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Weighted Norm Inequality for a Maximal Operator on Homogeneous Space

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Abstract. Let $X = G/H$ be a homogeneous space, $\widetilde{X} = X \times [0, \infty)$, μ a doubling measure on X induced by a Haar measure on the group G, β a positive measure on $\widetilde{X}$ and W a weight on X. Consider the maximal operator given by

$$\mathcal{M}f(x,r) = \sup_{s \geq r} \frac{1}{\mu(B(x,s))} \int_{B(x,s)} |f(y)|\, d\mu(y), \quad (x,r) \in \widetilde{X}.$$

In this paper, we obtain, for each $p, q, 1 < p \leq q < \infty$, a necessary and sufficient condition for the boundedness of the maximal operator $\mathcal{M}$ from $L^p(X, W d\mu)$ to $L^q(\widetilde{X}, d\beta)$. As an application, we obtain a necessary and sufficient condition for the boundedness of the Poisson integral of functions defined on the unit sphere S^n of the Euclidian space $\mathbb{R}^{n+1}$, from $L^p(S^n, W d\sigma)$ to $L^q(\mathbb{B}, d\nu)$, where σ is the Lebesgue measure on S^n, W is a weight on S^n and ν is a positive measure on the unit ball $\mathbb{B}$ of $\mathbb{R}^{n+1}$.

Keywords. Maximal function, Poisson integral, homogeneous space, A_p-weights, sphere

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1. Introduction

Let G be a locally compact Hausdorff topological group with unit element e, H a compact subgroup of G and $\pi : G \mapsto G/H$ the canonical map. Let dg denote a left Haar measure on G, which we assume to be normalized in the case of G to be compact. If A is a Borel subset of G we will denote by $|A|$ the Haar measure of A. The homogeneous space $X = G/H$ is the set of all left cosets $\pi(g) = gH, g \in G$, provided with the quotient topology. The Haar measure dg

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induces a measure μ on the Borel σ-field on X. For $f \in L^1(X)$,

$$\int_X f(x)\, d\mu(x) = \int_G f \circ \pi(g)\, dg.$$

We observe that the group G acts transitively on X by the map

$$(g, \pi(h)) \mapsto g\pi(h) = \pi(gh),$$

that is, for all $x, y \in X$, there exists $g \in G$ such that $gx = y$. We also observe that the measure μ on X is invariable on the action of G, that is, if $f \in L^1(X)$, $g \in G$ and $R_g f(x) = f(g^{-1}x)$, then

$$\int_X f(x)\, d\mu(x) = \int_X R_g f(x)\, d\mu(x).$$

A *quasi-distance* on X is a map $d : X \times X \mapsto [0, \infty)$ satisfying:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(gx, gy) = d(x, y)$ for all $g \in G,\ x, y \in X$;

(iv) there exists a constant $K \geq 1$ such that, for all $x, y, z \in X$,

$$d(x, y) \leq K[d(x, z) + d(z, y)];$$

(v) the balls $B(x, r) = \{y \in X : d(x, y) < r\},\ x \in X,\ r > 0$, are relatively compact and measurable, and the balls $B(\mathbb{1}, r), r > 0$, form a basis of neighborhoods of $\mathbb{1} = \pi(e)$;

(vi) (*doubling condition*) there exists a constant $A \geq 1$ such that, for all $r > 0$ and $x \in X$,

$$\mu(B(x, 2r)) \leq A\mu(B(x, r)).$$

Given a quasi-distance d on X, there exists a distance ρ on X and a positive real number γ such that d is equivalent to ρ^γ (see [4]). Therefore the family of d-balls is equivalent to the family of ρ^γ-balls and ρ^γ-balls are open sets.

It follows by (iii) in the definition of a quasi-distance that $B(gx, r) = gB(x, r)$ for all $g \in G, x \in X$ and $r > 0$, and hence $\mu(B(gx, r)) = \mu(B(x, r))$. Thus we can write $X = \bigcup_{j \geq 1} g_j B(x, r)$ where (g_j) is a sequence of elements of G and consequently $\mu(B(x, r)) > 0$. In particular, X is separable.

In this paper X will denote a homogeneous space provided with a quasi-distance d. We also write $\widetilde{X} = X \times [0, \infty)$ and if $B = B(x, r)$ we write $\widetilde{B} = B(x, r) \times [0, r]$. We define the maximal operator $\mathcal{M}$ by

$$\mathcal{M}f(x, r) = \sup_{s \geq r} \frac{1}{\mu(B(x, s))} \int_{B(x,s)} |f(y)|\, d\mu(y)$$

for all real-valued locally integrable function f on X and $(x,r) \in \widetilde{X}$. If $r = 0$ the above supremum is taken over all $s > 0$ and $\mathcal{M}f(x,0) = f^*(x)$ is the Hardy-Littlewood maximal function.

A weight is a positive locally integrable function $W(x)$ on the space X and we will write $W(A) = \int_A W d\mu$. We write $L^p(W) = L^p(X, Wd\mu)$ and $L^p(X) = L^p(X, d\mu)$, $1 \le p \le \infty$. If $1 < p < \infty$, p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. Throughout this paper β will denote a positive measure on the Borel subsets of $\widetilde{X}$.

The next theorem, which is proved in Section 3, is the main result of this paper. It gives a necessary and sufficient condition for the boundedness of the maximal operator $\mathcal{M}$ from $L^p(X, Wd\mu)$ to $L^q(\widetilde{X}, d\beta)$, for $1 < p \le q < \infty$.

Theorem 1.1. *Let $1 < p \le q < \infty$ and let W be a weight on X such that $W^{1-p'}d\mu$ is a doubling measure on X. Then the following conditions are equivalent:*

(i) *There exists a constant $C > 0$, such that, for all $f \in L^p(W)$,*

$$\left(\int_{\widetilde{X}} [\mathcal{M}f(x,r)]^q \, d\beta(x,r) \right)^{\frac{1}{q}} \le C \left(\int_X |f(x)|^p W(x) \, d\mu(x) \right)^{\frac{1}{p}}.$$

(ii) *There exists a constant $C > 0$, such that, for all balls $B = B(z,t), 0 \le t < \infty$,*

$$\left(\int_{\widetilde{B}} [\mathcal{M}(W^{1-p'}\chi_B)(x,r)]^q \, d\beta(x,r) \right)^{\frac{1}{q}} \le C \left(\int_B W^{1-p'}(x) \, d\mu(x) \right)^{\frac{1}{p}}.$$

The above result for $X = G/H$, where G is a compact or an Abelian group and $p = q$, was proved in Bordin–Fernandes–Tozoni [1]. For $W \equiv 1$, the condition (ii) of Theorem 1.1 is given by

$$\left(\int_{\widetilde{B}} [\mathcal{M}(\chi_B)(x,r)]^q \, d\beta(x,r) \right)^{\frac{1}{q}} \le C\big(\mu(B)\big)^{\frac{1}{p}} \tag{1}$$

for all balls B. Let us fix $B = B(z,t), 0 < t < \infty$. Then, it follows as in the proof of inequality (7) of Lemma 3.1 that there exists a constant $C > 0$ such that $C \le \mathcal{M}(\chi_B)(x,r) \le 1$ for all $(x,r) \in \widetilde{B}$. Therefore, from (1) we obtain

$$\left(C^q \beta(\widetilde{B}) \right)^{\frac{1}{q}} \le \left(\int_{\widetilde{B}} [\mathcal{M}(\chi_B)(x,r)]^q \, d\beta(x,r) \right)^{\frac{1}{q}} \le C\big(\mu(B)\big)^{\frac{1}{p}}.$$

Then, the condition (1) implies the condition

$$\left(\beta(\widetilde{B}) \right)^{\frac{1}{q}} \le C\big(\mu(B)\big)^{\frac{1}{p}} \tag{2}$$

for a constant $C > 0$ and all balls B. But, from the condition (2) we obtain

$$\left(\int_{\widetilde{B}} [\mathcal{M}(\chi_B)(x,r)]^q \, d\beta(x,r)\right)^{\frac{1}{q}} \leq \left(\beta(\widetilde{B})\right)^{\frac{1}{q}} \leq (C\mu(B))^{\frac{1}{p}},$$

and therefore the conditions (1) and (2) are equivalent. The condition (2) is the Carleson condition for the homogeneous space X, when $p = q$ (see Ruiz–Torrea [5]).

Let $B = B(z,t), 0 < t < \infty$ and $\nu = W^{1-p'}$. Then $C\frac{\nu(B)}{\mu(B)} \leq \mathcal{M}(\nu\chi_B)(x,r)$ for all $(x,r) \in \widetilde{B}$. Therefore, from the condition (ii) of Theorem 1.1 we obtain

$$\begin{aligned}
\beta(\widetilde{B})^{\frac{1}{q}} &= \frac{\mu(B)}{\nu(B)} \left[\int_{\widetilde{B}} \left(\frac{\nu(B)}{\mu(B)}\right)^q d\beta(x,r)\right]^{\frac{1}{q}} \\
&\leq C\frac{\mu(B)}{\nu(B)} \left[\int_{\widetilde{B}} [\mathcal{M}(\nu\chi_B)(x,r)]^q \, d\beta(x,r)\right]^{\frac{1}{q}} \\
&\leq C'\frac{\mu(B)}{\nu(B)} \, \nu(B)^{\frac{1}{p}}.
\end{aligned}$$

Then, the condition (ii) of Theorem 1.1 implies the condition

$$\frac{\beta(\widetilde{B})^{\frac{1}{q}}}{\mu(B)} \left(\int_B W^{1-p'}(x) \, d\mu(x)\right)^{\frac{1}{p'}} \leq C \tag{3}$$

for a constant $C > 0$ and all balls B. It was proved in [5] that the condition (3) is a necessary and sufficient condition for $\mathcal{M}$ to be a bounded operator from $L^p(X, W d\mu)$ into weak-$L^q(\widetilde{X}, d\beta)$.

Now, if $x \in \mathbb{R}^{n+1}$, we write $|x| = (x \cdot x)^{\frac{1}{2}}$ and $d(x,y) = |x - y|$, where $x \cdot y$ is the usual scalar product of x and y in $\mathbb{R}^{n+1}$. Here S^n will denote the unit n-sphere $\{y \in \mathbb{R}^{n+1} : |y| = 1\}$ in $\mathbb{R}^{n+1}$, σ the normalized Lebesgue measure on S^n and $h : [1 - \sqrt{2}, 1] \to [0, 2]$ will be the function defined by $h(r) = \sqrt{2}(1 - r)$. The Poisson kernel for the sphere S^n is given by

$$P_{ry}(x) = \frac{1}{\omega_n} \frac{1 - r^2}{|ry - x|^{n+1}}$$

for $x, y \in S^n$ and $0 \leq r < 1$, where ω_n is the area of the sphere S^n. For a real-valued integrable function f we denote by $u_f(ry)$ the Poisson integral

$$u_f(ry) = \int_{S^n} P_{ry}(x) f(x) \, d\sigma(x).$$

and we define the maximal function u_f^* by

$$u_f^*(ry) = \sup_{0 \leq s \leq r} |u_f(sy)|, \quad 0 \leq r < 1, \ y \in S^n.$$

We identify $S^n \times [0,1]$ with the ball $\mathbb{B} = \{y \in \mathbb{R}^{n+1} : \ |y| \le 1\}$ using the application $(y,r) \mapsto ry$. If f is a real-valued and integrable function on S^n we define $\overline{\mathcal{M}}f(y) = \mathcal{M}f(y', h(|y|))$ for $y \in \mathbb{B},\ y \neq 0,\ y' = y/|y|$. In Bordin–Fernandes–Tozoni [1] it was proved that there exist positive constants C_1 and C_2 such that, for all $f \in L^1(S^n)$, we have

$$\begin{aligned} u_f^*(y) &\le C_1 \overline{\mathcal{M}}f(y), && y \in \mathbb{B},\ 0 < |y| < 1 \\ u_f^*(y) &\ge C_2 \overline{\mathcal{M}}f(y), \quad f \ge 0, && y \in \mathbb{B},\ 0 < |y| < 1. \end{aligned}$$

If B is the open ball $B(z,t) = \{x \in S^n : \ |x-z| < t\}$, $0 < t \le 2$, we define

$$\bar{B} = \begin{cases} \{sx \,:\, h^{-1}(t) \le s \le 1,\, x \in B\} & \text{if } \ 0 < t \le \sqrt{2} \\ \{sx \,:\, 0 \le s \le 1,\, x \in B\} & \text{if } \ \sqrt{2} \le t \le 2. \end{cases}$$

We observe that $\bar{B}$ is a truncated cone in the ball $\mathbb{B} = \{y \in \mathbb{R}^{n+1} : |y| \le 1\}$ in $\mathbb{R}^{n+1}$ if $0 < t \le \sqrt{2}$ and a cone if $\sqrt{2} \le t \le 2$.

The next theorem is consequence of the above inequalities between $u_f^*(y)$ and $\overline{\mathcal{M}}f(y)$ and of Theorem 1.1.

Theorem 1.2. *Let $1 < p \le q < \infty$, let W be a weight on S^n such that $W^{1-p'}d\sigma$ is a doubling measure on S^n and let ν be a Borel positive measure on $\mathbb{B}$. Then the following conditions are equivalent:*

(i) *There exists a constant $C > 0$, such that, for all $f \in L^p(W)$,*

$$\left(\int_{\mathbb{B}} [u_f^*(y)]^q \, d\nu(y) \right)^{\frac{1}{q}} \le C \left(\int_{S^n} |f(x)|^p W(x) \, d\sigma(x) \right)^{\frac{1}{p}}.$$

(ii) *There exists a constant $C > 0$, such that, for all balls $B = B(z,t)$, $0 < t \le 2$,*

$$\left(\int_{\bar{B}} [u^*_{W^{1-p'}\chi_B}(y)]^q \, d\nu(y) \right)^{\frac{1}{q}} \le C \left(\int_B W^{1-p'}(x) \, d\sigma(x) \right)^{\frac{1}{p}}.$$

The Theorem 1.2 for $p = q$ was obtained in [1] and, for $W \equiv 1$, $p = q$ and $n = 1$, it was proved by L. Carleson in [2].

2. A maximal operator of dyadic type

In this section we study a weighted norm inequality for a maximal operator of dyadic type which we apply in the proof of Theorem 1.1, given in Section 3.

The following lemma presents the dyadic partitions for homogeneous spaces introduced by E. Sawyer and R. L. Wheeden in [6].

Lemma 2.1 ([6, Lemma 3.21]). *Let b be a positive integer and let $\lambda = 8K^5$. Then for each integer k, $-b \le k \le b$, there exists an enumerable Borel partition $\mathcal{A}_k^b$ of X, which elements will be called dyadic elements of X, and a positive constant C depending only on X, such that*

(i) *for all $Q \in \mathcal{A}_k^b, -b \le k \le b$, there exists $x_Q \in Q$ such that $B(x_Q, \lambda^k) \subset Q \subset B(x_Q, \lambda^{k+1})$ and $\mu(B(x_Q, \lambda^{k+1})) \le C\mu(Q)$;*

(ii) *if $-b \le k < b$, $Q_1 \in \mathcal{A}_{k+1}^b$, $Q_2 \in \mathcal{A}_k^b$ and $Q_1 \cap Q_2 \ne \emptyset$, then $Q_2 \subset Q_1$, and $0 < \mu(Q_1) \le C\mu(Q_2)$;*

Let b be a fixed positive integer. Given $Q \in \mathcal{A}^b = \bigcup_{-b\le k\le b} \mathcal{A}_k^b$, where $\mathcal{A}_k^b$ are the partitions of X in Lemma 2.1, $\widetilde{Q}$ will denote the subset $Q \times [0, \alpha^{-1}(\mu(Q))]$ of $\widetilde{X} = X \times [0,\infty)$, where $\alpha : [0,\infty) \to [0,\infty)$ is the function defined by $\alpha(r) = \mu(B(\mathbb{1}, r))$, $\mathbb{1} = \pi(e)$.

If f is a real-valued locally integrable function on X, we define, for each $(x,r) \in \widetilde{X}$,

$$\mathcal{M}_d^b f(x,r) = \sup_{\substack{x\in Q\in\mathcal{A}^b \\ \mu(Q)\ge\alpha(r)}} \frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y).$$

If $\mu(Q) < \alpha(r)$ for all $Q \in \mathcal{A}^b$ such that $x \in Q$, we define $\mathcal{M}_d^b f(x,r) = 0$.

Lemma 2.2 ([8, Lemma 3.8]). *Let V be a weight on X and let $1 < p \le q < \infty$. For each positive integer b, let $\{Q_i\}_{i\in I}$ be a enumerable collection of dyadic elements of $\mathcal{A}^b$, and for each $u \in G$, let $\{a_i(u)\}_{i\in I}$ and $\{b_i(u)\}_{i\in I}$ be positive numbers satisfying*

$$\int_{uQ_i} V(x)\, d\mu(x) \le Ca_i(u), \qquad i \in I,\ u \in G \tag{4}$$

$$\sum_{j:Q_j\subset Q_i} b_j(u) \le C\big(a_i(u)\big)^{\frac{q}{p}}, \quad i \in I,\ u \in G, \tag{5}$$

where C is a positive constant independent of b. Then

$$\left(\sum_{i\in I} b_i(u) \left(\frac{1}{a_i(u)} \int_{uQ_i} g(x)V(x)\, d\mu(x)\right)^q\right)^{\frac{1}{q}} \le C_{p,q} \left(\int_X (g(x))^p V(x)\, d\mu(x)\right)^{\frac{1}{p}}$$

for all positive measurable function g on X and $u \in G$, where $C_{p,q}$ is independent of u, b and g.

Theorem 2.3. *Given a weight W on X, a positive measure β on $\widetilde{X}$, and $1 < p \le q < \infty$, the following conditions are equivalent:*

(i) *There exists a constant $C > 0$, such that, for all $f \in L^p(W)$ and all positive integer b,*

$$\left(\int_{\widetilde{X}} [\mathcal{M}_d^b f(x,r)]^q\, d\beta(x,r)\right)^{\frac{1}{q}} \le C \left(\int_X |f(x)|^p W(x)\, d\mu(x)\right)^{\frac{1}{p}}.$$

(ii) *There exists a constant $C > 0$, such that, for all $Q \in \mathcal{A}^b$ and all positive integer b,*

$$\left(\int_{\widetilde{Q}} [\mathcal{M}_d^b(W^{1-p'}\chi_Q)(x,r)]^q \, d\beta(x,r)\right)^{\frac{1}{q}} \leq C \left(\int_Q W^{1-p'}(x)\, d\mu(x)\right)^{\frac{1}{p}}.$$

Proof. To prove (i) $\Rightarrow$ (ii) it is sufficient to choose $f(x) = W^{1-p'}\chi_Q$ in (i), where $Q \in \mathcal{A}^b$ and b is a positive integer.

Proof of (ii) $\Rightarrow$ (i): Let us fix a positive integer b, $f \in L^p(W)$ and for each $k \in \mathbb{Z}$, let Ω_k be the set $\Omega_k = \{(x,r) \in \widetilde{X} : \mathcal{M}_d^b(f\sigma)(x,r) > 2^k\}$, where $\sigma = W^{1-p'}$. For each $k \in \mathbb{Z}$, we denote by C_k^0 the family formed by all $Q \in \mathcal{A}^b$ such that $\frac{1}{\mu(Q)}\int_Q |f(y)|\sigma(y)d\mu(y) > 2^k$. Since for every $Q \in \mathcal{A}_k^b$, $-b \leq k < b$, there exists $Q' \in \mathcal{A}_{k+1}^b$ such that $Q \subset Q'$, then every element $Q \in C_k^0$ is contained in a maximal element $Q' \in C_k^0$. We denote by C_k the family $\{Q_j^k : j \in J_k\}$ formed by all maximal elements $Q \in C_k^0$. Since $\mathcal{A}_k^b$ is a partition of X and all elements of C_k are maximal, we can conclude that the sets $Q_j^k, j \in J_k$, are pairwise disjoint. Therefore the sets $\widetilde{Q}_j^k, j \in J_k$, are also pairwise disjoint and $\Omega_k = \bigcup_{j\in J_k} \widetilde{Q}_j^k$.

Now, for each $k \in \mathbb{Z}$ and each $j \in J_k$, let $E_j^k = \widetilde{Q}_j^k \setminus \Omega_{k+1}$. Then the sets E_j^k and E_i^h are disjoint for $(k,j) \neq (h,i)$ and

$$\{(x,r) : \mathcal{M}_d^b(f\sigma)(x,r) > 0\} = \bigcup_{k\in\mathbb{Z}} (\Omega_k \setminus \Omega_{k+1}) = \bigcup_{k\in\mathbb{Z}} \bigcup_{j\in J_k} E_j^k.$$

Therefore

$$\begin{aligned}
&\int_{\widetilde{X}} [\mathcal{M}_d^b(f\sigma)(x,r)]^q \, d\beta(x,r) \\
&\quad = \sum_{k,j} \int_{E_j^k} [\mathcal{M}_d^b(f\sigma)(x,r)]^q \, d\beta(x,r) \\
&\quad \leq 2^q \sum_{k,j} \beta(E_j^k)(2^k)^q \\
&\quad \leq 2^q \sum_{k,j} \beta(E_j^k)\left(\frac{1}{\mu(Q_j^k)}\int_{Q_j^k} |f(x)|\sigma(x)\, d\mu(x)\right)^q \\
&\quad = 2^q \sum_{k,j} \beta(E_j^k)\left(\frac{1}{\mu(Q_j^k)}\int_{Q_j^k} \sigma(x)\, d\mu(x)\right)^q \left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} |f(x)|\sigma(x)\, d\mu(x)\right)^q \\
&\quad \leq 2^q \sum_{k,j} \left(\int_{E_j^k} [\mathcal{M}_d^b(\sigma\chi_{Q_j^k})(x,t)]^q \, d\beta(x,t)\right)\left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} |f(x)|\sigma(x)\, d\mu(x)\right)^q \\
&\quad = 2^q \sum_{k,j} b_j^k \left(\frac{1}{a_j^k}\int_{Q_j^k} |f(x)|\sigma(x)\, d\mu(x)\right)^q, \qquad (6)
\end{aligned}$$

where $b_j^k = \int_{E_j^k} [\mathcal{M}_d^b(\sigma\chi_{Q_j^k})(x,t)]^q \, d\beta(x,t)$ and $a_j^k = \sigma(Q_j^k)$.

We will show that

$$\sum_{\substack{k,j \\ Q_j^k \subset Q_0}} b_j^k \leq C(a_0)^{\frac{q}{p}}$$

for all $Q_0 \in \mathcal{A}^b$, where b is a positive integer and $a_0 = \sigma(Q_0)$. Since the $\widetilde{Q}_j^k$'s are disjoint in k and j, then using (ii), we have

$$\sum_{\substack{k,j \\ Q_j^k \subset Q_0}} b_j^k \leq \int_{\widetilde{Q}_0} [\mathcal{M}_d^b(\sigma\chi_{Q_0})(x,t)]^q \, d\beta(x,t) \leq C \left(\int_{Q_0} W^{1-p'}(x) \, d\mu(x) \right)^{\frac{q}{p}} = C(a_0)^{\frac{q}{p}}.$$

Therefore all conditions of Lemma 2.2 are satisfied for the sequences $\{a_j^k\}$ and $\{b_j^k\}$, with $V = \sigma$. Then, by Lemma 2.2 and by (6) we obtain

$$\left(\int_{\widetilde{X}} [\mathcal{M}_d^b f(x,t)]^q \, d\beta(x,t) \right)^{\frac{1}{q}} \leq C_{p,q} \left(\int_X |f(x)|^p W(x) \, d\mu(x) \right)^{\frac{1}{p}}. \qquad \square$$

Remark 2.4. Let us fix $g \in G$ and let$g^{-1}\mathcal{A}_k^b = \{g^{-1}Q : Q \in \mathcal{A}_k^b\}$, $g^{-1}\mathcal{A}^b = \{g^{-1}Q : Q \in \mathcal{A}^b\}$. Then for each $-b \leq k \leq b$, $g^{-1}\mathcal{A}_k^b$ is a partition of X and Lemma 2.1 also holds, with the same constant C, when we change $\mathcal{A}_k^b$ for $g^{-1}\mathcal{A}_k^b$. If f is a real-valued locally integrable function on X, we define

$$\mathcal{M}_d^{b,g} f(x,r) = \sup_{\substack{x \in Q \in g^{-1}\mathcal{A}^b \\ \mu(Q) \geq \alpha(r)}} \frac{1}{\mu(Q)} \int_Q |f(y)| \, d\mu(y).$$

Then $\mathcal{M}_d^b(R_g f)(gx,r) = \mathcal{M}_d^{b,g} f(x,r)$ where $R_g f(x) = f(g^{-1}x)$. Theorem 2.3 also holds, with the same proof, when we change the operator $\mathcal{M}_d^b$ for $\mathcal{M}_d^{b,g}$ and the family $\mathcal{A}^b$ for $g^{-1}\mathcal{A}^b$.

3. The boundedness of the operator $\mathcal{M}$

Given a positive integer b and a real-valued locally integrable function f on X, we define for $(x,r) \in \widetilde{X}$,

$$\mathcal{M}^b f(x,r) = \sup_{\max\{\lambda^{-b-1}, r\} \leq s \leq \lambda^b} \frac{1}{\mu(B(x,s))} \int_{B(x,s)} |f(y)| \, d\mu(y).$$

We define $\mathcal{M}^b f(x,r) = 0$ if $r > \lambda^b$ and we observe that $\mathcal{M}^b f(x,r) \uparrow \mathcal{M} f(x,r)$ if $b \uparrow \infty$ for all $(x,r) \in \widetilde{X}$.

Lemma 3.1. *Let $\mathcal{M}_d^{b,g}$ be the maximal operator defined in Remark 2.4. Then there exists a positive constant C such that, for all positive integer b, $g \in G$, all real-valued locally integrable function f on X and $(x,r) \in \widetilde{X}$*

$$\mathcal{M}_d^{b,g} f(x,r) \leq C \mathcal{M} f(x,r). \tag{7}$$

Proof. Let us fix $(x,r) \in \widetilde{X}$ and $g \in G$. If $\mu(Q) < \alpha(r)$ for all $Q \in \mathcal{A}^b$ such that $x \in g^{-1}Q$, we have $\mathcal{M}_d^{b,g} f(x,r) = 0$. Thus to prove (7), it is enough to consider $Q \in \mathcal{A}_k^b$, $-b \leq k \leq b$, such that $x \in g^{-1}Q$ and $\mu(Q) \geq \alpha(r)$. By Lemma 2.1(i) there exist $x_Q \in Q$ such that $Q \subset B(x_Q, \lambda^{k+1})$ and $\mu(B(x_Q, \lambda^{k+1})) \leq C\mu(Q)$. For $t = 2K\lambda^{k+1}$ we have $B(g^{-1}x_Q, \lambda^{k+1}) \subset B(x,t)$ and hence

$$\alpha(t) = \mu(B(x,t)) \geq \mu(B(g^{-1}x_Q, \lambda^{k+1})) \geq \mu(Q) \geq \alpha(r).$$

If $2^{a-1} < K \leq 2^a$, it follows by (vi) of the definition of quasi-distance that

$$\mu(B(x,t)) \leq A^{a+1}\mu(B(x_Q, \lambda^{k+1})) \leq A^{a+1}C\mu(g^{-1}Q).$$

Therefore

$$\frac{1}{\mu(g^{-1}Q)} \int_{g^{-1}Q} |f(y)|\, d\mu(y) \leq \frac{A^{a+1}C}{\mu(B(x,t))} \int_{B(x,t)} |f(y)|\, d\mu(y) \leq A^{a+1}C\mathcal{M}f(x,r)$$

and hence we obtain (7). □

Lemma 3.2. *Let b be a positive integer. Then there exists a constant C, depending only on X, such that, for all real-valued locally integrable function f on X and all $(x,r) \in B(\mathbb{1}, \lambda^b) \times [0, \lambda^b]$, we have*

$$\mathcal{M}^b f(x,r) \leq \frac{C}{|\mathcal{G}_b|} \int_{\mathcal{G}_b} \mathcal{M}_d^{b,g} f(x,r)\, dg, \tag{8}$$

where $\mathcal{G}_b = \{g \in G : d(g\mathbb{1}, \mathbb{1}) < \lambda^{b+3}\}$.

Proof. First we observe that $|\mathcal{G}_b| = \mu(B(\mathbb{1}, \lambda^{b+3})) > 0$. Let $(x,r) \in B(\mathbb{1}, \lambda^b) \times[0, \lambda^b]$. Then there exists a ball $B = B(x,s)$, where $\max\{\lambda^{-b-1}, r\} \leq s \leq \lambda^b$, such that

$$\mathcal{M}^b f(x,r) \leq \frac{2}{\mu(B)} \int_B |f(y)|\, d\mu(y). \tag{9}$$

Let $-b \leq k \leq b$ such that $\lambda^{k-1} \leq s < \lambda^k$. If $s = \lambda^b$, we take $k = b$. We denote by Ω the set $\Omega = \left\{g \in \mathcal{G}_b : \text{ there exists } Q \in \mathcal{A}_{k+1}^b \text{ such that } B \subset g^{-1}Q\right\}$. Given $g \in \Omega$ let $Q \in \mathcal{A}_{k+1}^b$ such that $B \subset g^{-1}Q$. By Lemma 2.1(i) there exists $x_Q \in Q$ such that $B(x_Q, \lambda^{k+1}) \subset Q \subset B(x_Q, \lambda^{k+2})$ and hence $B \subset g^{-1}Q \subset B(g^{-1}x_Q, \lambda^{k+2})$. If l is the integer such that $2^{l-1} < \lambda^3 \leq 2^l$, then by the doubling condition we have $\mu(g^{-1}Q) \leq \mu(B(g^{-1}x_Q, \lambda^{k+2})) \leq A^l \mu(B)$ and thus

$$\frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y) \leq \frac{A^l}{\mu(g^{-1}Q)} \int_{g^{-1}Q} |f(y)|\, d\mu(y).$$

Therefore, since $x \in B = B(x,s) \subset g^{-1}Q$ and $\mu(g^{-1}Q) \geq \alpha(r)$, from (9) we get $\mathcal{M}^b f(x,r) \leq 2A^l M_d^{b,g} f(x,r)$, $g \in \Omega$. Now suppose that there exists a positive constant δ such that $|\Omega| \geq \delta|\mathcal{G}_b|$ for all positive integers b. Then integrating both sides of the above inequality with respect to the Haar measure dg and on Ω, we get (8) for $C = 2A^l\delta^{-1}$.

Now we will prove that there exists a positive constant δ, depending only on X, such that $|\Omega| \geq \delta|\mathcal{G}_b|$. Given $y \in X$ we denote by g_y an element in G such that $y = g_y \mathbb{1}$. Let $z \in g_{x_Q}\mathcal{G}_{k-3}g_x^{-1}$. Then $zx \in B(x_Q, \lambda^k)$ and hence for $y \in B$,

$$d(zy, x_Q) \leq K[d(zy, zx) + d(zx, x_Q)] \leq K[d(y, x) + \lambda^k] \leq \lambda^{k+1}.$$

Therefore $y \in z^{-1}Q$ and hence

$$B \subset z^{-1}Q, \quad z \in g_{x_Q}\mathcal{G}_{k-3}g_x^{-1}. \tag{10}$$

Let us denote by Γ the set $\Gamma = \left\{Q \in \mathcal{A}^b_{k+1} : Q \cap B(x, \lambda^{b+2}) \neq \emptyset\right\}$. Fix $Q \in \Gamma$ and let $u \in Q \cap B(x, \lambda^{b+2})$, $g \in g_{x_Q}\mathcal{G}_{k-3}$. Then $g\mathbb{1} \in B(x_Q, \lambda^k)$ and

$$\begin{aligned} d(g\mathbb{1}, \mathbb{1}) &\leq K[d(g\mathbb{1}, x_Q) + d(x_Q, \mathbb{1})] \\ &\leq K[\lambda^k + K[d(x_Q, u) + d(u, \mathbb{1})]] \\ &\leq K[\lambda^k + K[\lambda^{k+2} + K[d(u, x) + d(x, \mathbb{1})]]] \\ &\leq 4K^3\lambda^{b+2} \end{aligned}$$

and hence

$$d(gg_x^{-1}\mathbb{1}, \mathbb{1}) \leq K[d(g_x g^{-1}\mathbb{1}, g_x\mathbb{1}) + d(x, \mathbb{1})] \leq K[d(g\mathbb{1}, \mathbb{1}) + \lambda^b] < \lambda^{b+3}.$$

Thus $g \in \mathcal{G}_b g_x$ and hence $g_{x_Q}\mathcal{G}_{k-3}g_x^{-1} \subset \mathcal{G}_b$, $Q \in \Gamma$. Therefore from (10)

$$\bigcup_{Q\in\Gamma} g_{x_Q}\mathcal{G}_{k-3}g_x^{-1} \subset \Omega. \tag{11}$$

If $Q, Q' \in \mathcal{A}^b_{k+1}$ and $Q \neq Q'$ then $B(x_Q, \lambda^k) \cap B(x_{Q'}, \lambda^k) = \emptyset$ and hence $g_{x_Q}\mathcal{G}_{k-3}g_x^{-1} \cap g_{x_{Q'}}\mathcal{G}_{k-3}g_x^{-1} = \emptyset$. Then, since G is unimodular (see [3, p. 578]), it follows by (11) and by the doubling condition that

$$\begin{aligned} |\Omega| &\geq \left|\bigcup_{Q\in\Gamma} g_{x_Q}\mathcal{G}_{k-3}g_x^{-1}\right| \\ &= \sum_{Q\in\Gamma} |g_{x_Q}\mathcal{G}_{k-3}| \\ &\geq \sum_{Q\in\Gamma} A^{-l}\mu\big(B(x_Q, \lambda^{k+2})\big) \\ &\geq A^{-l}\mu\Big(\bigcap_{Q\in\Gamma} Q\Big) \\ &\geq A^{-l}\mu\big(B(x, \lambda^{b+2})\big) \\ &\geq A^{-2l}|\mathcal{G}_b|. \end{aligned}$$

□

Proof of Theorem 1.1. To prove the implication (i) $\Rightarrow$ (ii), it is sufficient to choose $f(x) = W^{1-p'}(x)\chi_B(x)$ in the hypothesis.

Let us prove (ii) $\Rightarrow$ (i). We fix a positive integer b, $g \in G$ and $Q \in \mathcal{A}_k^b$, $-b \le k \le b$. Then, by Lemma 2.1(i) there exists $x_Q \in Q$ such that $B(x_Q, \lambda^k) \subset Q \subset B(x_Q, \lambda^{k+1})$. We write $B = B(g^{-1}x_Q, \lambda^{k+1})$, $Q' = g^{-1}Q$, $\nu = W^{1-p'}$ and let a be a positive integer such that $2^{a-1} < \lambda \le 2^a$. Since $\nu d\mu$ is a doubling measure, there exists a positive constant C_ν such that

$$\nu(B) \le \nu(B(g^{-1}x_Q, 2^a\lambda^k)) \le C_\nu^a\, \nu(B(g^{-1}x_Q, \lambda^k)) \le C_1\nu(Q').$$

Then by the hypothesis and (7) we obtain

$$\begin{aligned}\left(\int_{\widetilde{Q'}} [\mathcal{M}_d^{b,g}(W^{1-p'}\chi_{Q'})(x,r)]^q\, d\beta(x,r)\right)^{\frac{1}{q}} &\le C_2\left(\int_{\widetilde{B}} [\mathcal{M}(\nu\chi_B)(x,r)]^q\, d\beta(x,r)\right)^{\frac{1}{q}}\\ &\le C_3\big(\nu(B)\big)^{\frac{1}{p}}\\ &\le C_4\left(\int_{Q'} W^{1-p'}(x)\, d\mu(x)\right)^{\frac{1}{p}}.\end{aligned}$$

Since the constant C_4 depends only on p, W and β, then by Theorem 2.3 and by Remark 2.4, there exists a constant C_5 such that,

$$\left(\int_{\widetilde{X}} [\mathcal{M}_d^{b,g} f(x,r)]^q\, d\beta(x,r)\right)^{\frac{1}{q}} \le C_5\left(\int_X |f(x)|^p W(x)\, d\mu(x)\right)^{\frac{1}{p}} \tag{12}$$

for all $f \in L^p(W)$ and all $g \in G$. Then, it follows by Lemma 3.2, (12) and by Jensen's inequality that

$$\begin{aligned}\left(\int_{B(\mathbb{1},\lambda^b)\times[0,\lambda^b]} [\mathcal{M}^b f(x,r)]^q d\beta(x,r)\right)^{\frac{1}{q}} &\le \left(\int_{\widetilde{X}} \left(\frac{C}{|\mathcal{G}_b|}\int_{\mathcal{G}_b} \mathcal{M}_d^{b,g} f(x,r)dg\right)^q d\beta(x,r)\right)^{\frac{1}{q}}\\ &\le C\left(\int_{\mathcal{G}_b}\int_{\widetilde{X}} [\mathcal{M}_d^{b,g} f(x,r)]^q d\beta(x,r)\frac{dg}{|\mathcal{G}_b|}\right)^{\frac{1}{q}}\\ &\le C\cdot C_5\left(\int_X |f(x)|^p W(x)\, d\mu(x)\right)^{\frac{1}{p}}.\end{aligned}$$

Letting $b \to \infty$ in the above inequality we obtain (i). □

Remark 3.3. Suppose that W is a weight on X, that β is a positive measure on $\widetilde{X}$ and that $1 < p \le q < \infty$. Now consider the following condition:

(iii) There exists a constant $C > 0$, such that, for all positive integer b, all $g \in G$ and all $Q \in g\mathcal{A}^b$,

$$\left(\int_{\widetilde{Q}} [\mathcal{M}(W^{1-p'}\chi_Q)(x,r)]^q\, d\beta(x,r)\right)^{\frac{1}{q}} \le C\left(\int_Q W^{1-p'}(x)\, d\mu(x)\right)^{\frac{1}{p}}.$$

It follows from the proof of Theorem 1.1 that the condition (iii) is a necessary and sufficient condition for the boundedness of the operator $\mathcal{M}$ as in Theorem 1.1 (i), without assuming that $W^{1-p'}d\mu$ is a doubling measure. Conditions of the type of (iii), that is, where an inequality must be satisfied for all dyadic elements of X, for others operators, can be found in [7].

An analogous of Theorem 1.1 for fractional maximal operators, when X has a group structure and $W^{1-p'}d\mu$ is a doubling measure, is the Theorem 4 in [8].

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