

Strong instability of solitary waves for nonlinear Klein–Gordon equations and generalized Boussinesq equations

Instabilité forte d'ondes solitaires pour des équations de Klein–Gordon non linéaires et des équations généralisées de Boussinesq

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Abstract

We study here instability problems of standing waves for the nonlinear Klein–Gordon equations and solitary waves for the generalized Boussinesq equations. It is shown that those special wave solutions may be strongly unstable by blowup in finite time, depending on the range of the wave's frequency or the wave's speed of propagation and on the nonlinearity.

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Résumé

On étudie des problèmes d'instabilité des ondes stationnaires pour des équations de Klein–Gordon non linéaires et des ondes solitaires pour des équations généralisées de Boussinesq. On établit que ces solutions d'ondes spéciales peuvent être fortement instables par explosion en temps fini, selon le rang de la fréquence des ondes ou la vitesse de propagation des ondes et la non-linéarité.

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1. Introduction

In this paper, we study strong instability of standing wave solutions $e^{i\omega t}\phi_\omega(x)$ for the nonlinear Klein–Gordon equation

$$\partial_t^2 u - \Delta u + u - |u|^{p-1}u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.1}$$

and of solitary wave solutions $\phi_\omega(x - \omega t)$ for the generalized Boussinesq equation

$$\partial_t^2 u - \partial_x^2 u + \partial_x^2(\partial_x^2 u + |u|^{p-1}u) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{1.2}$$

where $n \in \mathbb{N}$, $-1 < \omega < 1$, $p > 1$, $p < 1 + 4/(n - 2)$ if $n \geq 3$, and ϕ_ω is the ground state, i.e., the unique positive radially symmetric solution in $H^1(\mathbb{R}^n)$ of the equation

$$-\Delta\phi + (1 - \omega^2)\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n. \tag{1.3}$$

See Strauss [28] and Berestycki and Lions [3] for the existence, and Kwong [14] for the uniqueness of ϕ_ω .

The stability and instability of the ground state standing waves $e^{i\omega t}\phi_\omega(x)$ for (1.1) have been studied by many authors. Berestycki and Cazenave [1] proved that $e^{i\omega t}\phi_\omega(x)$ is strongly unstable by blowup (see Definition 1.2 below) when $\omega = 0$ (see also Payne and Sattinger [24] and Shatah [26]). Shatah [25] proved that $e^{i\omega t}\phi_\omega(x)$ is orbitally stable when $p < 1 + 4/n$ and $\omega_c < |\omega| < 1$, where

$$\omega_c = \sqrt{\frac{p - 1}{4 - (n - 1)(p - 1)}}.$$

On the other hand, Shatah and Strauss [27] proved that $e^{i\omega t}\phi_\omega(x)$ is orbitally unstable when $p < 1 + 4/n$ and $|\omega| < \omega_c$ or when $p \geq 1 + 4/n$ and $|\omega| < 1$. Ohta and Todorova [22] proved that $e^{i\omega t}\phi_\omega(x)$ is strongly unstable by blowup when $n \geq 3$ and $(p + 3)\omega^2 \leq (p - 1)$. Recently, it was proved by Ohta and Todorova [23] that $e^{i\omega t}\phi_\omega(x)$ is strongly unstable by blowup when $n \geq 2$, $p < 1 + 4/n$ and $|\omega| \leq \omega_c$ or when $n \geq 2$, $1 + 4/n \leq p < 1 + 4/(n - 1)$ and $|\omega| < 1$.

For related results for the nonlinear Schrödinger equations, see [1,9,29,30], and for general theory of orbital stability and instability of solitary waves, see Grillakis, Shatah and Strauss [12,13].

In view of the result of Ginibre and Velo [11], the Cauchy problem for (1.1) is locally well-posed in the energy space $X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, that is, for any $(u_0, u_1) \in X$ there exists a unique solution $\vec{u} = (u, \partial_t u) \in C([0, T_{\max}), X)$ of (1.1) with $\vec{u}(0) = (u_0, u_1)$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} \|\vec{u}(t)\|_X = \infty$. Moreover, the solution $\vec{u}(t)$ satisfies the conservation laws of energy and charge

$$E(\vec{u}(t)) = E(u_0, u_1), \quad Q(\vec{u}(t)) = Q(u_0, u_1), \quad t \in [0, T_{\max}),$$

where

$$E(u, v) = \int_{\mathbb{R}^n} \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 + \frac{1}{2} |v|^2 - \frac{1}{p+1} |u|^{p+1} \right\} dx$$

and

$$Q(u, v) = \text{Im} \int_{\mathbb{R}^n} \bar{u}v dx.$$

In what follows, we put $\vec{\phi}_\omega = (\phi_\omega, i\omega\phi_\omega)$. Then, note that

$$E'(\vec{\phi}_\omega) - \omega Q'(\vec{\phi}_\omega) = 0.$$

Orbital stability of standing waves for (1.1) refers to stability up to translations and phase shifts. More precisely,

Definition 1.1 (*Orbital stability and instability*). We say that a standing wave $e^{i\omega t}\phi_\omega(x)$ is orbitally stable for (1.1) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $(u_0, u_1) \in X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ with $\|(u_0, u_1) - \vec{\phi}_\omega\|_X < \delta$, the solution $\vec{u}(t)$ of (1.1) with initial value $\vec{u}(0) = (u_0, u_1)$ exists for all $t \in [0, \infty)$ and satisfies

$$\sup_{0 \leq t < \infty} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^n} \|\vec{u}(t) - e^{i\theta}\vec{\phi}_\omega(\cdot + y)\|_X < \varepsilon.$$

Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be orbitally unstable.

Definition 1.2 (*Strong instability by blowup*). We say that a standing wave $e^{i\omega t}\phi_\omega(x)$ is strongly unstable by blowup if for any $\varepsilon > 0$, there exists $(u_0, u_1) \in X$ such that $\|(u_0, u_1) - \vec{\phi}_\omega\|_X < \varepsilon$ and the solution $\vec{u}(t)$ of (1.1) with initial value $\vec{u}(0) = (u_0, u_1)$ blows up in finite time.

In view of the above definitions of instability, if the standing wave $e^{i\omega t}\phi_\omega(x)$ is strongly unstable by blowup, then it is orbitally unstable.

The principal result of the present paper for (1.1) is the following.

Theorem 1.3. Assume that $p > 1$, $p < 1 + 4/(n - 2)$ if $n \geq 3$, $-1 < \omega < 1$, and

$$\begin{cases} 0 < 2(p + 1)\omega^2 < p - 1 & \text{if } n = 1, \\ 0 < (p + 3)\omega^2 \leq p - 1 & \text{if } n \geq 2. \end{cases} \tag{1.4}$$

Let ϕ_ω be the ground state of (1.3). Then the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is strongly unstable by blowup.

As mentioned above, the strong instability by blowup of standing waves $e^{i\omega t}\phi_\omega(x)$ has been proved by [1] for the case $\omega = 0$ and $n \geq 1$, and by [22] for the case $(p + 3)\omega^2 \leq (p - 1)$ and $n \geq 3$. In the proof of Theorem 1.3, we need to assume that $\omega \neq 0$ for technical reasons (see (2.1) and the proof of Proposition 2.1 below). Theorem 1.3 gives a new result for the case $n = 1, 2$, which is a natural extension of the result for the case $n \geq 3$ by [22]. Although the result for the case $n \geq 3$ is not new, the proof is slightly simpler than the one in [22]. In fact, the essential point in the proof of [22] was to introduce two appropriate invariant sets for the flow of (1.1) (see (2.4) in [22]), while in the present paper we use only one invariant set Σ_1 which is defined by (2.5).

Now we turn attention to the Boussinesq equation (1.2). The original Boussinesq equation was the first model for the propagation of weakly nonlinear dispersive long surface and internal waves [5,7]. Eq. (1.2) has the equivalent form

$$\begin{cases} u_t = v_x, \\ v_t = (u - u_{xx} - |u|^{p-1}u)_x. \end{cases} \tag{1.5}$$

It is known in [17] that the Cauchy problem for (1.5) is locally well-posed in the space $X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Moreover, the solution $\vec{u}(t) = (u(t), v(t))$ with initial value (u_0, v_0) in $C([0, T_{\max}), X)$ satisfies the conservation laws

$$E(\vec{u}(t)) = E(u_0, v_0), \quad Q(\vec{u}(t)) = Q(u_0, v_0), \quad 0 \leq t < T_{\max},$$

where

$$E(u, v) = \int_{\mathbb{R}} \left\{ \frac{1}{2} |\partial_x u|^2 + \frac{1}{2} |u|^2 + \frac{1}{2} |v|^2 - \frac{1}{p+1} |u|^{p+1} \right\} dx$$

and

$$V(u, v) = \int_{\mathbb{R}} uv \, dx.$$

Put $\vec{\phi}_\omega = (\phi_\omega, -\omega\phi_\omega)$. Then, a simple computation shows that

$$E'(\vec{\phi}_\omega) + \omega V'(\vec{\phi}_\omega) = 0.$$

The stability of solitary wave $\vec{\phi}_\omega(x - \omega t)$ up to translations can be defined in the following.

Definition 1.4 (*Orbital stability and instability*). We say that a solitary wave solution $\vec{\phi}_\omega(x - \omega t)$ of (1.5) is orbitally stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $(u_0, v_0) \in X = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with

$$\|(u_0, u_1) - \vec{\phi}_\omega\|_X < \delta,$$

the solution $\vec{u}(t) = (u(t), v(t))$ of (1.5) with initial value $\vec{u}(0) = (u_0, v_0)$ satisfies

$$\sup_{0 \leq t < \infty} \inf_{y \in \mathbb{R}} \|\vec{u}(t) - \vec{\phi}_\omega(\cdot + y)\|_X < \varepsilon.$$

Otherwise, $\vec{\phi}_\omega(x - \omega t)$ is considered to be orbitally unstable.

Definition 1.5 (*Strong instability by blowup*). We say that a solitary wave solution $\vec{\phi}_\omega(x - \omega t)$ is strongly unstable by blowup if for any $\varepsilon > 0$, there exists $(u_0, v_0) \in X$ such that $\|(u_0, v_0) - \vec{\phi}_\omega\|_X < \varepsilon$ and the solution $\vec{u}(t)$ of (1.5) with initial value $\vec{u}(0) = (u_0, v_0)$ blows up in finite time.

The stability of solitary waves $\vec{\phi}_\omega(x - \omega t)$ with $|\omega| < 1$ has been the subject of a number of studies, and a satisfactory stability theory is now in hand. For example, Bona and Sachs [4] proved that the solitary wave $\vec{\phi}_\omega(x - \omega t)$ is orbitally stable if $1 < p < 5$, $(p-1)/4 < \omega^2 < 1$. Liu [17] proved the orbital instability under the conditions $1 < p < 5$ and $\omega^2 < (p-1)/4$ or $p \geq 5$ and $|\omega| < 1$. On the other hand, Liu [18] showed that solitary wave $\vec{\phi}_\omega(x - \omega t)$ is strongly unstable by blowup for the wave speed $\omega = 0$ (see also [19]).

The principal result for (1.5) is stated as follows.

Theorem 1.6. *Assume $1 < p < \infty$ and $0 < 2(p+1)\omega^2 < p-1$. Let ϕ_ω be the ground state of (1.3). Then the solitary wave solution $\vec{\phi}_\omega(x - \omega t) = (\phi_\omega(x - \omega t), -\omega\phi_\omega(x - \omega t))$ of (1.5) is strongly unstable by blowup.*

In next section, we give the proofs of Theorems 1.3 and 1.6. The method of the proofs is based on the idea by Berestycki and Cazenave [1] in the study of strong instability of standing waves for the nonlinear Schrödinger equation as well as the nonlinear Klein–Gordon equation. The crucial point in their proof is to construct some suitable invariant sets under the flow of the evolution equations. Then the strong instability results are obtained by use of the virial identities with the variational characterization of ground states. This method is recently developed by many authors [18–23]. To optimally use these virial identities, we need to construct some particular invariant sets of the flows of (1.1) or (1.5).

Notation. As above and henceforth, we denote the norm of the Lebesgue space $L^p(\mathbb{R}^n)$ by $\|\cdot\|_p$ for $1 \leq p \leq \infty$. The function space in which we shall work is the Sobolev space $X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

2. Proof of strong instability

In this section, we prove the main results, Theorems 1.3 and 1.6.

Define functionals J_ω and K_ω on $H^1(\mathbb{R}^n)$ by

$$J_\omega(v) = \frac{1}{2} \|\nabla v\|_2^2 + \frac{1-\omega^2}{2} \|v\|_2^2 - \frac{1}{p+1} \|v\|_{p+1}^{p+1},$$

$$K_\omega(v) = \frac{2\alpha+n-2}{2} \|\nabla v\|_2^2 + \frac{(1-\omega^2)(2\alpha+n)}{2} \|v\|_2^2 - \frac{(p+1)\alpha+n}{p+1} \|v\|_{p+1}^{p+1},$$

where we put

$$\alpha = \frac{(p-1) - (p+3)\omega^2}{2(p-1)\omega^2}. \quad (2.1)$$

Note that

$$K_\omega(v) = \partial_\lambda J_\omega(v_\lambda)|_{\lambda=1}, \quad v_\lambda(x) = \lambda^\alpha v(x/\lambda),$$

and by assumption (1.4) we have $2\alpha > 1$ if $n = 1$; $\alpha \geq 0$ if $n \geq 2$; and $2\alpha + n - 2 > 0$ except the case $n = 2$ and $(p+3)\omega^2 = p-1$.

Moreover, we put

$$d_\omega = \inf\{J_\omega(v) : v \in H^1(\mathbb{R}^n) \setminus \{0\}, K_\omega(v) = 0\},$$

$$\mathcal{S}_\omega = \{v \in H^1(\mathbb{R}^n) \setminus \{0\} : J'_\omega(v) = 0\},$$

$$\mathcal{G}_\omega = \{w \in \mathcal{S}_\omega : J_\omega(w) \leq J_\omega(v) \text{ for all } v \in \mathcal{S}_\omega\},$$

$$\mathcal{M}_\omega = \{v \in H^1(\mathbb{R}^n) \setminus \{0\} : J_\omega(v) = d_\omega, K_\omega(v) = 0\}$$

and

$$\begin{aligned} \tilde{J}_\omega(v) &= J_\omega(v) - \frac{1}{(p+1)\alpha + n} K_\omega(v) \\ &= \frac{(p-1)\alpha + 2}{(p+1)\alpha + n} \left\{ \frac{1}{2} \|\nabla v\|_2^2 + \frac{(p-1)\alpha}{(p-1)\alpha + 2} \cdot \frac{1-\omega^2}{2} \|v\|_2^2 \right\}. \end{aligned}$$

Then, we have

$$\tilde{J}_\omega(v) = \frac{(p-1)\alpha + 2}{(p+1)\alpha + n} \left\{ \frac{1}{2} \|\nabla v\|_2^2 + \alpha\omega^2 \|v\|_2^2 \right\}, \tag{2.2}$$

$$d_\omega = \inf \{ \tilde{J}_\omega(v) \mid v \in H^1(\mathbb{R}^n) \setminus \{0\}, K_\omega(v) = 0 \} \geq 0. \tag{2.3}$$

Our first goal is to show that the ground state ϕ_ω belongs to the set \mathcal{M}_ω under the assumption (1.4). To this end, we need to consider the case $n = 2$ and $(p + 3)\omega^2 = p - 1$ separately. For this case, see [2] and [8, Theorem 8.1.5]. In what follows, we assume that $(p + 3)\omega^2 < p - 1$ if $n = 2$.

Proposition 2.1. *If $n = 1$ or $n \geq 3$ assume (1.4); if $n = 2$ assume $(p + 3)\omega^2 < p - 1$. Then the set \mathcal{M}_ω is not empty.*

To prove Proposition 2.1, we need the following lemmas.

Lemma 2.2. *If $v \in H^1(\mathbb{R}^n)$ satisfies $K_\omega(v) < 0$, then $\tilde{J}_\omega(v) > d_\omega$.*

Proof. Let $v \in H^1(\mathbb{R}^n)$ satisfy $K_\omega(v) < 0$. Then, there exists $\lambda_1 \in (0, 1)$ such that $K_\omega(\lambda_1 v) = 0$. Since $v \neq 0$, we have $d_\omega \leq \tilde{J}_\omega(\lambda_1 v) = \lambda_1^2 \tilde{J}_\omega(v) < \tilde{J}_\omega(v)$. \square

Remark. In view of relation (2.3) and Lemma 2.2, it is found that

$$d_\omega = \inf \{ \tilde{J}_\omega(v) : v \in H^1(\mathbb{R}^n) \setminus \{0\}, K_\omega(v) \leq 0 \}.$$

The following compactness lemmas are obtained by Fröhlich, Lieb and Loss [10], Lieb [16] and Brezis and Lieb [6].

Lemma 2.3. [10,16] *Let $\{f_j\}$ be a bounded sequence in $H^1(\mathbb{R}^n)$. Assume that there exists $q \in (2, 2^*)$ such that $\limsup_{j \rightarrow \infty} \|f_j\|_q > 0$, where $2^* = \infty$ if $n = 1, 2$, and $2^* = 2n/(n - 2)$ if $n \geq 3$. Then, there exist $\{y_j\} \subset \mathbb{R}^n$ and $f \in H^1(\mathbb{R}^n) \setminus \{0\}$ such that $\{f_j(\cdot - y_j)\}$ has a subsequence that converges to f weakly in $H^1(\mathbb{R}^n)$.*

Lemma 2.4. [6] *Let $1 \leq q < \infty$ and $\{f_j\}$ be a bounded sequence in $L^q(\mathbb{R}^n)$. Assume that $f_j \rightarrow f$ a.e. in \mathbb{R}^n . Then we have*

$$\|f_j\|_q^q - \|f_j - f\|_q^q - \|f\|_q^q \rightarrow 0.$$

Proof of Proposition 2.1. Let $\{v_j\}$ be a minimizing sequence of (2.3). By (2.2), we see that $\{v_j\}$ is bounded in $H^1(\mathbb{R}^n)$. Indeed, when $n = 1, 2$, by the assumptions of Proposition 2.1 and (2.1), we have $\alpha\omega^2 > 0$, so $\{v_j\}$ is bounded in $H^1(\mathbb{R}^n)$. When $n \geq 3$, by (1.4) and (2.1), we have $\alpha \geq 0$, so $\{\|\nabla v_j\|_2\}$ is bounded. Since $2 < p + 1 < 2^*$, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $s^{p+1} \leq \varepsilon s^2 + C_\varepsilon s^{2^*}$ for all $s \geq 0$. Thus, by $K_\omega(v_j) = 0$ and the Sobolev inequality, we have

$$\frac{2\alpha + n - 2}{2} \|\nabla v_j\|_2^2 + \frac{(1 - \omega^2)(2\alpha + n)}{4} \|v_j\|_2^2 \leq C \|v_j\|_{2^*}^{2^*} \leq C \|\nabla v_j\|_2^{2^*}.$$

Since $\{\|\nabla v_j\|_2\}$ is bounded, we see that $\{v_j\}$ is bounded in $H^1(\mathbb{R}^n)$.

Moreover, by $K_\omega(v_j) = 0$ and the Sobolev inequality, there exist positive constants C_1, C_2 and C_3 such that $C_1 \|v_j\|_{H^1}^{p+1} \leq C_2 \|v_j\|_{p+1}^{p+1} \leq C_3 \|v_j\|_{H^1}^{p+1}$. Since $v_j \neq 0$, we have $C_1/C_3 \leq \|v_j\|_{H^1}^{p-1}$ and $\limsup_{j \rightarrow \infty} \|v_j\|_{p+1} > 0$. Therefore, by Lemma 2.3, there exist $\{y_j\} \subset \mathbb{R}^n$, a subsequence of $\{v_j(\cdot - y_j)\}$ (we denote it by $\{w_j\}$) and $w \in H^1(\mathbb{R}^n) \setminus \{0\}$ such that $w_j \rightharpoonup w$ weakly in $H^1(\mathbb{R}^n)$. By the weakly lower semicontinuity of \tilde{J}_ω , we have

$$\tilde{J}_\omega(w) \leq \liminf_{j \rightarrow \infty} \tilde{J}_\omega(w_j) = d_\omega.$$

Moreover, by Lemma 2.4, we have

$$K_\omega(w_j) - K_\omega(w_j - w) - K_\omega(w) \rightarrow 0,$$

which implies $K_\omega(w) \leq 0$. Indeed, suppose that $K_\omega(w) > 0$. Since $K_\omega(w_j) = 0$, we have $K_\omega(w_j - w) < 0$ for large j . By Lemma 2.2, we have $\tilde{J}_\omega(w_j - w) > d_\omega$, and

$$\tilde{J}_\omega(w) = \limsup_{j \rightarrow \infty} \{ \tilde{J}_\omega(w_j) - \tilde{J}_\omega(w_j - w) \} \leq 0.$$

On the other hand, by $w \neq 0$ and (2.2), we have $\tilde{J}_\omega(w) > 0$. This is a contradiction. Therefore, we see that $K_\omega(w) \leq 0$. Finally, by Lemma 2.2 and $\tilde{J}_\omega(w) \leq d_\omega$, we have $K_\omega(w) = 0$ and $d_\omega = \tilde{J}_\omega(w)$. Hence, $w \in \mathcal{M}_\omega$. \square

Proposition 2.5. *If $n = 1$ or $n \geq 3$ assume (1.4); if $n = 2$ assume $(p + 3)\omega^2 < p - 1$. Then we have*

$$\mathcal{M}_\omega = \mathcal{G}_\omega = \{ e^{i\theta} \phi_\omega(\cdot + y) : \theta \in \mathbb{R}, y \in \mathbb{R}^n \}$$

if translations and phase shifts are considered or

$$\mathcal{M}_\omega = \mathcal{G}_\omega = \{ \pm \phi_\omega(\cdot + y) : y \in \mathbb{R}^n \}$$

if only translations are considered, where ϕ_ω is the ground state of (1.3).

Proof. First, we show that $\mathcal{M}_\omega \subset \mathcal{G}_\omega$. Let $w \in \mathcal{M}_\omega$. Then, there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that $J'_\omega(w) = \eta K'_\omega(w)$. That is, w satisfies

$$-\{1 - (2\alpha + n - 2)\eta\} \Delta w + \{1 - (2\alpha + n)\eta\} (1 - \omega^2)w = (1 - \{(p + 1)\alpha + n\}\eta) |w|^{p-1} w \tag{2.4}$$

in $H^{-1}(\mathbb{R}^n)$. By $K_\omega(w) = 0$ and $\langle J'_\omega(w), w \rangle = \eta \langle K'_\omega(w), w \rangle$, we have

$$\frac{2(p + 1)}{p - 1} \|\nabla w\|_2^2 = (n - \eta(2\alpha + n)) \{(p + 1)\alpha + n\} \|w\|_{p+1}^{p+1}.$$

Since $w \neq 0$, we have

$$\eta < \frac{n}{(2\alpha + n)\{(p + 1)\alpha + n\}},$$

which implies $1 - (2\alpha + n - 2)\eta > 0$ and $1 - (2\alpha + n)\eta > 0$ in (2.4). Thus, by [8, Theorem 8.1.1], we have $x \cdot \nabla w \in H^1(\mathbb{R}^n)$, and we have

$$0 = K_\omega(w) = \partial_\lambda J_\omega(w_\lambda)|_{\lambda=1} = \langle J'_\omega(w), \alpha w - x \cdot \nabla w \rangle = \eta \langle K'_\omega(w), \alpha w - x \cdot \nabla w \rangle = \eta \partial_\lambda K_\omega(w_\lambda)|_{\lambda=1},$$

where $w_\lambda(x) = \lambda^\alpha w(x/\lambda)$. Moreover, by $K_\omega(w) = 0$, we have

$$\begin{aligned} \partial_\lambda K_\omega(w_\lambda)|_{\lambda=1} &= \frac{(2\alpha + n - 2)^2}{2} \|\nabla w\|_2^2 + \frac{(1 - \omega^2)(2\alpha + n)^2}{2} \|w\|_2^2 - \frac{\{(p + 1)\alpha + n\}^2}{p + 1} \|w\|_{p+1}^{p+1} \\ &= -\frac{(2\alpha + n - 2)\{(p - 1)\alpha + 2\}}{2} \|\nabla w\|_2^2 - \frac{(1 - \omega^2)(2\alpha + n)(p - 1)\alpha}{2} \|w\|_2^2 < 0. \end{aligned}$$

Thus, we have $\eta = 0$, and $w \in \mathcal{S}_\omega$. Moreover, for any $v \in \mathcal{S}_\omega$, we have $K_\omega(v) = 0$. By the definitions of d_ω and \mathcal{M}_ω , we have $J_\omega(w) = d_\omega \leq J_\omega(v)$. Therefore, we have $w \in \mathcal{G}_\omega$, and we conclude $\mathcal{M}_\omega \subset \mathcal{G}_\omega$.

On the other hand, by [8, Theorems 8.1.4 to 8.1.6], we have $\mathcal{G}_\omega = \{ e^{i\theta} \phi_\omega(\cdot + y) : \theta \in \mathbb{R}, y \in \mathbb{R}^n \}$. Since \mathcal{M}_ω is not empty by Proposition 2.1, we have $\mathcal{M}_\omega = \mathcal{G}_\omega$. \square

Define the set Σ_1 by

$$\Sigma_1 = \{ (u, v) \in X \mid E(u, v) - \omega Q(u, v) < d_\omega, K_\omega(u) < 0 \}. \tag{2.5}$$

Note that $\lambda(\phi_\omega, i\omega\phi_\omega) \in \Sigma_1$ for any $\lambda > 1$.

Lemma 2.6. *The set Σ_1 is invariant under the flow of (1.1). That is, if the data $(u_0, u_1) \in \Sigma_1$, then $\vec{u}(t) = (u(t), \partial_t u(t)) \in \Sigma_1$ for any $t \in [0, T_{\max})$, where $\vec{u}(t)$ is the solution of (1.1) with initial value (u_0, u_1) and T_{\max} is the life span of $\vec{u}(t)$.*

Proof. For the sake of convenience, we put

$$L_\omega(u, v) = E(u, v) - \omega Q(u, v).$$

It is immediately observed that

$$J_\omega(u) = L_\omega(u, v) - \frac{1}{2} \|v - i\omega u\|_2^2 \leq L_\omega(u, v). \tag{2.6}$$

From the conservation of energy and charge, we have $L_\omega(\vec{u}(t)) = L_\omega(u_0, u_1) < d_\omega$ for any $t \in [0, T_{\max})$. So, to conclude the proof of the lemma, it suffices to show that $K_\omega(u(t)) < 0$ for any $t \in [0, T_{\max})$. Suppose that there exists $t_0 \in (0, T_{\max})$ such that $K_\omega(u(t_0)) = 0$ and $K_\omega(u(t)) < 0$ for $t \in [0, t_0)$. It then follows from Lemma 2.2 that $\tilde{J}_\omega(u(t)) \geq d_\omega > 0$ for $t \in [0, t_0)$. Thus, we see that $u(t_0) \neq 0$. Since $K_\omega(u(t_0)) = 0$ and $u(t_0) \neq 0$, it follows from relation (2.6) and the definition of d_ω that $d_\omega \leq J_\omega(u(t_0)) \leq L_\omega(\vec{u}(t_0)) < d_\omega$, which is a contradiction. \square

For $\lambda > 1$, let $\vec{u}_\lambda(t) = (u_\lambda(t), \partial_t u_\lambda(t))$ be the solution of (1.1) with data $\vec{u}_\lambda(0) = \lambda \vec{\phi}_\omega$, where $\vec{\phi}_\omega = (\phi_\omega, i\omega \phi_\omega)$ and ϕ_ω is the ground state of (1.3). Let T_λ be the life span of $\vec{u}_\lambda(t)$. Define

$$I_\lambda(t) = \frac{1}{2} \|u_\lambda(t)\|_2^2, \quad 0 \leq t < T_\lambda.$$

The key lemma to prove the strong instability is the following lower estimate for the virial identity.

Lemma 2.7. *For any $\lambda > 1$, there exists a constant $a_\lambda > 0$ such that*

$$\frac{d^2}{dt^2} I_\lambda(t) \geq \frac{p+3}{2} \|\partial_t u_\lambda(t) - i\omega u_\lambda(t)\|_2^2 + a_\lambda, \quad 0 \leq t < T_\lambda.$$

Proof. By simple computations, we have

$$I'_\lambda(t) = \operatorname{Re} \int_{\mathbb{R}^n} \partial_t u_\lambda(t) \overline{u_\lambda(t)} \, dx = \operatorname{Re} \int_{\mathbb{R}^n} (\partial_t u_\lambda(t) - i\omega u_\lambda(t)) \overline{u_\lambda(t)} \, dx$$

and

$$\begin{aligned} I''_\lambda(t) &= \|\partial_t u_\lambda(t)\|_2^2 + \operatorname{Re} \int_{\mathbb{R}^n} \partial_t^2 u_\lambda(t, x) \overline{u_\lambda(t, x)} \, dx \\ &= \frac{p+3}{2} \|\partial_t u_\lambda(t)\|_2^2 + \frac{p-1}{2} \{ \|\nabla u_\lambda(t)\|_2^2 + (1-\omega^2) \|u_\lambda(t)\|_2^2 \} - (p+1)E(\vec{u}_\lambda(t)) \\ &= \frac{p+3}{2} \|i\omega u_\lambda(t) - \partial_t u_\lambda(t)\|_2^2 + (p-1) \left(\frac{1}{2} \|\nabla u_\lambda(t)\|_2^2 + \alpha\omega^2 \|u_\lambda(t)\|_2^2 \right) \\ &\quad - (p+1)L_\omega(\vec{u}_\lambda(t)) + 2\omega Q(\vec{u}_\lambda(t)) \\ &= \frac{p+3}{2} \|\partial_t u_\lambda(t) - i\omega u_\lambda(t)\|_2^2 + \frac{(p-1)((p+1)\alpha+n)}{(p-1)\alpha+2} \tilde{J}_\omega(u_\lambda(t)) \\ &\quad - (p+1)L_\omega(\lambda \vec{\phi}_\omega) + 2\omega Q(\lambda \vec{\phi}_\omega). \end{aligned} \tag{2.7}$$

Here, in the last equality, we have used the fact that L_ω and Q are conserved quantities. For any $\lambda > 1$, it is easy to see that

$$L_\omega(\lambda \vec{\phi}_\omega) = J_\omega(\lambda \phi_\omega) < J_\omega(\phi_\omega) = d_\omega \tag{2.8}$$

and

$$\omega Q(\lambda \vec{\phi}_\omega) = \omega^2 \lambda^2 \|\phi_\omega\|_2^2 > \omega^2 \|\phi_\omega\|_2^2. \tag{2.9}$$

On the other hand, it is found that

$$\|\phi_\omega\|_2^2 = \frac{(n+2) - (n-2)p}{(p-1)(1-\omega^2)} d_\omega. \tag{2.10}$$

Here, we put

$$a_\lambda = \frac{(p-1)((p+1)\alpha + n)}{(p-1)\alpha + 2} d_\omega - (p+1)L_\omega(\lambda\vec{\phi}_\omega) + 2\omega Q(\lambda\vec{\phi}_\omega).$$

Then, by (2.8)–(2.10), we have $a_\lambda > 0$. Moreover, by (2.7), we have

$$I''_\lambda(t) \geq \frac{p+3}{2} \|\partial_t u_\lambda(t) - i\omega u_\lambda(t)\|_2^2 + a_\lambda + \frac{(p-1)((p+1)\alpha + n)}{(p-1)\alpha + 2} \{ \tilde{J}_\omega(u_\lambda(t)) - d_\omega \} \tag{2.11}$$

for $0 \leq t < T_\lambda$. Since $\vec{u}_\lambda(t)$ is the solution of (1.1) with data $\lambda\vec{\phi}_\omega \in \Sigma_1$, it follows from Lemma 2.6 that $\vec{u}_\lambda(t) \in \Sigma_1$ for any $0 \leq t < T_\lambda$. Hence, it then follows from (2.11) and Lemma 2.2 that

$$I''_\lambda(t) \geq \frac{p+3}{2} \|\partial_t u_\lambda(t) - i\omega u_\lambda(t)\|_2^2 + a_\lambda$$

for $0 \leq t < T_\lambda$. This completes the proof of Lemma 2.7. \square

Proof of Theorem 1.3 then follows from Lemma 2.7 and concavity arguments due to Levine [15] as in Payne and Sattinger [24]. For the sake of completeness, we give the proof.

Proof of Theorem 1.3. We use the notation of Lemma 2.7. Since $\lambda\vec{\phi}_\omega \rightarrow \vec{\phi}_\omega$ in X as $\lambda \rightarrow 1$, it suffices to prove that $T_\lambda < \infty$ for any $\lambda > 1$. We prove this by contradiction. Assume that $T_\lambda = \infty$. By Lemma 2.7, we have $I''_\lambda(t) \geq a_\lambda > 0$ for any $t \in [0, \infty)$. This implies that there exists $t_1 \in (0, \infty)$ such that $I'_\lambda(t) > 0$ and $I_\lambda(t) > 0$ for any $t \in [t_1, \infty)$. Let $\beta = (p-1)/4$. Then by using Lemma 2.7 we obtain the following estimate

$$I''_\lambda(t)I_\lambda(t) - (\beta+1)I'_\lambda(t)^2 \geq \frac{p+3}{4} \left\{ \|\partial_t u_\lambda(t) - i\omega u_\lambda(t)\|_2^2 \|u_\lambda(t)\|_2^2 - \left(\operatorname{Re} \int_{\mathbb{R}^n} (\partial_t u_\lambda(t) - i\omega u_\lambda(t)) \overline{u_\lambda(t)} dx \right)^2 \right\} \geq 0.$$

Thus, for $t \in [t_1, \infty)$, we have

$$(I_\lambda(t)^{-\beta})' = -\beta I_\lambda(t)^{-\beta-1} I'_\lambda(t) < 0, \\ (I_\lambda(t)^{-\beta})'' = -\beta I_\lambda(t)^{-\beta-2} \{ I''_\lambda(t)I_\lambda(t) - (\beta+1)I'_\lambda(t)^2 \} \leq 0.$$

Therefore,

$$I_\lambda(t)^{-\beta} \leq I_\lambda(t_1)^{-\beta} - \beta I_\lambda(t_1)^{-\beta-1} I'_\lambda(t_1)(t-t_1), \quad t \in [t_1, \infty),$$

so there exists $t_2 \in (t_1, \infty)$ such that $I_\lambda(t_2)^{-\beta} \leq 0$. However, this is a contradiction. This completes the proof. \square

Having established the strong instability by blowup of standing waves for (1.1), attention is now given to the proof of Theorem 1.6, that is, strong instability of solitary waves for (1.5). The proof of Theorem 1.6 is similar to that of Theorem 1.3 and is approached via the following two main lemmas.

Lemma 2.8. *Let*

$$\Sigma_2 = \{ (u, v) \in X \mid E(u, v) + \omega V(u, v) < d_\omega, K_\omega(u) < 0 \}.$$

Then the set Σ_2 is invariant under the flow of (1.5). That is, if $(u_0, v_0) \in \Sigma_2$, then $\vec{u}(t) = (u(t), v(t)) \in \Sigma_2$ for any $t \in [0, T_{\max})$, where $\vec{u}(t)$ is the solution of (1.5) with initial value (u_0, v_0) and T_{\max} is the life span of $\vec{u}(t)$.

Proof. We omit the proof because it is similar to that of Lemma 2.6. \square

Note that $\lambda(\phi_\omega, -\omega\phi_\omega) \in \Sigma_2$ for any $\lambda > 1$.

Lemma 2.9. *The set $A = \{w \in H^1(\mathbb{R}) \mid \xi^{-1}\hat{w} \in L^2(\mathbb{R})\}$ is dense in $H^1(\mathbb{R})$, where \hat{w} is the Fourier transform of w .*

Proof. See Lemma 4.4 in [18]. \square

Let $\vec{\phi}_\omega = (\phi_\omega, -\omega\phi_\omega)$, where ϕ_ω is the ground state of (1.3). By Lemma 2.9, there is $\vec{w}_{\varepsilon_i} = (w_{\varepsilon_i}, -\omega w_{\varepsilon_i}) \in A$ such that $\vec{w}_{\varepsilon_i} \rightarrow \vec{\phi}_\omega$ in H^1 as $\varepsilon_i \rightarrow 0$. For $\lambda > 1$, let $\vec{u}_0^\lambda = \lambda\vec{w}_{\varepsilon_i}$. We claim that $\vec{u}_0^\lambda \in \Sigma_2$. In fact, we have

$$L_\omega(\vec{u}_0^\lambda) = E(\vec{u}_0^\lambda) + \omega V(\vec{u}_0^\lambda) = E(\lambda\vec{\phi}_\omega) + \omega V(\lambda\vec{\phi}_\omega) + \alpha(\varepsilon_i),$$

where $\alpha(\varepsilon_i) \rightarrow 0$ as $\varepsilon_i \rightarrow 0$. Since $\lambda\vec{\phi}_\omega \in \Sigma_2$, choose ε_i small enough such that $\alpha(\varepsilon_i) < d_\omega - L_\omega(\lambda\phi_\omega)$. It is then found that

$$L_\omega(\vec{u}_0^\lambda) < L_\omega(\vec{\phi}_\omega) = d_\omega. \tag{2.12}$$

Similarly, we have

$$K_\omega(\vec{u}_0^\lambda) = K_\omega(\lambda\phi_\omega) + \alpha(\varepsilon_i) < K_\omega(\vec{\phi}_\omega) = 0$$

and

$$-\omega V(\vec{u}_0^\lambda) = -\omega V(\lambda\vec{\phi}_\omega) + \alpha(\varepsilon_i) = \omega^2\lambda^2\|\phi_\omega\|_2^2 + \alpha(\varepsilon_i) > \omega^2\|\phi_\omega\|_2^2. \tag{2.13}$$

This in turn implies that $\vec{u}_0^\lambda \in \Sigma_2$. On the other hand, it is easy to see that

$$\|\phi_\omega\|_2^2 = \frac{3+p}{(p-1)(1-\omega^2)}d_\omega. \tag{2.14}$$

For $\lambda > 1$, let $\vec{u}_\lambda(t) = (u_\lambda(t), v_\lambda(t))$ be the solution of (1.5) with initial value \vec{u}_0^λ . Let T_λ be the life span of u_λ . Put

$$I_\lambda(t) = \frac{1}{2}\|\xi^{-1}\hat{u}_\lambda(t)\|_2^2, \quad 0 \leq t < T_\lambda.$$

Since $\vec{u}_0^\lambda \in A$, the function $I_\lambda(t)$ is well-defined. The following estimate of the virial identity can be obtained in a similar way as in Lemma 2.7.

Lemma 2.10. *For any $\lambda > 1$, there is a constant $a_\lambda > 0$ such that*

$$I_\lambda''(t) \geq \frac{p+3}{2}\|v_\lambda(t) + \omega u_\lambda(t)\|_2^2 + a_\lambda, \quad t \in [0, T_\lambda].$$

Proof. Since the proof of Lemma 2.10 is similar to that of Lemma 2.7, we only give an outline of the proof. A simple computation shows that

$$I_\lambda'(t) = \operatorname{Re} \int_{\mathbb{R}} \xi^{-1}\overline{\hat{u}_\lambda(t)}\hat{v}_\lambda(t) \, d\xi = \operatorname{Re} \int_{\mathbb{R}} \xi^{-1}\overline{\hat{u}_\lambda(t)}\{\hat{v}_\lambda(t) + \omega\hat{u}_\lambda(t)\} \, d\xi$$

and

$$\begin{aligned} I_\lambda''(t) &= \frac{p+3}{2}\|v_\lambda(t)\|_2^2 + \frac{p-1}{2}(\|\partial_x u_\lambda(t)\|_2^2 + (1-\omega^2)\|u_\lambda(t)\|_2^2) - (p+1)E(\vec{u}_\lambda(t)) \\ &= \frac{p+3}{2}\|v_\lambda(t) + \omega u_\lambda(t)\|_2^2 + (p-1)\left(\frac{1}{2}\|\partial_x u_\lambda(t)\|_2^2 + \alpha\omega^2\|u_\lambda(t)\|_2^2\right) \\ &\quad - (p+1)L_\omega(\vec{u}_\lambda(t)) - 2\omega V(\vec{u}_\lambda(t)) \\ &= \frac{p+3}{2}\|v_\lambda(t) + \omega u_\lambda(t)\|_2^2 + \frac{(p-1)((p+1)\alpha+1)}{(p-1)\alpha+2}\tilde{J}_\omega(u_\lambda(t)) - (p+1)L_\omega(\vec{u}_0^\lambda) - 2\omega V(\vec{u}_0^\lambda). \end{aligned}$$

Therefore, in view of (2.12)–(2.14), Lemma 2.10 can be obtained by Lemmas 2.2 and 2.8. \square

Proof of Theorem 1.6. The proof follows from Lemma 2.6 and the proof of Theorem 1.3. \square

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