

Supercritical elliptic problems in domains with small holes

Problèmes elliptiques supercritiques dans des domaines avec de petits trous

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Abstract

Let \mathcal{D} be a bounded, smooth domain in \mathbb{R}^N , $N \geq 3$, $P \in \mathcal{D}$. We consider the boundary value problem in $\Omega = \mathcal{D} \setminus B_\delta(P)$,

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

with p supercritical, namely $p > \frac{N+2}{N-2}$. We find a sequence

$$p_1 < p_2 < p_3 < \dots, \quad \text{with } \lim_{k \rightarrow +\infty} p_k = +\infty,$$

such that if p is given, with $p \neq p_j$ for all j , then for all $\delta > 0$ sufficiently small, this problem is solvable.

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Résumé

Soient \mathcal{D} un domaine borné régulier de \mathbb{R}^N , $N \geq 3$, et $P \in \mathcal{D}$. Nous définissons $\Omega = \mathcal{D} \setminus B_\delta(P)$ et nous nous intéressons au problème de Dirichlet

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{dans } \Omega,$$

$$u = 0 \quad \text{sur } \partial\Omega,$$

dans le cas où l'exposant p est supercritique, c'est à dire que $p > \frac{N+2}{N-2}$. Nous démontrons l'existence d'une suite d'exposants $p_1 < p_2 < p_3 < \dots$ qui tend vers l'infini et telle que, si p est fixé, $p \neq p_j$ pour tout j , il existe $\delta_p > 0$ assez petit pour lequel le problème ci-dessus admet une solution pour tout $\delta \leq \delta_p$.

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1. Introduction and statement of the main results

A basic model of nonlinear elliptic boundary problem is the Lane–Emden–Fowler equation,

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \Omega, \quad (1.1)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (1.2)$$

where Ω is a domain with smooth boundary in \mathbb{R}^N and $p > 1$.

A main characteristic of this problem is the role played by the critical exponent $p = \frac{N+2}{N-2}$ in the solvability question. When $1 < p < \frac{N+2}{N-2}$, a solution can be found as an extremal for the best constant in the compact embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, namely a minimizer of the variational problem

$$\inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2}{\left(\int_{\Omega} |u|^{p+1}\right)^{\frac{2}{p+1}}}.$$

When $p \geq \frac{N+2}{N-2}$, this minimization procedure fails, so does existence in general: Pohozaev [20] discovered that no solution exists in this case if the domain is strictly star-shaped. On the other hand Kazdan and Warner [13] observed that if Ω is an annulus, $\Omega = \{x \mid a < |x| < b\}$, compactness holds for any $p > 1$ within the class of radial functions, and a solution can again be found variationally without any constraint in p .

In the classical paper [3], Brezis and Nirenberg considered the critical case $p = \frac{N+2}{N-2}$ and proved that compactness, and hence solvability, is restored by the addition of a suitable linear term. Coron [4] used a variational approach to prove that (1.1), (1.2) is solvable for $p = \frac{N+2}{N-2}$ if Ω exhibits a *small hole*. Rey [22] established existence of multiple solutions if Ω exhibits several small holes. Bahri and Coron [1] established that solvability holds for $p = \frac{N+2}{N-2}$ whenever Ω has a non-trivial topology. The question by Rabinowitz, stated by Brezis in [2], whether the presence of non-trivial topology in the domain suffices for solvability in the supercritical case $p > \frac{N+2}{N-2}$, was answered negatively by Passaseo [18] by means of an example for $N \geq 4$ and $p > \frac{N+1}{N-3}$. If p is supercritical but close to critical, bubbling solutions are found in domains with small holes, see [8,9,14].

Except for results in domains involving symmetries or exponents close to critical, e.g. [7–10,15,16,19], solvability of (1.1), (1.2) in the supercritical case has been a widely open matter, particularly since variational machinery no longer applies, at least in its naturally adapted way for subcritical or critical problems.

In this paper we consider problem (1.1), (1.2) for exponents p above critical in a Coron's type domain: one exhibiting a small hole. Thus we assume in what follows that the domain Ω has the form

$$\Omega = \mathcal{D} \setminus B_{\delta}(Q) \quad (1.3)$$

where \mathcal{D} is a bounded domain with smooth boundary, $B_{\delta}(Q) \subset \mathcal{D}$ and $\delta > 0$ is to be taken small. Thus we consider the problem of finding classical solutions of

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \mathcal{D} \setminus B_{\delta}(Q), \quad (1.4)$$

$$u = 0 \quad \text{on } \partial\mathcal{D} \cup \partial B_{\delta}(Q). \quad (1.5)$$

Our main result states that there is a sequence of *resonant exponents*,

$$\frac{N+2}{N-2} < p_1 < p_2 < p_3 < \dots, \quad \text{with } \lim_{k \rightarrow +\infty} p_k = +\infty \quad (1.6)$$

such that if p is supercritical and differs from all elements of this sequence then problem (1.4), (1.5) is solvable whenever δ is sufficiently small.

Theorem 1. *There exists a sequence of the form (1.6) such that if $p > \frac{N+2}{N-2}$ and $p \neq p_j$ for all j , then there is a $\delta_0 > 0$ such that for any $\delta < \delta_0$, problem (1.4), (1.5) possesses at least one solution.*

While the min-max quantity yielding Coron’s solution [4], see also [22], suggests that the Morse index of that solution equals $N + 1$, our method of construction formally implies that the index for the solutions we find remains $N + 1$ as p grows until p_1 , while it grows to infinity as p increases more and more. This may indicate an obstruction in nature, besides the technical loss of Sobolev’s embedding, in obtaining general solvability results via min-max arguments for supercritical powers: not only geometry and topology of the domain are into play, but also their subtle interactions with special numerical values of the exponent.

In the background of our result is the problem

$$\Delta w + w^p = 0, \quad w > 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B_1(0)}, \tag{1.7}$$

$$w = 0 \quad \text{on } \partial B_1(0), \quad \limsup_{|x| \rightarrow +\infty} |x|^{2-N} w(x) < +\infty, \tag{1.8}$$

which is known to admit a unique radially symmetric solution $w(r)$ whenever $p > \frac{N+2}{N-2}$. The solutions we find have a profile similar to w suitably rescaled. More precisely, let us observe that

$$w_\delta(x) = \delta^{-\frac{2}{p-1}} w(\delta^{-1}|x - Q|) \tag{1.9}$$

solves uniquely the same problem with $B_1(0)$ replaced with $B_\delta(Q)$. The idea is to consider w_δ as a first approximation for a solution of problem (1.1), (1.2), provided that $\delta > 0$ is chosen small enough. What we shall prove is that an actual solution of the problem, which differs little from w_δ does exist. To this end, it is necessary to understand in rather fine terms the linearized operator around w_δ .

While we do not intend to express our result in most general forms, it is worthwhile to remark for instance that the result of Theorem 1 remains valid, with only minor modifications in the proof, for a problem of the form

$$\begin{aligned} \Delta u + u^p + \lambda u &= 0, \quad u > 0 \quad \text{in } \mathcal{D} \setminus B_\delta(Q), \\ u &= 0 \quad \text{on } \partial \mathcal{D} \cup \partial B_\delta(Q), \end{aligned}$$

where $\lambda < \lambda_1(\mathcal{D})$, the first eigenvalue of the Laplacian in \mathcal{D} . We can also get existence of multiple solutions in a domain of the form

$$\mathcal{D} \setminus \bigcup_{j=1}^m B_\delta(Q_j).$$

It is interesting to compare our result with one obtained recently in [6], for the exterior problem

$$\begin{aligned} \Delta u + u^p &= 0, \quad u > 0 \quad \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}}, \\ u &= 0 \quad \text{on } \partial \mathcal{D}, \quad \lim_{|x| \rightarrow +\infty} u(x) = 0 \end{aligned}$$

where \mathcal{D} is an arbitrary bounded domain, establishing in particular that this problem admits infinitely many solutions if $N \geq 4$ and $p > \frac{N+1}{N-3}$. These solutions are of a very different type from that of w : they are very small on bounded sets, while have slow decay, $u(x) \sim |x|^{-\frac{2}{p-1}}$. It is not expected that they can be used as approximations for an extra Dirichlet boundary condition taking place on the boundary of a large domain surrounding \mathcal{D} .

The question certainly opens on considering a non-spherical hole or, more generally, finding conditions which ensure solvability of rather general supercritical problems. A method beyond variational arguments or singular perturbations would be needed.

2. The operator $\Delta + pw^{p-1}$ on $\mathbb{R}^N \setminus B_1(0)$

The purpose of this section is to establish an invertibility theory for the linearized operator associated to w . We consider the problem

$$\Delta \phi + pw^{p-1}\phi = h \quad \text{in } \mathbb{R}^N \setminus \overline{B_1(0)}, \tag{2.1}$$

$$\phi = 0 \quad \text{on } \partial B_1(0), \quad \lim_{|x| \rightarrow +\infty} \phi(x) = 0. \tag{2.2}$$

2.1. Condition for non-resonance

We want to investigate under what conditions the homogeneous problem with $h = 0$ in (2.1), (2.2) admits only the trivial solution. To this end, let us consider the first eigenvalue of the problem

$$\psi'' + \frac{N-1}{r}\psi' + pw^{p-1}\psi + v\frac{\psi}{r^2} = 0, \tag{2.3}$$

$$\psi(1) = 0, \quad \psi(+\infty) = 0. \tag{2.4}$$

This eigenvalue is variationally characterized as

$$\nu(p) = \inf_{\psi \in \mathcal{E}} \frac{\int_1^\infty |\psi'|^2 r^{N-1} dr - p \int_1^\infty w^{p-1} |\psi|^2 r^{N-1} dr}{\int_1^\infty \psi^2 r^{N-3} dr}, \tag{2.5}$$

with

$$\mathcal{E} = \left\{ \psi \in C^1[1, \infty) \mid \psi(1) = 0, \int_1^\infty |\psi'(r)|^2 r^{N-1} dr < +\infty \right\}.$$

This quantity is well defined thanks to Hardy’s inequality,

$$\frac{(N-2)^2}{4} \int_1^\infty \psi^2 r^{N-3} dr \leq \int_1^\infty |\psi'|^2 r^{N-1} dr.$$

The number $\nu(p)$ is negative, since this Rayleigh quotient gets negative when evaluated at $\psi = w$. Using this fact, Hardy’s embedding and a simple compactness argument involving the fast decay of $w^{p-1} = o(r^{-4})$, yields the existence of an extremal for $\nu(p)$ which represents a positive solution to problem (2.3), (2.4) for $\nu = \nu(p)$. Let us consider now problem (2.1), (2.2) for $h = 0$, and assume that we have a solution ϕ . The symmetry of the domain $\mathbb{R}^N \setminus B_1(0)$ allows us to expand ϕ into spherical harmonics. We write ϕ as

$$\phi(x) = \sum_{k=0}^\infty \phi_k(r)\Theta_k(\theta), \quad r > 0, \theta \in S^{N-1},$$

where $\Theta_k, k \geq 0$, are the eigenfunctions of the Laplace–Beltrami operator $-\Delta_{S^{N-1}}$ on the sphere S^{N-1} , normalized so that they constitute an orthonormal system in $L^2(S^{N-1})$. We take Θ_0 to be a positive constant, associated to the eigenvalue 0 and $\Theta_i, 1 \leq i \leq N$, is an appropriate multiple of $\frac{x_i}{|x|}$ which has eigenvalue $\lambda_i = N - 1, 1 \leq i \leq N$. In general, λ_k denotes the eigenvalue associated to Θ_k , we repeat eigenvalues according to their multiplicity and we arrange them in an non-decreasing sequence. We recall that the set of eigenvalues is given by $\{j(N - 2 + j) \mid j \geq 0\}$.

The components ϕ_k then satisfy the differential equations

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2} \right) \phi_k = 0, \quad r \in (1, \infty), \tag{2.6}$$

$$\phi_k(1) = 0, \quad \phi_k(+\infty) = 0.$$

Let us consider first the radial mode $k = 0$, namely $\lambda_k = 0$. We observe that the function

$$Z_1(r) = rw'(r) + \frac{2}{p-1}w$$

satisfies

$$Z_1'' + \frac{N-1}{r}Z_1' + pw^{p-1}Z_1 = 0, \quad \forall r > 1,$$

but $Z_1(1) \neq 0$. We notice that Z_1 is one-signed for all large r . It follows then that a second generator of the solutions of this ODE is given, for large r , by the reduction of order formula,

$$Z_2 = Z_1(r) \int_R^r \frac{dr}{r^{N-1}Z_1^2}$$

but since at main order $Z_1(r) \sim cr^{2-N}$ we see that $Z_2(+\infty) \neq 0$. Since ϕ_0 is a linear combination of Z_1 and Z_2 it follows that the only possibility is $\phi_0 = 0$. Let us consider now mode 1, namely $k = 1, \dots, N - 1$, for which $\lambda_k = (N - 1)$. In this case we also have an explicit solution which does not vanish at $r = 1$ but it does at $r = +\infty$. Simply $Z_1(r) = w'(r)$. But the same argument as above gives us a second generator $Z_2(r) \sim r$ as $r \rightarrow +\infty$, hence again, the only possibility is that $\phi_k \equiv 0$ for all $k = 1, \dots, N$.

Let us consider now modes 2 or higher. Here unfortunately life is harder. Not only we do not have an explicit solution to the ODE to rely on, but it could be the case that a non-trivial solution exists. Let us assume this is the case for an arbitrary mode $k \geq N$. We claim that ϕ_k cannot change sign in $(1, \infty)$. In fact if it did, we begin by observing that it can only do it a finite number of times, since its behavior at infinity must be like eventually that of a decaying solution of the Euler's ODE

$$Z'' + \frac{N-1}{r}Z' - \frac{\lambda_k}{r^2}Z = 0$$

namely, at main order we must have

$$Z(r) = cr^{-\mu}(1 + o(1)), \quad \mu = -\frac{N-2}{2} - \frac{1}{2}\sqrt{(N-2)^2 + 4\lambda_k}.$$

Let $r_0 > 1$ be the last zero of ϕ_k , and let us assume that $\phi > 0$ on (r_0, ∞) . We observe now that since $\Delta w < 0$, $w'(r)$ has exactly one zero in $(1, \infty)$. Thanks to Sturm's theorem this zero must be less than r_0 . Hence $w' < 0$ in (r_0, ∞) . Let us observe now that

$$W(r) = r^{N-1}(w'\phi'_k - w''\phi_k)$$

satisfies in (r, ∞)

$$W'(r) = r^{N-3}(\lambda_k - \lambda_1)w'\phi_k < 0 \quad \text{in } (r_0, \infty),$$

while $W(r_0) < 0$ and $W(+\infty) = 0$, which is impossible. This shows that ϕ_k must be one-signed. Thus the only possibility for Eq. (2.6) to have a nontrivial solution for a given $k \geq N$ is that $\lambda_k = -\nu(p)$. Thus we have proven the following result

Lemma 2.1. *Assume that p is such that*

$$\nu(p) \neq -j(N - 2 + j) \quad \forall j = 2, 3, \dots \tag{2.7}$$

where $\nu(p)$ is the principal eigenvalue defined by (2.5). Then problem (2.3), (2.4) with $h = 0$ admits only the solution $\phi = 0$.

We will prove later that this non-resonance condition produces a good solvability theory for Eq. (2.1), (2.2). Before doing so we will describe the set of exponents p for which condition (2.7) fails. We will prove

Proposition 2.1. *For each $j \geq 2$ the set of numbers p for which $\nu(p) = -j(N - 2 + j)$ is non-empty and finite. In particular, there exists a sequence of the form*

$$\frac{N+2}{N-2} < p_1 < p_2 < p_3 < \dots; \quad p_j \rightarrow +\infty \text{ as } j \rightarrow +\infty, \tag{2.8}$$

such that condition (2.7) holds if and only if $p \neq p_j \forall j = 1, 2, \dots$

For the proof we need the following result, which contains elements of independent interest.

Lemma 2.2.

- (a) As $p \downarrow \frac{N+2}{N-2}$, we have that $\nu(p) \rightarrow -\lambda_1 = -(N - 1)$.
- (b) There exists a positive constant c_0 such that

$$\nu(p) = -c_0p^2(1 + o(1)) \quad \text{as } p \rightarrow +\infty. \tag{2.9}$$

- (c) The function $p \mapsto \nu(p)$ is real-analytic.

Proposition 2.1 is a direct consequence of this result. In fact, combining parts (a) and (b) we see that for each $j \geq 2$ the set of numbers p for which $\nu(p) = -j(N - 2 + j)$ is non-empty. Since $\nu(p)$ is a non-constant analytic function of p , this set can at most be finite. We actually believe that this set is a single point but have no proof of this.

Proof of Lemma 2.2 part (a). Let us set $p_0 = \frac{N+2}{N-2}$. An alternative way of writing Eq. (1.7), (1.8) and the eigenvalue problem (2.3), (2.4) is by means of the so-called Emden–Fowler transformation,

$$\tilde{w}(s) = r^{\frac{2}{p-1}} w(r), \quad \tilde{\psi}(s) = r^{\frac{2}{p-1}} \psi(r), \quad \text{where } r = e^s. \tag{2.10}$$

Then Eq. (1.7), (1.8) is converted into

$$\tilde{w}'' + \alpha \tilde{w}' - \beta \tilde{w} + \tilde{w}^p = 0, \quad \tilde{w}(0) = \tilde{w}(\infty) = 0, \quad s \in [0, \infty), \tag{2.11}$$

where

$$\alpha = N - 2 - \frac{4}{p - 1}, \quad \beta = \frac{2}{p - 1} \left(N - 2 - \frac{2}{p - 1} \right).$$

The eigenvalue problem (2.3), (2.4) becomes

$$\tilde{\psi}'' + \alpha \tilde{\psi}' - (\beta - \nu) \tilde{\psi} + p \tilde{w}^{p-1} \tilde{\psi} = 0, \quad \tilde{\psi}(0) = \tilde{\psi}(\infty) = 0, \quad s \in [0, \infty). \tag{2.12}$$

It is easy to see that as $p \rightarrow p_0$, $\alpha \rightarrow 0$, $\beta \rightarrow (N - 2)^2/4$ and

$$\tilde{w} = w_0(s - R_\alpha) + \text{lower order terms}, \tag{2.13}$$

where w_0 is the unique homoclinic solution of the limiting equation,

$$w_0'' - \frac{(N - 2)^2}{4} w_0 + w_0^{p_0} = 0, \quad w_0(0) = \max_{t \in \mathbb{R}} w_0(t), \quad w_0(\infty) = 0$$

and $R_\alpha \sim \log \frac{1}{|\alpha|} \rightarrow +\infty$ as $\alpha \rightarrow 0$. Therefore $\nu(p) \rightarrow -(N - 1)$ as $p \rightarrow p_0$, as desired. \square

In order to analyze the behavior of $\nu(p)$ for large p we need to understand the asymptotic behavior of w_p , the solution of (1.7), (1.8), where dependence on p is now emphasized. This can be done in exactly the same way as it was done in [11] in a fixed annulus $a < |x| < b$. In fact, by taking $a = 1$, $b = +\infty$ in [11], we can gather the following information.

Lemma 2.3. *As $p \rightarrow +\infty$, we have the validity of the following assertions.*

- (1) $w_p(x) \rightarrow w(x)$ in $C^0([1, +\infty))$, where setting $r_0 = 2^{\frac{1}{N-2}}$,
 $w(x) = 2(1 - |x|^{2-N})$ for $1 \leq |x| \leq r_0$, $w(|x|) = 2|x|^{2-N}$ for $|x| \geq r_0$.
- (2) $\|w_p\|_{L^\infty} = 1 + \frac{\log p}{p} + \frac{\gamma}{p} + o(\frac{1}{p})$, where γ is a generic constant.
- (3) $\frac{p}{\|w_p\|_{L^\infty}} (w_p(\epsilon_p r + r_p) - \|w_p\|_{L^\infty}) \rightarrow U(r) = \log \frac{4e^{\sqrt{2}r}}{(1 + e^{\sqrt{2}r})^2}$
 locally C^1 -uniformly in \mathbb{R} . Here
 $w_p(r_p) = \max_{r \geq 1} w_p(r)$.

Proof of Lemma 2.2 part (b). We shall split the proof into several steps:

Step 1. We have the following upper bound on $\nu(p)$.

$$\nu(p) \leq - \frac{(p - \frac{(p+1)^2}{4p}) \int_\Omega w_p^{2p}}{\int_\Omega w_p^{p+1}}. \tag{2.14}$$

In fact, testing the function $w_p^{\frac{p+1}{2}}$ in (2.5) and testing Eq. (1.7), (1.8) against w^p , we obtain (2.14).

Step 2. The following lower bound on $v(p)$ holds:

$$v(p) \geq - \frac{(p-1) \int_1^\infty w_p^{2p} r^{N+1} dr}{\int_1^\infty w_p^{p+1} r^{N-1} dr}. \tag{2.15}$$

In fact, since w_p is a minimizer for the radial energy functional

$$Q[u] = \frac{\int_1^\infty |u'|^2 r^{N-1} dr}{\left(\int_1^\infty u^{p+1} r^{N-1} dr\right)^{\frac{2}{p+1}}} \tag{2.16}$$

we obtain, by computing $Q[w_p + t\phi] - Q[w_p]$, that

$$\int_1^\infty [|\nabla\phi|^2 - pw_p^{p-1}\phi^2]r^{N-1} dr + (p-1) \frac{\left(\int_1^\infty w_p^p \phi r^{N-1} dr\right)^2}{\int_1^\infty w_p^{p+1} r^{N-1} dr} \geq 0, \quad \text{for all } \phi \in \mathcal{E}. \tag{2.17}$$

Inequality (2.15) then follows from (2.17) and Schwartz’s inequality.

Combining estimates (2.14) and (2.15), we get

Step 3. There exist two positive constants C_1 and C_2 such that

$$C_1 p^2 \leq -v_p \leq C_2 p^2. \tag{2.18}$$

In fact, by (3) of Lemma 2.3 and a similar argument as that of (4.13) of [11], we obtain

$$\int_1^\infty w_p^{p+1} r^{N-1} dr \sim 1, \quad \int_1^\infty w_p^{2p} r^{N-1} dr \sim p \tag{2.19}$$

from which, (2.18) follows.

Finally, we can prove (2.9): let us suppose for a subsequence $p_n \rightarrow \infty$, $v_{p_n}/p_n^2 \rightarrow c_0$. Let h_{p_n} be the corresponding eigenfunction. Without loss of generality, we may assume that $h_n(r_{p_n}) = 1$. Scaling

$$\epsilon_{p_n}^2 = \frac{P}{\|w_{p_n}\|_{L^\infty}}, \quad \tilde{h}_n(r) = h_n(\epsilon_{p_n} r + r_{p_n}).$$

By the same argument as in the proof of Theorem 1.5 of [11], we obtain a limiting equation for $h_\infty = \lim_{n \rightarrow +\infty} h_n$:

$$h_\infty'' - \lambda_\infty h_\infty + \operatorname{sech}^2\left(\frac{1}{\sqrt{2}}r\right)h = 0, \quad h_\infty(0) = 1, \quad h_\infty \leq 1, \tag{2.20}$$

where λ_∞ is defined by

$$\lambda_\infty = \lim_{n \rightarrow \infty} \frac{\epsilon_{p_n}^2 v_{p_n}}{r_{p_n}^2}. \tag{2.21}$$

Thus λ_∞ is a principal eigenvalue of (2.20) and $c_0 = \lambda_\infty r_0^2 \lim_{p \rightarrow \infty} (1/p \epsilon_p^2)$. (In fact, according to formula (5.8) of [23], we have $\lambda_\infty = \frac{1}{2}$.) This concludes the proof of part (b) of the lemma. \square

Next we will prove analyticity of $v(p)$. For this and also later purposes, it is convenient to carry out Kelvin’s transform to restate problem (1.7), (1.8) as an interior one in $B_1(0)$. Thus we set

$$w(r) = r^{2-N} w\left(\frac{1}{r}\right)$$

and find that w solves (1.7), (1.8) if and only if w is a classical, positive solution of

$$\Delta w + |x|^{p(N-2)-(N+2)} w^p = 0 \quad \text{in } B_1(0), \tag{2.22}$$

$$w = 0 \quad \text{on } \partial B_1(0). \tag{2.23}$$

Setting $\alpha = p(N - 2) - (N + 2)$, we observe that since $p > \frac{N+2}{N-2}$, then

$$p < \frac{N + 2 + 2\alpha}{N - 2},$$

a radial subcriticality condition which, by the way, ensures existence of a unique radial solution of (2.22), (2.23), and hence of (1.7), (1.8), see [17].

Naturally, Kelvin’s transform produces correspondence between linearized problems: If ϕ solves (2.1), (2.2) for a given h then setting

$$\tilde{\phi}(x) = |x|^{2-N}\phi\left(\frac{x}{|x|^2}\right), \quad \tilde{h}(x) = |x|^{-N-2}h\left(\frac{x}{|x|^2}\right), \tag{2.24}$$

we get the problem

$$\Delta \tilde{\phi} + |x|^{p(N-2)-(N+2)} p w^{p-1} \tilde{\phi} = \tilde{h} \quad \text{in } B_1(0), \tag{2.25}$$

$$\tilde{\phi} = 0 \quad \text{on } \partial B_1(0). \tag{2.26}$$

Proof of Lemma 2.2 part (c). We will show that the principal eigenvalue $v(p)$ in (2.5) is a real-analytic function.

We then need to analyze real-analyticity with respect to $p > 1$ of the radial solution $w(p)$. Of course this is not entirely obvious since the function $y \rightarrow y^p$ is not analytic if p is not an integer. Let ϕ_1 be a first positive eigenfunction of the Laplacian in $B_1(0)$ and consider the space C_1 of all radially symmetric continuous functions in $\bar{B}_1(0)$ for which $\|\phi_1^{-1}u\|_\infty < +\infty$. It was proven by Dancer [5] that if p_0 and u_0 are such that there exists a $\mu > 0$ for which $u_0 \geq \mu\phi$ then the map

$$(p, u) \in \mathbb{R} \times C_1 \mapsto (-\Delta)^{-1}(u^p) \in C_1$$

is analytic in a neighborhood of (p_0, u_0) (actually in a general domain). Dancer’s proof applies with no significant changes to establish that the same is true for the map

$$(p, u) \in \mathbb{R} \times C_\phi \mapsto (-\Delta)^{-1}(|x|^{p(N-2)-(N+2)}u^p) \in C_\phi.$$

The bottom line, is the fact that the application $\gamma > 0 \mapsto |x|^\gamma$ defines a real analytic map into $C(\bar{B}_1(0))$. Indeed we can expand

$$|x|^\gamma = \sum_{k=0}^{\infty} \frac{|x|^{\gamma_0} \log^k |x|}{k!} (\gamma - \gamma_0)^k.$$

Taking into account that for sufficiently large k ,

$$\sup_{|x|<1} |x|^{\gamma_0} |\log^k |x|| \leq \gamma_0^{-k} k^k e^{-k}$$

we see that the above power series is uniformly convergent on $|\gamma - \gamma_0|$ sufficiently small, thanks to Stirling’s formula. This fact is also in the background of Dancer’s proof to deal with the vanishing of u at the boundary in the proof of analyticity with respect to p . For analyticity with respect to u , we observe that

$$(u_0 + h)^p = u_0^p (1 + (h/u_0))^p$$

and a uniform convergent Taylor’s series can then be written for $\|h\|_{C_1}$ small. See Proposition 1 in [5] for the complete argument.

Now, $w = w(p)$ is the only solution of the problem

$$F(w, p) \equiv w - (-\Delta)^{-1}(|x|^{p(N-2)-(N+2)}w^p) = 0.$$

From what has been said, for each $p_0 > 1$ the map $F(u, p)$ is analytic into C_1 in a neighborhood of $(w(p_0), p_0)$. Besides, the map $F_u(w(p_0), p_0)$ is an isomorphism of C_1 since the linearized equation

$$\Delta \psi + |x|^{p(N-2)-(N+2)} p w^{p-1} \psi = 0 \quad \text{in } B_1(0),$$

$$\psi = 0 \quad \text{on } \partial B_1(0)$$

admits only the trivial radial solution, as it follows from Lemma 2.1. From the implicit function theorem in analytic version that the map $p \mapsto w(p)$ is analytic into C_ϕ . The same is the case with $p \mapsto |x|^{p(N-2)-(N+2)} w^{p-1}$. Finally, we observe that Kelvin’s transform implies that $v(p)$ can also be characterized as the first eigenvalue inside the class of radially symmetric functions of the problem

$$\begin{aligned} \Delta \psi + |x|^{p(N-2)-(N+2)} w^{p-1} \psi + \frac{v(p)}{|x|^2} \psi &= 0 \quad \text{in } B_1(0), \\ \psi &= 0 \quad \text{on } \partial B_1(0). \end{aligned}$$

Either a lengthy computation by hand or an application of the standard theory of eigenvalues for families of operators depending analytically on a parameter, as in [21,12] which can be adapted to our situation, yields that $v(p)$ is an analytic function. This finishes the proof. \square

2.2. Solvability of (2.1), (2.2)

We consider now the full problem (2.1), (2.2), namely

$$\begin{aligned} \Delta \phi + p w^{p-1} \phi &= h \quad \text{in } \mathbb{R}^N \setminus \overline{B_1(0)}, \\ \phi &= 0 \quad \text{on } \partial B_1(0), \quad \lim_{|x| \rightarrow +\infty} \phi(x) = 0. \end{aligned}$$

Our main result in this subsection concerns with solvability of this equation and estimates for the solution in appropriate norms. Let us fix a small number $\sigma > 0$ and consider the norms

$$\|\phi\|_* = \sup_{|x|>1} |x|^{N-2-\sigma} |\phi(x)| + \sup_{|x|>1} |x|^{N-1-\sigma} |\nabla \phi(x)| \tag{2.27}$$

and

$$\|h\|_{**} = \sup_{|x|>1} |x|^{N-\sigma} |h(x)|. \tag{2.28}$$

Proposition 2.2. *Assume that p satisfies condition (2.7). Then for any h with $\|h\|_{**} < +\infty$, problem (2.1), (2.2) has a unique solution $\phi = T(h)$ with $\|\phi\|_* < +\infty$. Besides, there exists a constant $C(p) > 0$ such that*

$$\|T(h)\|_* \leq C \|h\|_{**}.$$

Proof. The proof makes use of duality via Kelvin’s transform. Consider ϕ and h transformed into $\tilde{\phi}$ and \tilde{h} through the rule (2.24) into (2.25), (2.26),

$$\begin{aligned} \Delta \tilde{\phi} + |x|^{p(N-2)-(N+2)} w^{p-1} \tilde{\phi} &= \tilde{h} \quad \text{in } B_1(0), \\ \tilde{\phi} &= 0 \quad \text{on } \partial B_1(0). \end{aligned}$$

Then we have

$$|\tilde{h}(x)| \leq \|h\|_{**} |x|^{-2-\sigma}. \tag{2.29}$$

It follows in particular that, if σ is fixed small, $h \in L^q(B_1(0))$ for some $q > \frac{2N}{N+2}$, hence $h \in H^{-1}(B_1(0))$. From Lemma 2.1, it follows that only the trivial H_0^1 -solution is present for 0 right hand side. Existence of a unique weak solution $\tilde{\phi} \in H_0^1(B(0, 1))$ whose norm is controlled by a multiple of $\|h\|_{**}$. Let us now observe that

$$-\Delta |x|^{-\sigma} = \sigma(N - 2 - \sigma) |x|^{-2-\sigma},$$

hence, fixed σ we can find a $\rho(p, N, \sigma) > 0$ such that as well

$$-\Delta |x|^{-\sigma} - p |x|^{p(N-2)-(N+2)} w^{p-1} |x|^{-\sigma} \geq \frac{1}{2} \sigma(N - 2 - \sigma) r^{-2-\sigma}, \quad |x| < \rho. \tag{2.30}$$

Since h is bounded by a σ -dependent multiple of $\|h\|_{**}$ on, say, $\frac{\rho}{2} < |x| < 1$, elliptic estimates yield that

$$\|\phi\|_{L^\infty(|x| \geq \rho)} \leq C \|h\|_{**}$$

with C depending on N, p, σ . Then from (2.29), (2.30) and maximum principle in $|x| < \rho$, we deduce that

$$|\tilde{\phi}(x)| \leq C|x|^{-\sigma} \|h\|_{**}, \quad |x| < 1.$$

Hence

$$\| |x|^{N-2-\sigma} \phi \|_{\infty} = \| |x|^{\sigma} \tilde{\phi} \|_{\infty} \leq C \|h\|_{**}.$$

The conclusion desired for $\nabla \phi$ follows by scaling: consider a ball radius R centered at a point \bar{x} with $|\bar{x}| = 2R$, for $R > 5$. Set

$$\hat{\phi}(y) = R^{2-N+\sigma} \phi(\bar{x} + Ry).$$

Then

$$\Delta \hat{\phi} + pR^2 \hat{w}^{p-1} \hat{\phi} = R^{N-\sigma} \hat{h}, \quad y \in B(0, 1).$$

Clearly in this ball

$$\| R^N \hat{h} \|_{\infty} \leq C \|h\|_{**}, \quad R^2 \hat{w}^{p-1} = O(R^{-2}), \quad \| \hat{\phi} \|_{\infty} \leq C \|h\|_{**}.$$

Elliptic estimates then imply

$$|\nabla \hat{\phi}(0)| \leq C \|h\|_{**}$$

or

$$|\nabla \phi(\bar{x})| \leq C \|h\|_{**} |\bar{x}|^{1-N+\sigma}.$$

Since \bar{x} is arbitrary with $|\bar{x}| > 5$, the desired conclusions follow. This finishes the proof. \square

3. The operator $\Delta + pw^{p-1}$ in $\delta^{-1}\mathcal{D} \setminus B_1(0)$

In this section and in what follows we shall assume that $Q = 0$, and consider the large expanded domain $\mathcal{D}_{\delta} = \delta^{-1}\mathcal{D}$. We shall carry out a gluing procedure that will permit to establish the same conclusion of Proposition 2.2 in this domain, provided that δ is taken sufficiently small. Thus we consider now the linear problem

$$\Delta \phi + pw^{p-1} \phi = h \quad \text{in } \mathcal{D}_{\delta} \setminus \bar{B}_1(0), \tag{3.1}$$

$$\phi = 0 \quad \text{on } \partial B_1(0) \cup \partial \mathcal{D}_{\delta}. \tag{3.2}$$

We consider the same norms as in (2.27), (2.28) restricted to this domain.

Proposition 3.1. *Assume that p satisfies condition (2.7). Then there is a number δ_0 such that for all $\delta < \delta_0$ and any h with $\|h\|_{**} < +\infty$, problem (3.1), (3.2) has a unique solution $\phi = T_{\delta}(h)$ with $\|\phi\|_* < +\infty$. Besides, there exists a constant $C(p, \mathcal{D}) > 0$ such that*

$$\|T_{\delta}(h)\|_* \leq C \|h\|_{**}.$$

Proof. We assume with no loss of generality that the domain \mathcal{D} contains the ball $B_3(0)$. Let us consider a smooth, radial cut off $\eta(|y|)$ which equals one on $|y| < 2$ and vanishes identically for $|y| > 3$. We consider also a second cut-off $\zeta(|y|)$ which equals 1 on $|y| < 1$ and it is 0 for $|y| > 2$. In particular we have of course $\eta\zeta = \zeta$. Correspondingly, we also write

$$\eta_{\delta}(x) = \eta(\delta|x|), \quad \zeta_{\delta}(x) = \zeta(\delta|x|).$$

We look for a solution ϕ to problem (3.1), (2.26) in the form

$$\phi = \eta_{\delta}\varphi + \psi,$$

where ϕ and ψ are required to satisfy the following system.

$$\begin{cases} \Delta\varphi + pw^{p-1}\varphi = -p\zeta_\delta w^{p-1}\psi + \zeta_\delta h & \text{in } \mathbb{R}^N \setminus B_1(0), \\ \varphi = 0 & \text{on } \partial B_1(0), \\ \varphi(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty, \end{cases} \tag{3.3}$$

$$\begin{cases} \Delta\psi + p(1 - \zeta_\delta)w^{p-1}\psi = -2\nabla\eta_\delta\nabla\varphi - \varphi\Delta\eta_\delta + (1 - \zeta_\delta)h & \text{in } \mathcal{D}_\delta, \\ \psi = 0 & \text{on } \partial\mathcal{D}_\delta \cup \partial B_1(0). \end{cases} \tag{3.4}$$

We shall solve Eq. (3.4) for ψ in terms of φ and h . To do so, let us consider the linear problem

$$\begin{cases} \Delta\psi + p(1 - \zeta_\delta)w^{p-1}\psi = g & \text{in } \mathcal{D}_\delta \setminus B_1(0), \\ \psi = 0 & \text{on } \partial\mathcal{D}_\delta \cup \partial B_1(0) \end{cases} \tag{3.5}$$

for $g \in L^\infty(\mathcal{D}_\delta \cup \partial B_1(0))$. Scaling back δ by setting for any function ρ , $\tilde{\rho}(x) = \rho(\delta^{-1}x)$, we see that problem (3.5) is equivalent to

$$\begin{cases} \Delta\tilde{\psi} + p(1 - \zeta)\delta^{-2}\tilde{w}^{p-1}\psi = \delta^{-2}\tilde{g} & \text{in } \mathcal{D} \setminus B_\delta(0), \\ \psi = 0 & \text{on } \partial\mathcal{D} \cup \partial B_\delta(0). \end{cases}$$

We see that

$$p(1 - \zeta)\delta^{-2}\tilde{w}^{p-1} = o(\delta^2) < \lambda_1(\mathcal{D}) < \lambda_1(\mathcal{D} \setminus B_\delta(0))$$

if δ is taken sufficiently small. Hence this problem can be solved uniquely for $\tilde{\psi}$. In terms of ψ we get in addition the estimate

$$\|\psi\|_\infty \leq C\delta^{-2}\|g\|_\infty,$$

where C does not depend on δ . ψ defines of course a linear operator. Let us now go back to Eq. (3.4). Then this problem can be solved uniquely, as a linear operator of the pair (φ, h) , which we simply call $\psi(\varphi, h)$. Setting

$$g = -2\nabla\eta_\delta\nabla\varphi - \varphi\Delta\eta_\delta + (1 - \zeta_\delta)h$$

we easily obtain that

$$\|g\|_\infty \leq C[\delta^{N-\sigma}\|\varphi\|_* + \delta^{N-\sigma}\|h\|_{**}],$$

and hence

$$\|\psi(\varphi, h)\|_\infty \leq C[\delta^{N-2-\sigma}\|\varphi\|_* + \delta^{N-2-\sigma}\|h\|_{**}]. \tag{3.6}$$

Let us replaced this ψ into Eq. (3.3). We have thus a solution of the full system if we solve the fixed point problem

$$\varphi = T(-p\zeta_\delta w^{p-1}\psi(\varphi, h) + \zeta_\delta h), \tag{3.7}$$

where T is the linear operator defined by Proposition 3.1. We make now the observation that, assuming also $\sigma < (N - 2)(p - 1) - 4$,

$$\begin{aligned} |x|^{N-\sigma}w^{p-1}|\psi(\varphi, h)| &\leq |x|^{N-\sigma-(N-2)(p-1)}\delta^{N-2-\sigma}[\|\varphi\|_* + \|h\|_{**}] \\ &\leq |x|^{N-2\sigma-4}\delta^{N-2-\sigma}[\|\varphi\|_* + \|h\|_{**}] \leq |x|^{-2}\delta^\sigma[\|\varphi\|_* + \|h\|_{**}], \end{aligned}$$

so that

$$\|p\zeta_\delta w^{p-1}\psi(\varphi, h)\|_{**} \leq C\delta^\sigma[\|\varphi\|_* + \|h\|_{**}].$$

From here and contraction mapping principle, we get then that if δ is chosen sufficiently small, then (3.7) can be solved uniquely in the form $\varphi = T_\delta(h)$ where the bounds for T_δ are the same as those for T , independent of δ . This concludes the proof. \square

4. Proof of Theorem 1

Let us assume the validity of condition of condition (2.7) or, equivalently, that $p \neq p_j$ for all j , with p_j the sequence in (2.8). Problem (1.4), (1.5) is, after setting $v(x) = \delta^{\frac{2}{p-1}} u(\delta x)$, equivalent to

$$\Delta v + v^p = 0 \quad \text{in } \mathcal{D}_\delta \setminus \overline{B_1(0)}, \tag{4.1}$$

$$v = 0 \quad \text{on } \partial B_1(0) \cup \partial \mathcal{D}_\delta. \tag{4.2}$$

Let us consider the smooth cut-off function η_δ , introduced in the previous section, which equals 1 in $B(0, 2\delta^{-1})$ and 0 outside $B(0, 3\delta^{-1})$. We search for a solution v to problem (4.1), (4.2) of the form

$$v = \eta_\delta w + \phi,$$

which is equivalent to the following problem for ϕ :

$$\Delta \phi + p w^{p-1} \phi = N(\phi) + E \quad \text{in } \mathcal{D}_\delta \setminus \overline{B_1(0)}, \tag{4.3}$$

$$\phi = 0 \quad \text{on } \partial B_1(0) \cup \partial \mathcal{D}_\delta, \tag{4.4}$$

where

$$N(\phi) = N_1(\phi) + N_2(\phi),$$

$$N_1(\phi) = -(\eta_\delta w + \phi)^p + (\eta_\delta w)^p + p(\eta_\delta w)^{p-1} \phi,$$

$$N_2(\phi) = p(1 - \eta_\delta^{p-1}) w^{p-1} \phi,$$

and

$$E = -\Delta(\eta_\delta w) - (\eta_\delta w)^p.$$

According to Proposition 3.1 we thus have a solution to (4.1), (4.2) if ϕ solves the fixed point problem

$$\phi = T_\delta(N(\phi) + E). \tag{4.5}$$

Let us estimate E . We have, explicitly,

$$-E = \eta(\eta_\delta^{p-1} - 1)w^p + 2\nabla\eta_\delta\nabla w + w\Delta\eta_\delta.$$

We clearly have, globally, $|E(x)| \leq C\delta^N$ and hence

$$\|E\|_{**} \leq C\delta^\sigma. \tag{4.6}$$

Let us measure now $N(\phi)$. We observe that

$$\|N_2(\phi)\|_{**} = \|p(1 - \eta_\delta^{p-1})w^{p-1}\phi\|_{**} \leq C \sup_{|x| \geq \delta^{-1}} |x|^{N-\sigma} w(x)^{p-1} |\phi(x)| \leq C\delta^2 \|\phi\|_*. \tag{4.7}$$

Next we shall now estimate $\|N_1(\phi)\|_{**}$. Let us assume first $p < 2$. Then we estimate

$$|x|^{N-\sigma} |N_1(\phi)| \leq C|x|^{N-\sigma} |\phi(x)|^p \leq |x|^{N-\sigma} |x|^{-N-2} \|\phi\|_*^p \leq C\|\phi\|_*^p,$$

so that

$$\|N_1(\phi)\|_{**} \leq C\|\phi\|_*^p.$$

Let us assume now $p \geq 2$. In this case we have

$$|N_1(\phi)| \leq C(w^{p-2}\phi^2 + |\phi|^p).$$

Now, we directly check that

$$|x|^{N-\sigma} w^{p-2}\phi^2 \leq C|x|^{(p-2)(2-N)-N+4+\sigma} \|\phi\|_*^2.$$

The power of $|x|$ in the last expression is always negative. In fact, this is obvious if $N \geq 5$, while if $N = 3, 4$ supercriticality implies $p \geq 3$. On the other hand,

$$|x|^{N-\sigma} |\phi|^p \leq C|x|^{N-\sigma-p(N-2-\sigma)} \|\phi\|_*^p \leq |x|^{-2+(p-1)\sigma} \|\phi\|_*^p.$$

We conclude from these estimates that, for any $p > \frac{N+2}{N-2}$,

$$\|N_1(\phi)\|_{**} \leq C(\|\phi\|_*^p + \|\phi\|_*^2). \quad (4.8)$$

Let us consider now the operator

$$\mathcal{T}(\phi) = T_\delta(N(\phi) + E)$$

defined in the region

$$\mathcal{B} = \{\phi \in C^1(\overline{\mathcal{D}}_\delta \setminus B_1(0)) \mid \|\phi\|_* \leq \delta^{\frac{\sigma}{2}}\}.$$

Using estimates (4.6), (4.8), (4.7) we immediately get that $\mathcal{T}(\mathcal{B}) \subset \mathcal{B}$, provided that δ is sufficiently small. We observe that, in the bounded domain $\mathcal{D}_\delta \setminus B_1(0)$,

$$T_\delta = (\Delta + pw^{p-1})^{-1}$$

applies boundedly C^0 into $C^{1,\alpha}$, hence compactly into C^1 . It follows that the map \mathcal{T} is actually compact on the closed, bounded set of C^1 given by \mathcal{B} . The existence of a fixed point of \mathcal{T} on \mathcal{B} thus follows from Schauder's theorem. This concludes the proof of the theorem. \square

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