Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 24 (2007) 629–644
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Interior estimates for some semilinear elliptic problem with critical nonlinearity

Estimations à l'intérieur pour un problème elliptique semi-linéaire avec non-linéarité critique

Pierpaolo Esposito [1]

Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo S. Leonardo Murialdo, 1, 00146 Roma, Italy

Received 29 September 2005; accepted 14 April 2006

Available online 25 September 2006

Abstract

We study compactness properties for solutions of a semilinear elliptic equation with critical nonlinearity. For high dimensions, we are able to show that any solutions sequence with uniformly bounded energy is uniformly bounded in the interior of the domain. In particular, singularly perturbed Neumann equations admit pointwise concentration phenomena only at the boundary.

Résumé

On étudie les propriétés de compacité pour solutions d'une équation elliptique semi-linéaire avec non-linéarité critique. En hautes dimensions, on démontre qu'une suite de solutions avec énergie uniformément bornée est uniformément bornée dans l'intérieur du domaine. En particulier, les équations de Neumann perturbées singulièrement peuvent avoir des phénomènes de concentration seulement sur la frontière.

MSC: 35J20; 35J25; 35J60

Keywords: Compactness; Critical exponent; Singular perturbations; Blow-up analysis

1. Introduction and statement of the results

The starting point in our investigation has been the study of asymptotic properties for the problem:

$$\begin{cases} -\Delta u + \lambda u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

E-mail address: esposito@mat.uniroma3.it (P. Esposito).

[1] The author is supported by M.U.R.S.T., project "Variational methods and nonlinear differential equations".

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2006.04.004

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, $p = \frac{N+2}{N-2}$ is the critical exponent from the Sobolev viewpoint and $\lambda > 0$ is a large parameter. Here, $n(x)$ is the unit outward normal of Ω at $x \in \partial\Omega$.

Under the transformation $v(x) = \lambda^{-\frac{1}{p-1}} u(x)$, $d^2 = \frac{1}{\lambda}$, problem (1) reads equivalently as a singularly perturbed Neumann problem:

$$\begin{cases} -d^2 \Delta v + v = v^p & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $p = \frac{N+2}{N-2}$. For general exponent $p > 1$, problem (2) is related to the study of stationary solutions for a chemotaxis system (see [17]) proposed by Keller, Segel and Gierer, Meinhardt (see [18]).

Problem (1) for λ large has been widely studied in the subcritical case $p < \frac{N+2}{N-2}$. The asymptotic behaviour and the construction of blowing up solutions have been considered by several authors. In particular, there exist peak solutions which blow up at many finitely boundary and/or interior points of Ω.

The critical case $p = \frac{N+2}{N-2}$ has different features. Starting from the pioneering works of Adimurthi, Mancini and Yadava [3] (see also [1,2]), asymptotic analysis (see [13,15] for low energy solutions) and construction of solutions concentrating at boundary points of Ω have been considered by several authors (see for example [21]). We refer to [20] for an extensive list of references about subcritical and critical case.

As far as interior concentration, the situation is quite different since in literature no results are available and it is expected that in general such solutions should not exist. A first partial result in this direction is due to Cao, Noussair and Yan [6] for $N \geqslant 5$ and for isolated blow-up points. They show that any concentrating solutions sequence with bounded energy cannot have only interior peaks and so at least one blow-up point must lie on $\partial\Omega$. At the same time, Rey in [20] gets the same result for $N = 3$ by removing any assumption on the nature of interior blow-up points.

Using some techniques developed by Druet, Hebey and Vaugon in [12] for related problems on Riemannian manifolds, Castorina and Mancini in [7] were able to show, among other things, that the conclusion of previous papers holds without any restriction on the dimension. Namely, for $N \geqslant 3$ at least one blow-up point lies on $\partial\Omega$.

However, all these papers do not answer to the full question: do there exist blowing up solutions for (1) with bounded energy which do not remain bounded in the interior of Ω as $\lambda \to +\infty$? For $N > 6$ the answer is negative since we will show that ALL the blow-up points have to lie on $\partial\Omega$:

Theorem 1.1. *Let $N > 6$. Let $\lambda_n \to +\infty$ and u_n be a solutions sequence of*

$$\begin{cases} -\Delta u_n + \lambda_n u_n = N(N-2) u_n^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ \frac{\partial u_n}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

with uniformly bounded energy:

$$\sup_{n \in \mathbb{N}} \int_\Omega u_n^{\frac{2N}{N-2}} < +\infty.$$

Then, for any K compact set in Ω there exists C_K such that:

$$\max_{x \in K} u_n(x) \leqslant C_K$$

for any $n \in \mathbb{N}$.

Theorem 1.1 is based on a local description of possible compactness loss and does not need any boundary condition. In fact, we realized that Theorem 1.1 is a particular case of a more general interior compactness result, which is still more interesting that our initial question about singularly perturbed Neumann equations and becomes the main content of this paper. There holds:

Theorem 1.2. *Let $N > 6$. Let K be a compact set in Ω and $\Lambda > 0$. There exists a constant C, depending on K and Λ, such that any solution u of the problem*:

$$\begin{cases} -\Delta u + u = N(N-2)u^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \int_\Omega |\nabla u|^2 + u^2 \leqslant \Lambda, \end{cases}$$

satisfies the bound:

$$\max_{x \in K} u(x) \leqslant C.$$

Compactness properties of the type we are considering appear in a Riemannian context in [8,9] where a careful analysis based on the C^0-theory developed in [10,11] for Riemannian manifolds gives the Schoen compactness result in low dimensions and provides also in high dimensions results as in Theorem 1.2. However, in this context (without homogeneous Dirichlet boundary condition) the C^0-theory developed by Druet, Hebey and Robert is not available.

The paper is organized in the following way. In Section 2 we introduce the notion of (geometrical) blow-up set, we give a description of this set and we show by a rescaling argument that Theorem 1.1 is a particular case of Theorem 1.2. In Section 3, we provide the proof of Theorem 1.2: based on a technical result contained in [10,11] due to Druet, Hebey and Robert (which we report in Appendix A for the sake of completeness), for any interior blowing up solutions sequence we are able to prove an upper estimate (in terms of standard bubbles) which contradicts a related local Pohozaev identity.

2. The blow-up set

Let u_n be a solutions sequence of

$$\begin{cases} -\Delta u_n + \mu_n u_n = N(N-2)u_n^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ \sup_{n \in \mathbb{N}} \int_\Omega (|\nabla u_n|^2 + u_n^2) < +\infty, \end{cases} \tag{4}$$

where Ω is a domain in $\mathbb{R}^N$, $N \geqslant 3$, and $0 \leqslant \mu_n \to \mu \in [0, +\infty]$.

We define the (geometrical) blow-up set of u_n in Ω as

$$S = \left\{ x \in \Omega \colon \exists x_n \to x \text{ s.t. } \limsup_{n \to +\infty} u_n(x_n) = +\infty \right\},$$

and, by definition of S, clearly u_n is uniformly bounded in $C^0_{\mathrm{loc}}(\Omega \setminus S)$.

Further, define the set

$$\Sigma_c = \left\{ x \in \Omega \colon \limsup_{n \to +\infty} \int_{B_r(x)} u_n^{\frac{2N}{N-2}} \geqslant c \ \forall r > 0 \right\},$$

where $c > 0$. Let S_N be the best constant related to the immersion of $H_0^1(\Omega)$ into $L^{\frac{2N}{N-2}}(\Omega)$:

$$S_N = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2}{(\int_\Omega |u|^{\frac{2N}{N-2}})^{\frac{N-2}{N}}}. \tag{5}$$

By means of an iterative Moser-type scheme, we can describe the set S in the following way:

Proposition 2.1. *There exists $c = c_N > 0$ such that it holds $S = \Sigma_c$. In particular, S is a finite set and, if $\mu = +\infty$, we have that $u_n \to 0$ in $C^0_{\mathrm{loc}}(\Omega \setminus S)$ (up to a subsequence).*

Proof. First of all, we show the following implication:

$$\int_{B_{2r}(x)} u_n^{\frac{2N}{N-2}} \leqslant \left(\frac{S_N}{qN(N-2)} \right)^{\frac{N}{2}}, \ q \geqslant 2 \ \Rightarrow \left(\int_{B_r(x)} u_n^{\frac{N}{N-2}q} \right)^{\frac{N-2}{N}} \leqslant \frac{8}{S_N r^2} \int_{B_{2r}(x)} u_n^q$$

for any $r < \frac{1}{2}\,\mathrm{dist}(x, \mathbb{R}^N \setminus \Omega)$. Let $\varphi \in C_0^\infty(B_{2r}(x))$ be so that $0 \leqslant \varphi \leqslant 1$, $\varphi \equiv 1$ in $B_r(x)$ and $\|\nabla\varphi\|_\infty \leqslant \frac{2}{r}$. Multiplying (4) by $\varphi^2 u_n^{q-1}$ and integrating by parts, by (5) and Hölder's inequality we get that:

$$\int_\Omega \nabla u_n \nabla\big(\varphi^2 u_n^{q-1}\big) + \mu_n \int_\Omega \varphi^2 u_n^q = N(N-2)\int_\Omega u_n^{\frac{4}{N-2}}\big(\varphi u_n^{\frac{q}{2}}\big)^2$$
$$\leqslant \frac{N(N-2)}{S_N}\bigg(\int_{B_{2r}(x)} u_n^{\frac{2N}{N-2}}\bigg)^{\frac{2}{N}} \int_\Omega \big|\nabla\big(\varphi u_n^{\frac{q}{2}}\big)\big|^2.$$

On the other hand, we can write:

$$\int_\Omega \nabla u_n \nabla\big(\varphi^2 u_n^{q-1}\big) = (q-1)\int_\Omega \varphi^2 u_n^{q-2}|\nabla u_n|^2 + 2\int_\Omega \varphi u_n^{q-1}\nabla\varphi\nabla u_n$$
$$= \frac{2}{q}\int_\Omega \big|\nabla\big(\varphi u_n^{\frac{q}{2}}\big)\big|^2 + \frac{q-2}{2}\int_\Omega \varphi^2 u_n^{q-2}|\nabla u_n|^2 - \frac{2}{q}\int_\Omega |\nabla\varphi|^2 u_n^q$$
$$\geqslant \frac{2}{q}\int_\Omega \big|\nabla\big(\varphi u_n^{\frac{q}{2}}\big)\big|^2 - \frac{2}{q}\int_\Omega |\nabla\varphi|^2 u_n^q.$$

Combining these two estimates, we get that

$$\frac{2}{q}\int_\Omega \big|\nabla\big(\varphi u_n^{\frac{q}{2}}\big)\big|^2 \leqslant \frac{2}{q}\int_\Omega |\nabla\varphi|^2 u_n^q + \frac{N(N-2)}{S_N}\bigg(\int_{B_{2r}(x)} u_n^{\frac{2N}{N-2}}\bigg)^{\frac{2}{N}}\int_\Omega \big|\nabla\big(\varphi u_n^{\frac{q}{2}}\big)\big|^2$$
$$\leqslant \frac{8}{qr^2}\int_{B_{2r}(x)} u_n^q + \frac{1}{q}\int_\Omega \big|\nabla\big(\varphi u_n^{\frac{q}{2}}\big)\big|^2$$

in view of $\int_{B_{2r}(x)} u_n^{\frac{2N}{N-2}} \leqslant \big(\frac{S_N}{qN(N-2)}\big)^{\frac{N}{2}}$. Therefore, by (5) we obtain that

$$\bigg(\int_{B_r(x)} u_n^{\frac{N}{N-2}q}\bigg)^{\frac{N-2}{N}} \leqslant \bigg(\int_\Omega \big(\varphi u_n^{\frac{q}{2}}\big)^{\frac{2N}{N-2}}\bigg)^{\frac{N-2}{N}} \leqslant \frac{1}{S_N}\int_\Omega \big|\nabla\big(\varphi u_n^{\frac{q}{2}}\big)\big|^2 \leqslant \frac{8}{S_N r^2}\int_{B_{2r}(x)} u_n^q.$$

Since $\frac{N}{N-2}q > q$, we can iterate the procedure starting from $q = 2$ up to get a-priori bounds in L^p-norms around x for any $p > 2$ provided the $L^{\frac{2N}{N-2}}$-norm around x is sufficiently small. Namely, we find $0 < \delta < 1$, $p > \frac{N+2}{N-2}\frac{N}{2}$ and $c = c_N > 0$, depending only on N, such that, if $\int_{B_{2r}(x)} u_n^{\frac{2N}{N-2}} \leqslant c$, then

$$\bigg(\int_{B_{\delta r}(x)} u_n^p\bigg)^{\frac{2}{p}} \leqslant C(N,r)\int_{B_{2r}(x)} u_n^2,$$

for some constant $C(N,r)$ depending only on N and r. Let $u_n^{(1)}$ be the solution of

$$\begin{cases} -\Delta u_n^{(1)} = N(N-2)u_n^{\frac{N+2}{N-2}} & \text{in } B_{\delta r}(x),\\ u_n^{(1)} = 0 & \text{on } \partial B_{\delta r}(x), \end{cases}$$

and $u_n^{(2)}$ be an harmonic function such that $u_n^{(2)} = u_n$ on $\partial B_{\delta r}(x)$. Since

$$\big\|N(N-2)u_n^{\frac{N+2}{N-2}}\big\|_{L^s(B_{\delta r}(x))} = \mathrm{O}\bigg(\bigg(\int_{B_{2r}(x)} u_n^2\bigg)^{\frac{1}{2}}\bigg)$$

for some $s > \frac{N}{2}$, by elliptic regularity theory (cf. [14]) we get that

$$\|u_n^{(1)}\|_{C^0(B_{\delta r}(x))} = \mathrm{O}\bigg(\bigg(\int\limits_{B_{2r}(x)} u_n^2\bigg)^{\frac{1}{2}}\bigg)$$

provided $\int_{B_{2r}(x)} u_n^{\frac{2N}{N-2}} \leqslant c$. By the representation formula for harmonic function, we get that

$$\|u_n^{(2)}\|_{C^0(B_{\delta r/2}(x))} = \mathrm{O}\bigg(\int\limits_{\partial B_{\delta r}(x)} u_n\bigg).$$

Since by the maximum principle $0 < u_n \leqslant u_n^{(1)} + u_n^{(2)}$, we get that

$$\|u_n\|_{C^0(B_{\delta r/2}(x))} \leqslant C\bigg(\bigg(\int\limits_{B_{2r}(x)} u_n^2\bigg)^{\frac{1}{2}} + \int\limits_{\partial B_{\delta r}(x)} u_n\bigg) \tag{6}$$

for some $C > 0$.

By the continuous embedding of $H^1(\Omega)$ into $L^{\frac{2N}{N-2}}(\Omega)$ we get that $\sup_{n\in\mathbb{N}} \int_\Omega u_n^{\frac{2N}{N-2}} < +\infty$ and therefore, Σ_c is a finite set, where $c = c_N$ is as above. Moreover, up to a subsequence we can assume that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$, in view of the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$. Integrating (4) against φu_n, $\varphi \in C_0^\infty(\Omega)$, we get that $\mu_n u_n^2$ is uniformly bounded in $L^1_{\mathrm{loc}}(\Omega)$ and hence, $u = 0$ if $\mu = +\infty$.

By the compactness of the embedding of $H^1(B_{2r}(x))$ into $L^2(B_{2r}(x))$ and of $H^1(B_{\delta r}(x))$ into $L^1(\partial B_{\delta r}(x))$ in the sense of traces, in view of (6) we get that $S = \Sigma_{c_N}$ is a finite set and, if $\mu = +\infty$, $u_n \to 0$ in $C^0_{\mathrm{loc}}(\Omega \setminus S)$ (up to a subsequence). □

Remark 2.2. Blowing up the sequence u_n around a point $x \in S$, by means of the same techniques which we will exploit strongly in Appendix A, it is easy to show that:

$$\limsup_{n\to+\infty} \int\limits_{B_r(x)} u_n^{\frac{2N}{N-2}} \geqslant \left(\frac{S_N}{N(N-2)}\right)^{\frac{N}{2}}$$

for any $r > 0$. Hence, the value $c = c_N$ in Proposition 2.1 can be taken as $c = (\frac{S_N}{N(N-2)})^{\frac{N}{2}}$.

We are now in position to deduce Theorem 1.1 by Theorem 1.2.

Proof of Theorem 1.1. Multiplying (3) by u_n and integrating by parts, we get that:

$$\int\limits_\Omega \left(|\nabla u_n|^2 + \lambda_n u_n^2\right) \leqslant \Lambda := N(N-2) \sup_{n\in\mathbb{N}} \int\limits_\Omega u_n^{\frac{2N}{N-2}} < +\infty. \tag{7}$$

We can define the blow-up set S of the sequence u_n. By the validity of Theorem 1.2, we deduce that S has to be an empty set and therefore, u_n is uniformly bounded in $C^0_{\mathrm{loc}}(\Omega)$.

Otherwise, if $S \neq \emptyset$, up to a subsequence, we can assume that there exists $x_0 \in S$ such that $\max_{x\in B_r(x_0)} u_n(x) \to +\infty$ as $n \to +\infty$, for any $r > 0$. By Proposition 2.1, we know that S is a finite set. Let $0 < r < \mathrm{dist}(x_0, S \setminus \{x_0\})$ and x_n be such that $u_n(x_n) = \max_{x\in B_r(x_0)} u_n(x) \to +\infty$ as $n \to +\infty$. Clearly, since u_n is uniformly bounded in $C^0_{\mathrm{loc}}(\Omega \setminus S)$, $x_n \to x_0$ as $n \to +\infty$.

Introduce $\varepsilon_n = u_n(x_n)^{-\frac{2}{N-2}} \to 0$ and define $U_n(y) = \varepsilon_n^{\frac{N-2}{2}} u_n(\varepsilon_n y + x_n)$ for $y \in B_n := B_{\frac{r}{2\varepsilon_n}}(0)$. We have that

$$\begin{cases} -\Delta U_n + \mu_n U_n = N(N-2) U_n^{\frac{N+2}{N-2}} & \text{in } B_n,\\ 0 < U_n(y) \leqslant U_n(0) = 1, \end{cases}$$

where $\mu_n = \lambda_n \varepsilon_n^2$. Assume that $\mu_n = \lambda_n \varepsilon_n^2 \to \mu \in [0, +\infty]$. Since U_n is uniformly bounded in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ and in $C^0_{\mathrm{loc}}(\mathbb{R}^N)$, if $\mu = +\infty$, Proposition 2.1 implies that $U_n \to 0$ in $C^0_{\mathrm{loc}}(\mathbb{R}^N)$ (up to a subsequence) contradicting $U_n(0) = 1$.

So, $\mu < +\infty$. By standard elliptic estimates (cf. [14]), we have that $U_n \to U$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ where $U \in H^1(\mathbb{R}^N)$ is a solution of

$$\begin{cases} -\Delta U + \mu U = N(N-2) U^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ 0 < U(y) \leqslant U(0) = 1 \end{cases} \tag{8}$$

(in view of (7)). By a Pohozaev identity on $\mathbb{R}^N$ (see [19]), we must have that

$$\mu_n = \lambda_n \varepsilon_n^2 \to \mu = 0.$$

Now, we do the following rescaling. Let $v_n(x) = \lambda_n^{-\frac{N-2}{4}} u_n(x/\sqrt{\lambda_n} + x_n)$ be defined for $x \in B_1(0)$. The function v_n satisfies:

$$\begin{cases} -\Delta v_n + v_n = N(N-2) v_n^{\frac{N+2}{N-2}} & \text{in } B_1(0), \\ v_n > 0 & \text{in } B_1(0), \\ \int_{B_1(0)} (|\nabla v_n|^2 + v_n^2) \leqslant \Lambda, \end{cases}$$

since

$$\int_{B_1(0)} \left(|\nabla v_n|^2 + v_n^2\right) = \int_{B_{1/\sqrt{\lambda_n}}(x_n)} \left(|\nabla u_n|^2 + \lambda_n u_n^2\right) \leqslant \int_{\Omega} \left(|\nabla u_n|^2 + \lambda_n u_n^2\right) \leqslant \Lambda.$$

By Theorem 1.2 we get that there exists $C > 0$ such that

$$\max_{x \in B_{1/2}(0)} v_n(x) \leqslant C.$$

So, we reach a contradiction since we have already shown that

$$v_n(0) = \lambda_n^{-\frac{N-2}{4}} u_n(x_n) = \left(\frac{1}{\lambda_n \varepsilon_n^2}\right)^{\frac{N-2}{4}} \to +\infty$$

as $n \to +\infty$. The proof is now complete. □

3. Nonexistence of interior blow-up points

The proof of Theorem 1.2 is based on a contradiction argument. In view of Proposition 2.1, let us assume the existence of a solutions sequence u_n of the following problem:

$$\begin{cases} -\Delta u_n + u_n = N(N-2) u_n^{\frac{N+2}{N-2}} & \text{in } B_1(0), \\ u_n > 0 & \text{in } B_1(0), \\ \sup_{n \in \mathbb{N}} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} < +\infty, \end{cases}$$

which blows up in $B_1(0)$ only at 0: $\max_{x \in B_1(0)} u_n(x) \to +\infty$ as $n \to +\infty$ and u_n is uniformly bounded in $C^0_{\mathrm{loc}}(B_1(0) \setminus \{0\})$.

By means of Propositions A.1, A.2 and by elliptic regularity theory (cf. [14]), up to a subsequence, we will assume throughout this section the existence of sequences $x_n^1, \ldots, x_n^k \to 0$, $\varepsilon_n^1, \ldots, \varepsilon_n^k \to 0$ and $x_1, \ldots, x_k \in \mathbb{R}^N$ such that for any $i = 1, \ldots, k$:

$$U_n^i(y) = \left(\varepsilon_n^i\right)^{\frac{N-2}{2}} u_n\left(\varepsilon_n^i y + x_n^i\right) \to \frac{1}{(1 + |y - x_i|^2)^{\frac{N-2}{2}}} \quad \text{in } C^2_{\mathrm{loc}}\left(\mathbb{R}^N \setminus S_i\right) \text{ as } n \to +\infty, \tag{9}$$

$$u_n \to u_0 \quad \text{in } C^0_{\mathrm{loc}}\left(B_1(0) \setminus \{0\}\right) \quad \text{as } n \to +\infty, \tag{10}$$

$$d_k(x)^{\frac{N-2}{2}} u_n(x) \leqslant C \quad \text{for any } n \in \mathbb{N},\ |x| < 1, \tag{11}$$

$$\lim_{R\to+\infty} \limsup_{n\to+\infty} \max_{x\in B_R^n} \left(d_k(x)^{\frac{N-2}{2}} \left|u_n(x) - u_0(x)\right|\right) = 0, \tag{12}$$

for some constant $C > 0$ and for some smooth solution $u_0 \geqslant 0$ of the equation:

$$-\Delta u_0 + u_0 = N(N-2)u_0^{\frac{N+2}{N-2}} \quad \text{in } B_1(0),$$

where $d_k(x) = \min\{|x - x_n^i|\colon i = 1, \ldots, k\}$, $B_R^n = \{|x| < 1\colon |x - x_n^i| \geqslant R\varepsilon_n^i\ \forall i = 1, \ldots, k\}$ and

$$S_i = \left\{ y_j = \lim_{n\to+\infty} \frac{x_n^j - x_n^i}{\varepsilon_n^i}\colon\ j < i \text{ s.t. } \frac{|x_n^j - x_n^i|}{\varepsilon_n^i} = \mathrm{O}(1) \right\}.$$

Let now x be so that $|x - x_n^i| = R\varepsilon_n^i$ for some $i = 1, \ldots, k$. We have that $y = \frac{x - x_n^i}{\varepsilon_n^i}$ satisfies: $|y| = R$ and $|y - y_j| \geqslant R - |y_j| \geqslant 1$, for R large. Hence, by (9) we get that for any $R > 0$ large and $C > 1$ there exists N_0 such that for any $n \geqslant N_0$ and $|x - x_n^i| = R\varepsilon_n^i$

$$u_n(x) \leqslant C U_{\varepsilon_n^i, x_n^i + \varepsilon_n^i x_i}(x),$$

where

$$U_{\varepsilon,y}(x) = \varepsilon^{-\frac{N-2}{2}} U\left(\frac{x-y}{\varepsilon}\right) = \frac{\varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + |x-y|^2)^{\frac{N-2}{2}}}.$$

Since $|x - (x_n^i + \varepsilon_n^i x_i)| \geqslant (1 - \frac{\max_{i=1,\ldots,k} |x_i|}{R})|x - x_n^i|$ for $|x - x_n^i| = R\varepsilon_n^i$, we obtain that for any $R > 0$ large there exists N_0 such that

$$u_n(x) \leqslant 2U_{\varepsilon_n^i, x_n^i}(x) \tag{13}$$

for any $n \geqslant N_0$ and $|x - x_n^i| = R\varepsilon_n^i$.

The aim will be to estimate from above $u_n(x)$ in terms of the standard bubbles $U_{\varepsilon_n^i, x_n^i}(x)$, $i = 1, \ldots, k$, in $B_\delta(0) \setminus \bigcup_{i=1}^k B_{R\varepsilon_n^i}(x_n^i)$, $0 < \delta < 1$. By performing some simple asymptotic analysis we get the following result (see also the techniques developed by Schoen in [22] and exploited in [15,16]):

Lemma 3.1. *Let $\alpha \in (0, \frac{N-2}{2})$. There exist $R > 0$, $0 < \delta < 1$ and $N_0 \in \mathbb{N}$ such that*

$$u_n(x) \leqslant \sum_{i=1}^k \left(\left(\varepsilon_n^i\right)^{\frac{N-2}{2}-\alpha} \left|x - x_n^i\right|^{2-N+\alpha} + M_n \left|x - x_n^i\right|^{-\alpha} \right),$$

for any $n \geqslant N_0$ and $|x| \leqslant \delta$ with $|x - x_n^i| \geqslant R\varepsilon_n^i$, $i = 1, \ldots, k$, where $M_n = 2\delta^\alpha \sup_{|x|=\delta} u_n(x)$.

Proof. Let us introduce the operator $L_n = -\Delta + 1 - N(N-2)u_n^{\frac{4}{N-2}}$. Since u_n is a positive solution of $L_n u_n = 0$ in $B_\delta(0)$, we have that L_n satisfies the minimum principle in $B_\delta(0)$ for any $0 < \delta < 1$ (see [14]). Since u_0 is a smooth function, by (12) we have that there exist $R > 2^{\frac{1}{\alpha}}$, $0 < \delta < 1$ and $N_0 \in \mathbb{N}$ such that

$$d_k(x)^{\frac{N-2}{2}} u_n(x) \leqslant \left(\frac{\alpha(N-2-\alpha)}{kN(N-2)}\right)^{\frac{N-2}{4}} \tag{14}$$

for any $n \geqslant N_0$ and $x \in B_\delta(0)$: $|x - x_n^i| \geqslant R\varepsilon_n^i$, $i = 1, \ldots, k$.

Define now a comparison function φ_n in the form $\varphi_n = \sum_{i=1}^k \varphi_n^i$, where

$$\varphi_n^i(x) = \left(\varepsilon_n^i\right)^{\frac{N-2}{2}-\alpha} \left|x - x_n^i\right|^{2-N+\alpha} + M_n \left|x - x_n^i\right|^{-\alpha},$$

and compute L_n on $\varphi_n - u_n$:

$$L_n(\varphi_n - u_n) = \sum_{i=1}^k L_n \varphi_n^i = \sum_{i=1}^k \big(\alpha(N-2-\alpha)\big|x - x_n^i\big|^{-2} + 1 - N(N-2)u_n^{\frac{4}{N-2}}\big)\varphi_n^i.$$

Let $x \in B_\delta(0)$ be such that $|x - x_n^i| \geqslant R\varepsilon_n^i$, $i = 1, \dots, k$. There exists $j \in \{1, \dots, k\}$ so that $|x - x_n^j| = \min\{|x - x_n^i|: i = \dots, k\}$. Since $|x - x_n^j| \leqslant |x - x_n^i|$, we have that $\varphi_n^j(x) \geqslant \varphi_n^i(x)$ for any $i = 1, \dots, k$ and therefore,

$$\begin{aligned} L_n(\varphi_n - u_n)(x) &= \sum_{i=1}^k \big(\alpha(N-2-\alpha)\big|x - x_n^i\big|^{-2} + 1 - N(N-2)u_n(x)^{\frac{4}{N-2}}\big)\varphi_n^i(x) \\ &\geqslant \alpha(N-2-\alpha)\big|x - x_n^j\big|^{-2}\varphi_n^j(x) - N(N-2)u_n(x)^{\frac{4}{N-2}} \sum_{i=1}^k \varphi_n^i(x) \\ &\geqslant \Big[\alpha(N-2-\alpha) - kN(N-2)\big(\big|x - x_n^j\big|^{\frac{N-2}{2}} u_n(x)\big)^{\frac{4}{N-2}}\Big]\big|x - x_n^j\big|^{-2}\varphi_n^j(x) \geqslant 0 \end{aligned}$$

in view of (14), for any $n \geqslant N_0$ and $x \in B_\delta(0)$: $|x - x_n^i| \geqslant R\varepsilon_n^i$, $i = 1, \dots, k$. In view of the validity of (13) on $\partial B_{R\varepsilon_n^i}(x_n^i)$, we can always assume that R and N_0 are such that

$$u_n(x) \leqslant 2\big(\varepsilon_n^i\big)^{\frac{N-2}{2}}\big|x - x_n^i\big|^{2-N}$$

for any $n \geqslant N_0$ and $|x - x_n^i| = R\varepsilon_n^i$. Therefore, we have that

$$u_n(x) \leqslant 2R^{-\alpha}\big(\varepsilon_n^i\big)^{\frac{N-2}{2}-\alpha}\big|x - x_n^i\big|^{2-N+\alpha} \leqslant \big(\varepsilon_n^i\big)^{\frac{N-2}{2}-\alpha}\big|x - x_n^i\big|^{2-N+\alpha} \leqslant \varphi_n^i(x) \leqslant \varphi_n(x)$$

for $n \geqslant N_0$ and $|x - x_n^i| = R\varepsilon_n^i$ for some $i = 1, \dots, k$. Since

$$u_n(x) \leqslant \frac{1}{2\delta^\alpha} M_n \leqslant M_n \sum_{i=1}^k \big|x - x_n^i\big|^{-\alpha} \leqslant \varphi_n(x)$$

for $|x| = \delta$ and n large, by the minimum principle for L_n we get the desired estimate in the region $x \in B_\delta(0)$ with $|x - x_n^i| \geqslant R\varepsilon_n^i$ $\forall i = 1, \dots, k$. □

We have to combine the estimate contained in Lemma 3.1 with the following Pohozaev-type inequality ("essentially" proved in [7], for the Pohozaev identity refer to [19]):

Lemma 3.2. *There exists $C > 0$, depending only on the dimension N, such that for any $|x| < 1$ and $0 < h < \frac{1-|x|}{4}$*

$$\int_{B_h(x)} u_n^2 \leqslant C \int_{B_{2h}(x)\setminus B_h(x)} \left(\frac{1}{h^2}u_n^2 + u_n^{\frac{2N}{N-2}}\right). \tag{15}$$

Proof. Since Lemma 3.2 is written in a slightly different way with respect to [7], let us outline why some difference appears. By [7] we get that:

$$\int_{B_h(x)} u_n^2 \leqslant -\frac{2}{N}\int_\Omega \langle x, \nabla\varphi\rangle \varphi u_n^{\frac{2N}{N-2}} + R_n,$$

where $\varphi \in C_0^\infty(B_{2h}(x))$ is such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ on $B_h(x)$, and

$$R_n = -\int_\Omega u_n^2\left[\big\langle \nabla\varphi, \nabla\langle x, \nabla\varphi\rangle\big\rangle + \frac{N}{2}\varphi\Delta\varphi + \frac{1}{2}\big\langle x, \nabla(\varphi\Delta\varphi)\big\rangle + \frac{N-2}{2}|\nabla\varphi|^2\right].$$

Assuming that $|\nabla\varphi| \leqslant \frac{2}{h}$, we have that $\langle x, \nabla\varphi\rangle\varphi$ vanishes outside $B_{2h}(x) \setminus B_h(x)$ and

$$\big|\langle x, \nabla\varphi\rangle\varphi\big| \leqslant 4$$

in $B_{2h}(x) \setminus B_h(x)$. Hence, we get that

$$\left| \int_\Omega \langle x, \nabla\varphi \rangle \varphi u_n^{\frac{2N}{N-2}} \right| \leqslant 4 \int_{B_{2h}(x)\setminus B_h(x)} u_n^{\frac{2N}{N-2}}.$$

Similarly, assuming that $|\nabla^2 \varphi| \leqslant \frac{2}{h^2}$ we show that

$$|R_n| \leqslant \frac{C}{h^2} \int_{B_{2h}(x)\setminus B_h(x)} u_n^2$$

for some constant $C > 0$. The proof is now complete. □

Let $N > 6$ and fix $0 < \alpha < \frac{N-6}{3}$. Let us define the following sequence:

$$r_n = \max\{\varepsilon_n^i \colon i = \dots, k\}^{\frac{2+\alpha}{N-2-2\alpha}}.$$

Up to a subsequence and a re-labeling, let us assume that $\varepsilon_n^k = \max\{\varepsilon_n^i \colon i = \dots, k\}$ and that, for some integer $s \in \{1, \dots, k-1\}$, there hold:

$$\frac{|x_n^i - x_n^k|}{r_n} \to +\infty, \quad i = 1, \dots, s, \qquad \frac{|x_n^i - x_n^k|}{r_n} \leqslant D - 1, \quad i = s+1, \dots, k, \tag{16}$$

as $n \to +\infty$, where $D > 1$ is a constant. By (16), we obtain that for $x \in B_{2Dr_n}(x_n^k) \setminus B_{Dr_n}(x_n^k)$ there hold:

$$\begin{cases} |x - x_n^i| \geqslant |x_n^i - x_n^k| - |x - x_n^k| \geqslant |x_n^i - x_n^k| - 2Dr_n \geqslant r_n & \text{if } i = 1, \dots, s \\ |x - x_n^i| \geqslant |x - x_n^k| - |x_n^k - x_n^i| \geqslant Dr_n - (D-1)r_n = r_n & \text{if } i = s+1, \dots, k. \end{cases} \tag{17}$$

We apply now (15) on $B_{Dr_n}(x_n^k)$. Let $r = \frac{1}{2}\min\{|y_j| \colon y_j \in S_k,\ y_j \neq 0\}$ if $S_k \setminus \{0\} \neq \emptyset$ and $r = 1$ otherwise. Since $r_n \gg \varepsilon_n^k$ for n large in view of $\alpha < \frac{N-4}{3}$, we have that

$$\int_{B_{Dr_n}(x_n^k)} u_n^2 \geqslant \int_{B_{r\varepsilon_n^k}(x_n^k)} u_n^2 = (\varepsilon_n^k)^2 \int_{B_r(0)} (U_n^k)^2 \geqslant (\varepsilon_n^k)^2 \int_{B_r(0)\setminus B_{r/2}(0)} (U_n^k)^2.$$

Since $S_k \cap \{\frac{r}{2} \leqslant |y| \leqslant r\} = \emptyset$, by (9) we get that

$$\int_{B_r(0)\setminus B_{r/2}(0)} (U_n^k)^2 \to \int_{B_r(0)\setminus B_{r/2}(0)} \frac{1}{(1 + |y - x_k|^2)^{N-2}} > 0$$

and hence,

$$\int_{B_{Dr_n}(x_n^k)} u_n^2 \geqslant \frac{1}{2} (\varepsilon_n^k)^2 \int_{B_r(0)\setminus B_{r/2}(0)} \frac{1}{(1 + |y - x_k|^2)^{N-2}}$$

for n large. Let us remark that for n large $\frac{1}{4D}\delta > r_n \geqslant R\varepsilon_n^k \geqslant R\varepsilon_n^i$ for any $i = 1, \dots, k$. Therefore, we can use Lemma 3.1 and (17) to provide that

$$u_n(x) \leqslant \sum_{i=1}^k \left((\varepsilon_n^i)^{\frac{N-2}{2} - \alpha} |x - x_n^i|^{2-N+\alpha} + M_n |x - x_n^i|^{-\alpha} \right) \leqslant k (\varepsilon_n^k)^{\frac{N-2}{2} - \alpha} r_n^{2-N+\alpha} + k M_n r_n^{-\alpha}, \tag{18}$$

for any $n \geqslant N_0$ and $x \in B_{2Dr_n}(x_n^k) \setminus B_{Dr_n}(x_n^k)$. Hence, by (18) we get that

$$\int_{B_{2Dr_n}(x_n^k)\setminus B_{Dr_n}(x_n^k)} \left(\frac{1}{D^2 r_n^2} u_n^2 + u_n^{\frac{2N}{N-2}}\right)$$
$$\leqslant C\left(\left(\frac{\varepsilon_n^k}{r_n}\right)^{N-2-2\alpha} + M_n^2 r_n^{N-2-2\alpha} + \left(\frac{\varepsilon_n^k}{r_n}\right)^{N-\alpha\frac{2N}{N-2}} + M_n^{\frac{2N}{N-2}} r_n^{N-\alpha\frac{2N}{N-2}}\right)$$
$$\leqslant C'\left(\left(\frac{\varepsilon_n^k}{r_n}\right)^{N-2-2\alpha} + M_n^2 r_n^{N-2-2\alpha}\right) = C'\left(\left(\varepsilon_n^k\right)^{N-4-3\alpha} + M_n^2\left(\varepsilon_n^k\right)^{2+\alpha}\right)$$

since $\alpha < \frac{N-2}{2}$ and M_n is bounded, in view of (10). Finally, by Lemma 3.2 we obtain that

$$1 \leqslant C\left(\left(\varepsilon_n^k\right)^{N-6-3\alpha} + \left(\varepsilon_n^k\right)^{\alpha}\right),$$

which gives a contradiction for n large, since $\varepsilon_n^k \to 0$ as $n \to +\infty$.

Appendix A

In this appendix, we want to give a detailed local description of the blow-up phenomenon for a solutions sequence u_n of the following problem:

$$\begin{cases} -\Delta u_n + u_n = N(N-2)u_n^{\frac{N+2}{N-2}} & \text{in } B_1(0),\\ u_n > 0 & \text{in } B_1(0),\\ \sup_{n\in\mathbb{N}} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} < +\infty. \end{cases}$$

We assume that 0 is the only blow-up point of u_n in $B_1(0)$:

$$\max_{x\in B_1(0)} u_n(x) \to +\infty \quad \text{as } n \to +\infty, \tag{19}$$
$$u_n \to u_0 \quad \text{in } C^0_{\text{loc}}\left(B_1(0)\setminus\{0\}\right) \tag{20}$$

(up to a subsequence), where $u_0 \geqslant 0$ is a smooth solution of the limit equation:

$$-\Delta u_0 + u_0 = N(N-2)u_0^{\frac{N+2}{N-2}} \quad \text{in } B_1(0).$$

First of all, the following classical result holds (see for example [10,11] and [4,22]):

Proposition A.1. *Up to a subsequence of u_n, there exist $s \in \mathbb{N}^*$ and sequences $x_n^1, \ldots, x_n^s \to 0$ such that for any $i, j = 1, \ldots, s$, $i \neq j$, we have*:

$$\varepsilon_n^i := u_n\left(x_n^i\right)^{-\frac{2}{N-2}} \to 0, \quad \frac{\varepsilon_n^i + \varepsilon_n^j}{|x_n^i - x_n^j|} \to 0 \quad \textit{as } n \to +\infty, \tag{21}$$
$$u_n\left(x_n^i\right) = \max_{x\in B_{\varepsilon_n^i}(x_n^i)} u_n(x) \quad \textit{for } n \textit{ large}, \tag{22}$$
$$U_n^i(y) = \left(\varepsilon_n^i\right)^{\frac{N-2}{2}} u_n\left(\varepsilon_n^i y + x_n^i\right) \to \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}} \quad \textit{in } C^2_{\text{loc}}\left(\mathbb{R}^N\right) \quad \textit{as } n \to +\infty, \tag{23}$$
$$d_s(x)^{\frac{N-2}{2}} u_n(x) \leqslant C \quad \textit{for any } n \in \mathbb{N},\ |x| < 1, \tag{24}$$

where $d_j(x) = \min\{|x - x_n^i|:\ i = 1, \ldots, j\}$, $1 \leqslant j \leqslant s$. In particular, there holds

$$\liminf_{n\to+\infty} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} \geqslant s\left(\frac{S_N}{N(N-2)}\right)^{\frac{N}{2}} + \int_{B_1(0)} u_0^{\frac{2N}{N-2}}, \tag{25}$$

where S_N is the Sobolev constant.

Proof. Let x_n^1 be the maximum point of u_n in $B_1(0)$. By (19), (20) we deduce that $x_n^1 \to 0$ and $\varepsilon_n^1 = u_n(x_n^1)^{-\frac{2}{N-2}} \to 0$. Define $U_n^1(y) = (\varepsilon_n^1)^{\frac{N-2}{2}} u_n(\varepsilon_n^1 y + x_n^1)$ for $y \in B_n := B_{(\varepsilon_n^1)^{-1}}(-x_n^1/\varepsilon_n^1)$. We have that

$$\begin{cases} -\Delta U_n^1 + (\varepsilon_n^1)^2 U_n^1 = N(N-2)(U_n^1)^{\frac{N+2}{N-2}} & \text{in } B_n, \\ 0 < U_n^1(y) \leqslant U_n^1(0) = 1. \end{cases}$$

By standard elliptic estimates (cf. [14]) and a diagonal process, (up to a subsequence) we have that $U_n^1 \to U$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ where U is a solution of

$$\begin{cases} -\Delta U = N(N-2)U^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ 0 < U(y) \leqslant U(0) = 1. \end{cases} \tag{26}$$

By the classification in [5], problem (26) admits only the solution

$$U(y) = \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}}.$$

Hence, we get that

$$U_n^1(y) \to \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}} \quad \text{in } C^2_{\mathrm{loc}}\big(\mathbb{R}^N\big).$$

Moreover, by taking the liminf as $n \to +\infty$ and the limit as $\delta \to 0$, $R \to +\infty$ in the following inequality chain:

$$\int_{B_1(0)} u_n^{\frac{2N}{N-2}} \geqslant \int_{B_{R\varepsilon_n^1}(x_n^1)} u_n^{\frac{2N}{N-2}} + \int_{B_1(0)\setminus B_\delta(0)} u_n^{\frac{2N}{N-2}} = \int_{B_R(0)} \big(U_n^1\big)^{\frac{2N}{N-2}} + \int_{B_1(0)\setminus B_\delta(0)} u_n^{\frac{2N}{N-2}},$$

which is true for $\delta,\ R > 0$ and $n \geqslant N = N(\delta, R)$, we deduce that

$$\liminf_{n\to+\infty} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} \geqslant \int_{\mathbb{R}^N} \frac{dy}{(1+|y|^2)^N} + \int_{B_1(0)} u_0^{\frac{2N}{N-2}} = \left(\frac{S_N}{N(N-2)}\right)^{\frac{N}{2}} + \int_{B_1(0)} u_0^{\frac{2N}{N-2}}.$$

If (24) holds true for x_n^1, we take $s = 1$ and the proof is complete since (21)–(23) are already verified. Otherwise, by induction assume that there exist j sequences $x_n^1, \ldots, x_n^j \to 0$ as $n \to +\infty$ satisfying (21)–(23). If (24) holds true, we have done.

Otherwise, there exists a sequence $y_n \in B_1(0)$ such that

$$d_j(y_n)^{\frac{N-2}{2}} u_n(y_n) = \max_{x \in B_1(0)} d_j(x)^{\frac{N-2}{2}} u_n(x) \to +\infty \quad \text{as } n \to +\infty. \tag{27}$$

Since $u_n \to u_0$ away from 0, by (27) we have that $y_n \to 0$ and, for $\mu_n = u_n(y_n)^{-\frac{2}{N-2}}$, property (27) reads equivalently as:

$$\frac{\mu_n}{d_j(y_n)} \to 0 \quad \text{as } n \to +\infty. \tag{28}$$

Since (21)–(23) imply $d_j(x)^{\frac{N-2}{2}} u_n(x) \leqslant 1$ for $|x - x_n^i| \leqslant R\varepsilon_n^i$, $i = 1, \ldots, j$, and $n \geqslant N = N(R)$, we deduce that y_n must be outside this region, i.e.

$$\frac{\varepsilon_n^i}{|y_n - x_n^i|} \to 0 \quad \text{as } n \to +\infty,\ \forall i = 1, \ldots, j. \tag{29}$$

Introduce the function $U_n(y) = \mu_n^{\frac{N-2}{2}} u_n(\mu_n y + y_n)$ for $y \in B_n$, where the balls $B_n = \{y \in \mathbb{R}^N\colon\ |y| \leqslant \frac{d_j(y_n)}{2\mu_n}\}$ expand to $\mathbb{R}^N$ as $n \to +\infty$ in view of (28). Moreover, since

$$\big|\mu_n y + y_n - x_n^i\big| \geqslant \big|y_n - x_n^i\big| - \mu_n|y| \geqslant \frac{1}{2}\big|y_n - x_n^i\big|$$

for any $y \in B_n$ and $i = 1, \ldots, j$, we have the following estimate:

$$U_n(y) = \left(\frac{d_j(y_n)}{d_j(\mu_n y + y_n)}\right)^{\frac{N-2}{2}} \frac{d_j(\mu_n y + y_n)^{\frac{N-2}{2}} u_n(\mu_n y + y_n)}{d_j(y_n)^{\frac{N-2}{2}} u_n(y_n)} \leqslant 2^{\frac{N-2}{2}}$$

for any $y \in B_n$, in view of the maximality property in the definition (27) of y_n. Hence, by standard elliptic estimates and a diagonal process, (up to a subsequence) we get that $U_n \to U$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$, where U is a solution of

$$\begin{cases} -\Delta U = N(N-2)U^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ 0 < U(y) \leqslant 2^{\frac{N-2}{2}}, & U(0) = 1. \end{cases}$$

By the classification in [5], the function U satisfies:

$$U(y) = \frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |y - y_0|^2)^{\frac{N-2}{2}}},$$

for some $\mu > 0$ and $y_0 \in \mathbb{R}^N$. Since $U(y)$ has a nondegenerate maximum point at $y = y_0$, for n large $U_n(y)$ has a unique maximum point z_n close to y_0. Hence, for the original sequence, the point $x_n^{j+1} := \mu_n z_n + y_n$ is a maximum point of u_n on $B_{2\mu_n}(x_n^{j+1})$ for n large. Since $2^{\frac{N-2}{2}} \geqslant U_n(z_n) = \mu_n^{\frac{N-2}{2}} u_n(x_n^{j+1}) \geqslant U_n(0) = 1$, we get that, for $\varepsilon_n^{j+1} = u_n(x_n^{j+1})^{-\frac{2}{N-2}}$, there holds:

$$\frac{\varepsilon_n^{j+1}}{\mu_n} \to U(y_0)^{-\frac{2}{N-2}} = \mu \in \left[\frac{1}{2}, 1\right].$$

By (28), (29) we get that

$$\left|x_n^{j+1} - x_n^i\right| \geqslant \left|y_n - x_n^i\right| - \left|y_n - x_n^{j+1}\right| = \left|y_n - x_n^i\right| - \mu_n|z_n| \gg \begin{cases} 2\mu_n \geqslant \varepsilon_n^{j+1}, \\ \varepsilon_n^i \end{cases}$$

for any $i = 1, \ldots, j$ and hence (21), (22) hold. Moreover, there holds:

$$\left(\varepsilon_n^{j+1}\right)^{\frac{N-2}{2}} u_n\left(\varepsilon_n^{j+1} y + x_n^{j+1}\right) = \left(\frac{\varepsilon_n^{j+1}}{\mu_n}\right)^{\frac{N-2}{2}} U_n\left(\frac{\varepsilon_n^{j+1}}{\mu_n} y + z_n\right) \to \mu^{\frac{N-2}{2}} U(\mu y + y_0) = \frac{1}{(1 + |y|^2)^{\frac{N-2}{2}}}$$

in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ and, so (23) holds. So, the inductive step holds for $j + 1$. By (21) there holds

$$\int_{B_1(0)} u_n^{\frac{2N}{N-2}} \geqslant \sum_{i=1}^{j+1} \int_{B_{R\varepsilon_n^i}(x_n^i)} u_n^{\frac{2N}{N-2}} + \int_{B_1(0)\setminus B_\delta(0)} u_n^{\frac{2N}{N-2}} = \sum_{i=1}^{j+1} \int_{B_R(0)} (U_n^i)^{\frac{2N}{N-2}} + \int_{B_1(0)\setminus B_\delta(0)} u_n^{\frac{2N}{N-2}}$$

for $\delta,\ R > 0$ and $n \geqslant N = N(\delta, R)$, and hence, by (23) we deduce that

$$\begin{aligned} \liminf_{n\to+\infty} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} &\geqslant (j+1) \int_{\mathbb{R}^N} \frac{dy}{(1+|y|^2)^N} + \int_{B_1(0)} u_0^{\frac{2N}{N-2}} \\ &= (j+1)\left(\frac{S_N}{N(N-2)}\right)^{\frac{N}{2}} + \int_{B_1(0)} u_0^{\frac{2N}{N-2}}. \end{aligned} \tag{30}$$

Since $\sup_{n\in\mathbb{N}} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} < +\infty$, by (30) the induction process has to stop after s steps giving that also (24) holds for the sequences $x_n^1, \ldots, x_n^s$. Further, there holds (25) by means of the validity of (30) for $s = j + 1$. □

Now, in the spirit of the blow-up techniques developed by Druet, Hebey and Robert in [10,11], we can perform a finer analysis. Proposition A.2 below has already been showed in [10] for compact Riemannian manifolds. A careful reader could observe that, in fact, their analysis is quite local and does not use any geometric features and/or compactness of the underlying space, and therefore it extends directly to our situation. For the sake of completeness, we re-write their proof in the simpler context of an Euclidean domain:

Proposition A.2. *Up to a subsequence of u_n, there exist $k \in \mathbb{N}$, $k \geqslant s$, sequences $x_n^{s+1}, \dots, x_n^k \to 0$, $\varepsilon_n^{s+1}, \dots, \varepsilon_n^k \to 0$ and points $x_{s+1}, \dots, x_k \in \mathbb{R}^N$ such that for any $i = s+1, \dots, k$, $1 \leqslant j < i$, we have*:

$$\frac{\varepsilon_n^j}{|x_n^i - x_n^j|} \to 0, \quad \frac{\varepsilon_n^j}{\varepsilon_n^i} \to 0 \quad \text{as } n \to +\infty, \tag{31}$$

$$U_n^i(y) = (\varepsilon_n^i)^{\frac{N-2}{2}} u_n(\varepsilon_n^i y + x_n^i) \to \frac{1}{(1 + |y - x_i|^2)^{\frac{N-2}{2}}} \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^N \setminus S_i) \text{ as } n \to +\infty, \tag{32}$$

$$\lim_{R \to +\infty} \limsup_{n \to +\infty} \max_{x \in B_R^n} \left(d_k(x)^{\frac{N-2}{2}} |u_n(x) - u_0(x)| \right) = 0, \tag{33}$$

where $x_n^1, \dots, x_n^s$ are given in Proposition A.1, *$d_k(x) = \min\{|x - x_n^i|\colon i = 1, \dots, k\}$, $B_R^n = \{|x| < 1\colon |x - x_n^i| \geqslant R\varepsilon_n^i \ \forall i = 1, \dots, k\}$ and*

$$S_i = \left\{ y_j = \lim_{n \to +\infty} \frac{x_n^j - x_n^i}{\varepsilon_n^i}\colon\ j < i \text{ s.t. } \frac{|x_n^j - x_n^i|}{\varepsilon_n^i} = \mathrm{O}(1) \right\}$$

is a nonempty set. In particular, there holds

$$\liminf_{n \to +\infty} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} \geqslant k \left(\frac{S_N}{N(N-2)} \right)^{\frac{N}{2}} + \int_{B_1(0)} u_0^{\frac{2N}{N-2}}. \tag{34}$$

Proof. If (33) holds already true for $x_n^1, \dots, x_n^s$, we take $k = s$ and the proof is complete. Moreover, since $\int_{\bigcup_{i=1}^s B_{R\varepsilon_n^i}(x_n^i)} u_n^{\frac{2N}{N-2}} = \sum_{i=1}^s \int_{B_R(0)} (U_n^i)^{\frac{2N}{N-2}}$ for $R > 0$ and $n \geqslant N = N(R)$, by (23) we deduce that

$$\liminf_{n \to +\infty} \int_{\bigcup_{i=1}^s B_{R\varepsilon_n^i}(x_n^i)} u_n^{\frac{2N}{N-2}} \geqslant s \int_{B_R(0)} \frac{dy}{(1 + |y|^2)^N}$$

for any $R > 0$. In particular, we get that

$$\liminf_{n \to +\infty} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} \geqslant s \int_{B_R(0)} \frac{dy}{(1 + |y|^2)^N} + \int_{B_1(0) \setminus B_\delta(0)} u_0^{\frac{2N}{N-2}}$$

for any $\delta,\ R > 0$, and hence (34) holds.

Otherwise, by induction assume that there exist $j - s$ sequences $x_n^{s+1}, \dots, x_n^j \to 0$, $\varepsilon_n^{s+1}, \dots, \varepsilon_n^k \to 0$ as $n \to +\infty$ such that (31), (32) hold and

$$\liminf_{n \to +\infty} \int_{\bigcup_{i=1}^j B_{R\varepsilon_n^i}(x_n^i)} u_n^{\frac{2N}{N-2}} \geqslant \sum_{i=1}^j \int_{B_R(0)} \frac{dy}{(1 + |y - x_i|^2)^N}, \tag{35}$$

where we take $x_i = 0$ if $i = 1, \dots, s$. If (33) holds true, we take $k = j$ and, as before, (35) implies the validity of (34).

Otherwise, (up to a subsequence) there exist $R_n \to +\infty$ and $y_n \in B_{R_n}^n$ such that

$$d_j(y_n)^{\frac{N-2}{2}} |u_n(y_n) - u_0(y_n)| = \max_{x \in B_{R_n}^n} \left(d_j(x)^{\frac{N-2}{2}} |u_n(x) - u_0(x)| \right) \geqslant (4\delta_0)^{\frac{N-2}{2}} \tag{36}$$

for some $\delta_0 > 0$, where $B_{R_n}^n = \{|x| < 1\colon |x - x_n^i| \geqslant R_n \varepsilon_n^i \ \forall i = 1, \dots, j\}$. Since $u_n \to u_0$ away from 0, by (36) we have that $y_n \to 0$ and $d_j(y_n)^{\frac{N-2}{2}} u_n(y_n) \geqslant (2\delta_0)^{\frac{N-2}{2}}$ for n large. Hence, there exists a sequence $x_n^{j+1} \in B_{R_n}^n$ so that

$$d_j\big(x_n^{j+1}\big)^{\frac{N-2}{2}}u_n\big(x_n^{j+1}\big)=\max_{x\in B^n_{R_n}}\big(d_j(x)^{\frac{N-2}{2}}u_n(x)\big)\geqslant(2\delta_0)^{\frac{N-2}{2}}.$$

For $\varepsilon_n^{j+1}=u_n(x_n^{j+1})^{-\frac{2}{N-2}}$ there holds:

$$\frac{|x_n^{j+1}-x_n^i|}{\varepsilon_n^{j+1}}\geqslant 2\delta_0,\qquad \frac{\varepsilon_n^i}{|x_n^{j+1}-x_n^i|}\to 0\quad\text{as } n\to+\infty \tag{37}$$

for any $i=1,\dots,j$. In particular, $\varepsilon_n^{j+1}\to 0$ as $n\to+\infty$. Moreover, observing that property (24) is still true if we add the points $x_n^{s+1},\dots,x_n^j$:

$$d_j(x)^{\frac{N-2}{2}}u_n(x)\leqslant C,\quad |x|<1,\ n\in\mathbb{N}, \tag{38}$$

we have that $|x_n^{j+1}-x_n^i|/\varepsilon_n^{j+1}\leqslant C^{\frac{2}{N-2}}$ for some $i=1,\dots,j$ (up to a subsequence) and so, $S_{j+1}\neq\emptyset$ and (31) is satisfied. Up to a subsequence, assume that the limits $y_i=\lim_{n\to+\infty}x_n^i-x_n^{j+1}/\varepsilon_n^{j+1}$ exist for any $i=1,\dots,j$ so that $|x_n^i-x_n^{j+1}|/\varepsilon_n^{j+1}=\mathrm{O}(1)$. Introduce the set $S_{j+1}=\{y_i\colon i\leqslant j\}$ and remark that by (37) we have that $S_{j+1}\cap B_{\delta_0}(0)=\emptyset$.

Consider the function $U_n(y)=(\varepsilon_n^{j+1})^{\frac{N-2}{2}}u_n(\varepsilon_n^{j+1}y+x_n^{j+1})$ for $y\in B_n$, where the balls $B_n=\{y\in\mathbb{R}^N\colon |y|\leqslant\frac{1}{\varepsilon_n^{j+1}}\}$ expand to $\mathbb{R}^N$ as $n\to+\infty$. We want to show that U_n is uniformly bounded in $C^0_{\mathrm{loc}}(\mathbb{R}^N\setminus S_{j+1})$. Let $y\in B_R(0)$ be so that $|y-y_i|\geqslant\frac{1}{R}$ for any $y_i\in S_{j+1}$. Since for any $i=1,\dots,j$:

$$\big|\varepsilon_n^{j+1}y+x_n^{j+1}-x_n^i\big|=\varepsilon_n^{j+1}\left|y-\frac{x_n^i-x_n^{j+1}}{\varepsilon_n^{j+1}}\right|\geqslant\begin{cases}\varepsilon_n^{j+1}\left(|y-y_i|-\left|y_i-\dfrac{x_n^i-x_n^{j+1}}{\varepsilon_n^{j+1}}\right|\right)\geqslant\dfrac{\varepsilon_n^{j+1}}{2R} & \text{if } y_i\in S_{j+1},\\[2ex] \varepsilon_n^{j+1}\left(\left|\dfrac{x_n^i-x_n^{j+1}}{\varepsilon_n^{j+1}}\right|-|y|\right)\geqslant\dfrac{\varepsilon_n^{j+1}}{2R} & \text{otherwise}\end{cases}$$

for $n\geqslant N=N(R)$, we get that $d_j(\varepsilon_n^{j+1}y+x_n^{j+1})\geqslant\varepsilon_n^{j+1}/(2R)$ and by (38) we get that

$$U_n(y)=\big(\varepsilon_n^{j+1}\big)^{\frac{N-2}{2}}u_n\big(\varepsilon_n^{j+1}y+x_n^{j+1}\big)\leqslant(2R)^{\frac{N-2}{2}}d_j\big(\varepsilon_n^{j+1}y+x_n^{j+1}\big)^{\frac{N-2}{2}}u_n\big(\varepsilon_n^{j+1}y+x_n^{j+1}\big)\leqslant C(2R)^{\frac{N-2}{2}}$$

for $n\geqslant N=N(R)$. By standard elliptic estimates and a diagonal process, (up to a subsequence) we get that $U_n\to U$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N\setminus S_{j+1})$, where U is a nonnegative solution of

$$\begin{cases}-\Delta U=N(N-2)U^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N\setminus S_{j+1},\\ U(0)=1,\end{cases}$$

since $B_{\delta_0}(0)\subset\mathbb{R}^N\setminus S_{j+1}$. By the result of Caffarelli, Gidas and Spruck [5], we have that:

$$U(y)=\frac{\mu^{\frac{N-2}{2}}}{(\mu^2+|y-x_i|^2)^{\frac{N-2}{2}}},$$

for some $\mu>0$ and $x_i\in\mathbb{R}^N$. Since

$$\big(\mu\varepsilon_n^{j+1}\big)^{\frac{N-2}{2}}u_n\big(\mu\varepsilon_n^{j+1}y+x_n^{j+1}\big)=\mu^{\frac{N-2}{2}}U_n(\mu y)\to\mu^{\frac{N-2}{2}}U(\mu y)=\frac{1}{(1+|y-\mu^{-1}x_i|^2)^{\frac{N-2}{2}}},$$

we can replace ε_n^{j+1} with $\mu\varepsilon_n^{j+1}$, x_i with $\mu^{-1}x_i$ and S_{j+1} with respect to ε_n^{j+1} with S_{j+1} with respect to $\mu\varepsilon_n^{j+1}$ to obtain the validity of (32). Now, we want to show the validity of (35) with j replaced by $j+1$. Let $I=\{i=1,\dots,j\colon y_i\in S_{j+1}\}$ and let $C>0$ be such that $|x_n^i-x_n^{j+1}|\leqslant C\varepsilon_n^{j+1}$ for any $i\in I$. Clearly, we have that

$$\bigcup_{i\in I}B_{\delta\varepsilon_n^{j+1}}\big(x_n^i\big)\subset B_{(C+\delta)\varepsilon_n^{j+1}}\big(x_n^{j+1}\big)$$

for any $\delta>0$, and by (31) we also deduce that

$$\bigcup_{i\in I} B_{R\varepsilon_n^i}\big(x_n^i\big) \subset B_{2C\varepsilon_n^{j+1}}\big(x_n^{j+1}\big)$$

for $n \geqslant N = N(R)$. Similarly, there holds

$$\Big(\bigcup_{i=1,\dots,j,\ i\notin I} B_{R\varepsilon_n^i}\big(x_n^i\big)\Big) \cap B_{R\varepsilon_n^{j+1}}\big(x_n^{j+1}\big) = \emptyset$$

for $n \geqslant N = N(R)$. Hence, we have that for $R \geqslant 2C$ and $0 < \delta < C$

$$\begin{aligned}\bigcup_{i=1}^{j+1} B_{R\varepsilon_n^i}\big(x_n^i\big) &= B_{R\varepsilon_n^{j+1}}\big(x_n^{j+1}\big) \oplus \bigcup_{i=1,\dots,j,\ i\notin I} B_{R\varepsilon_n^i}\big(x_n^i\big)\\ &= \Big(B_{R\varepsilon_n^{j+1}}\big(x_n^{j+1}\big)\Big\backslash \bigcup_{i\in I} B_{\delta\varepsilon_n^{j+1}}\big(x_n^i\big)\Big) \oplus \bigcup_{i\in I} B_{\delta\varepsilon_n^{j+1}}\big(x_n^i\big) \oplus \bigcup_{i=1,\dots,j,\ i\notin I} B_{R\varepsilon_n^i}\big(x_n^i\big)\\ &\supset \Big(B_{R\varepsilon_n^{j+1}}\big(x_n^{j+1}\big)\Big\backslash \bigcup_{i\in I} B_{\delta\varepsilon_n^{j+1}}\big(x_n^i\big)\Big) \oplus \bigcup_{i=1}^{j} B_{R\varepsilon_n^i}\big(x_n^i\big)\end{aligned}$$

for $n \geqslant N = N(\delta, R)$ in view of (31), where $\oplus$ stands for the union of disjoint sets. Hence, we can write

$$\begin{aligned}\int_{\bigcup_{i=1}^{j+1} B_{R\varepsilon_n^i}(x_n^i)} u_n^{\frac{2N}{N-2}} &\geqslant \int_{B_{R\varepsilon_n^{j+1}}(x_n^{j+1})\backslash\bigcup_{i\in I} B_{\delta\varepsilon_n^{j+1}}(x_n^i)} u_n^{\frac{2N}{N-2}} + \int_{\bigcup_{i=1}^{j} B_{R\varepsilon_n^i}(x_n^i)} u_n^{\frac{2N}{N-2}}\\ &= \int_{B_R(0)\backslash\bigcup_{i\in I} B_\delta(\frac{x_n^i - x_n^{j+1}}{\varepsilon_n^{j+1}})} \big(U_n^{j+1}\big)^{\frac{2N}{N-2}} + \int_{\bigcup_{i=1}^{j} B_{R\varepsilon_n^i}(x_n^i)} u_n^{\frac{2N}{N-2}}\end{aligned}$$

for $n \geqslant N = N(\delta, R)$. Passing to the liminf as $n \to +\infty$, by the validity of (35), (32) for $j+1$ and $(x_n^i - x_n^{j+1})/\varepsilon_n^{j+1} \to y_i \in S_{j+1}$ as $n \to +\infty$ for any $i \in I$, we get that

$$\liminf_{n\to+\infty} \int_{\bigcup_{i=1}^{j+1} B_{R\varepsilon_n^i}(x_n^i)} u_n^{\frac{2N}{N-2}} \geqslant \int_{B_R(0)\backslash\bigcup_{i\in I} B_\delta(y_i)} \frac{\mathrm{d}y}{(1+|y-x_{j+1}|^2)^N} + \sum_{i=1}^{j} \int_{B_R(0)} \frac{\mathrm{d}y}{(1+|y-x_i|^2)^N}$$

for any R large and δ small. Letting $\delta \to 0^+$, we obtain that

$$\liminf_{n\to+\infty} \int_{\bigcup_{i=1}^{j+1} B_{R\varepsilon_n^i}(x_n^i)} u_n^{\frac{2N}{N-2}} \geqslant \sum_{i=1}^{j+1} \int_{B_R(0)} \frac{\mathrm{d}y}{(1+|y-x_i|^2)^N}. \tag{39}$$

So, the inductive step holds for $j+1$. Since $\sup_{n\in\mathbb{N}} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} < +\infty$, by (39) the induction process has to stop after $k-s$ steps giving that also (33) holds for the sequences $x_n^1, \dots, x_n^s, x_n^{s+1}, \dots, x_n^k$. Moreover, arguing as before, (39) for $k = j+1$ implies the validity of

$$\liminf_{n\to+\infty} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} \geqslant \sum_{i=1}^{k} \int_{B_R(0)} \frac{\mathrm{d}y}{(1+|y-x_i|^2)^N} + \int_{B_1(0)\backslash B_\delta(0)} u_0^{\frac{2N}{N-2}}$$

for any $\delta,\ R > 0$, and hence, taking $\delta \to 0$ and $R \to +\infty$,

$$\liminf_{n\to+\infty} \int_{B_1(0)} u_n^{\frac{2N}{N-2}} \geqslant k\Big(\frac{S_N}{N(N-2)}\Big)^{\frac{N}{2}} + \int_{B_1(0)} u_0^{\frac{2N}{N-2}}.$$

Hence, (34) holds and the proof is complete. □

Acknowledgements

I would like to thank Prof. Giovanni Mancini and Dr. Daniele Castorina which introduced me to the study of this topic. I am indebted to them for many stimulating discussions and useful comments during the completion of this work.

References

[1] Adimurthi, G. Mancini, Geometry and topology of the boundary in the critical Neumann problem, J. Reine Angew. Math. 456 (1994) 1–18.
[2] Adimurthi, G. Mancini, The Neumann problem for elliptic equations with critical nonlinearity, in: Nonlinear Analysis, in: Sc. Norm. Super. di Pisa Quaderni, Scuola Norm. Sup., Pisa, 1991, pp. 9–25.
[3] Adimurthi, G. Mancini, S.L. Yadava, The role of the mean curvature in semilinear Neumann problem involving critical exponent, Comm. Partial Differential Equations 20 (3–4) (1995) 591–631.
[4] A. Bahri, Critical Points at Infinity in Some Variational Problems, Pitman Research Notes in Mathematics Series, vol. 182, Longman Scientific & Technical, Harlow, 1989, copublished in the United States with John Wiley & Sons, Inc., New York.
[5] L.A. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (3) (1989) 271–297.
[6] D. Cao, E.S. Noussair, S. Yan, Existence and nonexistence of interior-peaked solution for a nonlinear Neumann problem, Pacific J. Math. 200 (1) (2001) 19–41.
[7] D. Castorina, G. Mancini, Non existence of bounded-energy solutions for some semilinear elliptic equations with a large parameter, Rend. Sem. Mat. Univ. Padova 110 (2003) 147–160.
[8] O. Druet, From one bubble to several bubbles: the low-dimensional case, J. Differential Geom. 63 (3) (2003) 399–473.
[9] O. Druet, Compactness for Yamabe metrics in low dimensions, Int. Math. Res. Not. 23 (2004) 1143–1191.
[10] O. Druet, E. Hebey, F. Robert, Blow-Up Theory for Elliptic PDEs in Riemannian Geometry, Mathematical Notes, vol. 45, Princeton University Press, Princeton, NJ, 2004.
[11] O. Druet, E. Hebey, F. Robert, A C^0-theory for the blow-up of second order elliptic equations of critical Sobolev growth, Electron. Res. Announc. Amer. Math. Soc. 9 (2003) 19–25 (electronic).
[12] O. Druet, E. Hebey, M. Vaugon, Pohozaev type obstructions and solutions of bounded energy for quasilinear elliptic equations with critical Sobolev growth. The conformally flat case, Nonlinear Anal. 51 (1) (2002) 79–94.
[13] N. Ghoussoub, C. Gui, M. Zhu, On a singularly perturbed Neumann problem with the critical exponent, Comm. Partial Differential Equations 26 (11–12) (2001) 1929–1946.
[14] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer-Verlag, 1983.
[15] C. Gui, C.S. Lin, Estimates for boundary-bubbling solutions to an elliptic Neumann problem, J. Reine Angew. Math. 546 (2002) 201–235.
[16] Y.Y. Li, Prescribing scalar curvature on S^n and related problems. I, J. Differential Equations 120 (2) (1995) 319–410.
[17] C.S. Lin, W.M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system, J. Differential Equations 72 (1) (1988) 1–27.
[18] H. Meinhardt, Models of Biological Pattern Formation, Academic Press, London, 1982.
[19] S.I. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6 (1965) 1408–1411. Translated from the Russ. Dokl. Acad. Nauk SSSR 165 (1965) 33–36.
[20] O. Rey, The question of interior blow-up points for an elliptic Neumann problem: the critical case, J. Math. Pures Appl. (9) 81 (7) (2002) 655–696.
[21] O. Rey, Boundary effect for an elliptic Neumann problem with critical nonlinearity, Comm. Partial Differential Equations 22 (7–8) (1997).
[22] R. Schoen, On the number of constant scalar curvature metrics in a conformal class, in: Differential Geometry, in: Pitman Monogr. Surveys Pure Appl. Math., vol. 52, Longman Sci. Tech., Harlow, 1991, pp. 311–320.