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Prescribing scalar curvature on S^3

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Abstract

We give existence results for solutions of the prescribed scalar curvature equation on S^3, when the curvature function is a positive Morse function and satisfies an index-count condition.

Résumé

Nous démontrons l'existence de solutions de l'équation courbure scalaire sur S^3, quand la courbure scalaire est une fonction de Morse positive satisfaisant à une condition d'indice.

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1. Introduction

Let S^3 be the standard sphere with round metric g_0 induced by $S^3 = \partial B_1(0) \subset \mathbb{R}^4$. We study the problem: Which functions K on S^3 occur as scalar curvature of metrics g conformally equivalent to g_0? Writing $g = \varphi^4 g_0$ and $k(\theta) := \frac{1}{6}(K(\theta) - 6)$ this is equivalent to solving for $t = 1$ (see [3])

$$-8\Delta_{S^3}\varphi + 6\varphi = 6\big(1 + tk(\theta)\big)\varphi^5, \quad \varphi > 0 \quad \text{in } S^3. \tag{1.1}$$

In stereographic coordinates $\mathcal{S}_\theta(\cdot)$ centered at some point $\theta \in S^3$, i.e. $\mathcal{S}_\theta(0) = \theta$, Eq. (1.1) is equivalent to

$$-\Delta u = \big(1 + tk_\theta(x)\big)u^5 \quad \text{in } \mathbb{R}^3, \quad u > 0, \tag{1.2}$$

where $k_\theta(x) := k \circ \mathcal{S}_\theta(x)$ and

$$u(x) = \mathcal{R}_\theta(\varphi)(x) := 3^{\frac{1}{4}}\big(1 + |x|^2\big)^{-\frac{1}{2}}\varphi \circ \mathcal{S}_\theta(x). \tag{1.3}$$

An obvious necessary condition for the existence of solutions to (1.1) is that the function K has to be positive somewhere. Moreover, there are the *Kazdan–Warner* obstructions [14,7], which imply in particular, that a monotone function of the coordinate function x_1 cannot be realized as the scalar curvature of a metric conformal to g_0.

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Numerous studies have been made on Eq. (1.1) and its higher dimensional analogue and various sufficient conditions for its solvability have been found (see [2,16,15,11,12,6,4] and the reference therein), usually under a non-degeneracy assumption on K. On S^3 a positive function K is non-degenerate, if

$$\Delta K(\theta) \neq 0 \quad \text{if } \nabla K(\theta) = 0. \tag{nd}$$

For positive Morse functions K on S^3 it is shown in [18,5,10] that (1.1) is solvable if K satisfies (nd) and

$$d := -\Bigg(1 + \sum_{\substack{\nabla K(\theta)=0,\\ \Delta K(\theta)<0}} (-1)^{\operatorname{ind}(K,\theta)}\Bigg) \neq 0, \tag{1.4}$$

where $\operatorname{ind}(K,\theta)$ is the Morse index of K at θ. We are interested in the case when the non-degeneracy assumption (nd) is not satisfied.

As in [10] we use a continuity method and join the curvature function K to the constant function $K_0 \equiv 6$ by a one parameter family $K_t(\theta) := 6(1 + tk(\theta))$.

The positive solutions of (1.2) for $t = 0$ are completely known (see [9,13]) and given by a non-compact manifold

$$Z := \left\{ z_{\mu,y}(x) := \mu^{-\frac{1}{2}} 3^{\frac{1}{4}} \left(1 + \left|\frac{x-y}{\mu}\right|^2\right)^{-\frac{1}{2}}:\ y \in \mathbb{R}^3,\ \mu > 0 \right\},$$

where $z_{\mu,y}(y) \to \infty$ as $\mu \to 0$.

Thus, in general, there are no a priori L^∞-estimates for (1.1). For $t = 0$ this lack of compactness stems from the fact that the noncompact group of conformal transformation of S^3 acts on solutions. In particular the dilations allow solutions to concentrate in a single point with large L^∞-norm. One expects that a non-constant k breaks this symmetry leading to a priori estimates for solutions. Indeed, in [10,11] it is shown that if $K \in C^2(S^3)$ is a positive function and satisfies (nd) then for $\delta > 0$ there is $C = C(K, \delta) > 0$ such that for all $t \in [\delta, 1]$ and solutions φ_t of (1.1) we have

$$C^{-1} \leqslant \varphi_t(\theta) \leqslant C \quad \text{and} \quad \|\varphi_t\|_{C^{2,\alpha}(S^3)} \leqslant C.$$

Furthermore, Chang et al. [10] compute the Leray–Schauder degree for (1.1) for $t > 0$ small, and show that it equals $-\deg(G, B_1(0), 0)$, which is given by d in (1.4), when K is a Morse function. The map G is associated to K and defined on $B_1(0) \subset \mathbb{R}^4$. The a priori estimate implies the invariance of the degree as the parameter t moves to 1 and gives a solution to (1.1) if $d \neq 0$.

Hence, if (nd) fails, we face two problems: Is the a priori bound still valid and how do critical points of K with $\Delta K = 0$ occur in the index count condition (1.4)? A priori bounds, when (nd) fails, are given in [17]. Here, we will mainly deal with the second question and give a generalized version of (1.4).

In the following, unless otherwise stated, we will assume that $K = 6(1 + k) \in C^6(S^3)$ is positive. To give our main results we need the following notation. We write $k_\theta = k \circ \mathcal{S}_\theta$ and for a critical point θ of k we let

$$\begin{aligned}
a_0(\theta) &:= \oint_{\mathbb{R}^3} \big(k_\theta(x) - T^2_{k_\theta,0}(x)\big)|x|^{-6},\\
a_1(\theta) &:= \Delta^2 k_\theta(0) + \nabla\big(\Delta k_\theta(0)\big) \cdot \big(D^2 k_\theta(0)\big)^{-1} \nabla\big(\Delta k_\theta(0)\big),\\
a_2(\theta) &:= k_\theta(0) a_1(\theta) - \frac{15}{8\pi} \int_{\partial B_1(0)} \big|D^2 k_\theta(0)(x)^2\big|^2,\\
a_3(\theta) &:= \frac{12}{\pi^2} \big(D^2 k_\theta(0)\big)^{-1} \nabla\big(\Delta k_\theta(0)\big) \cdot \oint_{\mathbb{R}^3} \big(\nabla k_\theta(x) - T^2_{\nabla k_\theta,0}(x)\big)|x|^{-6}\\
&\quad + \frac{48}{\pi^2} \big(D^2 k_\theta(0)\big)^{-1} \nabla\big(\Delta k_\theta(0)\big) \cdot \oint_{\mathbb{R}^3} \big(k_\theta(x) - T^3_{k_\theta,0}(x)\big)\frac{x_i}{|x|^8} + \frac{120}{\pi^2} \oint_{\mathbb{R}^3} \big(k_\theta(x) - T^4_{k_\theta,0}(x)\big)\frac{1}{|x|^8},
\end{aligned}$$

where all differentiations are done in $\mathbb{R}^3$, the mth Taylor polynomial of k in y is abbreviated by

$$T_{k,y}^m(x) := \sum_{\ell=0}^{m} \frac{1}{\ell!} D^\ell k(y)(x-y)^\ell,$$

and $\oint$ is the Cauchy principal value of the integral,

$$\oint_{\mathbb{R}^3} f(x) := \lim_{r\to 0} \int_{\mathbb{R}^3\setminus B_r(0)} f(x).$$

Denote by M, M_+, and M_0 the sets,

$$M := \{\theta \in S^3\colon \nabla k(\theta) = 0, \Delta k_\theta(0) = a_0(\theta) = 0, \text{ and } a_2(\theta) \neq 0\},$$
$$M_+ := \{\theta \in M\colon 0 < -a_1(\theta)/a_2(\theta) \leqslant 1\},$$
$$M_0 := \{\theta \in M\colon a_1(\theta) = 0\}.$$

We fix $\theta \in S^3$ and define for $0 < \mu < \infty$ and $y \in \mathbb{R}^3$ the Melnikov function Γ_θ by

$$\Gamma_\theta(\mu, y) := \frac{1}{6} \int_{\mathbb{R}^3} k_\theta(x)(z_{\mu,y})^6 \, dx = \frac{1}{6} \int_{\mathbb{R}^3} k_\theta(\mu x + y)(z_{1,0})^6 \, dx.$$

Theorem 1.1. *Suppose $1 + k \in C^6(S^3)$ is positive and satisfies*

$$D^2 k_\theta(0) \text{ is invertible, if } \theta \in \mathcal{A} := \{\theta \in S^3\colon \nabla k(\theta) = 0 \text{ and } \Delta_{S^3} k(\theta) = 0\},$$

the set M_+ is empty, and $a_3(\theta) \neq 0$ if $\theta \in M_0$. Then there is $R_0 > 0$ such that for any $R \geqslant R_0$ and $\theta \in S^3$ the degree $\deg(\Gamma_\theta', \Omega_R, 0)$ is well defined and independent of θ and R, where

$$\Omega_R := \{(\mu, y) \in \mathbb{R}^3\colon R^{-1} < \mu < R \text{ and } |y| < R\},$$

and (1.1) *is solvable, if for some* (*and hence for any*) $\theta \in S^3$ *and* $R \geqslant R_0$

$$0 \neq d := -\deg(\Gamma_\theta', \Omega_R, 0) + \sum_{\theta\in M_0\colon a_3(\theta)>0} (-1)^{\operatorname{ind}(k,\theta)}. \tag{1.5}$$

The number d is the Leray–Schauder degree of the problem (1.1). *If, moreover, k is a Morse function, then*

$$d = -\left(1 + \sum_{\theta\in \mathrm{Crit}_-(k)} (-1)^{\operatorname{ind}(k,\theta)}\right), \tag{1.6}$$

where

$$\mathrm{Crit}_-(k) := \Big\{\theta \in S^3\colon \nabla k(\theta) = 0, \ \Delta k(\theta)^2 + a_0(\theta)^2 + a_1(\theta)^2 \neq 0, \text{ and } \lim_{\mu\to 0^+} \operatorname{sgn}\big(\Delta k(\theta) + a_0(\theta)\mu - a_1(\theta)\mu^2\big) = -1\Big\}.$$

Under the assumptions of Theorem 1.1 the set M is finite. Thus we need only consider and sum over a finite number of points θ.

In [17] it is shown for Morse functions K that the condition $M_+ = \emptyset$ is equivalent to the compactness in $C^2(S^3)$ of the set of solutions to (1.1), when $t \in [\delta, 1+\delta]$ for small $\delta > 0$. In the general case this condition is only sufficient. Hence, the Leray–Schauder degree of (1.1) is invariant with respect to $t \in (0, 1]$. By simply replacing k by sk for some $s > 0$ we obtain from Theorem 1.1, if we abandon the condition $M_+ = \emptyset$, that the degree of (1.1) is given by d for any t such that

$$0 < t < \min\left\{-\frac{a_1(\theta)}{a_2(\theta)}\colon \theta \in M \text{ and } -\frac{a_1(\theta)}{a_2(\theta)} > 0\right\}. \tag{1.7}$$

The first addend in (1.5) gives the degree of the solutions that remain uniformly bounded as $t \to 0^+$, whereas the second addend is the degree of the solutions that blow up as $t \to 0^+$. It is part of the proof to show that the family of solutions splits in this way. Here, the assumption $a_3(\theta)\neq 0$ at points $\theta\in M$, where $a_1(\theta)=0$, assures that the blowing-up solutions lie on C^1-curves (t,φ_t) emanating from $t=0$ (see [17]).

If k satisfies (nd) the solutions are uniformly bounded with respect to $t\in(0,1]$, the set M is empty, and one recovers (1.4). The derivative of the Melnikov function Γ_θ is closely related to the above mentioned map G in [10] (see Remark 5.1 below).

It is interesting to note that if K is a Morse function, then the formula for the degree in (1.6) is independent of the coefficients $a_3(\theta)$. This gives the perspective that for Morse functions and t satisfying (1.7) the degree is always given by (1.6).

We sketch the strategy of the proof of Theorem 1.1 and outline the remaining part of the paper. The transformations in (1.3) gives rise to a Hilbert space isomorphism between $H^{1,2}(S^3)$ and $\mathcal{D}^{1,2}(\mathbb{R}^3)$, where $\mathcal{D}^{1,2}(\mathbb{R}^3)$ denotes the closure of $C_c^\infty(\mathbb{R}^3)$ with respect to

$$\|u\|^2=\int\limits_{\mathbb{R}^3}|\nabla u|^2.$$

Due to elliptic regularity (see [8]) and Harnack's inequality it is enough to find a weak nonnegative solution of (1.1) in $H^{1,2}(S^3)$, or of the equivalent equation. We take advantage of both formulations: We fix $\theta\in S^3$, consider (1.2), and find solutions as critical points of $f_t^\theta:\mathcal{D}^{1,2}(\mathbb{R}^3)\to\mathbb{R}$, where

$$f_t^\theta(u):=\frac{1}{2}\int|\nabla u|^2-\frac{1}{6}\int\big(1+tk_\theta(x)\big)|u|^6.$$

To avoid cumbrous indexing we will suppress the dependence θ and write f_t instead of f_t^θ. For $t=0$ the functional f_0 possesses, as seen above, a $(3+1)$-dimensional manifold of critical points Z. We recall some facts about the spectrum of $f_0''(z)$ in Section 2. We use a finite dimensional reduction of Melnikov type developed in [1,2]. In Section 3 we recall without proof that a sequence of solutions to (1.1) can only blow-up in a single point (see [18,15]) and fit this result into our framework. Section 4 contains the finite dimensional reduction of our problem, where we sum up the results in [17] and obtain a one-dimensional function that describes the blow-up behavior of solutions. The computation of the Leray–Schauder degree is done in Section 5. Appendix A contains the proof of differentiability of the curve of blowing-up solutions, which is done briefly by using the computations and estimates in [17].

2. The unperturbed problem

We define for $\mu>0$ and $y\in\mathbb{R}^3$ the maps $\mathcal{U}_\mu,\mathcal{T}_y:\mathcal{D}^{1,2}(\mathbb{R}^3)\to\mathcal{D}^{1,2}(\mathbb{R}^3)$ by

$$\mathcal{U}_\mu(u):=\mu^{-\frac{1}{2}}u\left(\frac{\cdot}{\mu}\right)\quad\text{and}\quad\mathcal{T}_y(u):=u(\cdot-y).$$

With this notation the critical manifold Z is given by

$$Z=\big\{z_{\mu,y}=\mathcal{T}_y\circ\mathcal{U}_\mu(z_{1,0}):\ y\in\mathbb{R}^3,\ \mu>0\big\}.$$

It is easy to check that the dilation $\mathcal{U}_\mu$ and the translation $\mathcal{T}_y$ conserve the norms $\|\cdot\|$ and $\|\cdot\|_{L^6}$. Thus for every $\mu>0$ and $y\in\mathbb{R}^3$

$$(\mathcal{U}_\mu)^{-1}=(\mathcal{U}_\mu)^t=U_{\mu^{-1}},\quad(\mathcal{T}_y)^{-1}=(\mathcal{T}_y)^t=\mathcal{T}_{-y},\quad\text{and}\quad f_0=f_0\circ\mathcal{U}_\mu=f_0\circ\mathcal{T}_y\tag{2.1}$$

where $(\cdot)^t$ denotes the adjoint. Twice differentiating the identities for f_0 in (2.1) yields

$$f_0''(v)=(\mathcal{T}_y\circ\mathcal{U}_\mu)^{-1}\circ f_0''\big(\mathcal{T}_y\circ\mathcal{U}_\mu(v)\big)\circ(\mathcal{T}_y\circ\mathcal{U}_\mu)\quad\forall v\in\mathcal{D}^{1,2}\big(\mathbb{R}^3\big).\tag{2.2}$$

Moreover, we see that $U(\mu,y,z):=\mathcal{T}_y\circ\mathcal{U}_\mu(z)$ maps $(0,\infty)\times\mathbb{R}^3\times Z$ into Z, hence

$$\frac{\partial U}{\partial z}(\mu,y,z)=\mathcal{T}_y\circ\mathcal{U}_\mu:T_zZ\to T_{\mathcal{T}_y\circ\mathcal{U}_\mu(z)}Z\quad\text{and}\quad\mathcal{T}_y\circ\mathcal{U}_\mu:(T_zZ)^\perp\to(T_{\mathcal{T}_y\circ\mathcal{U}_\mu(z)}Z)^\perp.\tag{2.3}$$

The tangent space $T_{z_{\mu,y}}Z$ at a point $z_{\mu,y}\in Z$ is spanned by 4 orthonormal functions,

$$T_{z_{\mu,y}}Z=\big\langle(\dot{\xi}_{\mu,y})_i:\ i=0\dots3\big\rangle,$$

where $(\dot{\xi}_{\mu,y})_i$ denotes for $i = 0$ the normalized tangent vector $\frac{\mathrm{d}}{\mathrm{d}\mu} z_{\mu,y}$ and for $1 \leqslant i \leqslant 3$ the normalized tangent vector $\frac{\mathrm{d}}{\mathrm{d}y_i} z_{\mu,y} = -\frac{\partial}{\partial x_i} z_{\mu,y}$. By (2.3) we obtain

$$(\dot{\xi}_{\mu,y})_i = \mathcal{T}_y \circ \mathcal{U}_\mu\big((\dot{\xi}_{1,0})_i\big).$$

An explicit calculation gives for $1 \leqslant i \leqslant 3$

$$(\dot{\xi}_{1,0})_i(x) = \frac{8}{\pi\sqrt{15}}\big(1 + |x|^2\big)^{-\frac{1}{2}} x_i.$$

For $i = 0$ we find

$$(\dot{\xi}_{1,0})_0(x) = \frac{4}{\pi\sqrt{15}}\big(1 + |x|^2\big)^{-\frac{1}{2}}\big(1 - 2\big(1 + |x|^2\big)^{-1}\big).$$

Using the canonical identification of the Hilbert space $\mathcal{D}^{1,2}(\mathbb{R}^3)$ with its dual induced by the scalar-product and denoted by $\mathcal{K} : (\mathcal{D}^{1,2}(\mathbb{R}^3))' \to \mathcal{D}^{1,2}(\mathbb{R}^3)$,

$$\big(\mathcal{K}(\varphi), u\big) = \varphi(u) \quad \forall(\varphi, u) \in \big(\mathcal{D}^{1,2}\big(\mathbb{R}^3\big)\big)' \times \mathcal{D}^{1,2}\big(\mathbb{R}^3\big),$$

we shall consider $f_t'(u)$ as an element of $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $f_t''(u)$ as one of $\mathcal{L}(\mathcal{D}^{1,2}(\mathbb{R}^3))$. With this identification $f_t''(u)$ is of the form *identity–compact* (see [2]) and hence a Fredholm operator of index zero.

Since $f_0''(z_{\mu,y})$ is a self-adjoint, compact perturbation of the identity map in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, its spectrum $\sigma(f_0''(z_{\mu,y}))$ consists of point-spectrum, possibly accumulating at 1. In [17] the spectrum and the eigenfunctions of $f_0''(z_{\mu,y})$ are computed,

$$\sigma\big(f_0''(z_{\mu,y})\big) = \left\{\lambda_{i,j} := 1 - \frac{15}{4(1 + i + j)^2 - 1} : i, j \in \mathbb{N}_0\right\}, \tag{2.4}$$

and the dimension of the eigenspace, $\langle \Phi_{i,j,l}^{\mu,y}\rangle$, corresponding to the eigenvalue $\lambda_{i,j}$ is given by $\binom{i+2}{2} - \binom{i}{2}$, the dimension of the spherical harmonics of degree i. Since Z is a manifold of critical points of f_0', the tangent space $T_z Z$ at a point $z \in Z$ is contained in the kernel $N(f_0''(z))$ of $f_0''(z)$. As $\lambda_{i,j} = 0$ if and only if $i + j = 1$, the dimension of $N(f_0''(z))$ is 4, which implies that

$$T_z Z = N\big(f_0''(z)\big) \quad \text{for all } z \in Z. \tag{2.5}$$

If (2.5) holds the critical manifold Z is called non-degenerate (see [1]) and the self-adjoint Fredholm operator $f_0''(z)$ maps the space $\mathcal{D}^{1,2}(\mathbb{R}^3)$ into $T_z Z^\perp$ and is invertible in $\mathcal{L}(T_z Z^\perp)$. From (2.2) and (2.3), we obtain in this case

$$\big\|\big(f_0''(z_{1,0})\big)^{-1}\big\|_{\mathcal{L}(T_{z_{1,0}} Z^\perp)} = \big\|\big(f_0''(z)\big)^{-1}\big\|_{\mathcal{L}(T_z Z^\perp)} \quad \forall z \in Z. \tag{2.6}$$

Moreover, $f_0''(z)$ and $f_0''(z)|_{T_z Z^\perp}$ have precisely one negative eigenvalue -4 with one-dimensional eigenspace $\langle z\rangle$.

3. Blow up analysis

Based on the results in [18,15] we have the following lemma (see [17, Corollary 3.2])

Lemma 3.1. *Suppose* $1 + k \in C^1(S^3)$ *is positive. If* $(t_i, \varphi_i) \in [0, 1] \times C^2(S^3)$ *solve (1.1) with* $t = t_i$*, then after passing to a subsequence either* (φ_i) *is uniformly bounded in* $L^\infty(S^3)$ (*and hence in* $C^{2,\alpha}(S^3)$ *by standard elliptic regularity*) *or there exist* $\theta \in S^3$ *and sequences* $(\mu_i) \in (0, \infty)$, $(y_i) \in \mathbb{R}^3$ *satisfying* $\lim_{i\to\infty} \mu_i = 0$ *and* $\lim_{i\to\infty} y_i = 0$*, such that* (*in stereographic coordinates* $\mathcal{S}_\theta(\cdot)$ *about* θ)

$$\mathcal{R}_\theta(\varphi_i) - \big(1 + t_i k_\theta(y_i)\big)^{-\frac{1}{4}} z_{\mu_i,y_i} \text{ is orthogonal to } T_{z_{\mu_i,y_i}} Z,$$

$$\big\|\mathcal{R}_\theta(\varphi_i) - \big(1 + t_i k_\theta(y_i)\big)^{-\frac{1}{4}} z_{\mu_i,y_i}\big\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \mathrm{o}(1).$$

4. Finite dimensional reduction

For the rest of the paper, unless otherwise indicated, integration extends over $\mathbb{R}^3$ and is done with respect to the variable x.

In this section we state without proof results obtained in [17], which yield a finite dimensional reduction of our problem.

Lemma 4.1. [17, Lemma 4.2] *Suppose $1+k \in C^5(S^3)$ is positive and $\theta \in S^3$. Then there exist $\rho_0 = \rho_0(k) > 0$, $t_0 = t_0(k) > 0$, $b = b(k) > 0$, an upper continuous function $\mu_0 : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$, depending only on k, and two functions $w : \Omega \to \mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\vec{\alpha} : \Omega \to \mathbb{R}^4$ depending on k and θ, where*

$$\Omega := \{(t,\mu,y) \in [-b, 1+b] \times (0,+\infty) \times \mathbb{R}^3 : \ 0 < \mu < \mu_0(t)\},$$
$$\mu_0(t) = +\infty \quad \text{if } |t| \leqslant t_0,$$

such that for any $(t,\mu,y) \in \Omega$

$$w(t,\mu,y) \quad \text{is orthogonal to } T_{z_{\mu,y}}Z \tag{4.1}$$
$$f_t'\big(z_{\mu,y} + w(t,\mu,y)\big) = \vec{\alpha}(t,\mu,y)\cdot \dot{\xi}_{\mu,y} \in T_{z_{\mu,y}}Z, \tag{4.2}$$
$$\big\|w(t,\mu,y) - w_0(t,\mu,y)\big\| + \big\|\vec{\alpha}(t,\mu,y)\big\| < \rho_0, \tag{4.3}$$

where $\{(\dot{\xi}_{\mu,y})_i : \ i = 0\ldots 3\}$ denotes the basis of $T_{z_{\mu,y}}Z$ given in (1.6) and

$$w_0(t,\mu,y) := \big(\big(1+tk_\theta(y)\big)^{-\frac{1}{4}} - 1\big) z_{\mu,y}.$$

The functions w and $\vec{\alpha}$ are of class C^2 and unique in the sense that if $(v,\vec{\beta})$ satisfies (4.1)–(4.3) for some $(t,\mu,y) \in \Omega$ then $(v,\vec{\beta})$ is given by $(w(t,\mu,y), \vec{\alpha}(t,\mu,y))$.

Moreover, we have for $1 \leqslant j \leqslant 3$

$$\left\|\vec{\alpha}(t,\mu,y)_0 - \sum_{i=1}^{2} \vec{\alpha}_i(t,\mu,y)_0\right\| \leqslant \mathrm{O}\big(t\big(\big|\nabla k_\theta(y)\big|^2 \min\big(1,\mu^2\big) + \min\big(1,\mu^{\frac{9}{4}}\big)\big)\big),$$
$$\left\|\vec{\alpha}(t,\mu,y)_j - \sum_{i=1}^{2} \vec{\alpha}_i(t,\mu,y)_j\right\| \leqslant \mathrm{O}\big(t \min\big(1,\mu^{\frac{9}{4}}\big)\big), \tag{4.4}$$

where

$$\vec{\alpha}_1(t,\mu,y) := -t\min(1,\mu)\big(1+tk_\theta(y)\big)^{-\frac{5}{4}} \frac{\pi}{3^{\frac{1}{4}}\sqrt{5}} \begin{pmatrix} 0 \\ \nabla k_\theta(y) \end{pmatrix},$$
$$\vec{\alpha}_2(t,\mu,y) := -t\min\big(1,\mu^2\big)\big(1+tk_\theta(y)\big)^{-\frac{5}{4}} \frac{\pi}{3^{\frac{1}{4}}\sqrt{5}} \begin{pmatrix} \Delta k_\theta(y) \\ \vec{0} \end{pmatrix}.$$

Remark 4.2. From the proof of Lemma 4.1 in [17] and (4.4) we see that

$$\frac{1}{t\mu}\vec{\alpha}(t,\mu,y)$$

is a well defined, continuous function for $(t,\mu,y) \in \Omega$.

Lemma 4.3. [17, Lemma 5.1] *Under the assumptions of Lemma 4.1 we have for all $(t,\mu,y) \in \Omega$ with $|\mu| \leqslant 1$ and $1 \leqslant i, j \leqslant 3$*

$$\frac{1}{t\mu}\frac{\partial \alpha(t,\mu,y)_i}{\partial y_j} = -\frac{\pi}{3^{\frac{1}{4}}\sqrt{5}}\big(1+tk(y)\big)^{-\frac{5}{4}} \frac{\partial^2 k_\theta(y)}{\partial x_i \partial x_j} + \mathrm{O}\big(\big|\nabla k_\theta(y)\big|^2 + \mu^{1+\frac{1}{4}}\big).$$

Under the assumptions of Lemma 4.1 suppose $(t_i,\varphi_i)\in(0,1]\times C^2(S^3)$ solve (1.1) with $t=t_i$ and blow up at $\theta\in S^3$, i.e. $t_i\to t_0\in[0,1]$ and there is a sequence (θ_i) in S^3 such that $\theta_i\to\theta$ and $\varphi_i(\theta_i)\to\infty$ as $i\to\infty$. Setting

$$u_i:=\mathcal{R}_\theta(\varphi_i)$$

we obtain from Lemma 3.1 that for large i the tuple $(u_i-z_{\mu_i,y_i},\vec{0})$ satisfies (4.1)–(4.3) for $(t_i,\mu_i,y_i)\in\Omega$. From the uniqueness part in Lemma 4.1 we get that for large i

$$\frac{1}{t_i\mu_i}\vec{\alpha}(t_i,\mu_i,y_i)=0.$$

Consequently to exclude or to construct blow-up sequences it is enough to exclude or construct zeros of $\vec{\alpha}(t,\mu,y)$ for small μ. From the expansion of $\vec{\alpha}$ in Lemma 4.1 we immediately get that $\nabla k_\theta(0)=0$ and $\Delta k_\theta(0)=0$ if $\theta\in S^3$ is a blow-up point.

Thus the remaining possible blow-up points are the critical points θ of k where $\Delta k_\theta(0)=0=\Delta_{S^3}k(\theta)$. If θ is a nondegenerate critical point of k, then the determination of blow-up points can be reduced to a one dimensional problem.

Lemma 4.4. [17, Lemma 6.1] *Under the assumptions of Lemma* 4.1 *suppose* 0 *is a nondegenerate critical point of* k_θ*, i.e.*

$$\nabla k_\theta(0)=0\quad\textit{and}\quad D^2k_\theta(0)\ \textit{is invertible.}$$

Moreover, assume $\Delta k_\theta(0)=0$*. Consider the function* $\hat{\alpha}$*, defined by*

$$\hat{\alpha}(t,\mu,y):=\frac{3^{\frac{1}{4}}\sqrt{5}}{t\mu\pi}\big(1+tk(\theta)\big)^{\frac{5}{4}}\big(\vec{\alpha}(t,\mu,y)_1,\dots,\vec{\alpha}(t,\mu,y)_3\big)^T,$$

which is well defined and continuous in Ω (*see Remark* 4.2)*. Then there are* $\delta_1=\delta_1(k)>0$ *and a* C^2*-function* β*,*

$$\beta:\big\{(t,\mu):\ t\in[-b,1+b],\ 0<\mu<\delta_1\big\}\to\mathbb{R}^3,$$

such that $\beta(t,\mu)=\mathrm{O}(\mu^2)$ *as* $\mu\to 0$ *and*

$$\hat{\alpha}\big(t,\mu,\beta(t,\mu)\big)=0\quad\textit{for all } t\in[-b,1+b],\ 0<\mu<\delta_1.$$

Moreover, β *is unique in the sense that, if* $y\in B_{\delta_1}(0)$ *satisfies* $\hat{\alpha}(t,\mu,y)=0$ *for some* $t\in[-b,1+b]$ *and* $0<\mu<\delta_1$*, then* $y=\beta(t,\mu)$*.*

Hence, to exclude or to construct blow-up sequences, which blow-up at a nondegenerate critical point θ of k with $\Delta k_\theta(0)=0$ it suffices to study $\alpha(t,\mu,\beta(t,\mu))_0$.

Lemma 4.5. [17, Lemma 6.2] *Under the assumptions of Lemma* 4.4 *and* $k\in C^6(S^3)$ *we have*

$$\begin{aligned}\big(\alpha\big(t,\mu,\beta(t,\mu)\big)\big)_0 &= -t\mu^3\big(1+tk(\theta)\big)^{-\frac{5}{4}}\frac{3^{\frac{3}{4}}4}{\pi\sqrt{5}}a_0(\theta)\\ &\quad+t\mu^4\big(1+tk(\theta)\big)^{-\frac{9}{4}}\frac{\pi 3^{\frac{3}{4}}\sqrt{5}}{30}\big(a_1(\theta)+ta_2(\theta)\big)\\ &\quad+t\mu^5\big(1+tk(\theta)\big)^{-\frac{5}{4}}\frac{\pi 3^{\frac{3}{4}}\sqrt{5}}{30}a_3(\theta)+\mathrm{O}\big(t\mu^{5+\frac{1}{2}}\big)+\mathrm{O}\big(t^2\mu^{4+\frac{1}{4}}\big).\end{aligned}$$

Consequently, under the assumptions of Lemma 4.4, if (t_i,φ_i) blow up at $\theta\in S^3$ then necessarily $\Delta k_\theta(0)=0=a_0(\theta)$ and $a_2(\theta)\neq 0$. The necessary condition $a_2(\theta)\neq 0$ follows from that fact, that if $a_2(\theta)=0$ then from its definition $a_1(\theta)\neq 0$, which is impossible by Lemma 4.5, if θ is a blow-up point. Moreover, the expansion in Lemma 4.5 leads to restrictions on (t_i), that is (t_i) has to converge to $-a_1(\theta)/a_2(\theta)$.

It remains the question if there exist (t_i,φ_i) that blow up at $\theta\in S^3$, if $\Delta k_\theta(0)=0=a_0(\theta)$ and $a_2(\theta)\neq 0$. This is true if $a_1(\theta)\neq 0$, in the case $a_1(\theta)=0$ one needs to assume $a_3(\theta)\neq 0$.

Theorem 4.6. *Suppose* $1+k \in C^6(S^3)$ *is positive and satisfies*

$$D^2k_\theta(0) \text{ is invertible, if } \theta \in \mathcal{A} := \{\theta \in S^3\colon\ \nabla k(\theta) = 0 \text{ and } \Delta_{S^3}k(\theta) = 0\}$$

and

$$a_1(\theta)^2 + a_3(\theta)^2 \neq 0 \quad \text{for all } \theta \in M.$$

Then there is $\delta > 0$ *such that for any* $\theta \in M_+ \cup M_0$ *there exists a unique* C^1*-curve*

$$\{0 < \mu < \delta\} \ni \mu \mapsto \big(t^\theta(\mu), \varphi^\theta(\mu,\cdot)\big) \in \big((-\delta, 1+\delta) \setminus \{0\}\big) \times C^{2,\alpha}\big(S^3\big),$$

such that as $\mu \to 0$

$$t^\theta(\mu) = -\frac{1}{a_2(\theta)} \begin{cases} a_1(\theta) + \mathrm{O}(\mu^{\frac14}) & \text{if } a_1(\theta) \neq 0, \\ a_3(\theta)\mu + \mathrm{O}(\mu^{1+\frac14}) & \text{if } a_1(\theta) = 0 \end{cases}$$

and $\varphi^\theta(\mu,\cdot)$ *solves* (1.1) *for* $t = t^\theta(\mu)$ *and blows up like*

$$\big\|\mathcal{R}_\theta\big(\varphi^\theta(\mu,x)\big) - \big(1+t^\theta(\mu)k(\theta)\big)^{-\frac14} z_{\mu,0}(x)\big\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)\cap C^2(B_1(0))} = \mathrm{O}\big(\mu^2\big).$$

The curves are unique, in the sense that, if $(t_i,\varphi_i) \in ((-\delta, 1+\delta)\setminus\{0\}) \times C^{2,\alpha}(S^3)$ *blow up at some* $\theta \in S^3$ *then* $\theta \in M_+ \cup M_0$ *and there is a sequence of positive numbers* (μ_i) *converging to zero such that* $(t_i,\varphi_i) = (t^\theta(\mu_i), \varphi^\theta(\mu_i,\cdot))$ *for all but finitely many* $i \in \mathbb{N}$.

Proof. The claim follows directly from [17, Theorems 1.2, 1.3, 7.1]; we only need to note that if (t_i,φ_i) blow up at some $\theta \in S^3$ with $\lim_{i\to\infty} t_i = 0$ then by our assumptions $a_3(\theta) \neq 0$ and θ has to be in M_0. □

In order to compute the degree of the concentrating solutions, when $t \to 0^+$, we need to compute the derivative of $t^\theta(\cdot)$ as $\mu \to 0^+$.

Lemma 4.7. *Under the assumptions of Theorem* 4.6 *suppose* $\theta \in M_0$. *Then*

$$\frac{\partial t^\theta(\mu)}{\partial\mu} = -\frac{a_3(\theta)}{a_2(\theta)} + \mathrm{O}\big(\mu^{\frac14}\big).$$

The proof is given in Appendix A. The next observation is important for the calculation of the degree of solutions which remain uniformly bounded as $t \to 0^+$.

Lemma 4.8. *Under the assumptions of Theorem* 4.6 *suppose* $\theta \in \mathcal{A}$ *and let* $\vec{\alpha}$ *be as in Lemma* 4.1. *Then there is* $\delta_2 > 0$ *such that for all* $0 < \delta \leqslant \delta_2$ *exist* $t_2(\delta) > 0$ *and* $d_2(\delta) > 0$ *satisfying*

$$\inf_{|t|<t_2(\delta),\, \delta\leqslant\mu\leqslant\delta_2,\, |y|\leqslant\delta_2} \left|\frac{1}{t\mu}\vec{\alpha}(t,\mu,y)\right| \geqslant d_2(\delta) > 0.$$

Proof. The assumptions of Theorem 4.6 imply $a_0(\theta)^2 + a_1(\theta)^2 + a_3(\theta)^2 \neq 0$. In the case $a_0(\theta)^2 + a_1(\theta)^2 \neq 0$ we may choose $t_2(\delta)$ independently of δ: If $a_0(\theta) \neq 0$ we set $t_2(\delta) := 1$ and if $a_0(\theta) = 0$ and $a_1(\theta) \neq 0$ we let $t_2(\delta) := \frac12 \max(1, |a_2(\theta)|)^{-1}|a_1(\theta)|$ such that

$$\big|a_1(\theta) + t a_2(\theta)\big| > \frac12 \big|a_1(\theta)\big| \quad \text{for } |t| \leqslant t_2.$$

If $a_0(\theta)^2 + a_1(\theta)^2 = 0$ we set

$$t_2(\delta) := \min\left(\frac{|a_3(\theta)|}{4|a_2(\theta)|}, 1\right)\delta.$$

From the expansion in Lemma 4.5 there exists $0 < \delta_2 < \delta_1$ such that

$$h_1(t,\mu) := \frac{1}{t\mu}\vec{\alpha}\big(t,\mu,\beta(t,\mu)\big)_0 \neq 0 \quad \text{for } \delta \leqslant \mu \leqslant \delta_2 \text{ and } |t| \leqslant t_2(\delta). \tag{4.5}$$

To obtain a contradiction assume there are $(t_n, \mu_n, y_n) \in (-t_1, t_1) \times (\delta, \delta_2) \times B_{\delta_2}(0)$ such that

$$\frac{1}{t_n\mu_n}\vec{\alpha}(t_n, \mu_n, y_n) \to 0 \quad \text{as } n \to \infty.$$

We may assume $(t_n, \mu_n, y_n) \to (\bar{t}, \bar{\mu}, \bar{y})$ as $n \to \infty$, where $\delta \leqslant \bar{\mu} \leqslant \delta_2$. Thus, $\hat{\alpha}(\bar{t}, \bar{\mu}, \bar{y}) = 0$. The uniqueness part of Lemma 4.4 gives $\bar{y} = \beta(\bar{t}, \bar{\mu})$. Hence, $h_1(\bar{t}, \bar{\mu}) = 0$ contradicting (4.5). □

5. Leray–Schauder degree

We recall that for $\theta \in S^3$ the Melnikov function $\Gamma_\theta : (0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$\Gamma_\theta(\mu, y) = \frac{1}{6}\int k_\theta(x)(z_{\mu,y})^6 = \frac{1}{6}\int k_\theta(\mu x + y)(z_{1,0})^6.$$

It is known (see [1,2]) that Γ_θ extends via

$$\Gamma_\theta(-\mu, y) := \Gamma_\theta(\mu, y) \quad \text{and} \quad \Gamma_\theta(0, y) := k_\theta(y)\frac{1}{6}\int (z_{1,0})^6$$

to a function in $C^2(\mathbb{R} \times \mathbb{R}^3)$ and

$$\begin{aligned}
&\frac{\partial \Gamma_\theta}{\partial y_i}(0, y) = \frac{\partial k}{\partial x_i}(y)\frac{1}{6}\int (z_{1,0})^6 = \frac{\pi^2\sqrt{3}}{8}\frac{\partial k}{\partial x_i}(y), \qquad \frac{\partial \Gamma_\theta}{\partial \mu}(0, y) = 0,\\
&\frac{\partial^2 \Gamma_\theta}{\partial y_i \partial y_j}(0, y) = \frac{\pi^2\sqrt{3}}{8}\frac{\partial^2 k}{\partial x_i \partial x_j}(y), \qquad \frac{\partial^2 \Gamma_\theta}{\partial \mu \partial y_j}(0, y) = 0,\\
&\frac{\partial^2 \Gamma_\theta}{\partial \mu^2}(0, y) = \Delta k_\theta(y)\frac{1}{18}\int |x|^2 (z_{1,0})^6 = \frac{\pi^2\sqrt{3}}{8}\Delta k_\theta(y).
\end{aligned} \tag{5.1}$$

Using the Kelvin transform $z_{\mu,y} \mapsto |x|^{-1} z_{\mu,y}(x/|x|^2)$, we see

$$\Gamma_\theta\big((\mu, y)\big(\mu^2 + |y|^2\big)^{-1}\big) = \frac{1}{6}\int k_\theta\big(x/|x|^2\big) z_{\mu,y}^6.$$

Consequently, we may extend Γ_θ to a function in $C^2(S^4)$ by identifying $\mathbb{R} \times \mathbb{R}^3$ with $S^4 \setminus \{(0, -\theta)\}$ via $\mathcal{S}_{(0,\theta)}$ and setting

$$\Gamma_\theta\big((0, -\theta)\big) := k(-\theta)\frac{1}{6}\int (z_{1,0})^6 = \frac{\pi^2\sqrt{3}}{8}k(-\theta).$$

Hence there is a function $\Gamma \in C^2(S^4, \mathbb{R})$ such that Γ_θ is the function Γ in stereographic coordinates centered at $(0, \theta) \in S^4$.

Remark 5.1. The function Γ is related to the map G, which was used in [10] to compute the degree of (1.1), via (up to an unimportant constant)

$$\frac{\partial \Gamma_\theta}{\partial y_i}(\mu, y) = \mu^{-1} G(\mu, y)_i \quad \text{and} \quad \frac{\partial \Gamma_\theta}{\partial \mu}(\mu, y) = \mu^{-1} G(\mu, y)_4,$$

when the set of conformal transformations of S^3 is parametrized by $\mu \in (0,\infty)$ and $y \in \mathbb{R}^3$ using the coordinates $\mathcal{S}_\theta$. Obviously the factor μ^{-1} does not change the degree. We make use of Γ as it is convenient to have a potential and because it fits perfectly in our perturbative approach.

By standard elliptic regularity the operator L_t, defined by

$$L_t : \varphi \mapsto (-8\Delta_{S^3} + 6)^{-1}\big(6\big(1 + tk(\theta)\big)\varphi^5\big),$$

is compact from $C^2(S^3)$ into $C^2(S^3)$. Under the assumptions of Theorem 1.1 for any $\delta > 0$ there is a positive constant $C_{k,\delta}$ such that the Leray–Schauder degree $\deg(\mathrm{Id} - L_t, \mathcal{B}_{k,\delta}, 0)$ is well-defined and independent of $t \in [\delta, 1]$ (see [17]), where

$$\mathcal{B}_{k,\delta} := \big\{\varphi \in C^2\big(S^3\big):\ \|\varphi\|_{C^2(S^3)} < C_{k,\delta} \text{ and } C_{k,\delta}^{-1} < \varphi\big\}.$$

We will compute $\deg(\mathrm{Id} - L_t, \mathcal{B}_{k,\delta}, 0)$ for small t using the Melnikov function Γ. To this end we use the transformation (1.3) and get for any $\theta \in S^3$

$$\deg(\mathrm{Id} - L_t, \mathcal{B}_{k,\delta}, 0) = \deg\big(f_t', \mathcal{R}_\theta(\mathcal{B}_{k,\delta}), 0\big).$$

In the sequel we will denote by ∂_0 the derivation with respect to μ and by ∂_i the derivation with respect to y_i for $1 \leqslant i \leqslant 3$. A direct calculation gives $\partial_i z_{\mu,y} = \mu^{-1} c_\xi (\dot{\xi}_{\mu,y})_i$, where

$$c_\xi = \frac{\pi 3^{\frac14}\sqrt{15}}{8}.$$

We first show that there is an open neighborhood $\mathcal{U}$ of the equator $\{(0,\theta):\ \theta \in S^3\}$ in S^4 such that all critical points of Γ in $\mathcal{U}$ lie on the equator.

Lemma 5.2. *Under the assumptions of Theorem* 4.6 *there is* $\delta_3 = \delta_3(k) > 0$ *such that* $\Gamma'(s,\theta) \neq 0$ *for all* $(s,\theta) \in U_{\delta_3}$*, where*

$$U_{\delta_3} := \big\{(s,\theta) \in S^4:\ 0 < \operatorname{dist}\big((s,\theta), \big\{(0,\theta):\ \theta \in S^3\big\}\big) < \delta_3\big\}.$$

Proof. Fix $\theta \in S^3$. We have by (5.1)

$$\Gamma_\theta'(\mu, y) = \frac{\pi^2\sqrt{3}}{8}\begin{pmatrix} \Delta k_\theta(0)\mu \\ \nabla k_\theta(0) + D^2 k_\theta(0) y \end{pmatrix} + \mathrm{o}\big(\mu^2 + |y|^2\big).$$

Hence, for $\theta \notin \mathcal{A}$ there is $\delta(\theta) > 0$ such that $\Gamma_\theta'(\mu, y) \neq 0$ for all (μ, y) satisfying $0 < |\mu| < \delta$ and $|y| < \delta$.

As $T_{z_{\mu,y}} Z = \ker(f_0''(z_{\mu,y}))$ by (2.5) we obtain for any $v \in T_{z_{\mu,y}} Z^\perp$

$$5\int (z_{\mu,y})^4 v \partial_i z_{\mu,y} = \langle \partial_i z_{\mu,y}, v\rangle - f_0''(z_{\mu,y})\partial_i z_{\mu,y} v = 0 \quad \text{for } i = 0, \ldots, 3. \tag{5.2}$$

Suppose $\theta \in \mathcal{A}$. From Lemma 4.1, the fact that $w(t,\mu,y)$ and $z_{\mu,y}$ are orthogonal to $\partial_i z_{\mu,y}$, and (5.2) we see for $|\mu| + |y| \leqslant 1$ in coordinates $\mathcal{S}_\theta$

$$\begin{aligned} c_\xi \mu^{-1} \vec{\alpha}(t,\mu,y)_i &= f_t'\big(z_{\mu,y} + w(t,\mu,y)\big)\partial_i z_{\mu,y} \\ &= -\int \big(1 + tk_\theta(x)\big)\big(z_{\mu,y} + w(t,\mu,y)\big)^5 \partial_i z_{\mu,y} \\ &= -\int (z_{\mu,y})^4 z_{\mu,y}\partial_i z_{\mu,y} - t\int k_\theta(x)(z_{\mu,y})^5 \partial_i z_{\mu,y} - 5\int (z_{\mu,y})^4 w(t,\mu,y)\partial_i z_{\mu,y} + \mathrm{O}\big(t^2\big) \\ &= -t\partial_i \Gamma_\theta(\mu,y) + \mathrm{O}\big(t^2\big). \end{aligned} \tag{5.3}$$

We apply Lemma 4.8 and obtain from (5.3)

$$\big|\Gamma_\theta'(\mu,y)\big| \geqslant d_2(\delta) > 0$$

for all $\delta < \mu < \delta_2$ and $|y| \leqslant \delta_2$, which gives the claim. □

Thus, under the assumptions of Theorem 4.6 there is $R_\Gamma > 0$ independent of $\theta \in S^3$ such that

$$(\Gamma_\theta')^{-1}(0) \cap (0,\infty) \times \mathbb{R}^3 \subset \Omega_{R_\Gamma},$$

where Ω_R is defined by

$$\Omega_R := \left(R^{-1}, R\right) \times B_R(0).$$

Lemma 5.3. *Under the assumptions of Theorem* 4.6 *we fix some point* $\vartheta \in S^3$ *such that* $\nabla k(-\vartheta) \neq 0$ *and use coordinates* $\mathcal{S}_\vartheta$ *and the transformation in* (1.3). *Then there is* $t_1 = t_1(k, \vartheta) > 0$ *and* $R_0 = R_0(k, \vartheta) \geqslant R_\Gamma > 0$ *such that if* $0 < t \leqslant t_1$ *then any solution* u_t *of* (1.2) *is of the form* $z_{\mu,y} + w(t, \mu, y)$, *where* $(\mu, y) \in (0, R_0) \times B_{R_0}(0)$. *Moreover, either* $(\mu, y) \in \Omega_{R_0}$ *or* $u_t = \mathcal{R}_\vartheta(\varphi^\theta(\mu, \cdot))$ *and* $t = t^\theta(\mu)$ *for some* $0 < \mu < (R_0)^{-1}/2$ *and* $\theta \in M_0$, *where* φ^θ *and* t^θ *are given in Theorem* 4.6.

Proof. From the expansion of t^θ in Theorem 4.6 we may take $\tilde{R}_1 > 0$ such that we have for all $\theta \in M_0$ and $0 < \mu \leqslant (\tilde{R}_1)^{-1}$

$$\mu \leqslant 2\frac{|a_2(\theta)|}{|a_3(\theta)|}\left|t^\theta(\mu)\right|. \tag{5.4}$$

We first show that the solutions of (1.1), which do not belong to

$$L_0 := \left\{\varphi^\theta(\mu, \cdot)\colon\ 0 < \mu < (\tilde{R}_1)^{-1},\ \theta \in M_0\right\},$$

remain bounded as $t \to 0$. Let (t_i, φ_i) be a sequence such that φ_i solves (1.1) with $t = t_i$,

$$0 < t_i \to 0 \quad \text{and} \quad \|\varphi_i\|_\infty \to \infty \quad \text{as } i \to \infty.$$

The uniqueness part of Theorem 4.6 shows that (t_i, φ_i) equals $(t^\theta(\mu_i), \varphi^\theta(\mu_i, \cdot))$ for some $\theta \in M_0$ and (μ_i) tending to 0.

Consequently, there are two types of solutions to (1.1), the solutions which lie in L_0 and blow up as $t \to 0$ and solutions which remain uniformly bounded above (and by Remark 5.4 below) as $t \to 0$. From our analysis above there is $\tilde{t}_1 > 0$ such that the solutions in L_b, defined by

$$L_b := \left\{\varphi\colon\ \varphi \text{ solves (1.1) for some } 0 < t \leqslant \tilde{t}_1 \text{ and } \varphi \notin L_0\right\},$$

are uniformly bounded below and above.

Using the transformation in (1.3) with coordinates $\mathcal{S}_\vartheta$ there is $R_0 > 0$ such that for any solution $u_t \in \mathcal{R}_\vartheta(L_b)$ of (1.2) with $0 < t < \tilde{t}_1$

$$\|u_t - z_{\mu,y}\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \geqslant \frac{1}{2}\|z_{1,0}\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \quad \text{if } (\mu, y) \notin \Omega_{R_0}.$$

Due to the classical result of Caffarelli et al. [9], Gidas et al. [13] all positive solutions of (1.2) are given by Z. Hence, using the uniform bound, there exists $0 < t_1 \leqslant \tilde{t}_1$ such that any solution $u_t \in \mathcal{R}_\vartheta(L_b)$ of (1.2) with $0 < t < t_1$ satisfies $\operatorname{dist}(u_t, Z) < \rho_0$, where ρ_0 is given in Lemma 4.1. Shrinking t_1 if necessary, we see that for any solution $u_t \in \mathcal{R}_\vartheta(L_b)$ of (1.2) with $0 < t < t_1$ there is $z_{\mu,y}$ such that $(\mu, y) \in \Omega_{R_0}$ and

$$u_t - z_{\mu,y} \text{ is orthogonal to } Z_{z_{\mu,y}}Z, \quad \|u_t - z_{\mu,y}\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} < \rho_0.$$

For $f_t'(z_{\mu,y} + u_t - z_{\mu,y}) = 0$ the uniqueness in Lemma 4.1 yields for

$$u_t = z_{\mu,y} + w(t, \mu, y), \quad \text{where } (\mu, y) \in \Omega_{R_0}.$$

The solutions in $\mathcal{R}_\vartheta(L_0)$ given by Theorem 4.6 are by construction of the form $z_{\mu,y} + w(t, \mu, y)$, where $y(\mu) \to \mathcal{S}_\vartheta(\theta)$ for some $\theta \in M_0$ as $\mu \to 0$. Enlarging R_0 depending on $\operatorname{dist}(-\vartheta, M_0)$ and shrinking t_1 we infer from (5.4) that any solution in $u_t \in \mathcal{R}_\vartheta(L_0)$ of (1.2) with $0 < t < t_1$ satisfies,

$$u_t = z_{\mu,y} + w(t, \mu, y), \quad \text{where } 0 < \mu < \frac{1}{2}(R_0)^{-1} \text{ and } |y| < R_0.$$

This finishes the proof. □

Remark 5.4. We note that the uniform lower bound of positive solutions follows directly from Harnack's inequality and the upper bound. To see this we multiply (1.1) by φ and use Sobolev's inequality to get

$$\int_{S^3} 8|\nabla\varphi|^2 + 6\varphi^2 = 6\int_{S^3}\big(1+tk(\theta)\big)\varphi^6 \leqslant \text{const}(k)\|\varphi\|^6_{H^{1,2}(S^3)},$$

which shows that $\|\varphi\|_{H^{1,2}(S^3)}$ is bounded below by a positive constant $c(k)$. Harnack's inequality then implies

$$c(k)^2 \leqslant \int_{S^3} 8|\nabla\varphi|^2 + 6\varphi^2 = 6\int_{S^3}\big(1+tk(\theta)\big)\varphi^6 \leqslant \text{const}(k)\inf_{\theta\in S^3}\varphi(\theta)^6,$$

and leads to the desired lower bound.

For the computation of the degree we will use tubular coordinates, when we are close to the critical manifold Z.

Lemma 5.5. *There are $\rho_1 > 0$ and a differentiable homeomorphism $Q : B_{\rho_1}(Z) \to NZ(\rho_1)$, where $NZ(\rho_1)$ denotes the normal disk bundle of Z in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ with radius ρ_1 and*

$$B_{\rho_1}(Z) := \big\{u \in \mathcal{D}^{1,2}\big(\mathbb{R}^3\big):\ \text{dist}(u, Z) < \rho_1\big\}.$$

Proof. We consider the map $q : Z \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}^4 \times \mathcal{D}^{1,2}(\mathbb{R}^3)$, defined by

$$q(z, w) := \big(\big(w, (\dot{\xi}_z)_i\big)_i, z + w\big).$$

For its derivative we find

$$Dq|_{(z,0)}\begin{pmatrix}\vec{h}\cdot(\dot{\xi}_z)\\ v\end{pmatrix} = \begin{pmatrix}(v, (\dot{\xi}_z)_i)_i\\ \vec{h}\cdot(\dot{\xi}_z) + v\end{pmatrix}$$

and

$$Dq|_{(\mathcal{T}_y\circ\mathcal{U}_\mu z,0)} = \begin{pmatrix}\text{id} & 0\\ 0 & \mathcal{T}_y\circ\mathcal{U}_\mu\end{pmatrix} Dq|_{(z,0)}\begin{pmatrix}(\mathcal{T}_y\circ\mathcal{U}_\mu)^{-1} & 0\\ 0 & (\mathcal{T}_y\circ\mathcal{U}_\mu)^{-1}\end{pmatrix}.$$

Hence, $Dq|_{(z,0)}$ is uniformly invertible and we may apply the inverse function theorem to deduce the existence of ρ_1 and $Q(u) := q^{-1}(0, u)$. □

By Lemma 5.3 there are two types of solutions to (1.1) as $t \to 0^+$: the solutions in L_b remain uniformly bounded as $t \to 0^+$ and the solutions in L_0 blow up as $t \to 0^+$ and are isolated by Theorem 4.6 and Lemma 4.7 for each fixed small $t > 0$. From Lemma 5.5 we obtain for any $0 < \delta < t_0$ and $t \in [\delta, t_1]$ using the additivity of the degree

$$\deg\big(f_t', \mathcal{R}_\vartheta(\mathcal{B}_{k,\delta}), 0\big) = \deg\big(f_t', B_{\rho_2,R_0}(Z), 0\big) + \sum_{\substack{\theta\in M_0:\\ \exists\mu^\theta>0:\ t^\theta(\mu^\theta)=t}} \deg_{\text{loc}}\big(f_t', \mathcal{R}_\vartheta\big(\varphi^\theta\big(\mu^\theta,\cdot\big)\big)\big),$$

where $\rho_2 := \min(\rho_0, \rho_1)$ and

$$B_{\rho_2,R_0}(Z) := Q^{-1}\big(\big\{(z_{\mu,y}, w):\ (z_{\mu,y}, w) \in NZ(\rho_2), (\mu, y) \in \Omega_{R_0}\big\}\big) \cap \mathcal{R}_\vartheta(\mathcal{B}_{k,\delta}).$$

We first compute the degree of the blowing-up solutions

Lemma 5.6. *Under the assumptions of Lemma 5.3 there holds for $0 < t \leqslant t_1$*

$$\sum_{\substack{\theta\in M_0:\\ \exists\mu^\theta>0:\ t^\theta(\mu^\theta)=t}} \deg_{\text{loc}}\big(f_t', \mathcal{R}_\vartheta\big(\varphi^\theta\big(\mu^\theta,\cdot\big)\big)\big) = \sum_{\theta\in M_0:\ a_3(\theta)>0}(-1)^{\text{ind}(k,\theta)}.$$

Proof. Note that if $a_1(\theta)=0$ then the definition of $a_2(\theta)$ implies $a_2(\theta)<0$. From Theorem 4.6 for $\theta\in M_0$ there exists a $\mu^\theta=\mu^\theta(t)>0$ such that $t^\theta(\mu^\theta)=t$ if and only if $-\frac{a_3(\theta)}{a_2(\theta)}>0$. As $a_2(\theta)<0$, this is equivalent to $a_3(\theta)>0$. Lemma 4.7 then shows that $\mu^\theta(t)>0$ is unique.

Fix $\theta\in M_0$, such that $a_3(\theta)>0$, $0<\delta<t_0$, and $t\in[\delta,t_0]$. By Theorem 4.6, Lemmas 4.1 and 5.3 we know that

$$u_{\mu^\theta}:=\mathcal{R}_\vartheta\bigl(\varphi^\theta\bigl(\mu^\theta,\cdot\bigr)\bigr)=z_{\mu^\theta,y(\mu^\theta)}+w\bigl(t,\mu^\theta,y\bigl(\mu^\theta\bigr)\bigr),$$
$$\bigl\|w\bigl(t,\mu^\theta,y\bigl(\mu^\theta\bigr)\bigr)\bigr\|=\mathrm{O}\bigl(\mu^\theta\bigr)=\mathrm{O}(t),\qquad y\bigl(\mu^\theta\bigr)=\beta\bigl(t,\mu^\theta\bigr),$$

and the solution u_{μ^θ} remains uniform isolated for $t\in(0,t_0]$. Consequently after possibly shrinking t_0 we have for $\varepsilon=\sqrt{t}$

$$\deg_{\mathrm{loc}}(f_t',u_{\mu^\theta})=\deg\bigl(f_t',U_\varepsilon(u_{\mu^\theta}),0\bigr),$$

where

$$U_\varepsilon(u_{\mu^\theta}):=\bigl\{z_{\mu,y}+w\colon\ \bigl|\mu-\mu^\theta\bigr|\leqslant\varepsilon^4,\ \bigl|y-y\bigl(\mu^\theta\bigr)\bigr|\leqslant\varepsilon,\ w\in T_{z_{\mu,y}}^\perp,\ \|w\|\leqslant\varepsilon\bigr\}.$$

We note that by Lemma 5.5 we may work with tubular coordinates that is any $u\in U_\varepsilon(u_{\mu^\theta})$ splits into the sum of $Q(u)=z_{Q(u)_\mu,Q(u)_y}$ and $u-Q(u)=w\in T_{Q(u)}Z^\perp$.

To obtain a contradiction, assume

$$sf_t'(z_{\mu,y}+w)+(1-s)f_t'\bigl(z_{\mu,y}+w(t,\mu,y)\bigr)+(1-s)\mathrm{Proj}_{T_{z_{\mu,y}}Z^\perp}\bigl(f_t'(z_{\mu,y}+w)\bigr)=0$$

for some $s\in[0,1]$ and $z_{\mu,y}+w\in\partial U_\varepsilon(u_{\mu^\theta})$. By Lemma 4.1 we have $f_t'(z_{\mu,y}+w(t,\mu,y))\in T_{z_{\mu,y}}Z$, which yields

$$\mathrm{Proj}_{T_{z_{\mu,y}}Z^\perp}\bigl(f_t'(z_{\mu,y}+w)\bigr)=0.$$

The uniqueness part of Lemma 4.1 implies $w=w(t,\mu,y)$, which gives the contradiction $f_t'(z_{\mu,y}+w)=0$ for some $z_{\mu,y}+w\in\partial U_\varepsilon(u_{\mu^\theta})$.

Consequently

$$\begin{aligned}\deg\bigl(f_t',U_\varepsilon(u_{\mu^\theta}),0\bigr)&=\deg\bigl(f_t'\bigl(z_{\mu,y}+w(t,\mu,y)\bigr)+\mathrm{Proj}_{T_{z_{\mu,y}}Z^\perp}\bigl(f_t'(z_{\mu,y}+w)\bigr),U_\varepsilon(u_{\mu^\theta}),0\bigr)\\&=\deg\left(\mathrm{Proj}_{T_{z_{\mu,y}}Z^\perp}\bigl(f_t'(z_{\mu,y}+w)\bigr)+\sum_{i=0}^{3}\alpha(t,\mu,y)_i(\dot\xi_{\mu,y})_i,U_\varepsilon(u_{\mu^\theta}),0\right),\end{aligned}$$

where we used again Lemma 4.1. For $w\in T_{z_{\mu,y}}Z^\perp$ with $\|w\|=\varepsilon$ we may estimate

$$\begin{aligned}f_t'(z_{\mu,y}+w)&=f_t'(z_{\mu,y})+f_t''(z_{\mu,y})w+\mathrm{O}\bigl(\varepsilon^2\bigr)\\&=f_0''(z_{\mu,y})w+\mathrm{O}\bigl(\varepsilon^2\bigr)+\mathrm{O}(t)=f_0''(z_{\mu,y})w+\mathrm{O}\bigl(\varepsilon^2\bigr),\end{aligned}$$

where $\|f_0''(z_{\mu,y})w\|\geqslant\mathrm{const}\,\varepsilon$ due to (2.6). Note that $f_0''(z_{\mu,y})w\in T_{z_{\mu,y}}^\perp$. Hence

$$\begin{aligned}&\deg\left(\mathrm{Proj}_{T_{z_{\mu,y}}Z^\perp}\bigl(f_t'(z_{\mu,y}+w)\bigr)+\sum_{i=0}^{3}\alpha(t,\mu,y)_i(\dot\xi_{\mu,y})_i,U_\varepsilon(u_{\mu^\theta}),0\right)\\&\quad=\deg\left(f_0''(z_{\mu,y})w+\sum_{i=0}^{3}\alpha(t,\mu,y)_i(\dot\xi_{\mu,y})_i,U_\varepsilon(u_{\mu^\theta}),0\right).\end{aligned}$$

By Lemma 4.4 there holds in $U_\varepsilon(u_{\mu^\theta})$

$$\alpha(t,\mu,y)_i=0\quad\text{for } i=1\ldots3\quad\Longleftrightarrow\quad y=\beta(t,\mu).$$

As above we may deduce

$$\deg\left(f_0''(z_{\mu,y})w+\sum_{i=0}^{3}\alpha(t,\mu,y)_i(\dot{\xi}_{\mu,y})_i,U_\varepsilon(u_{\mu^\theta}),0\right)$$
$$=\deg\left(f_0''(z_{\mu,y})w+\alpha\big(t,\mu,\beta(t,\mu)\big)_0(\dot{\xi}_{\mu,y})_0+\sum_{i=1}^{3}\alpha(t,\mu,y)_i(\dot{\xi}_{\mu,y})_i,U_\varepsilon(u_{\mu^\theta}),0\right).$$

From Lemma 4.3 we have for $|y-y(\mu^\theta)|=\varepsilon$

$$\frac{1}{t\mu^\theta}\big(\alpha(t,\mu,y)_i\big)_{1\leqslant i\leqslant 3}=-\frac{\pi}{3^{\frac14}\sqrt{5}}\big(1+tk(y)\big)^{-\frac54}D^2k\big(y\big(\mu^\theta\big)\big)\big(y-y\big(\mu^\theta\big)\big)+\mathrm{O}\big(\varepsilon^{1+\frac12}\big).$$

As $D^2k(y(\mu^\theta))$ is non-degenerated we may replace $\sum_{i=1}^{3}\alpha(t,\mu,y)_i(\dot{\xi}_{\mu,y})_i$ by

$$-D^2k\big(y\big(\mu^\theta\big)\big)\big(y-y\big(\mu^\theta\big)\big)\cdot(\dot{\xi}_{\mu,y})_{1\leqslant i\leqslant 3},$$

without changing the degree and get

$$\deg_{\mathrm{loc}}(f_t',u_{\mu^\theta})=\deg\big(f_0''(z_{\mu,y})w+\alpha\big(t,\mu,\beta(t,\mu)\big)_0(\dot{\xi}_{\mu,y})_0$$
$$-D^2k\big(y\big(\mu^\theta\big)\big)\big(y-y\big(\mu^\theta\big)\big)\cdot(\dot{\xi}_{\mu,y})_{1\leqslant i\leqslant 3},U_\varepsilon(u_{\mu^\theta}),0\big).$$

From the proof of Lemma 4.7 in Appendix A we know

$$\frac{\partial\gamma}{\partial\mu}(t,\mu)=a_3(\theta)+\mathrm{O}\big(\mu^{\frac14}\big),$$

where

$$\gamma(t,\mu)=\frac{1}{t\mu^4}\big(1+tk(\theta)\big)^{\frac94}\frac{30}{\pi 3^{\frac34}\sqrt{5}}\big(\alpha\big(t,\mu,\beta(t,\mu)\big)\big)_0.$$

By (2.4)–(2.6) the self-adjoint operator $f_0''(z_{\mu,y})$ restricted to $T_{z_{\mu,y}}Z^\perp$ is invertible with only one negative eigenvalue. Hence, we finally see

$$\deg_{\mathrm{loc}}(f_t',u_{\mu^\theta})=\deg\big(f_0''(z_{\mu,y})w+a_3(\theta)\big(\mu-\mu^\theta\big)(\dot{\xi}_{\mu,y})_0$$
$$-D^2k\big(y\big(\mu^\theta\big)\big)\big(y-y\big(\mu^\theta\big)\big)\cdot(\dot{\xi}_{\mu,y})_{1\leqslant i\leqslant 3},U_\varepsilon(u_{\mu^\theta}),0\big)$$
$$=(-1)\cdot 1\cdot(-1)^3\cdot(-1)^{\mathrm{ind}(k,\theta)}.$$

This ends the proof. □

We shall show that the degree $\deg(f_t',B_{\rho_2,R_0}(Z),0)$ of the solutions that remain bounded as $t\to 0^+$ is given by $-\deg(\Gamma_\theta',\Omega_R,0)$ for $\theta\in S^3$ and large R. We prove the identity by comparing local degrees. Since we cannot assume that the critical points of Γ_θ are isolated, we use a transversality argument and consider small perturbations of Γ_θ' and f_t'.

Lemma 5.7. *Under the assumptions of Lemma* 5.3 *let $\vec{\varepsilon}\in\mathbb{R}^4$ such that $|\vec{\varepsilon}|<\rho_0$ and $\Gamma_\vartheta'-\vec{\varepsilon}\neq 0$ in $\partial\Omega_{R_0}$. Suppose $\Gamma_\vartheta'-\vec{\varepsilon}$ has only non-degenerate zeros in Ω_{R_0}, such that*

$$A_2\geqslant\sup\big\{\big\|\big(\Gamma_\vartheta''(\mu,y)\big)^{-1}\big\|:\ \Gamma_\theta'(\mu,y)=\vec{\varepsilon}\text{ and }(\mu,y)\in\Omega_{R_0}\big\}.$$

Then there exists $t_1=t_1(k,A_2)>0$ such that for any $0<t<t_1$ any solution $u_t\in B_{\rho_2,R_0}(Z)$ of

$$0=f_{t,\vec{\varepsilon}}'(u_t):=f_t'(u_t)+(c_\xi)^{-1}tQ(u_t)_\mu\vec{\varepsilon}\cdot(\dot{\xi}_{Q(u_t)})$$

is of the form $u_t=z_{\mu_t,y_t}+w(t,\mu_t,y_t)$, where $\vec{\alpha}(t,\mu_t,y_t)=-(c_\xi)^{-1}t\mu_t\vec{\varepsilon}$ and $(\mu_t,y_t)\in\Omega_{R_0}$. Moreover, there is $C_1=C_1(k,A_2)>0$ such that if $(u_{t_n},t_n)\in B_{\rho_2,R_0}(Z)\times(0,t_1)$ satisfy $f_{t_n,\vec{\varepsilon}}'(u_{t_n})=0$ and $t_n\to 0^+$ then there is $(\bar{\mu},\bar{y})\in\Omega_{R_0}$ such that, up to a subsequence,

$$|\mu_{t_n}-\bar{\mu}|+|y_{t_n}-\bar{y}|\leqslant C_1t_n,\tag{5.5}$$

where $\Gamma'_\vartheta(\bar\mu,\bar y)=\vec\varepsilon$. Vice-versa, for any zero $(\bar\mu,\bar y)\in\Omega_{R_0}$ of $\Gamma'_\vartheta-\vec\varepsilon$ and for $0<t<t_1$ there exists one and only one point (μ_t,y_t) such that $\vec\alpha(t,\mu_t,y_t)=-(c_\xi)^{-1}t\mu\vec\varepsilon$ and (5.5) holds.

Proof. From the uniqueness part in Lemma 4.1 we have $u_t=z_{\mu_t,y_t}+w(t,\mu_t,y_t)$ and $c_\xi\vec\alpha(t,\mu_t,y_t)=-t\mu_t\vec\varepsilon$. As $u_t\in B_{\rho_2,R_0}(Z)$ there holds $(\mu_t,y_t)\in\Omega_{R_0}$.

Fix a sequence (u_{t_n}) with (t_n) converging to 0. Since (μ_{t_n},y_{t_n}) is bounded, we may assume that (μ_{t_n},y_{t_n}) converges to $(\bar\mu,\bar y)$. From expansion (5.3)

$$-t_n\vec\varepsilon=c_\xi\mu_{t_n}^{-1}\vec\alpha(t_n,\mu_{t_n},y_{t_n})=-t_n\big(\Gamma'_\vartheta(\mu_{t_n},y_{t_n})+\mathrm{O}(t_n)\big),$$

hence $\Gamma'_\vartheta(\bar\mu,\bar y)=\vec\varepsilon$. A further expansion yields

$$\begin{aligned}0&=c_\xi\mu_{t_n}^{-1}\vec\alpha(t_n,\mu_{t_n},y_{t_n})+t_n\vec\varepsilon\\&=-t_n\left(\Gamma''_\vartheta(\bar\mu,\bar y)\begin{pmatrix}\mu_{t_n}-\bar\mu\\y_{t_n}-\bar y\end{pmatrix}+\mathrm{o}\big((\mu_{t_n},y_{t_n})-(\bar\mu,\bar y)\big)\right)+\mathrm{O}\big(t_n^2\big),\end{aligned}$$

which gives as $n\to\infty$

$$\big(\Gamma''_\theta(\bar\mu,\bar y)+\mathrm{o}(1)\big)\begin{pmatrix}\mu_{t_n}-\bar\mu\\y_{t_n}-\bar y\end{pmatrix}=\mathrm{O}(t_n)$$

proving (5.5) for $\Gamma''_\theta(\bar\mu,\bar y)$ is invertible.

Let $(\bar\mu,\bar y)$ be a zero of $\Gamma'_\vartheta-\vec\varepsilon$. Arguing as above we see as $(\mu,y)\to(\bar\mu,\bar y)$ and for any $0<t<t_1$

$$c_\xi\mu^{-1}\vec\alpha(t,\mu,y)+t\vec\varepsilon=-t\big(\Gamma''_\theta(\bar\mu,\bar y)+\mathrm{o}(1)\big)\begin{pmatrix}\mu-\bar\mu\\y-\bar y\end{pmatrix}+\mathrm{O}\big(t^2\big).$$

Using the degree in a $\mathrm{O}(t)$-neighborhood of $(\bar\mu,\bar y)$, we find (μ_t,y_t) such that

$$\vec\alpha(t,\mu_t,y_t)=-(c_\xi)^{-1}t\mu_t\vec\varepsilon$$

and (5.5) holds. To prove uniqueness of (μ_t,y_t), we use the fact that by (A.8) below and the results in [17, Lemma 5.1]

$$\big\|\partial_j w(t,\mu,y)\big\|=\mathrm{O}(t)\quad\text{for }0\leqslant j\leqslant 3\text{ and }(\mu,y)\in\Omega_{R_0}.\tag{5.6}$$

We obtain

$$\begin{aligned}&\partial_j\big(c_\xi\mu_t^{-1}\vec\alpha(t,\mu_t,y_t)_i+t\vec\varepsilon_i\big)\\&\quad=\partial_j\big(f'_t\big(z_{\mu_t,y_t}+w(t,\mu_t,y_t)\big)\partial_i z_{\mu_t,y_t}+t\vec\varepsilon_i\big)\\&\quad=f''_t\big(z_{\mu_t,y_t}+w(t,\mu_t,y_t)\big)\partial_j\big(z_{\mu_t,y_t}+w(t,\mu_t,y_t)\big)\partial_i z_{\mu_t,y_t}+f'_t\big(z_{\mu_t,y_t}+w(t,\mu_t,y_t)\big)\partial_j\partial_i z_{\mu_t,y_t}\\&\quad=f''_0\big(z_{\mu_t,y_t}+w(t,\mu_t,y_t)\big)\partial_j\big(z_{\mu_t,y_t}+w(t,\mu_t,y_t)\big)\,\partial_i z_{\mu_t y_t}-5t\int k(x)(z_{\mu_t,y_t})^4\partial_j z_{\mu_t,y_t}\,\partial_i z_{\mu_t,y_t}\\&\qquad+f'_0\big(z_{\mu_t,y_t}+w(t,\mu_t,y_t)\big)\partial_j\partial_i z_{\mu_t,y_t}-t\int k(x)(z_{\mu_t,y_t})^5\partial_j\partial_i z_{\mu_t,y_t}+\mathrm{O}\big(t^2\big)\\&\quad=f'''_0(z_{\mu_t,y_t})w(t,\mu_t,y_t)\,\partial_j z_{\mu_t,y_t}\,\partial_i z_{\mu_t,y_t}+f''_0(z_{\mu_t,y_t})\partial_j\partial_i z_{\mu_t,y_t}\,w(t,\mu_t,y_t)-t\partial_j\partial_i\Gamma_\theta(\mu_t,y_t)+\mathrm{O}\big(t^2\big).\end{aligned}$$

Differentiating $f''_0(z_{\mu,y})\partial_i z_{\mu,y}=0$ with ∂_j and testing with $w(t,\mu_t,y_t)$ we obtain

$$0=f'''_0(z_{\mu_t,y_t})w(t,\mu_t,y_t)\,\partial_j z_{\mu_t,y_t}\,\partial_i z_{\mu_t,y_t}+f''_0(z_{\mu_t,y_t})\partial_j\partial_i z_{\mu_t,y_t}\,w(t,\mu_t,y_t)$$

and finally

$$\partial_j\big(c_\xi\mu_t^{-1}\vec\alpha(t,\mu_t,y_t)_i+t\vec\varepsilon_i\big)=-t\partial_j\partial_i\Gamma_\theta(\mu_t,y_t)+\mathrm{O}\big(t^2\big).\tag{5.7}$$

To prove uniqueness, we choose $\delta>0$ such that $\operatorname{sgn}\det\Gamma''_\theta(\mu,y)=\operatorname{sgn}\det\Gamma''_\theta(\bar\mu,\bar y)\neq 0$ for any $|(\mu,y)-(\bar\mu,\bar y)|<\delta$ and $(\bar\mu,\bar y)$ is the only zero of $\Gamma'_\vartheta-\vec\varepsilon$ in $B_\delta(\bar\mu,\bar y)$. From (5.7), there exists $t(\delta)>0$ such that if $0<t<t(\delta)$ and $(\mu_t,y_t)\in B_\delta(\bar\mu,\bar y)$ solves $c_\xi\vec\alpha(t,\mu_t,y_t)=-t\mu\vec\varepsilon$, then

$$\operatorname{sgn}\det\big(\partial_j\big(c_\xi\mu_t^{-1}\vec\alpha(t,\mu_t,y_t)_i+t\vec\varepsilon_i\big)\big)=\operatorname{sgn}\det\Gamma''_\theta(\bar\mu,\bar y).$$

From (5.3) after possibly shrinking $t(\delta)$ we get for $0 < t < t(\delta)$

$$\begin{aligned}\operatorname{sgn}\det \Gamma_\theta''(\bar\mu,\bar y) &= \deg\big(\Gamma_\theta' - \vec\varepsilon, B_\delta(\bar\mu,\bar y),0\big) = \deg\big(-c_\xi\mu^{-1}\vec\alpha(t,\mu,y) - t\vec\varepsilon, B_\delta(\bar\mu,\bar y),0\big)\\ &= \sum_{\substack{(\mu_t,y_t)\in B_\delta(\bar\mu,\bar y)\\ c_\xi\vec\alpha(t,\mu_t,y_t)=-t\mu_t\vec\varepsilon}} \operatorname{sgn}\det\big(\partial_j\big(-\mu_t^{-1}\vec\alpha(t,\mu_t,y_t)_i\big) - (c_\xi)^{-1}\vec\varepsilon t\big)\\ &= \#\big\{(\mu_t,y_t)\in B_\delta(\bar\mu,\bar y):\ c_\xi\vec\alpha(t,\mu_t,y_t) = -t\mu_t\vec\varepsilon\big\}\operatorname{sgn}\det\Gamma_\theta''(\bar\mu,\bar y).\end{aligned}$$

Hence $\#\{(\mu_t,y_t)\in B_\delta(\bar\mu,\bar y):\ c_\xi\vec\alpha(t,\mu_t,y_t) = -t\mu_t\vec\varepsilon\} = 1$, proving uniqueness. □

Lemma 5.8. *Under the assumptions of Lemma 5.7 let $(\bar\mu,\bar y)\in\Omega_{R_0}$ be a zero of $\Gamma_\theta - \vec\varepsilon$ and (t,μ_t,y_t) such that $0<t<t_1$, $c_\xi\vec\alpha(t,\mu_t,y_t) = -t\mu_t\vec\varepsilon$ and* (5.5) *holds. Then we have the identity of local degrees*

$$\deg_{\mathrm{loc}}\big(f'_{t,\vec\varepsilon}, z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big) = -\deg_{\mathrm{loc}}\big(\Gamma_\theta' - \vec\varepsilon,(\bar\mu,\bar y)\big).$$

Proof. We have

$$f''_{t,\vec\varepsilon}(u) := \frac{\mathrm{d}f'_{t,\vec\varepsilon}}{\mathrm{d}u}(u) = f_t''(u) + (c_\xi)^{-1}t\big\langle(\dot\xi_{Q(u)})_0,\cdot\big\rangle\vec\varepsilon\cdot\dot\xi_{Q(u)} + (c_\xi)^{-1}tQ(u)_\mu\sum_{i=0}^{3}\vec\varepsilon_i\sum_{j=0}^{3}\partial_j(\dot\xi_{Q(u)})_i\big\langle(\dot\xi_{Q(u)})_j,\cdot\big\rangle,$$

where we used Lemma 5.5 to see

$$\frac{\partial Q(u)_j}{\partial u} = \big\langle(\dot\xi_{Q(u)})_j,\cdot\big\rangle.$$

Since $\|\partial_j w(t,\mu_t,y_t)\| = \mathrm{O}(t)$ we have for small t that

$$\mathcal{D}^{1,2}\big(\mathbb{R}^3\big) = \big\langle\partial_j\big(z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big):\ 0\leqslant j\leqslant 3\big\rangle\oplus T_{z_{\mu_t,y_t}}Z^\perp.$$

We recall that the eigenvectors $\Phi^{\mu,y}_{i,j,l}$ of $f_0''(z_{\mu,y})$ with $i+j\neq 1$ are spanning $T_{z_{\mu_t,y_t}}Z^\perp$ (see Section 2). Differentiating $f'_{t,\vec\varepsilon}(z_{\mu,y} + w(t,\mu,y))\Phi^{\mu,y}_{i,j,l}\equiv 0$ we get for $i+j\neq 1$

$$\big\langle f''_{t,\vec\varepsilon}\big(z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big)\partial_j\big(z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big),\Phi^{\mu,y}_{i,j,l}\big\rangle = 0.$$

Furthermore, by (5.6) and (5.7)

$$\begin{aligned}&\big\langle f''_{t,\vec\varepsilon}\big(z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big)\partial_j\big(z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big),\partial_i\big(z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big)\big\rangle\\ &\quad= \partial_j\big(f'_{t,\vec\varepsilon}\big(z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big)\partial_i z_{\mu_t,y_t}\big)\\ &\qquad+ \big\langle f''_{t,\vec\varepsilon}\big(z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big)\partial_j\big(z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big),\partial_i w(t,\mu_t,y_t)\big\rangle\\ &\quad= \partial_j\big(c_\xi\mu_t^{-1}\vec\alpha(t,\mu_t,y_t)_i + t\vec\varepsilon_i\big) + f_0''(z_{\mu_t,y_t})\partial_j z_{\mu_t,y_t}\partial_i w(t,\mu_t,y_t) + \mathrm{O}\big(t^2\big)\\ &\quad= -t\partial_j\partial_i\Gamma_\theta(\mu_t,y_t) + \mathrm{O}\big(t^2\big).\end{aligned}$$

For $i+j\neq 1$ we find

$$f''_{t,\vec\varepsilon}\big(z_{\mu_t,y_t} + w(t,\mu_t,y_t)\big)\Phi^{\mu_t,y_t}_{i,j,l} = f_0''(z_{\mu_t,y_t})\Phi^{\mu_t,y_t}_{i,j,l} + \mathrm{O}(t).$$

Consequently, in the above decomposition the map $f''_{t,\vec\varepsilon}(z_{\mu_t,y_t} + w(t,\mu_t,y_t))$ looks like

$$\begin{pmatrix} -t\partial_j\partial_i\Gamma_\theta(\mu_t,y_t) + \mathrm{O}(t^2) & \mathrm{O}(t)\\ 0 & f_0''(z_{\mu_t,y_t})|_{T_{z_{\mu_t,y_t}}Z^\perp} + \mathrm{O}(t)\end{pmatrix}.$$

From (2.4)–(2.6) we know that $f_0''(z_{\mu_t,y_t})|_{T_{z_{\mu_t,y_t}}Z^\perp}$ is invertible with only one negative eigenvalue. Shrinking t_1 if necessary we may assume in view of (5.5)

$$\operatorname{sgn}\det\Gamma_\theta''(\mu_t,y_t) = \operatorname{sgn}\det\Gamma_\theta''(\bar\mu,\bar y)$$

and the claim follows. □

Lemma 5.9. *Under the assumptions of Lemma* 5.3 *there is $t_0 > 0$ such that for all $t \in (0, t_0]$, $\theta \in S^3$, and $R \geqslant R_0$ we have*

$$\deg\big(f_t', B_{\rho_2, R_0}(Z), 0\big) = -\deg(\Gamma_\vartheta', \Omega_{R_0}, 0) = -\deg(\Gamma_\theta', \Omega_R, 0).$$

Proof. By transversality and Lemmas 5.2–5.3 we can choose $\vec{\varepsilon}$ small that $\Gamma_\vartheta' - \vec{\varepsilon}$ has only non-degenerate zeros and

$$\deg(\Gamma_\vartheta', \Omega_{R_0}, 0) = \deg(\Gamma_\vartheta' - \vec{\varepsilon}, \Omega_{R_0}, 0),$$
$$\deg\big(f_t', B_{\rho_2, R_0}(Z), 0\big) = \deg\big(f_{t,\vec{\varepsilon}}', B_{\rho_2, R_0}(Z), 0\big).$$

By Lemmas 5.2–5.8 we have for small t

$$\begin{aligned}\deg\big(f_t', B_{\rho_2, R_0}(Z), 0\big) &= \deg\big(f_{t,\vec{\varepsilon}}', B_{\rho_2, R_0}(Z), 0\big)\\ &= \sum_{\substack{(\mu,y)\in\Omega_{R_0}\\ c_\xi\vec{\alpha}(t,\mu,y) = -t\mu\vec{\varepsilon}}} \deg_{\mathrm{loc}}\big(f_{t,\vec{\varepsilon}}', z_{\mu,y} + w(t,\mu,y)\big)\\ &= -\sum_{\substack{(\mu,y)\in\Omega_{R_0}\\ \Gamma_\vartheta'(\mu,y) = \vec{\varepsilon}}} \deg_{\mathrm{loc}}\big(\Gamma_\vartheta' - \vec{\varepsilon}, (\mu, y)\big)\\ &= -\deg(\Gamma_\vartheta' - \vec{\varepsilon}, \Omega_{R_0}, 0) = -\deg(\Gamma_\vartheta', \Omega_{R_0}, 0).\end{aligned}$$

From Lemma 5.2, after possibly enlarging R_0, and since Γ_θ and Γ_ϑ are just the map Γ in different charts the degree is invariant with respect to $\theta \in S^3$ and $R \geqslant R_0$. □

If K is a Morse function we may compute $\deg(\Gamma_\theta', \Omega_R, 0)$ explicitly, using the Poincaré–Hopf index formula as in [10].

Lemma 5.10. *Suppose $1 + k \in C^6(S^3)$ is a positive Morse function such that $a_1(\theta)^2 + a_3(\theta)^2 \neq 0$ for all $\theta \in M$. We define* $\mathrm{Crit}_-(k) \subset S^3$ *by*

$$\mathrm{Crit}_-(k) := \Big\{\theta \in S^3\colon\ \nabla k(\theta) = 0,\ \Delta k(\theta)^2 + a_0(\theta)^2 + a_1(\theta)^2 \neq 0,\ \text{and}\ \lim_{\mu\to 0^+} \operatorname{sgn}\big(\Delta k(\theta) + a_0(\theta)\mu - a_1(\theta)\mu^2\big) = -1\Big\}.$$

Then we have for all $\vartheta \in S^3$

$$\deg(\Gamma_\vartheta', \Omega_{R_0}, 0) = 1 + \sum_{\theta\in\mathrm{Crit}_-(k)} (-1)^{\mathrm{ind}(k,\theta)} + \sum_{\theta\in M_0\colon a_3(\theta)>0} (-1)^{\mathrm{ind}(k,\theta)}.$$

Proof. Since Γ_ϑ is even in μ, the Poincaré–Hopf index formula for the Euler characteristic and the additivity of the degree give for every $\vartheta \in S^3$

$$\begin{aligned}2 = \deg\big(\Gamma', S^4, 0\big) &= \deg(\Gamma', \mathcal{U}_{\delta_3}, 0) + \deg\big(\Gamma', S^4 \setminus \mathcal{U}_{\delta_3}, 0\big)\\ &= \deg(\Gamma', \mathcal{U}_{\delta_3}, 0) + 2\deg(\Gamma_\vartheta', \Omega_{R_0}, 0),\end{aligned}$$

where

$$\mathcal{U}_{\delta_3} = U_{\delta_3} \cup \big\{(0,\theta) \in S^4\colon\ \theta \in S^3\big\}.$$

We set

$$\sigma_\theta := \lim_{\mu\to 0^+} \operatorname{sgn}\big(\Delta k(\theta) + a_0(\theta)\mu - a_1(\theta)\mu^2 - a_3(\theta)\mu^3\big),$$

which is well defined because at least one of the four coefficients does not vanish.

As k is a Morse function and by Lemma 5.2 each critical point of Γ' in $\mathcal{U}_{\delta_3}$ is isolated and lies on the equator. Consequently,

$$\deg(\Gamma',\mathcal{U}_{\delta_3},0)=\sum_{\nabla k(\theta)=0}\deg_{\mathrm{loc}}\bigl(\Gamma',(0,\theta)\bigr).$$

Fix $\theta\in S^3$ such that $\nabla k(\theta)=0$ and use stereographic coordinates $\mathcal{S}_\theta$. If $\Delta k(\theta)\neq 0$ we easily get from (5.1)

$$\deg_{\mathrm{loc}}\bigl(\Gamma',(0,\theta)\bigr)=(-1)^{\mathrm{ind}(k,\theta)}\operatorname{sgn}\bigl(\Delta k(\theta)\bigr)=(-1)^{\mathrm{ind}(k,\theta)}\sigma_\theta.$$

Hence we may assume $\Delta k(\theta)=0$. We obtain by (5.3) for small t and r

$$\deg_{\mathrm{loc}}\bigl(\Gamma',(0,\theta)\bigr)=\deg\bigl(\Gamma'_\theta,B_r(0,0),0\bigr)=\deg\biggl(-\frac{1}{t\mu}\vec{\alpha}(t,\mu,y),B_r(0,0),0\biggr),$$

where we extend $\frac{1}{t\mu}\vec{\alpha}(t,\mu,y)$ for $\mu\leqslant 0$ to a continuous function in $B_r(0,0)$ by Lemma 4.1 via

$$\frac{1}{t(-\mu)}\vec{\alpha}(t,-\mu,y)_i=\frac{1}{t\mu}\vec{\alpha}(t,\mu,y)_i=\hat{\alpha}(t,\mu,y)_i\quad\text{for }1\leqslant i\leqslant 3,$$
$$\frac{1}{t(-\mu)}\vec{\alpha}(t,-\mu,y)_0=-\frac{1}{t\mu}\vec{\alpha}(t,\mu,y)_0.$$

By Lemma 4.4 we may extend β to a continuous function for $|\mu|<r$ by

$$\beta(t,-\mu)=\beta(t,\mu).$$

As $\hat{\alpha}(t,\mu,y)_i=0$ if and only if $y=\beta(t,\mu,y)$, we see

$$\deg\biggl(-\frac{1}{t\mu}\vec{\alpha}(t,\mu,y),B_r(0,0),0\biggr)=\deg\biggl(-\begin{pmatrix}\frac{1}{t\mu}\vec{\alpha}(t,\mu,\beta(t,\mu))_0\\ \hat{\alpha}(t,\mu,y)\end{pmatrix},B_r(0,0),0\biggr).$$

From Lemma 4.1 we see for $(\mu,y)\in B_r(0)$

$$\begin{aligned}-\hat{\alpha}(t,\mu,y)&=\bigl(1+tk_\theta(y)\bigr)^{-\frac{5}{4}}\frac{\pi}{3^{\frac14}\sqrt{5}}\nabla k_\theta(y)+\mathrm{O}\bigl(\mu^2\bigr)\\ &=\bigl(1+tk_\theta(0)\bigr)^{-\frac{5}{4}}\frac{\pi}{3^{\frac14}\sqrt{5}}D^2k_\theta(0)y+\mathrm{O}\bigl(r^2\bigr).\end{aligned}$$

By Lemma 4.5 after shrinking t and r if necessary we then get

$$\begin{aligned}&\deg\biggl(\begin{pmatrix}-\frac{1}{t\mu}\vec{\alpha}(t,\mu,\beta(t,\mu))_0\\ -\hat{\alpha}(t,\mu,y)_i\end{pmatrix},B_r(0,0),0\biggr)\\ &\quad=\deg\biggl(\begin{pmatrix}\sigma_\theta\mu\\ -\hat{\alpha}(t,\mu,y)_i\end{pmatrix},B_r(0,0),0\biggr)=\deg\biggl(\begin{pmatrix}\sigma_\theta\mu\\ D^2k_\theta(0)y\end{pmatrix},B_r(0,0),0\biggr)\\ &\quad=(-1)^{\mathrm{ind}(k,\theta)}\sigma_\theta.\end{aligned}$$

This gives

$$\deg(\Gamma',\mathcal{U}_{\delta_3},0)=\sum_{\nabla k(\theta)=0}(-1)^{\mathrm{ind}(k,\theta)}\sigma_\theta.$$

By the Poincaré–Hopf index formula we have

$$\sum_{\nabla k(\theta)=0}(-1)^{\mathrm{ind}(k,\theta)}=0,$$

hence

$$\sum_{\nabla k(\theta)=0}(-1)^{\mathrm{ind}(k,\theta)}\sigma_\theta=-2\sum_{\nabla k(\theta)=0,\ \sigma_\theta<0}(-1)^{\mathrm{ind}(k,\theta)},$$

which gives the claim. □

Proof of Theorem 1.1. From [17, Theorem 1.1] the condition $M_+ = \emptyset$ implies that for any $\delta > 0$ the Leray–Schauder degree $\deg(\mathrm{Id} - L_t, \mathcal{B}_{k,\delta}, 0)$ of the problem (1.1) is well-defined and independent of $t \in [\delta, 1]$. Hence it is enough to compute the degree for small $t > 0$. Lemma 5.3 shows that for small $t > 0$ the total degree is the sum of the degree of the blowing-up solutions, computed in Lemma 5.6, and the degree of the solutions, which remain bounded as $t \to 0^+$, given in Lemmas 5.9 and 5.10, if K is a Morse function. Summing up yields the claim. □

Appendix A. Derivatives of $\vec{\alpha}$ and t^θ

The appendix is devoted to the computation of the derivative of t^θ with respect to μ in the case when $a_1(\theta) = 0$ and $a_3(\theta) \neq 0$. From the results in [17] we know in this case as $\mu \to 0$

$$t^\theta(\mu) = -\frac{a_3(\theta)}{a_2(\theta)}\mu + \mathrm{O}\big(\mu^{1+\frac{1}{4}}\big).$$

We shall prove the corresponding expansion for the derivative as stated in Lemma 4.7 above. Since we proceed as in [17], where derivatives with respect to t and y are computed, we will only sketch the computations and arguments that lead to the desired result.

We first recall the expansion of $\vec{\alpha}$, w and β as $\mu \to 0$ given in [17, Section 4,6]

Lemma A.1. [17, Section 4] *Under the assumptions of Lemma* 4.1 *we have as* $\mu \to 0$

$$\left|\vec{\alpha}(t,\mu,y) - \sum_{j=1}^{4}\vec{\alpha}_j(t,\mu,y)\right| = t\mathrm{O}\big(\mu^{4+\frac{1}{2}}\big) + t^2\mathrm{O}\big(\mu^2\big|\nabla k_\theta(y)\big|^2 + \mu^3\big|\nabla k_\theta(y)\big| + \mu^4\big|\Delta k_\theta(y)\big| + \mu^{4+\frac{1}{4}}\big),$$

where α_1, α_2 *are given in Lemma* 4.1, *for* $1 \leqslant i \leqslant 3$

$$\vec{\alpha}_3(t,\mu,y)_i := -t\mu^3\frac{3^{\frac{1}{4}}\pi}{2\sqrt{15}}\big(1+tk_\theta(y)\big)^{-\frac{5}{4}}\frac{\partial}{\partial x_i}\Delta k_\theta(y),$$

$$\vec{\alpha}_4(t,\mu,y)_i := -t\mu^4\big(1+tk_\theta(y)\big)^{-\frac{5}{4}}\frac{3^{\frac{3}{4}}8}{\pi\sqrt{5}}\oint\big(k_\theta(x+y) - T^3_{k_\theta(\cdot+y),0}(x)\big)\frac{x_i}{|x|^8}$$

and

$$\vec{\alpha}_3(t,\mu,y)_0 := -t\mu^3\big(1+tk_\theta(y)\big)^{-\frac{5}{4}}\frac{3^{\frac{3}{4}}4}{\pi\sqrt{5}}\oint\big(k_\theta(x+y) - T^2_{k_\theta(\cdot+y),0}(x)\big)\frac{1}{|x|^6},$$

$$\vec{\alpha}_4(t,\mu,y)_0 := t\mu^4\big(1+tk_\theta(y)\big)^{-\frac{5}{4}}\frac{3^{\frac{3}{4}}\pi\sqrt{5}}{30}\Delta^2 k_\theta(y) - t^2\mu^4\big(1+tk_\theta(y)\big)^{-\frac{9}{4}}\frac{3^{\frac{3}{4}}\sqrt{5}}{16}\left(\int_{\partial B_1(0)}\big|D^2k_\theta(y)(x)^2\big|^2\right).$$

If, moreover, $k \in C^6(S^3)$ *then*

$$\left|\vec{\alpha}(t,\mu,y)_0 - \sum_{j=1}^{5}\vec{\alpha}_j(t,\mu,y)_0\right| = \mathrm{O}\big(t\mu^6\big) + t^2\mathrm{O}\big(\mu^2\big|\nabla k_\theta(y)\big|^2 + \mu^3\big|\nabla k_\theta(y)\big| + \mu^4\big|\Delta k_\theta(y)\big| + \mu^{4+\frac{1}{4}}\big),$$

where

$$\vec{\alpha}_5(t,\mu,y)_0 := \frac{t\mu^5}{(1+tk_\theta(y))^{\frac{5}{4}}}\frac{3^{\frac{3}{4}}4\sqrt{5}}{\pi}\oint\big(k_\theta(y+x) - T^4_{k_\theta(\cdot+y),0}(x)\big)\frac{1}{|x|^8}.$$

Lemma A.2. [17, Section 4] *Under the assumptions of Lemma* 4.1 *we have as* $\mu \to 0$

$$w(t,\mu,y) = w_0(t,\mu,y) + w_2(t,\mu,y) + t\mathrm{O}\big(\big|\nabla k_\theta(y)\big|\mu + \mu^{2+\frac{1}{4}}\big),$$

where w_0 *is given in Lemma* 4.1 *and*

$$w_2(t,\mu,y) := t\min\bigl(1,\mu^2\bigr)\bigl(1+tk_\theta(y)\bigr)^{-\frac{5}{4}}\mathcal{T}_y\circ\mathcal{U}_\mu\bigl(\tilde{w}_2(y)\bigr),$$
$$\tilde{w}_2(y) := \mathcal{F}_0^{-1}\left(\frac{1}{2}\int D^2k_\theta(y)x^2(z_{1,0})^5\cdot\right).$$

The operator $\mathcal{F}_0^{-1}\in\mathcal{L}(\mathcal{D}^{1,2}(\mathbb{R}^3),T_{z_{1,0}}Z^\perp)$ is defined by

$$\mathcal{F}_0^{-1} := \bigl(f_0''(z_{1,0})|_{T_{z_{1,0}}Z^\perp}\bigr)^{-1}\circ\mathrm{Proj}_{T_{z_{1,0}}Z^\perp}.$$

The definition of w_2 is equivalent to $w_2\in T_{z_{\mu,y}}Z^\perp$ and

$$f_0''(z_{\mu,y})w_2 = \frac{1}{2}t\int D^2k_\theta(y)(x-y)^2(z_{\mu,y}+w_0)^5\cdot+\vec{\alpha}_2(t,\mu,y)\cdot\dot{\xi}_{\mu,y}. \tag{A.1}$$

Lemma A.3. [17, Section 6] *Under the assumptions of Lemma* 4.4 *we have*

$$\beta(t,\mu) = 0+\bigl(D^2k_\theta(0)\bigr)^{-1}\left(\sum_{j=3}^{4}\hat{\alpha}_j(t,\mu,0)\right)+\mathrm{O}\bigl(\mu^{3+\frac{1}{4}}\bigr),$$

where $\hat{\alpha}_j(t,\mu,y)$ is defined analogously to $\hat{\alpha}$ by

$$\hat{\alpha}_j(t,\mu,y) := \frac{3^{\frac{1}{4}}\sqrt{5}}{t\mu\pi}\bigl(1+tk(\theta)\bigr)^{\frac{5}{4}}\bigl(\vec{\alpha}_j(t,\mu,y)_1,\ldots,\vec{\alpha}_j(t,\mu,y)_3\bigr)^T.$$

We begin by computing the derivative of $\vec{\alpha}$ with respect to μ.

Lemma A.4. *Under the assumptions of Lemma* 4.1 *and $k\in C^6(S^3)$ we have*

$$\frac{\partial(\vec{\alpha})_0}{\partial\mu} = \sum_{j=1}^{5}\frac{\partial(\vec{\alpha}_j)_0}{\partial\mu}+t\mathrm{O}\bigl(\mu^{4+\frac{1}{2}}\bigr)+t^2\mathrm{O}\bigl(\bigl|\nabla k_\theta(y)\bigr|^2\mu+\bigl|\nabla k_\theta(y)\bigr|\mu^2+\bigl|\Delta k_\theta(y)\bigr|\mu^3+\mu^{3+\frac{1}{4}}\bigr),$$

and for $1\leqslant i\leqslant 3$

$$\frac{\partial(\vec{\alpha})_i}{\partial\mu} = \sum_{j=1}^{4}\frac{\partial(\vec{\alpha}_j)_i}{\partial\mu}t\mathrm{O}\bigl(\mu^{3+\frac{1}{2}}\bigr)+t^2\mathrm{O}\bigl(\bigl|\nabla k_\theta(y)\bigr|^2\mu+\bigl|\nabla k_\theta(y)\bigr|\mu^2+\bigl|\Delta k_\theta(y)\bigr|\mu^3+\mu^{3+\frac{1}{4}}\bigr).$$

Proof. As in [17] we define $H:\mathbb{R}\times(0,\infty)\times\mathbb{R}^3\times\mathcal{D}^{1,2}(\mathbb{R}^3)\times\mathbb{R}^4\to\mathcal{D}^{1,2}(\mathbb{R}^3)\times\mathbb{R}^4$ by

$$H(t,\mu,y,w,\vec{\alpha}) := \bigl(f_t'(z_{\mu,y}+w)-\vec{\alpha}\cdot\dot{\xi}_{\mu,y},\bigl(\bigl\langle w,(\dot{\xi}_{\mu,y})_l\bigr\rangle\bigr)_l\bigr).$$

The functions $\vec{\alpha}$ and w are implicitly defined via $H(t,\mu,y,w,\vec{\alpha})=(0,0)$. We have

$$\left(\frac{\partial H}{\partial(w,\vec{\alpha})}(t,\mu,y,w,\vec{\alpha})\right)\begin{pmatrix}\varphi\\ \vec{\beta}\end{pmatrix} = \bigl(f_t''(z_{\mu,y}+w)\varphi-\vec{\beta}\dot{\xi}_{\mu,y},\bigl(\bigl\langle\varphi,(\dot{\xi}_{\mu,y})_l\bigr\rangle\bigr)_l\bigr).$$

From [17, Remark 4.4] we may assume that $(\frac{\partial H}{\partial(w,\vec{\alpha})}(t,\mu,y,w(t,\mu,y),\vec{\alpha}(t,\mu,y)))$ is uniformly invertible with respect to $(t,\mu,y)\in\Omega$, i.e.

$$\left\|\left(\frac{\partial H}{\partial(w,\vec{\alpha})}\bigl(t,\mu,y,w(t,\mu,y),\vec{\alpha}(t,\mu,y)\bigr)\right)^{-1}\right\|\leqslant C_*\quad\text{for all }(t,\mu,y)\in\Omega. \tag{A.2}$$

In the sequel we will suppress the dependence of w and $\vec{\alpha}$ on t, μ and y, when there is no possibility of confusion. Moreover, we always assume $0<\mu\leqslant 1$.

Differentiating $H(t,\mu,y,w,\vec{\alpha})=(0,0)$ with respect to μ we get

$$-\frac{\partial H}{\partial \mu} = -\left(f_t''(z_{\mu,y}+w)\frac{\partial z_{\mu,y}}{\partial \mu} - \vec{\alpha}\cdot \frac{\partial \dot{\xi}_{\mu,y}}{\partial \mu}, \left(\left\langle w, \frac{\partial (\dot{\xi}_{\mu,y})_l}{\partial \mu}\right\rangle\right)_l \right)$$
$$= \frac{\partial H}{\partial (w,\vec{\alpha})}\begin{pmatrix} \frac{\partial w}{\partial \mu} \\ \frac{\partial \vec{\alpha}}{\partial \mu}\end{pmatrix}. \tag{A.3}$$

A direct calculation gives for $0 \leqslant l \leqslant 3$

$$\left\| \frac{\partial (\dot{\xi}_{\mu,y})_l}{\partial \mu} \right\| + \left\| \frac{\partial z_{\mu,y}}{\partial \mu}\right\| \leqslant \operatorname{const} \mu^{-1}. \tag{A.4}$$

Differentiating $\langle w_i(t,\mu,y), (\dot{\xi}_{\mu,y})_l\rangle \equiv 0$ leads to

$$\left\langle w_i(t,\mu,y), \frac{\partial (\dot{\xi}_{\mu,y})_l}{\partial \mu}\right\rangle = \left\langle \frac{\partial w_i(t,\mu,y)}{\partial \mu}, (\dot{\xi}_{\mu,y})_l\right\rangle. \tag{A.5}$$

Differentiating (A.1) we see

$$f_0'''(z_{\mu,y})\frac{\partial z_{\mu,y}}{\partial \mu} w_2 + f_0''(z_{\mu,y})\frac{\partial w_2}{\partial \mu}$$
$$= 5t \int \frac{1}{2} D^2 k_\theta(y)(x-y)^2 (z_{\mu,y}+w_0)^4 \frac{\partial (z_{\mu,y}+w_0)}{\partial \mu}\cdot$$
$$+ \frac{\partial \vec{\alpha}_2(t,\mu,y)}{\partial \mu}\cdot \dot{\xi}_{\mu,y} + \vec{\alpha}_2(t,\mu,y)\cdot \frac{\partial (\dot{\xi}_{\mu,y})}{\partial \mu}. \tag{A.6}$$

From (A.2)–(A.6) we obtain

$$\left\| \frac{\partial w_2}{\partial \mu}\right\| \leqslant \operatorname{const} \mu. \tag{A.7}$$

Estimating $f_t''(z_{\mu,y}+w)\frac{\partial z_{\mu,y}}{\partial \mu}$ and $f_t''(z_{\mu,y}+w)\frac{\partial (w_0+w_2)}{\partial \mu}$ as in [17, Section 5] and using (A.4) and (A.5) we see

$$-\frac{\partial H}{\partial \mu}(t,\mu,y,w,\vec{\alpha}) = \frac{\partial H}{\partial (w,\vec{\alpha})}\begin{pmatrix} \frac{\partial (w_0+w_2)}{\partial \mu} \\ \frac{\partial \vec{\alpha}_2}{\partial \mu}\end{pmatrix} + t\mathrm{O}\left(\left|\nabla k_\theta(y)\right| + \mu^{1+\frac{1}{4}}\right).$$

Consequently, by (A.2),

$$\left\| \frac{\partial w}{\partial \mu} - \frac{\partial (w_0+w_2)}{\partial \mu}\right\| + \left\| \frac{\partial \vec{\alpha}}{\partial \mu} - \frac{\partial \vec{\alpha}_2}{\partial \mu}\right\| = t\mathrm{O}\left(\left|\nabla k_\theta(y)\right| + \mu^{1+\frac{1}{4}}\right). \tag{A.8}$$

It is easy to see that for $0 \leqslant i,j \leqslant 3$

$$\left\langle \frac{\partial (\dot{\xi}_{\mu,y})_i}{\partial \mu}, (\dot{\xi}_{\mu,y})_j\right\rangle = 0, \tag{A.9}$$

which implies, if we test (A.3) with $(\dot{\xi}_{\mu,y})_j$,

$$\frac{\partial (\vec{\alpha})_j}{\partial \mu} = f_t''(z_{\mu,y}+w)\left(\frac{\partial z_{\mu,y}}{\partial \mu} + \frac{\partial w}{\partial \mu}\right)(\dot{\xi}_{\mu,y})_j. \tag{A.10}$$

By (A.8) we estimate $f_t''(z_{\mu,y}+w)(\frac{\partial z_{\mu,y}}{\partial \mu} + \frac{\partial w}{\partial \mu})(\dot{\xi}_{\mu,y})_j$ as in [17, Section 5] and arrive at

$$\frac{\partial (\vec{\alpha})_j}{\partial \mu} = -5t \int \left(k_\theta(x) - k_\theta(y)\right)(z_{\mu,y}+w_0)^4 \frac{\partial (z_{\mu,y}+w_0)}{\partial \mu}(\dot{\xi}_{\mu,y})_j$$
$$- t \int \left(k_\theta(x) - k_\theta(y)\right)(z_{\mu,y}+w_0)^5 \frac{\partial (\dot{\xi}_{\mu,y})_j}{\partial \mu}$$
$$- 5t \frac{\mathrm{d}}{\mathrm{d}\mu}\left(\int \frac{1}{2} D^2 k_\theta(y)(x-y)^2 (z_{\mu,y}+w_0)^4 w_2 (\dot{\xi}_{\mu,y})_j\right)$$

$$-10\frac{\mathrm{d}}{\mathrm{d}\mu}\biggl(\int\bigl(1+tk_\theta(y)\bigr)(z_{\mu,y}+w_0)^3(w_2)^2(\dot{\xi}_{\mu,y})_j\biggr)$$
$$+t^2\mathrm{O}\bigl(\bigl|\nabla k_\theta(y)\bigr|^2\mu+\bigl|\nabla k_\theta(y)\bigr|\mu^2+\mu^{3+\frac14}\bigr). \tag{A.11}$$

The computations in the proof of Lemma 4.7 in [17] show, that

$$-5t\int\frac12 D^2k_\theta(y)(x-y)^2(z_{\mu,y}+w_0)^4w_2(\dot{\xi}_{\mu,y})_j-10\int\bigl(1+tk_\theta(y)\bigr)(z_{\mu,y}+w_0)^3(w_2)^2(\dot{\xi}_{\mu,y})_j$$
$$=\begin{cases}-\frac{t^2\mu^4}{(1+tk_\theta(y))^{\frac94}}\frac{3^{\frac34}\sqrt5}{16}\bigl(\int_{\partial B_1(0)}\bigl|D^2k_\theta(y)(x)^2\bigr|^2+\mathrm{O}(\Delta k_\theta(y))\bigr) & \text{if } j=0,\\ 0 & \text{if } 1\leqslant j\leqslant 3,\end{cases}$$

where the term $\mathrm{O}(\Delta k_\theta(y))$ does not depend on μ.

Since $z_{\mu,y}$, $(\dot{\xi}_{\mu,y})_j$, w_0, and their derivatives with respect to μ are known functions the remaining integrals may be computed explicitly as in [17, Section 4]. This leads to the desired result and finishes the proof. □

Lemma A.5. *Under the assumptions of Lemma* 4.1 *we have for all* $(t,\mu,y)\in\Omega$ *with* $|\mu|\leqslant 1$ *and* $1\leqslant j\leqslant 3$

$$\frac{\partial\alpha(t,\mu,y)_0}{\partial y_j}=-\frac{t\mu^2\pi}{3^{\frac14}\sqrt5}\bigl(1+tk_\theta(y)\bigr)^{-\frac54}\frac{\partial}{\partial x_j}\Delta k_\theta(y)$$
$$-t\mu^3\bigl(1+tk_\theta(y)\bigr)^{-\frac54}\frac{3^{\frac34}4}{\pi\sqrt5}\oint\biggl(\frac{\partial k}{\partial x_j}(x+y)-\sum_{\ell=0}^{2}\frac{1}{\ell!}D^\ell\frac{\partial k}{\partial x_j}(y)(x)^\ell\biggr)\frac{1}{|x|^6}$$
$$+t\mathrm{O}\bigl(\bigl|\nabla k_\theta(y)\bigr|^2\mu^2+\mu^{3+\frac12}\bigr)+t^2\mathrm{O}\bigl(\bigl|\nabla k_\theta(y)\bigr|^2\mu+\bigl|\nabla k_\theta(y)\bigr|\mu^2+\mu^3\bigr). \tag{A.12}$$

Proof. Proceeding as in [17, Section 4] we find

$$\frac{\partial\alpha(t,\mu,y)_0}{\partial y_j}=f_t''(z_{\mu,y}+w)\frac{\partial(z_{\mu,y}+w)}{\partial y_j}(\dot{\xi}_{\mu,y})_0-\sum_{l=0}^{3}\vec{\alpha}(t,\mu,y)_l\biggl\langle\frac{\partial(\dot{\xi}_{\mu,y})_l}{\partial y_j},(\dot{\xi}_{\mu,y})_0\biggr\rangle, \tag{A.13}$$

and

$$f_t''(z_{\mu,y}+w)\frac{\partial(z_{\mu,y}+w)}{\partial y_j}(\dot{\xi}_{\mu,y})_0$$
$$=-5t\int\frac{k_\theta(x)-k_\theta(y)}{(1+tk_\theta(y))^{\frac54}}(z_{\mu,y})^4\frac{\partial z_{\mu,y}}{\partial y_j}(\dot{\xi}_{\mu,y})_0-20\int(z_{\mu,y})^3(w-w_0)\frac{\partial z_{\mu,y}}{\partial y_j}(\dot{\xi}_{\mu,y})_0$$
$$+t^2\mathrm{O}\bigl(\bigl|\nabla k_\theta(y)\bigr|^2\mu+\bigl|\nabla k_\theta(y)\bigr|\mu^2+\mu^3\bigr). \tag{A.14}$$

The tricky part is the term containing $(w-w_0)$. Differentiating

$$f_0''(z_{\mu,y})(w-w_0)(\dot{\xi}_{\mu,y})_0=0$$

with respect to y_j and using the expansion (see [17, (4.17)])

$$f_t'(z_{\mu,y}+w)=f_0''(z_{\mu,y})(w-w_0)-t\int\bigl(k_\theta(x)-k_\theta(y)\bigr)(z_{\mu,y}+w_0)^5\cdot$$
$$+t^2\mathrm{O}\bigl(\bigl|\nabla k_\theta(y)\bigr|^2\mu^2+\bigl|\nabla k_\theta(y)\bigr|\mu^3+\mu^4\bigr)$$

we obtain

$$-20\int(z_{\mu,y})^3(w-w_0)\frac{\partial z_{\mu,y}}{\partial y_j}(\dot{\xi}_{\mu,y})_0$$
$$=f_0'''(z_{\mu,y})(w-w_0)\frac{\partial z_{\mu,y}}{\partial y_j}(\dot{\xi}_{\mu,y})_0=-f_0''(z_{\mu,y})(w-w_0)\frac{\partial(\dot{\xi}_{\mu,y})_0}{\partial y_j}$$

$$= -\sum_{l=0}^{3} \vec{\alpha}(t,\mu,y)_l \left\langle (\dot{\xi}_{\mu,y})_l, \frac{\partial(\dot{\xi}_{\mu,y})_0}{\partial y_j} \right\rangle - t \int \frac{k_\theta(x) - k_\theta(y)}{(1+tk_\theta(y))^{\frac{5}{4}}} (z_{\mu,y})^5 \frac{\partial(\dot{\xi}_{\mu,y})_0}{\partial y_j}$$
$$+ t^2 \mathrm{O}\big(\big|\nabla k_\theta(y)\big|^2 \mu + \big|\nabla k_\theta(y)\big|\mu^2 + \mu^3\big). \tag{A.15}$$

Inserting (A.15) and (A.14) into (A.13) leads to

$$\begin{aligned}\frac{\partial \alpha(t,\mu,y)_0}{\partial y_j} &= \frac{-t}{(1+tk_\theta(y))^{\frac{5}{4}}} \int \big(k_\theta(x) - k_\theta(y)\big) \frac{\mathrm{d}}{\mathrm{d}y_j} \big((z_{\mu,y})^5 (\dot{\xi}_{\mu,y})_0\big) \\ &\quad + t^2 \mathrm{O}\big(\big|\nabla k_\theta(y)\big|^2 \mu + \big|\nabla k_\theta(y)\big|\mu^2 + \mu^3\big) \\ &= \frac{-t}{(1+tk_\theta(y))^{\frac{5}{4}}} \int \frac{\partial k}{\partial x_j}(x) (z_{\mu,y})^5 (\dot{\xi}_{\mu,y})_0 \\ &\quad + t^2 \mathrm{O}\big(\big|\nabla k_\theta(y)\big|^2 \mu + \big|\nabla k_\theta(y)\big|\mu^2 + \mu^3\big),\end{aligned}$$

where we used partial integration and the fact that $\frac{\partial}{\partial y_i} z_{\mu,y}(x) = -\frac{\partial}{\partial x_i} z_{\mu,y}(x)$ and $\frac{\partial}{\partial y_i}(\dot{\xi}_{\mu,y})_j(x) = -\frac{\partial}{\partial x_i}(\dot{\xi}_{\mu,y})_j(x)$. The latter integral may be computed as in [17, Section 4]. This finishes the proof. □

Proof of Lemma 4.7. The function t^θ is implicitly defined by $\gamma(t^\theta(\mu),\mu) \equiv 0$, where

$$\gamma(t,\mu) := \frac{1}{t\mu^4} \big(1 + tk_\theta(0)\big)^{\frac{9}{4}} \frac{30}{\pi 3^{\frac{3}{4}} \sqrt{5}} \alpha\big(t,\mu,\beta(t,\mu)\big)_0.$$

Hence

$$\frac{\partial t^\theta}{\partial \mu} = -\left(\frac{\partial \gamma}{\partial t}\right)^{-1} \frac{\partial \gamma}{\partial \mu}.$$

The derivative of γ with respect to t is computed in [17, Section 6] and is given by

$$\frac{\partial \gamma}{\partial t}(t,\mu) = a_2(\theta) + \mathrm{O}\big(\mu^{\frac{1}{4}}\big).$$

From the definition of γ we find for $t = \tilde{t}$

$$\frac{\partial \gamma}{\partial \mu}(t,\mu) = \frac{30(1+tk_\theta(0))^{\frac{9}{4}}}{t\mu^4 \pi 3^{\frac{3}{4}} \sqrt{5}} \left(\frac{\partial(\alpha)_0}{\partial \mu}\big(t,\mu,\beta(t,\mu)\big) + \frac{\partial(\alpha)_0}{\partial y}\big(t,\mu,\beta(t,\mu)\big) \frac{\partial \beta}{\partial \mu}(t,\mu) \right). \tag{A.16}$$

In view of Lemma A.3 and the fact that $\nabla k_\theta(0) = 0$ we may estimate functions F of $\beta = \beta(t,\mu)$ and of $k_\theta(\beta) = k_\theta(\beta(t,\mu))$ as follows

$$F(\beta) = F(0) + F'(0)\big(D^2 k_\theta(0)\big)^{-1} \left(\sum_{j=3}^{4} \hat{\alpha}_j(t,\mu,0) \right) + \mathrm{O}\big(\mu^{3+\frac{1}{4}}\big),$$
$$F\big(k_\theta(\beta)\big) = F\big(k_\theta(0)\big) + \mathrm{O}\big(\mu^4\big). \tag{A.17}$$

For instance, we have $|\nabla k_\theta(\beta)| + |\Delta k_\theta(\beta)| = \mathrm{O}(\mu^2)$. Moreover, from Theorem 4.6 we may estimate t^2 by $t\mathrm{O}(\mu)$.

To compute the derivative of β we use that β is defined by $\hat{\alpha}(t,\mu,\beta(t,\mu)) \equiv 0$ and get from Lemmas 4.3 and A.4

$$\frac{\partial \beta}{\partial \mu}(t,\mu) = t^{-1}\mu^{-2} \frac{3^{\frac{1}{4}}\sqrt{5}}{\pi} \big(1 + tk_\theta(0)\big)^{\frac{5}{4}} \big(D^2 k_\theta(0)\big)^{-1} \left(\sum_{j=3}^{4} (j-1)\alpha_j(t,\mu,\beta)_i \right)_{i=1\ldots3} + \mathrm{O}\big(\mu^{2+\frac{1}{4}}\big).$$

Thus we see by Lemma A.5

$$\frac{\partial(\alpha)_0}{\partial y}(t,\mu,\beta) \frac{\partial \beta}{\partial \mu}(t,\mu) = -\nabla \Delta k_\theta(0) \big(D^2 k_\theta(0)\big)^{-1} \left(\sum_{j=3}^{4} (j-1)\alpha_j(t,\mu,0)_i \right)_{i=1\ldots3}$$

$$-\mu\frac{24}{\pi^2}\left(\oint\left(\nabla k_\theta(x)-\sum_{\ell=0}^{2}\frac{D^\ell\nabla k_\theta(0)}{\ell!}(x)^\ell\right)\frac{1}{|x|^6}\right)$$
$$\times\left(D^2k_\theta(0)\right)^{-1}\left(\alpha_3(t,\mu,0)_i\right)_{i=1\ldots3}$$
$$+t\mathrm{O}\left(\mu^{4+\frac{1}{4}}\right).$$

Moreover,

$$\frac{\partial(\alpha)_0}{\partial\mu}(t,\mu,\beta)=\mu^{-1}\sum_{j=3}^{5}(j-2)\alpha_j(t,\mu,\beta)_0+t\mathrm{O}\left(\mu^{4+\frac{1}{4}}\right).$$

We use the estimate in (A.17) to see as in Lemma 4.5 in [17]

$$\alpha_3(t,\mu,\beta)_0=\alpha_3(t,\mu,0)_0+t\mu^5\left(1+tk_\theta(0)\right)^{-\frac{5}{4}}\frac{3^{\frac{3}{4}}2}{\pi\sqrt{5}}$$
$$\times\oint\left(\nabla k_\theta(x)-\sum_{\ell=0}^{2}\frac{1}{\ell!}D^\ell\nabla k_\theta(0)(x)^\ell\right)\frac{1}{|x|^6}\left(D^2k_\theta(0)\right)^{-1}\nabla\Delta k_\theta(0)$$
$$+t\mathrm{O}\left(\mu^6\right)$$

and

$$\alpha_4(t,\mu,\beta)_0+\alpha_5(t,\mu,\beta)_0=\alpha_4(t,\mu,0)_0+\alpha_5(t,\mu,0)_0+t\mathrm{O}\left(\mu^6\right).$$

Summing up and inserting the result into (A.16) we find

$$\frac{\partial\gamma}{\partial\mu}(t,\mu)=\frac{2}{\mu}\left(a_1(\theta)+ta_2(\theta)\right)+3a_3(\theta)+\mathrm{O}\left(\mu^{\frac{1}{4}}\right).$$

As $a_1(\theta)=0$ and $t=-a_3(\theta)/a_2(\theta)\mu+\mathrm{O}(\mu^{1+\frac{1}{4}})$ we get

$$\frac{\partial\gamma}{\partial\mu}(t,\mu)=a_3(\theta)+\mathrm{O}\left(\mu^{\frac{1}{4}}\right),$$

which yields the claim. □

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