)u_0 \\
&\quad + M\sum_{i=1}^k \left\{L^i_\varepsilon(u_n)_+^{\frac12} - \varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\right\}_+ L^i_\varepsilon(u_n)_+^{-\frac12} \\
&\quad \times \left(\int_{\tilde\Lambda_i} \left(\varepsilon^2|\nabla u_0|^2 + V(x)u_n u_0\right) - \int_{\Lambda_i} K(x)\left(u_n^+\right)^p u_0 - \int_{\tilde\Lambda_i\setminus\Lambda_i} f_\varepsilon(x,u_n)u_0\right) \\
&= o_n(1).
\end{aligned} \tag{2.15}$$

In addition, from $\langle J'_\varepsilon(u_n), u_n\rangle = o_n(1)\|u_n\|$ and the boundedness of $\{u_n\}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\varepsilon^2|\nabla u_n|^2 + V(x)u_n^2\right) - \int_{\Lambda} K(x)\left(u_n^+\right)^{p+1} - \int_{\mathbb{R}^N\setminus\Lambda} f_\varepsilon(x,u_n)u_n \\
&\quad + M\sum_{i=1}^k \left\{L^i_\varepsilon(u_n)_+^{\frac12} - \varepsilon^{\frac N2}(c_i+\sigma_i)^{\frac12}\right\}_+ L^i_\varepsilon(u_n)_+^{-\frac12} \\
&\quad \times \left(\int_{\tilde\Lambda_i} \left(\varepsilon^2|\nabla u_n|^2 + V(x)u_n^2\right) - \int_{\Lambda_i} K(x)\left(u_n^+\right)^{p+1} - \int_{\tilde\Lambda_i\setminus\Lambda_i} f_\varepsilon(x,u_n)u_n\right) \\
&= o_n(1).
\end{aligned} \tag{2.16}$$

On the other hand, we find

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)u_n^2 = \lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)u_n u_0, \tag{2.17}$$

$$\lim_{n\to\infty}\int_{\Lambda} K(x)\left(u_n^+\right)^{p+1} = \lim_{n\to\infty}\int_{\Lambda} K(x)\left(u_n^+\right)^p u_0, \tag{2.18}$$

$$\lim_{n\to\infty}\int_{\tilde\Lambda_i} V(x)u_n^2 = \lim_{n\to\infty}\int_{\tilde\Lambda_i} V(x)u_n u_0, \tag{2.19}$$

$$\lim_{n\to\infty}\int_{\Lambda_i} K(x)\left(u_n^+\right)^{p+1} = \lim_{n\to\infty}\int_{\Lambda_i} K(x)\left(u_n^+\right)^p u_0, \tag{2.20}$$

$$\lim_{n\to\infty}\int_{\tilde\Lambda_i\setminus\Lambda_i} f_\varepsilon(x,u_n)u_n = \lim_{n\to\infty}\int_{\tilde\Lambda_i\setminus\Lambda_i} f_\varepsilon(x,u_n)u_0, \tag{2.21}$$

and for any fixed large $R>0$ such that $\Lambda \subset B_R(0)$,

$$\lim_{n\to\infty}\int_{B_R(0)\setminus\Lambda} f_\varepsilon(x,u_n)u_n = \lim_{n\to\infty}\int_{B_R(0)\setminus\Lambda} f_\varepsilon(x,u_n)u_0.$$

We will prove that for any given $\delta > 0$, there exists $R > 0$ such that for all n

$$\left| \int\limits_{\mathbb{R}^N \setminus B_R(0)} f_\varepsilon(x, u_n) u_0 \right| < \delta, \qquad \left| \int\limits_{\mathbb{R}^N \setminus B_R(0)} f_\varepsilon(x, u_n) u_n \right| < \delta.$$

We just check the first inequality since the second one can be checked in a similar way. As in the proof of (2.6), we have

$$\begin{aligned} \left| \int\limits_{\mathbb{R}^N \setminus B_R(0)} f_\varepsilon(x, u_n) u_0 \right| &\leqslant \varepsilon^3 \int\limits_{\mathbb{R}^N \setminus B_R(0)} \frac{|u_n||u_0|}{1 + |x|^{\theta_0}} \\ &\leqslant \varepsilon^3 \Bigg(\int\limits_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{(1 + |x|^{\theta_0})^{\frac{N}{2}}} \Bigg)^{\frac{2}{N}} \|u_n\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \|u_0\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \\ &\leqslant \frac{\varepsilon}{R^{\theta_0 - 2}} \|u_n\|_\varepsilon \|u_0\|_\varepsilon \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

So

$$\lim_{n \to \infty} \int\limits_{\mathbb{R}^N \setminus \Lambda} f_\varepsilon(x, u_n) u_n = \lim_{n \to \infty} \int\limits_{\mathbb{R}^N \setminus \Lambda} f_\varepsilon(x, u_n) u_0. \tag{2.22}$$

From the boundedness of $\{u_n\}$, we have

$$M\big(L_\varepsilon^i(u_n)_+^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}}(c_i + \sigma_i)^{\frac{1}{2}}\big)_+ L_\varepsilon^i(u_n)_+^{-\frac{1}{2}} = a_i + o_n(1), \tag{2.23}$$

where $a_i \geqslant 0$, $i = 1, \ldots, k$ are constants. So inserting (2.17)–(2.23) into (2.15) and (2.16), we obtain

$$o_n(1) = \int\limits_{\mathbb{R}^N} \varepsilon^2\big(|\nabla u_n|^2 - |\nabla u_0|^2\big) + \sum_{i=1}^k \big(a_i + o_n(1)\big) \int\limits_{\tilde{\Lambda}_i} \varepsilon^2\big(|\nabla u_n|^2 - |\nabla u_0|^2\big) \geqslant o_n(1).$$

Thus, we have $\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 = \int_{\mathbb{R}^N} |\nabla u_0|^2$ and hence

$$\lim_{n \to \infty} \int\limits_{\mathbb{R}^N} \big(\varepsilon^2 |\nabla u_n|^2 + V(x) u_n^2\big) = \int\limits_{\mathbb{R}^N} \big(\varepsilon^2 |\nabla u_0|^2 + V(x) u_0^2\big).$$

So, the proof of Lemma 2.2 is completed. □

The previous lemma makes it possible to use the critical point theory to find critical points of the functional J_ε. We will formulate an appropriate minimax problem for J_ε.

As in [17], we define a class of functions Γ. A continuous function $\gamma : [0, 1]^k \to E_\varepsilon$ is in Γ if there are continuous functions $g_i : [0, 1] \to E_\varepsilon$ for $i = 1, \ldots, k$ satisfying

(i) $\operatorname{supp}\{g_i(\tau)\} \subset \Lambda_i$ for all $\tau \in [0, 1]$;
(ii) $\gamma(\tau_1, \ldots, \tau_k) = \sum_{i=1}^k g_i(\tau_i)$ for all $(\tau_1, \ldots, \tau_k) \in \partial[0, 1]^k$;
(iii) $g_i(0) = 0$ and $L_\varepsilon(g_i(1)) < 0$;
(iv) $J_\varepsilon(\gamma(t)) \leqslant \varepsilon^N(\sum_{i=1}^k c_i - \sigma)$ for all $t \in \partial[0, 1]^k$,

where $0 < \sigma < \frac{1}{2} \min\{c_i \mid i = 1, \ldots, k\}$ is a fixed number.

Define

$$C_\varepsilon = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]^k} J_\varepsilon\big(\gamma(t)\big).$$

As in [19], we can prove that Γ is non-empty and

Lemma 2.3.

$$C_\varepsilon = \varepsilon^N\left(\sum_{i=1}^k c_i + o(1)\right).$$

Proof. The proof is similar to Lemma 2.3 in [15] (see also [19, Lemma 1.2]), and thus we omit it. □

From Lemma 2.2 and Lemma 2.3, we conclude that there exists a critical point $u_\varepsilon \in E_\varepsilon$ of J_ε such that $J_\varepsilon(u_\varepsilon) = C_\varepsilon$. We define the local weights

$$\rho_\varepsilon^i = M\left\{\left(L_\varepsilon^i(u_\varepsilon)_+\right)^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}}(c_i+\sigma_i)^{\frac{1}{2}}\right\}_+\left(L_\varepsilon^i(u_\varepsilon)_+\right)^{-\frac{1}{2}},$$

and then the function

$$\rho_\varepsilon = \sum_{i=1}^k \rho_\varepsilon^i \chi_{\tilde{\Lambda}_i}.$$

The critical point u_ε satisfies in the weak sense

$$\varepsilon^2 \operatorname{div}\big((1+\rho_\varepsilon)\nabla u_\varepsilon\big) - (1+\rho_\varepsilon)V(x)u_\varepsilon + (1+\rho_\varepsilon)g_\varepsilon(x,u_\varepsilon) = 0, \tag{2.24}$$

and

$$\int_{\mathbb{R}^N}(1+\rho_\varepsilon)\big(\varepsilon^2\nabla u_\varepsilon\nabla\varphi + V(x)u_\varepsilon\varphi\big) = \int_{\mathbb{R}^N}(1+\rho_\varepsilon)g_\varepsilon(x,u_\varepsilon)\varphi, \quad \forall\varphi\in E_\varepsilon. \tag{2.25}$$

Set $\Lambda_i^\varepsilon = \{y\in\mathbb{R}^N \mid \varepsilon y\in\Lambda_i\}$ and $\tilde{\Lambda}_i^\varepsilon = \{y\in\mathbb{R}^N \mid \varepsilon y\in\tilde{\Lambda}_i\}$. Let $v_\varepsilon(y) = u_\varepsilon(\varepsilon y)$ for $y\in\mathbb{R}^N$, then we see

$$\operatorname{div}\big(\big(1+\rho_\varepsilon(\varepsilon y)\big)\nabla v_\varepsilon\big) - \big(1+\rho_\varepsilon(\varepsilon y)\big)V(\varepsilon y)v_\varepsilon + \big(1+\rho_\varepsilon(\varepsilon y)\big)g_\varepsilon(\varepsilon y, v_\varepsilon) = 0. \tag{2.26}$$

Finally, proceeding as in the first part of the proof of Lemma 2.2, we obtain from the estimates on C_ε given in Lemma 2.3 that

$$\int_{\mathbb{R}^N}\big(\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\big) \leqslant C\varepsilon^N, \qquad \int_{\mathbb{R}^N}\big(|\nabla v_\varepsilon|^2 + V(\varepsilon y)v_\varepsilon^2\big) \leqslant C. \tag{2.27}$$

3. Proof of Theorem 1.1

In this section, we will prove that $\rho_\varepsilon \equiv 0$ and u_ε is indeed a solution of the original equation (1.1).

Given $R>0$ and set $A\subset\mathbb{R}^N$, we denote by $N_R(A)$ the set $\{y \mid \operatorname{dist}(y,A) < R\}$. The following lemma means that v_ε is small away from the set $\Lambda^\varepsilon = \bigcup_{i=1}^k \Lambda_i^\varepsilon$.

Lemma 3.1. *There exists a $C>0$ such that, for any given $R>0$, one has*

$$\int_{\mathbb{R}^N\setminus N_R(\Lambda^\varepsilon)}\big(|\nabla v_\varepsilon|^2 + V(\varepsilon y)v_\varepsilon^2\big) \leqslant \frac{C}{R}, \tag{3.1}$$

for ε sufficiently small.

Proof. For any given $R>0$, $\varepsilon>0$, we may choose smooth cut-off functions $0\leqslant\psi_{i,R}^\varepsilon\leqslant 1$ such that

$$\psi_{i,R}^\varepsilon(y) = \begin{cases} 1, & \text{if } \operatorname{dist}(y,\Lambda_i^\varepsilon) < R/2,\\ 0, & \text{if } \operatorname{dist}(y,\Lambda_i^\varepsilon) > R,\end{cases} \tag{3.2}$$

and $|\nabla\psi_{i,R}^\varepsilon| \leqslant 4/R$. Then set $\eta_R^\varepsilon = 1 - \sum_{i=1}^k \psi_{i,R}^\varepsilon$.

Similarly to (2.6), we have

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\left(\eta_R^\varepsilon\right)^2 f_\varepsilon(\varepsilon y, v_\varepsilon)v_\varepsilon &\leqslant \varepsilon^3\int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\frac{(\eta_R^\varepsilon v_\varepsilon)^2}{1+|\varepsilon y|^{\theta_0}}\\
&\leqslant \varepsilon^3\left(\int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\left(\eta_R^\varepsilon v_\varepsilon\right)^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}\left(\int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\left(\frac{1}{1+|\varepsilon y|^{\theta_0}}\right)^{\frac{N}{2}}\right)^{\frac{2}{N}}\\
&\leqslant \varepsilon^3 C\left(\int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\left|\nabla\left(\eta_R^\varepsilon v_\varepsilon\right)\right|^2\right)\left(\int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\frac{1}{(1+|\varepsilon y|^{\theta_0})^{\frac{N}{2}}}\right)^{\frac{2}{N}}\\
&= C\varepsilon\left(\int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\left|\nabla\left(\eta_R^\varepsilon v_\varepsilon\right)\right|^2\right)\left(\int_{\mathbb{R}^N\setminus\Lambda}\frac{1}{(1+|x|^{\theta_0})^{\frac{N}{2}}}\right)^{\frac{2}{N}}\\
&= C\varepsilon\int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\left|\nabla\left(\eta_R^\varepsilon v_\varepsilon\right)\right|^2\\
&\leqslant C\varepsilon\int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\left(\eta_R^\varepsilon\right)^2|\nabla v_\varepsilon|^2 + C\varepsilon\int_{N_R(\Lambda^\varepsilon)\setminus\Lambda^\varepsilon}\left|\nabla\eta_R^\varepsilon\right|^2 v_\varepsilon^2.\end{aligned}$$

Using the test function $(\eta_R^\varepsilon)^2 v_\varepsilon$ in (2.26), one gets

$$\begin{aligned}C\int_{\mathbb{R}^N\setminus N_R(\Lambda^\varepsilon)}\left(|\nabla v_\varepsilon|^2+V(\varepsilon y)v_\varepsilon^2\right) &\leqslant \int_{\mathbb{R}^N\setminus N_R(\Lambda^\varepsilon)}\left(|\nabla v_\varepsilon|^2+V(\varepsilon y)v_\varepsilon^2\right) - C\varepsilon\int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\left(\eta_R^\varepsilon\right)^2|\nabla v_\varepsilon|^2\\
&\quad - C\varepsilon\int_{N_R(\Lambda^\varepsilon)\setminus\Lambda^\varepsilon}\left|\nabla\eta_R^\varepsilon\right|^2 v_\varepsilon^2\\
&\leqslant \int_{\mathbb{R}^N\setminus\Lambda^\varepsilon}\left(1+\rho_\varepsilon(\varepsilon y)\right)\left(\eta_R^\varepsilon\right)^2\left(|\nabla v_\varepsilon|^2+V(\varepsilon y)v_\varepsilon^2-f_\varepsilon(\varepsilon y, v_\varepsilon)v_\varepsilon\right)\\
&= -2\int_{N_R(\Lambda^\varepsilon)\setminus\Lambda^\varepsilon}\left(1+\rho_\varepsilon(\varepsilon y)\right)v_\varepsilon\eta_R^\varepsilon\nabla v_\varepsilon\nabla\eta_R^\varepsilon. \end{aligned}\tag{3.3}$$

Note that ρ_ε is uniformly bounded by a constant possibly depending on M. Using this, the choice of η_R^ε, (H_1), (H_2) and (2.27), we find that (3.1) follows immediately from (3.3). □

Before we proceed further, we give a preliminary lemma.

Lemma 3.2. *Assume that $v\in H^1(\mathbb{R}^N)\cap C(\mathbb{R}^N)$ is nonnegative and satisfies the equation*

$$\Delta v - v + \chi_{\{x_1<0\}}v^p = 0,\quad x\in\mathbb{R}^N.$$

Then $v\equiv 0$.

Proof. Standard regularity arguments yield $v\in C^1(\mathbb{R}^N)$ and $v\to 0, \nabla v\to 0$ as $|x|\to+\infty$. Using $\frac{\partial v}{\partial x_1}$ as a test function, we see

$$\frac{1}{2}\int_{\mathbb{R}^{N-1}}dx'\int_{-\infty}^{+\infty}\frac{\partial}{\partial x_1}\left(|\nabla v|^2+v^2\right)dx_1 - \frac{1}{p+1}\int_{\mathbb{R}^{N-1}}v^{p+1}\left(0,x'\right)dx' = 0.$$

Noting that the first integral is zero, we obtain $v(0,x') \equiv 0$. Now, to prove that $v \equiv 0$ for $x_1 > 0$, we use $v\chi_{\{x_1>0\}} \in H^1(\mathbb{R}^N)$ as a test function, and obtain that $v\chi_{\{x_1>0\}} \equiv 0$. This contradicts the strong maximum principle if $v \not\equiv 0$.

As a result, we complete the proof. □

Lemma 3.3. *Let M be as in* (2.10). *Assume that $\varepsilon_j \to 0$ as $j \to +\infty$ and* $\lim_{j\to+\infty} L^i_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} > c_i + \sigma_i$. *Then, for M large enough,*

$$\lim_{j\to+\infty} L^i_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geqslant 2c_i.$$

Proof. We first prove that there exist numbers $S > 0$ and $\rho > 0$, such that

$$\sup_{y\in\Lambda_i^{\varepsilon_j}} \int_{B_S(y)} v_{\varepsilon_j}^2 \geqslant \rho, \quad \forall j \geqslant j_0. \tag{3.4}$$

Indeed, since $\lim_{j\to+\infty} L^i_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} > c_i + \sigma_i$, there is a $\lambda > 0$ such that

$$\int_{\tilde{\Lambda}_i^{\varepsilon_j}} \big(|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2\big) \geqslant \lambda, \quad \forall j \geqslant j_0,$$

then, Lemma 3.1 implies that for all $R > 0$ large enough

$$\int_{N_R(\Lambda_i^{\varepsilon_j})} \big(|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2\big) \geqslant \frac{\lambda}{2}. \tag{3.5}$$

Now assume that (3.4) is false. Then we may assume that for all $S > 0$, we have

$$\sup_{y\in\Lambda_i^{\varepsilon_j}} \int_{B_S(y)} v_{\varepsilon_j}^2 \to 0 \quad \text{as } j \to \infty. \tag{3.6}$$

Let $v_j^R = \psi_R^j v_{\varepsilon_j}$, where $\psi_R^j = \psi_{i,2R}^{\varepsilon_j}$ is given by (3.2). Then (3.6) implies

$$\sup_{y\in\mathbb{R}^N} \int_{B_S(y)} \big(v_j^R\big)^2 \to 0 \quad \text{as } j \to \infty$$

for all $S > 0$. Moreover, $\{v_j^R\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$. Then applying the concentration compactness principle (see Lemma I.1 in [29] or Lemma 2.18 in [17]), we obtain that

$$\int_{\mathbb{R}^N} \big(v_j^R\big)^q \to 0, \quad \text{for all } 2 < q < 2N/(N-2),$$

for each $R > 0$. In particular

$$\int_{N_R(\Lambda_i^{\varepsilon_j})} v_{\varepsilon_j}^{p+1} \to 0.$$

Using v_j^R as a test function in (2.26), we get

$$\int_{\tilde{\Lambda}_i^{\varepsilon_j}} \big(|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2\big)\psi_R^j$$
$$= -\int_{\tilde{\Lambda}_i^{\varepsilon_j}} v_{\varepsilon_j}\nabla v_{\varepsilon_j}\nabla\psi_R^j + \int_{\tilde{\Lambda}_i^{\varepsilon_j}} g_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j})v_{\varepsilon_j}\psi_R^j$$

$$\leqslant \frac{C}{R} \int\limits_{N_{2R}(\Lambda_i^{\varepsilon_j})\setminus N_R(\Lambda_i^{\varepsilon_j})} |\nabla v_{\varepsilon_j}||v_{\varepsilon_j}| + \int\limits_{\Lambda_i^{\varepsilon_j}} K(\varepsilon_j y) v_{\varepsilon_j}^{p+1} + \varepsilon_j^3 \int\limits_{N_{2R}(\Lambda_i^{\varepsilon_j})\setminus \Lambda_i^{\varepsilon_j}} \frac{|v_{\varepsilon_j}|^2}{1+|\varepsilon_j y|^{\theta_0}}$$

$$\leqslant C\left(\frac{1}{R}+\varepsilon_j^3\right)\int\limits_{\mathbb{R}^N}\left(|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2\right) + C\int\limits_{\Lambda_i^{\varepsilon_j}} v_{\varepsilon_j}^{p+1},$$

which contradicts (3.5) if we choose R and j large enough. This shows the validity of (3.4). Thus, we may assume that there is a sequence $y_j \in \Lambda_i^{\varepsilon_j}$ such that

$$\int\limits_{B_S(y_j)} v_{\varepsilon_j}^2 \geqslant \rho > 0, \quad \text{for all } j \geqslant j_0. \tag{3.7}$$

Let us now set $v_j(y) = v_{\varepsilon_j}(y_j + y)$ and $\Lambda_{i,y_j}^{\varepsilon_j} = \{y \in \mathbb{R}^N \mid y + y_j \in \Lambda_i^{\varepsilon_j}\}$. Then v_j is a bounded sequence in $H^1(\Lambda_{i,y_j}^{\varepsilon_j})$, and hence we may assume that it converges weakly to a $v \in H^1(\mathbb{R}^N)$.

Assume first that

$$\operatorname{dist}\big(y_j, \partial\Lambda_i^{\varepsilon_j}\big) \to \infty.$$

Set $x_j = \varepsilon_j y_j \in \Lambda_i$ and assume that $x_j \to \xi \in \bar{\Lambda}_i$. Then v satisfies in $\mathbb{R}^N$

$$\Delta v - V(\xi)v + K(\xi)v^p = 0, \tag{3.8}$$

and $v \not\equiv 0$, due to (3.7).

If

$$\operatorname{dist}\big(y_j, \partial\Lambda_i^{\varepsilon_j}\big) \leqslant C < \infty,$$

we will have that v satisfies an equation of the form

$$\Delta v - V(\xi)v + \chi_{\{x_1<0\}}K(\xi)v^p = 0, \quad \text{in } \mathbb{R}^N. \tag{3.9}$$

But, we know $v \equiv 0$ in this case by Lemma 3.2. Hence $v > 0$ solves (3.8) and is the unique critical point of the functional I_ξ. Thus we have

$$c_i \leqslant I_\xi(v) \leqslant c_i + \sigma_i.$$

Now, using the elliptic regularity theory we see that v_{ε_j} converges strongly in the H^1-sense over any compact set. Passing to a further subsequence if necessary, we may find a sequence of positive numbers $R_j \to +\infty$ such that

$$\lim_{j\to\infty} \int\limits_{B_{R_j}(y_j)} \left[\frac{1}{2}\left(|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2\right) - G_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j})\right] = I_\xi(v) \leqslant c_i + \sigma_i. \tag{3.10}$$

Thus, combining the assumption $\lim_{j\to+\infty} L_{\varepsilon_j}^i(u_{\varepsilon_j})\varepsilon_j^{-N} > c_i + \sigma_i$ and (3.10), we find that there exists $\eta > 0$ such that for all large j,

$$\int\limits_{\tilde{\Lambda}_i^{\varepsilon_j}\setminus B_{R_j}(y_j)} \left(|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2\right) > \eta > 0. \tag{3.11}$$

Hence, similar to the proof of (3.4), we can find an $S > 0$ and a sequence $\tilde{y}_j \in \Lambda_i^{\varepsilon_j} \setminus B_{R_j}(y_j)$ such that

$$\int\limits_{B_S(\tilde{y}_j)} v_{\varepsilon_j}^2 \geqslant \rho > 0.$$

Thus, we have again, after passing to a subsequence, the weak convergence of $v_{\varepsilon_j}(\cdot + \tilde{y}_j)$ to a nonzero $\hat{v} \in H^1(\mathbb{R}^N)$. Moreover, $\hat{v}$ satisfies the equation

$$\Delta\hat{v} - V(\hat{\xi})\hat{v} + K(\hat{\xi})\hat{v}^p = 0,$$

with $\hat{\xi} \in \bar{\Lambda}_i$ and $I_{\hat{\xi}}(\hat{v}) \geqslant c_i$.

Next, we verify that

$$\lim_{j\to\infty} L^i_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geqslant I_\xi(v) + I_{\hat\xi}(\hat v) \geqslant 2c_i. \tag{3.12}$$

We recall that v_{ε_j} satisfies on $\tilde\Lambda_i^{\varepsilon_j}$ the equation

$$\Delta v_{\varepsilon_j} - V(\varepsilon_j y)v_{\varepsilon_j} + g_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j}) = 0. \tag{3.13}$$

To prove (3.12), firstly, we show that

$$\max_{x\in\partial\tilde\Lambda_i^{\varepsilon_j}} v_{\varepsilon_j}(x) \to 0 \quad \text{as } j\to\infty. \tag{3.14}$$

It suffices to show that $\max_{x\in\partial\tilde\Lambda_i} u_{\varepsilon_j}(x) \to 0$ as $j\to\infty$. Suppose on the contrary that there exist subsequences, still denoted by $\{\varepsilon_j\}$, and $\{\bar y_j\}\subset\partial\tilde\Lambda_i$, such that $\varepsilon_j\to 0$, $\bar y_j \to y_0\in\partial\tilde\Lambda_i$ as $j\to\infty$ and $u_{\varepsilon_j}(\bar y_j)\geqslant\delta>0$. Choose $\rho>0$ such that $B_\rho(y_0)\subset\mathbb{R}^N\setminus(\bigcup_{i=1}^k\Lambda_i)$. We may assume $\{\bar y_j\}\subset B_\rho(y_0)$. Using the above scaling technique on $B_\rho(y_0)$, it is easy to prove that $w_j(x):=u_{\varepsilon_j}(\bar y_j+\varepsilon_j x)$ converges in C^2 on any compact set to some function $w\in H^1(\mathbb{R}^N)$. Moreover w satisfies

$$-\Delta w + V(y_0)w = 0, \quad \text{in } \mathbb{R}^N,$$

which implies $w\equiv 0$. This contradicts the fact that $\max w(x)\geqslant\delta$, and therefore (3.14) holds.

We use in (3.13) a test function of the form

$$\phi = v_{\varepsilon_j}\big[\psi\big(|y-y_j|/R\big) + \psi\big(|y-\tilde y_j|/R\big)\big],$$

where ψ is a C^∞ function with $\psi(s)=0$ for $s\leqslant 1$ and $\psi(s)=1$ for $s\geqslant 2$. Denoting $N_R(y_j,\tilde{y_j}) = B(y_j,R)\cup B(\tilde y_j,R)$, by (3.14), we conclude that

$$\begin{aligned}
&\int_{\tilde\Lambda_i^{\varepsilon_j}\setminus N_R(y_j,\tilde{y_j})} \big(|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2\big)\\
&\quad\geqslant \int_{\tilde\Lambda_i^{\varepsilon_j}\setminus N_R(y_j,\tilde{y_j})} g_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j})v_{\varepsilon_j} - \int_{N_{2R}(y_j,\tilde{y_j})\setminus N_R(y_j,\tilde{y_j})} g_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j})v_{\varepsilon_j} + \frac{C}{R} + o_{\varepsilon_j}(1)\\
&\quad\geqslant 2\int_{\Lambda_i^{\varepsilon_j}\setminus N_R(y_j,\tilde{y_j})} G_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j}) + \int_{\tilde\Lambda_i^{\varepsilon_j}\setminus\Lambda_i^{\varepsilon_j}} g_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j})v_{\varepsilon_j} + \frac{C}{R} + C(R) + o_{\varepsilon_j}(1)\\
&\quad= 2\int_{\tilde\Lambda_i^{\varepsilon_j}\setminus N_R(y_j,\tilde{y_j})} G_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j}) + \int_{\tilde\Lambda_i^{\varepsilon_j}\setminus\Lambda_i^{\varepsilon_j}} g_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j})v_{\varepsilon_j} - 2\int_{\tilde\Lambda_i^{\varepsilon_j}\setminus\Lambda_i^{\varepsilon_j}} G_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j}) + \frac{C}{R} + C(R) + o_{\varepsilon_j}(1)\\
&\quad\geqslant 2\int_{\tilde\Lambda_i^{\varepsilon_j}\setminus N_R(y_j,\tilde{y_j})} G_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j}) - \varepsilon_j^3\int_{\tilde\Lambda_i^{\varepsilon_j}\setminus\Lambda_i^{\varepsilon_j}} \frac{1}{1+|\varepsilon_j y|^{\theta_0}} v_{\varepsilon_j}^2 + \frac{C}{R} + C(R) + o_{\varepsilon_j}(1)\\
&\quad= 2\int_{\tilde\Lambda_i^{\varepsilon_j}\setminus N_R(y_j,\tilde{y_j})} G_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j}) + \frac{C}{R} + C(R) + o_{\varepsilon_j}(1),
\end{aligned}$$

where $C(R)\to 0$ as $R\to+\infty$, and we have used

$$\int_{\tilde\Lambda_i^{\varepsilon_j}\setminus\Lambda_i^{\varepsilon_j}} \frac{1}{1+|\varepsilon_j y|^{\theta_0}} v_{\varepsilon_j}^2 \leqslant C\int_{\tilde\Lambda_i^{\varepsilon_j}\setminus\Lambda_i^{\varepsilon_j}} V(\varepsilon_j y)v_{\varepsilon_j}^2 \leqslant C\int_{\mathbb{R}^N} V(\varepsilon_j y)v_{\varepsilon_j}^2. \tag{3.15}$$

Thus, it gives

$$L^i_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geqslant \int\limits_{N_R(y_j,\tilde{y_j})} \left[\frac{1}{2}\left(|\nabla v_{\varepsilon_j}|^2 + V(\varepsilon_j y)v_{\varepsilon_j}^2\right) - G_{\varepsilon_j}(\varepsilon_j y, v_{\varepsilon_j})\right] + O\left(\frac{1}{R}\right) + C(R) + o_{\varepsilon_j}(1),$$

and

$$\lim_{j\to\infty} L^i_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geqslant \int\limits_{B_R(0)} \left[\frac{1}{2}\left(|\nabla v|^2 + V(\xi)v^2\right) - \frac{1}{p+1}K(\xi)v^{p+1}\right]$$
$$+ \int\limits_{B_R(0)} \left[\frac{1}{2}\left(|\nabla \hat{v}|^2 + V(\hat{\xi})\hat{v}^2\right) - \frac{1}{p+1}K(\hat{\xi})\hat{v}^{p+1}\right] + O\left(\frac{1}{R}\right) + C(R).$$

Consequently, choosing R large, we get (3.12) and complete the proof. □

Lemma 3.4. *For $M > 0$ sufficiently large,*

$$\limsup_{\varepsilon\to 0} L^i_\varepsilon(u_\varepsilon)\varepsilon^{-N} \leqslant c_i + \sigma_i, \quad \forall i = 1, \ldots, k. \tag{3.16}$$

Proof. Firstly, we prove that

$$\liminf_{\varepsilon\to 0} L_\varepsilon(u_\varepsilon)\varepsilon^{-N} \geqslant 0. \tag{3.17}$$

Choose a smooth cut-off function $0 \leqslant \psi_R \leqslant 1$ such that

$$\psi_R(y) = \begin{cases} 1, & \text{if } y \in N_R(\Lambda^\varepsilon), \\ 0, & \text{if } y \in \mathbb{R}^N \setminus N_{2R}(\Lambda^\varepsilon), \end{cases}$$

and $|\nabla\psi_R| \leqslant 2/R$.

Using in (3.13) the test function $v_\varepsilon\psi_R$ and noting (3.1) and (3.15), we have

$$\int\limits_{N_R(\Lambda^\varepsilon)} \left(|\nabla v_\varepsilon|^2 + V(\varepsilon y)v_\varepsilon^2\right) + \frac{C}{R} = \int\limits_{N_{2R}(\Lambda^\varepsilon)} g_\varepsilon(\varepsilon y, v_\varepsilon)v_\varepsilon\psi_R$$
$$\geqslant \int\limits_{\Lambda^\varepsilon} K(\varepsilon y)v_\varepsilon^{p+1} - \varepsilon^3 C \int\limits_{\mathbb{R}^N} V(\varepsilon y)v_\varepsilon^2.$$

Similarly, using (2.6) and (3.1), we see

$$L_\varepsilon(v_\varepsilon) = \frac{1}{2}\int\limits_{N_R(\Lambda^\varepsilon)} \left(|\nabla v_\varepsilon|^2 + V(\varepsilon y)v_\varepsilon^2\right) - \int\limits_{\Lambda^\varepsilon} G_\varepsilon(\varepsilon y, v_\varepsilon) - \int\limits_{\mathbb{R}^N\setminus\Lambda^\varepsilon} F_\varepsilon(\varepsilon y, v_\varepsilon) + \frac{C}{R}$$
$$\geqslant \frac{1}{2}\int\limits_{N_R(\Lambda^\varepsilon)} \left(|\nabla v_\varepsilon|^2 + V(\varepsilon y)v_\varepsilon^2\right) - \frac{1}{p+1}\int\limits_{\Lambda^\varepsilon} K(\varepsilon y)v_\varepsilon^{p+1} - \varepsilon C + \frac{C}{R}.$$

Now, combining these two inequalities, we find that for R large and ε small,

$$L_\varepsilon(v_\varepsilon) \geqslant \left(\frac{1}{2} - \frac{1}{p+1}\right)\int\limits_{N_R(\Lambda^\varepsilon)} \left(|\nabla v_\varepsilon|^2 + V(\varepsilon y)v_\varepsilon^2\right) - \frac{C}{R} - C\varepsilon > 0,$$

which implies (3.17).

Suppose that (3.16) is not true, then it follows from Lemma 3.3 that

$$\lim_{j\to+\infty} L^i_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geqslant 2c_i,$$

which, together with (3.17), implies that

$$\liminf_{j\to\infty} J_{\varepsilon_j}(u_{\varepsilon_j})\varepsilon_j^{-N} \geqslant M\{(2c_i)^{1/2} - (c_i+\sigma_i)^{1/2}\}_+^2.$$

Using the upper estimate for the critical value $C_{\varepsilon_j} = J_{\varepsilon_j}(u_{\varepsilon_j})$, we obtain

$$M\{(2c_i)^{1/2} - (c_i+\sigma_i)^{1/2}\}_+^2 \leqslant \sum_{i=1}^k c_i.$$

Therefore, if M is such that

$$M > \frac{\sum_{i=1}^k c_i}{\min\{((2c_i)^{1/2} - (c_i+\sigma_i)^{1/2})^2 \mid i=1,\dots,k\}},$$

then we can get a contradiction to Lemma 2.3. Hence (3.16) is true for large M. This concludes the proof of Lemma 3.4. □

Lemma 3.4 implies that $\rho_\varepsilon \equiv 0$ if M is chosen large enough. In the sequel we fix M so large that Lemma 3.4 holds true. Now, using the standard arguments (see [15, Lemma 3.4]) we can prove

Lemma 3.5.

$$\lim_{\varepsilon\to 0} L_\varepsilon^i(u_\varepsilon)\varepsilon^{-N} = c_i, \quad \forall i=1,\dots,k.$$

Lemma 3.5 implies that the concentration of u_ε must occur around some $\bar{x}_i \in \Lambda_i$ with $G(\bar{x}_i) = c_i$. The concentration implies the presence of at least one local maximum x_ε^i in each Λ_i, and also $\lim_{\varepsilon\to 0} G(x_\varepsilon^i) = c_i$.

Next we show the uniqueness of the maxima x_ε^i in Λ_i. We argue by contradiction. Assume the existence of a sequence $\varepsilon_j \to 0$, such that u_{ε_j} possesses two local maxima $\bar{x}_j^1, \bar{x}_j^2 \in \Lambda_i$. Then $u_{\varepsilon_j}(\bar{x}_j^i) \geqslant b$ $(i=1,2)$ for some constant $b>0$. But, the fact that $\lim_{j\to\infty} G(\bar{x}_j^i) = c_i < \inf_{\partial\Lambda_i} G(x)$ implies that these sequences stay away from the boundary of Λ_i, so, if we let $v_j(x) = u_{\varepsilon_j}(\bar{x}_j^1 + \varepsilon_j x)$, then after passing to a subsequence v_j converges in the C^2 sense over compact sets to a solution v in $H^1(\mathbb{R}^N)$ of $\Delta v - V(\bar{\xi}^1)v + K(\bar{\xi}^1)v^p = 0$, where $\lim_{j\to\infty} \bar{x}_j^1 = \bar{\xi}^1$. The function v has a local maximum at zero and is radially symmetric and radially decreasing, which implies that $\bar{x}_j = \varepsilon_j^{-1}(\bar{x}_j^2 - \bar{x}_j^1)$ satisfies $|\bar{x}_j| \to \infty$. Now repeating the process in the proof of Lemma 3.3, we have

$$\liminf_{n\to\infty} L_{\varepsilon_j}^i(u_{\varepsilon_j})\varepsilon_j^{-N} \geqslant 2c_i,$$

which is obviously a contradiction to Lemma 3.5. So, in Λ_i, the maxima x_ε^i of u_ε are unique and in Λ, u_ε has exactly k peaks.

The above procedure indeed proves the following proposition.

Proposition 3.1. *The sequences $\{x_\varepsilon^i\} \subset \Lambda_i$, $i=1,\dots,k$ satisfy that, for any $\nu>0$, there exist $\varepsilon_1(\nu), \rho_1(\nu) > 0$ such that for $\varepsilon < \varepsilon_1(\nu)$*

$$\varepsilon^{-N} \int\limits_{\mathbb{R}^N \setminus \bigcup_{i=1}^k B_{\varepsilon\rho_1(\nu)}(x_\varepsilon^i)} \left(\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\right) < \nu, \tag{3.18}$$

and

$$\operatorname{dist}\left(x_\varepsilon^i, M^i\right) < \nu, \tag{3.19}$$

here $M^i = \{\xi_i \in \Lambda_i \mid G(\xi_i) = c_i\}$, $i=1,\dots,k$.

In the rest of this section, we will prove that u_ε is indeed a solution of the original problem (1.1). Let $d_0 = \min\{\operatorname{dist}(\partial\Lambda_i, M^i),\ i=1,\dots,k\} > 0$, and $V_1 = \min_{x\in\Lambda} V(x)/2 > 0$. Fix two positive numbers $K_0 > \max\{128, 2d_0\}$ and $c>0$ such that $c^2 \geqslant (128K_0^2)/(d_0^2 V_1)$.

Set $\nu_0 = \min\{d_0/K_0, (8C_1)^{-\frac{2}{p-1}}\}$ and $\varepsilon_2 = \min\{\varepsilon_1(\nu_0), d_0/(K_0\rho_1(\nu_0)), (\ln 2)/c\}$, where C_1 is defined in (2.5), $\varepsilon_1(\nu_0)$ and $\rho_1(\nu_0)$ are given in (3.18) and (3.19) respectively. In the sequel, we assume $\varepsilon < \varepsilon_2$ and $\nu < \nu_0$. Hence,

$$\operatorname{dist}\bigl(x_\varepsilon^i, \partial\Lambda_i\bigr) > \frac{d_0}{2}, \quad i = 1, \ldots, k \quad \text{and} \quad \varepsilon\rho_1(\nu_0) < \frac{d_0}{K_0}. \tag{3.20}$$

Define $\Omega_{n,\varepsilon} = \mathbb{R}^N \setminus \bigcup_{i=1}^k B_{R_{n,\varepsilon}}(x_\varepsilon^i)$ with $R_{n,\varepsilon} = e^{c\varepsilon n}$ and let $\tilde{n} > \hat{n}$ be integers such that

$$R_{\hat{n}-1,\varepsilon} < \frac{d_0}{K_0} \leqslant R_{\hat{n},\varepsilon}, \qquad R_{\tilde{n}+2,\varepsilon} \leqslant \frac{d_0}{2} < R_{\tilde{n}+3,\varepsilon}. \tag{3.21}$$

By (3.20), one gets $R_{n,\varepsilon} \geqslant R_{\hat{n},\varepsilon} \geqslant d_0/K_0 > \varepsilon\rho_1(\nu_0)$ for $n \geqslant \hat{n}$, and hence

$$\Omega_{n,\varepsilon} \cap \left(\bigcup_{i=1}^k B_{\varepsilon\rho_1(\nu_0)}\bigl(x_\varepsilon^i\bigr)\right) = \emptyset. \tag{3.22}$$

Let $\chi_{n,\varepsilon}(x)$ be smooth cut-off functions such that $\chi_{n,\varepsilon}(x) = 0$ in $\bigcup_{i=1}^k B_{R_{n,\varepsilon}}(x_\varepsilon^i)$, $\chi_{n,\varepsilon}(x) = 1$ in $\Omega_{n+1,\varepsilon}$, $0 \leqslant \chi_{n,\varepsilon} \leqslant 1$ and $|\nabla\chi_{n,\varepsilon}| \leqslant 2/(R_{n+1,\varepsilon} - R_{n,\varepsilon})$.

Lemma 3.6. *Assume that* (H_1) *and* (H_2) *hold true. If* $\hat{n} \leqslant n \leqslant \tilde{n}$, *then*

$$\int_{\mathbb{R}^N} A_{n,\varepsilon} \leqslant \frac{1}{2} \int_{\Omega_{n,\varepsilon}} \bigl(\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\bigr), \tag{3.23}$$

where $A_{n,\varepsilon}(x) = \varepsilon^2|\nabla(\chi_{n,\varepsilon}u_\varepsilon)|^2 + V(x)(\chi_{n,\varepsilon}u_\varepsilon)^2$.

Proof. Calculating $\langle L'_\varepsilon(u_\varepsilon), \chi_{n,\varepsilon}^2 u_\varepsilon\rangle = 0$ directly, one gets

$$\begin{aligned}\int_{\mathbb{R}^N} A_{n,\varepsilon} &= \int_{\Omega_{n,\varepsilon}} \varepsilon^2|\nabla\chi_{n,\varepsilon}|^2 u_\varepsilon^2 + \int_{\Lambda\cap\Omega_{n,\varepsilon}} \chi_{n,\varepsilon}^2 K(x)\bigl(u_\varepsilon^+\bigr)^{p+1} + \int_{(\mathbb{R}^N\setminus\Lambda)\cap\Omega_{n,\varepsilon}} f_\varepsilon(x,u_\varepsilon)\chi_{n,\varepsilon}^2 u_\varepsilon \\ &:= I + II + III.\end{aligned}$$

Observing that

$$\varepsilon^2|\nabla\chi_{n,\varepsilon}|^2 \leqslant \frac{4\varepsilon^2}{|R_{n+1,\varepsilon} - R_{n,\varepsilon}|^2} \leqslant \frac{16}{c^2 R_{n+1,\varepsilon}^2}$$

and that, for $\hat{n} \leqslant n \leqslant \tilde{n}$ and $x \in (\bigcup_{i=1}^k B_{R_{n+1,\varepsilon}}(x_\varepsilon^i)) \setminus (\bigcup_{i=1}^k B_{R_{n,\varepsilon}}(x_\varepsilon^i))$,

$$\frac{128}{c^2 R_{n+1,\varepsilon}^2} \leqslant \frac{128}{\frac{128K_0^2}{d_0^2 V_1} \cdot \frac{d_0^2}{K_0^2}} = V_1 \leqslant V(x),$$

we obtain

$$\varepsilon^2|\nabla\chi_{n,\varepsilon}|^2 \leqslant \frac{1}{8}V(x), \quad \text{in } \mathbb{R}^N$$

and hence

$$I \leqslant \frac{1}{8} \int_{\Omega_{n,\varepsilon}} \bigl(\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2\bigr). \tag{3.24}$$

Now, we estimate II.

Clearly, we only need to consider the case $\Lambda \cap \Omega_{n,\varepsilon} \neq \emptyset$. Similarly to (2.5), we obtain

$$\int_{\Lambda\cap\Omega_{n,\varepsilon}} K(x)\big(u_\varepsilon^+\big)^{p+1} \leqslant \int_{\Lambda\cap\Omega_{n,\varepsilon}} K(x)|u_\varepsilon|^{p+1}$$
$$\leqslant C_1\varepsilon^{-\frac{N(p-1)}{2}}\bigg(\int_{\Lambda\cap\Omega_{n,\varepsilon}}\big(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)u_\varepsilon^2\big)\bigg)^{\frac{p+1}{2}}.$$

On the other hand, from (3.22), we see $\Lambda\cap\Omega_{n,\varepsilon}\subset\mathbb{R}^N\setminus(\bigcup_{i=1}^k B_{\varepsilon\rho_1(\nu_0)}(x_\varepsilon^i))$, for $n\geqslant\hat{n}$. Thus, it follows from (3.18) that

$$II\leqslant C_1\varepsilon^{-\frac{N(p-1)}{2}}\bigg(\int_{\mathbb{R}^N\setminus(\bigcup_{i=1}^k B_{\varepsilon\rho_1(\nu_0)}(x_\varepsilon^i))}\big(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)u_\varepsilon^2\big)\bigg)^{\frac{p-1}{2}}\int_{\Lambda\cap\Omega_{n,\varepsilon}}\big(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)u_\varepsilon^2\big)$$
$$\leqslant C_1\nu_0^{\frac{p-1}{2}}\int_{\Omega_{n,\varepsilon}}\big(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)u_\varepsilon^2\big)$$
$$\leqslant\frac{1}{8}\int_{\Omega_{n,\varepsilon}}\big(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)u_\varepsilon^2\big). \tag{3.25}$$

Finally, to estimate III, we only need to use the method for (2.6) and obtain

$$III\leqslant\int_{\Omega_{n,\varepsilon}}\frac{\varepsilon^3}{1+|x|^{\theta_0}}u_\varepsilon^2\leqslant\frac{1}{8}\int_{\Omega_{n,\varepsilon}}\big(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)u_\varepsilon^2\big). \tag{3.26}$$

Consequently, (3.23) can be proved by combining (3.24)–(3.26). □

Lemma 3.7. *Under the assumptions of Lemma* 3.6*, for small $\varepsilon<\varepsilon_2$, one has*

$$\int_{\mathbb{R}^N}\big|\nabla(\chi_{\tilde{n},\varepsilon}u_\varepsilon)\big|^2\leqslant C\varepsilon^{N-2}2^{-\frac{\ln 2}{c\varepsilon}}.$$

Proof. Lemma 3.6 implies that

$$\int_{\mathbb{R}^N}A_{n,\varepsilon}\leqslant\frac{1}{2}\int_{\Omega_{n,\varepsilon}}\big(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)u_\varepsilon^2\big)\leqslant\frac{1}{2}\int_{\mathbb{R}^N}A_{n-1,\varepsilon}.$$

Iterating the above process, we see

$$\int_{\mathbb{R}^N}A_{\tilde{n},\varepsilon}\leqslant\bigg(\frac{1}{2}\bigg)^{\tilde{n}-\hat{n}}\int_{\mathbb{R}^N}A_{\hat{n},\varepsilon}\leqslant\bigg(\frac{1}{2}\bigg)^{\tilde{n}-\hat{n}+1}\int_{\Omega_{\hat{n},\varepsilon}}\big(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)u_\varepsilon^2\big)$$
$$\leqslant\bigg(\frac{1}{2}\bigg)^{\tilde{n}-\hat{n}+1}\int_{\mathbb{R}^N\setminus(\bigcup_{i=1}^k B_{\varepsilon\rho_1(\nu_0)}(x_\varepsilon^i))}\big(\varepsilon^2|\nabla u_\varepsilon|^2+V(x)u_\varepsilon^2\big)$$
$$\leqslant C\varepsilon^N\bigg(\frac{1}{2}\bigg)^{\tilde{n}-\hat{n}+1}=C\varepsilon^N e^{-(\tilde{n}-\hat{n}+1)\ln 2}\leqslant C\varepsilon^N 2^{-\frac{\ln 2}{c\varepsilon}},$$

where, in the last inequality, we have used

$$e^{c\varepsilon(\tilde{n}-\hat{n}+1)}=e^{c\varepsilon\tilde{n}}e^{-c\varepsilon(\hat{n}-1)}=e^{c\varepsilon\tilde{n}}R_{\hat{n}-1,\varepsilon}^{-1}\geqslant\frac{K_0}{d_0}\geqslant 2.$$

As a result, we obtain

$$\int_{\mathbb{R}^N} \left|\nabla(\chi_{\tilde{n},\varepsilon} u_\varepsilon)\right|^2 \leqslant \varepsilon^{-2} \int_{\mathbb{R}^N} A_{\tilde{n},\varepsilon} \leqslant C\varepsilon^{N-2} 2^{-\frac{\ln 2}{c\varepsilon}}. \qquad \square$$

Lemma 3.8. *Under the assumptions of Lemma* 3.6*, there holds*

$$u_\varepsilon(x) \leqslant C 2^{-\frac{\ln 2}{2c\varepsilon}}, \quad \forall x \in \mathbb{R}^N \Bigg\backslash \left(\bigcup_{i=1}^{k} B_{\frac{d_0}{2}}\left(x_\varepsilon^i\right) \right).$$

Proof. Define

$$c_\varepsilon(x) = \chi_\varepsilon(x) K(\varepsilon x) v_\varepsilon^{p-1} + \left(1 - \chi_\varepsilon(x)\right) \frac{\varepsilon^3}{1 + |\varepsilon x|^{\theta_0}},$$

where χ_ε is a characteristic function of $\Lambda^\varepsilon = \{\varepsilon^{-1} x \mid x \in \Lambda\}$. We know $v_\varepsilon = u_\varepsilon(\varepsilon x) \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ is a nonnegative weak subsolution of $\Delta v + c_\varepsilon(x) v = 0$.

Fix $s \in (\frac{N}{2}, \frac{2N}{(p-1)(N-2)})$. Since $\theta_0 > 2$, we find

$$\begin{aligned}
\left\|c_\varepsilon(x)\right\|_{L^s} &\leqslant \left\|\chi_\varepsilon(x) K(\varepsilon x) v_\varepsilon^{p-1}\right\|_{L^s} + \left\|\left(1 - \chi_\varepsilon(x)\right) \frac{\varepsilon^3}{1 + |\varepsilon x|^{\theta_0}}\right\|_{L^s} \\
&\leqslant C \left(\int_{\Lambda^\varepsilon} \left(|\nabla v_\varepsilon|^2 + V(\varepsilon x) v_\varepsilon^2\right) \right)^{\frac{p-1}{2}} + C\varepsilon^{3-\frac{N}{s}} \left(\int_{\mathbb{R}^N \setminus \Lambda} \frac{1}{(1 + |y|^{\theta_0})^s} \right)^{\frac{1}{s}} \leqslant C,
\end{aligned}$$

which shows that $\|c_\varepsilon(x)\|_{L^s}$ is uniformly bounded with respect to ε. By Theorem 8.17 on page 194 in [25], there is a constant C depending only on d_0, the dimension N and the L^s bound of $c_\varepsilon(x)$ such that for $z \in \mathbb{R}^N$,

$$v_\varepsilon(z) \leqslant C \left(\int_{B_{\frac{cd_0}{4}}(z)} v_\varepsilon^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}}. \tag{3.27}$$

Noting that for ε small,

$$B_{\frac{\varepsilon c d_0}{4}}(x) \subset \Omega_{\tilde{n}+1,\varepsilon}, \quad \forall x \in \mathbb{R}^N \Bigg\backslash \left(\bigcup_{i=1}^{k} B_{\frac{d_0}{2}}\left(x_\varepsilon^i\right) \right),$$

we see that, for any $x \in \mathbb{R}^N \setminus (\bigcup_{i=1}^k B_{\frac{d_0}{2}}(x_\varepsilon^i))$,

$$\begin{aligned}
u_\varepsilon(x) = v_\varepsilon\left(\varepsilon^{-1} x\right) &\leqslant C \left(\int_{B_{\frac{cd_0}{4}}(\varepsilon^{-1} x)} v_\varepsilon^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}} \\
&= C \left(\varepsilon^{-N} \int_{B_{\frac{\varepsilon c d_0}{4}}(x)} u_\varepsilon^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}} \leqslant C\varepsilon^{-\frac{N-2}{2}} \left(\int_{\mathbb{R}^N} (\chi_{\tilde{n},\varepsilon} u_\varepsilon)^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}} \\
&\leqslant C\varepsilon^{-\frac{N-2}{2}} \left(\int_{\mathbb{R}^N} \left|\nabla(\chi_{\tilde{n},\varepsilon} u_\varepsilon)\right|^2 \right)^{\frac{1}{2}} \\
&\leqslant C 2^{-\frac{\ln 2}{2c\varepsilon}}. \qquad \square
\end{aligned}$$

Now we are ready to prove Theorem 1.1.

Proof. Since $p \in ((N+\beta_1)/(N-2), (N+2)/(N-2))$, we can choose σ_0 less than but close to $N-2$ such that

$$2 < \theta_0 < (p-1)\sigma_0 - \beta_1, \qquad \sigma_0 p - \beta_1 > N. \tag{3.28}$$

Define the following comparison function

$$U(x) = \sum_{i=1}^{k} \frac{1}{|x - x_\varepsilon^i|^{\sigma_0}} \quad \text{in } \mathbb{R}^N \Bigg\backslash \left(\bigcup_{i=1}^{k} B_{\frac{d_0}{2}}\left(x_\varepsilon^i\right) \right).$$

It is easy to know that $Z(x) = U(x) - \varepsilon^2 u_\varepsilon(x) \geqslant 0$ on $\partial(\bigcup_{i=1}^k B_{\frac{d_0}{2}}(x_\varepsilon^i))$ for small ε and $Z(x)$ vanishes at infinity due to (3.27).

On the other hand, noting $\sigma_0 < N-2$, we conclude from Lemma 3.8 that for ε sufficiently small and for all $x \in \mathbb{R}^N \setminus (\bigcup_{i=1}^k \overline{B_{\frac{d_0}{2}}(x_\varepsilon^i)})$,

$$\begin{aligned} -\Delta Z &= -\Delta U + \varepsilon^2 \Delta u_\varepsilon \\ &= \sigma_0(N-2-\sigma_0) \sum_{i=1}^{k} \frac{1}{|x - x_\varepsilon^i|^{\sigma_0+2}} + V(x)u_\varepsilon - g_\varepsilon(x, u_\varepsilon) \\ &\geqslant \sigma_0(N-2-\sigma_0) \sum_{i=1}^{k} \frac{1}{|x - x_\varepsilon^i|^{\sigma_0+2}} - \chi_\Lambda(x)\varepsilon - \frac{\varepsilon(1-\chi_\Lambda(x))}{1+|x|^N} \\ &\geqslant 0. \end{aligned}$$

Thus, the maximum principle ensures that $u_\varepsilon \leqslant U/\varepsilon^2$ in $\mathbb{R}^N \setminus (\bigcup_{i=1}^k \overline{B_{\frac{d_0}{2}}(x_\varepsilon^i)})$ and hence

$$u_\varepsilon(x) \leqslant \sum_{i=1}^{k} \frac{1}{\varepsilon^2 |x - x_\varepsilon^i|^{\sigma_0}} \leqslant \frac{C}{\varepsilon^2(1+|x|^{\sigma_0})} \quad \text{in } \mathbb{R}^N \setminus \Lambda. \tag{3.29}$$

Next we verify that u_ε actually solves Eq. (1.1). Indeed, it follows from (H_3), Lemma 3.8, (3.28) and (3.29) that we can choose γ larger than but close to 1 such that, for small ε and all $x \in \mathbb{R}^N \setminus \Lambda$,

$$K(x)u_\varepsilon^p \leqslant k_1\left(1+|x|^{\beta_1}\right)\left(\frac{C}{\varepsilon^2(1+|x|^{\sigma_0})}\right)^{p-\gamma} 2^{-\frac{(\gamma-1)\ln 2}{2c\varepsilon}} u_\varepsilon \leqslant \frac{\varepsilon^3}{1+|x|^{\theta_0}} u_\varepsilon,$$

and

$$K(x)u_\varepsilon^p \leqslant k_1\left(1+|x|^{\beta_1}\right)\left(\frac{C}{\varepsilon^2(1+|x|^{\sigma_0})}\right)^{p+1-\gamma} 2^{-\frac{(\gamma-1)\ln 2}{2c\varepsilon}} \leqslant \frac{\varepsilon}{1+|x|^N}.$$

Therefore, $g_\varepsilon(x, u_\varepsilon) \equiv K(x)u_\varepsilon^p$ holds true in $\mathbb{R}^N \setminus \Lambda$ and u_ε solves the original problem (1.1). Noting that σ_0 is close to $N-2$, (3.29) leads to $u_\varepsilon \in L^2(\mathbb{R}^N)$ for $N \geqslant 5$.

Finally, Proposition 3.1, Lemma 3.8 and (3.29) imply that u_ε has exactly k local maxima $x_\varepsilon^i \in \Lambda_i$, $i = 1, \dots, k$, which satisfy that $G(x_\varepsilon^i) \to \inf_{x \in \Lambda_i} G(x)$ as $\varepsilon \to 0$. □

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Appendix A. The smoothness of $J_\varepsilon(u)$

Proposition A.1. *Let $J_\varepsilon(u)$ be as in Section 2. Then $J_\varepsilon(u) \in C^1$.*

Proof. Since $J_\varepsilon(u) = L_\varepsilon(u) + P_\varepsilon(u)$, we divide the proof into two steps.

Step 1. The functions $L_\varepsilon(u)$ is of class C^1.

We recall

$$f_\varepsilon(x,t) = \min\left\{K(x)\left(t^+\right)^p, \frac{\varepsilon^3}{1+|x|^{\theta_0}}t^+, \frac{\varepsilon}{1+|x|^N}\right\}. \tag{A.1}$$

It is easy to check that $f_\varepsilon(x,t)$ is continuous in both x and t. As a result, $F_\varepsilon(x,t)$ is C^1 in t.

We claim that $I(u) := \int_{\mathbb{R}^N\setminus\Lambda} F_\varepsilon(x,u)$ is C^1. Indeed, it follows from direct calculation that, for any $h \in E_\varepsilon$

$$\frac{d}{dt}I(u+th)\bigg|_{t=0} = \int\limits_{\mathbb{R}^N\setminus\Lambda} f_\varepsilon(x,u)h.$$

Thus, $I(u)$ is Gateaux differentiable. To show that $I(u)$ is Fréchet differentiable and $I(u)$ is C^1, we just need to show that $\int_{\mathbb{R}^N\setminus\Lambda} f_\varepsilon(x,u)h$ is continuous in E_ε.

Suppose that $u_n \to u$ in E_ε. We have

$$\begin{aligned}\left|\int\limits_{\mathbb{R}^N\setminus\Lambda}\bigl(f_\varepsilon(x,u_n) - f_\varepsilon(x,u)\bigr)h\right| &\leqslant \int\limits_{\mathbb{R}^N\setminus\Lambda}\bigl|f_\varepsilon(x,u_n) - f_\varepsilon(x,u)\bigr||h| \\ &\leqslant \left(\int\limits_{\mathbb{R}^N\setminus\Lambda}\bigl|f_\varepsilon(x,u_n) - f_\varepsilon(x,u)\bigr|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}}\left(\int\limits_{\mathbb{R}^N\setminus\Lambda}|h|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}} \\ &\leqslant C\left(\int\limits_{\mathbb{R}^N\setminus\Lambda}\bigl|f_\varepsilon(x,u_n) - f_\varepsilon(x,u)\bigr|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}}\|h\|_\varepsilon.\end{aligned} \tag{A.2}$$

Using (A.1), we see that for any set $S \subset \mathbb{R}^N\setminus\Lambda$,

$$\begin{aligned}\int\limits_S\bigl|f_\varepsilon(x,u_n)\bigr|^{\frac{2N}{N+2}} &\leqslant C\int\limits_S\left(\frac{|u_n|}{1+|x|^{\theta_0}}\right)^{\frac{2N}{N+2}} \leqslant \left(\int\limits_{\mathbb{R}^N}|u_n|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N+2}}\left(\int\limits_S\frac{1}{1+|x|^{\frac{N\theta_0}{2}}}\right)^{\frac{4}{N+2}} \\ &\leqslant C\left(\int\limits_S\frac{1}{1+|x|^{\frac{N\theta_0}{2}}}\right)^{\frac{4}{N+2}}.\end{aligned}$$

Since $\theta_0 > 2$, we see that if $meas(S) \to 0$, or S moves to infinity, then

$$\int\limits_S\bigl|f_\varepsilon(x,u_n)\bigr|^{\frac{2N}{N+2}} \to 0,$$

uniformly in n. It follows from the Vitalli theorem that

$$\int\limits_{\mathbb{R}^N\setminus\Lambda}\bigl|f_\varepsilon(x,u_n) - f_\varepsilon(x,u)\bigr|^{\frac{2N}{N+2}} \to 0,$$

as $n \to +\infty$. As a consequence, $\int_{\mathbb{R}^N\setminus\Lambda} F_\varepsilon(x,u)$ and hence $L_\varepsilon(u)$ are C^1.

Step 2. $P_\varepsilon(u)$ is C^1.

Recall

$$P_\varepsilon(u) = M\sum_{i=1}^k\left\{\left(L_\varepsilon^i(u)_+\right)^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}}(c_i+\sigma_i)^{\frac{1}{2}}\right\}_+^2.$$

Direct calculations show that

$$\frac{d}{dt}P_\varepsilon(u+th)\bigg|_{t=0} = M\sum_{i=1}^{k}\big\{\big(L_\varepsilon^i(u)_+\big)^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}}(c_i+\sigma_i)^{\frac{1}{2}}\big\}_+ \frac{\langle (L_\varepsilon^i(u))', h\rangle}{(L_\varepsilon^i(u)_+)^{\frac{1}{2}}}.$$

Since $\{(L_\varepsilon^i(u)_+)^{\frac{1}{2}} - \varepsilon^{\frac{N}{2}}(c_i+\sigma_i)^{\frac{1}{2}}\}_+ \frac{1}{(L_\varepsilon^i(u)_+)^{\frac{1}{2}}}$ is continuous, we see that each term in functional $P_\varepsilon(u)$ is C^1. So, we complete the proof. □

References

[1] A. Ambrosetti, V. Felli, A. Malchiodi, Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. Eur. Math. Soc. 7 (2005) 117–144.

[2] A. Ambrosetti, A. Malchiodi, Concentration phenomena for NLS: recent results and new perspectives, in: Perspectives in Nonlinear Partial Differential Equations, in: Contemp. Math., vol. 446, Amer. Math. Soc., Providence, RI, 2007, pp. 19–30.

[3] A. Ambrosetti, A. Malchiodi, D. Ruiz, Bound states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. Anal. Math. 98 (2006) 317–348.

[4] A. Ambrosetti, A. Malchiodi, S. Secchi, Multiplicity results for some nonlinear Schrödinger equations with potentials, Arch. Ration. Mech. Anal. 159 (2001) 253–271.

[5] A. Ambrosetti, Z.Q. Wang, Nonlinear Schrödinger equations with vanishing and decaying potentials, Diff. Int. Equats. 18 (2005) 1321–1332.

[6] T. Bartsch, S. Peng, Symmetric Schrödinger equations: existence of solutions concentrating simultaneously on several spheres, Z. Angew. Math. Phys. 58 (2007) 778–804.

[7] F.A. Berezin, M.A. Shubin, The Schrödinger Equation, Kluwer Acad. Publ., 1991.

[8] J. Byeon, Z.Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, Arch. Ration. Mech. Anal. 165 (2002) 295–316.

[9] J. Byeon, Z.Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations. II, Calc. Var. Partial Differential Equations 18 (2003) 207–219.

[10] J. Byeon, Z.Q. Wang, Standing wave for nonlinear Schrödinger equations with singular potentials, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009) 943–958.

[11] J. Byeon, Z.Q. Wang, Spherical semiclassical states of a critical frequency for Schrödinger equations with decaying potentials, J. Eur. Math. Soc. 8 (2006) 217–228.

[12] D. Cao, H.P. Heinz, Uniqueness of positive multi-lump bound states of nonlinear Schrödinger equations, Math. Z. 243 (2003) 599–642.

[13] D. Cao, E.S. Noussair, Multi-bump standing waves with a critical frequency for nonlinear Schrödinger equations, J. Diff. Equats. 203 (2004) 292–312.

[14] D. Cao, E.S. Noussair, S. Yan, Multiscale-bump standing waves with a critical frequency for nonlinear Schrödinger equations, Trans. Amer. Math. Soc. 360 (2008) 3813–3837.

[15] D. Cao, S. Peng, Multi-bump bound states of Schrödinger equations with a critical frequency, Math. Ann. 336 (2006) 925–948.

[16] D. Cao, S. Peng, Semi-classical bound states for Schrödinger equations with potentials vanishing or unbounded at infinity, Comm. Partial Differential Equations 34 (2009) 1566–1591.

[17] V. Coti Zelati, P. Rabinowitz, Homoclinic type solutions for semilinear elliptic PDE on $\mathbb{R}^N$, Comm. Pure Appl. Math. 45 (1992) 1217–1269.

[18] E.N. Dancer, S. Yan, Singularly perturbed elliptic problems in exterior domains, Diff. Int. Equats. 13 (2000) 747–777.

[19] M. del Pino, P. Felmer, Multi-peak bound states for nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998) 127–149.

[20] M. del Pino, P. Felmer, Semi-classical states for nonlinear Schrödinger equations, J. Funct. Anal. 149 (1997) 245–265.

[21] M. del Pino, P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differential Equations 4 (1996) 121–137.

[22] M. Fei, H. Yin, Existence and concentration of bound states of nonlinear Schrödinger equations with compactly supported and competing potentials, Pacific J. Math. 244 (2010) 261–296.

[23] A. Floer, A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, J. Funct. Anal. 69 (1986) 397–408.

[24] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, Adv. Math. Suppl. Stud. 7A (1981) 369–402.

[25] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second edition, Grundlehren Math. Wiss., vol. 224, Springer, Berlin, Heidelberg, New York, 1998.

[26] M. Grossi, On the number of single-peak solutions of the nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002) 261–280.

[27] C. Gui, Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method, Comm. Partial Differential Equations 21 (1996) 787–820.

[28] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - \lambda u + u^p = 0$ in $\mathbb{R}^N$, Arch. Ration. Mech. Anal. 105 (1989) 243–266.

[29] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. Part II, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 223–283.

[30] E.S. Noussair, S. Yan, On positive multipeak solutions of a nonlinear elliptic problem, J. London Math. Soc. 62 (2000) 213–227.

[31] Y.G. Oh, Existence of semi-classical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_\alpha$, Comm. Partial Differential Equations 13 (1988) 1499–1519.
[32] Y.G. Oh, On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential, Comm. Math. Phys. 131 (1990) 223–253.
[33] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992) 270–291.
[34] Y. Sato, Multi-peak positive solutions for nonlinear Schrödinger equations with critical frequency, Calc. Var. Partial Differential Equations 29 (2007) 365–395.
[35] X. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, Comm. Math. Phys. 153 (1993) 229–244.
[36] X. Wang, B. Zeng, On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions, SIAM J. Math. Anal. 28 (1997) 633–655.
[37] Z.Q. Wang, Existence and symmetry of multi-bump solutions for nonlinear Schrödinger equations, J. Diff. Equats. 159 (1999) 102–137.